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Abstract

We study a model of consumption choice and portfolio allocation that captures, in two different interpretations, the combined effect of local substitution and habit formation and the combined effect of durability of consumption goods and habit formation over service flows from those goods. In a third interpretation, the model captures the idea of a dual purpose commodity. The optimal allocation problem is from the class of free boundary singular control problems. We discuss, formally, necessary and sufficient conditions for a consumption and portfolio policy to be optimal. We also introduce a numerical technique based on approximating the original program by a sequence of discrete parameter Markov chain control problems. We provide convergence results of the value function, the optimal investment policy, and the optimal consumption regions in the approximating discrete control problems to those in the original continuous time dynamic program. We construct numerically the consumption boundary that divides the state space into two regions— one of immediate consumption and the other of abstinence. We show that both the wealth required to start consuming and the optimal fraction of wealth invested in the risky asset are cyclical functions in both the stock of the durable good and the standard of living. This is due to the interaction between the durability and habit formation effects. We also study the effect of the cyclical investment behavior on the equilibrium risk premium in a representative consumer economy.

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1 Introduction and Summary

We study the problem of optimal consumption and portfolio rules for an agent whose preferences over consumption are given as

$$(1) \quad \mathbf{E}\Big[\int_0^\infty e^{-\delta t} u(z(t), y(t))\, dt\Big],$$

where

$$(2) \quad z(t) = z(0^-)e^{-\beta t} + \beta \int_{0^-}^t e^{-\beta(t-s)}\, dC(s) \quad \text{and}$$

$$(3) \quad y(t) = y(0^-)e^{-\lambda t} + \lambda \int_{0^-}^t e^{-\lambda(t-s)}\, dC(s).$$

In this formulation, the consumption process $C(t)$ denotes the *total* amount of consumption till time t. The processes $z(t)$ and $y(t)$ are derived from consumption using the weighting factors β and λ, respectively, with $\beta > \lambda$. Both $z(0^-) \geq 0$ and $y(0^-) \geq 0$ are given constants and $\delta > 0$ captures the impatience of the agent. The felicity function u is continuous and strictly increasing. Furthermore, u is strictly concave and has the property that $u_{12} > 0$, where u_{12} is the second cross partial derivative of u with respect to z and y.

We entertain three different economic ideas in three different interpretations of the model specified in (1), (2), and (3). In one interpretation, preferences given by (1) exhibit the notions of local substitution and habit formation. Agents with such preferences treat consumptions at adjacent dates as close substitutes and consumptions at distant dates as complements. In a second interpretation, the model represents habit forming preferences over the service flows from irreversible purchases of a durable good that decays over time. In the third interpretation, the model represents preferences for consumption of a dual purpose commodity that provides the agent with two sources of utility. The two components of such a composite good, however, have different half-lives.

Local substitution in continuous time, analyzed in Hindy, Huang, and Kreps (1992), Hindy and Huang (1992) and Heaton (1993), is the notion that consumption at one time reduces marginal utility at nearby times. Habit formation, studied since Marshall (1920), is the notion that agents develop tastes because of past consumption experience. Specifically, a high standard of living in the past increases the appetite of the agent for current consumption. Habit formation has been studied by many authors. Ryder and Heal (1973) introduced the notion of adjacent versus distant complementariness. Stigler and Becker (1977) emphasize the importance of analyzing the endogenous development of preferences in the search for factors that explain differences in tastes. Abel (1990), Constantinides (1990), DeTemple and Zapatero (1991), Heaton (1993), and Sundaresan (1989) study habit formation models.

In this paper, we study preferences that exhibit a combination of local substitution and habit formation. The distinguishing feature of the preference specification in (1) is that the felicity function u depends only on exponentially weighted averages of past consumption. In particular, the current consumption rate does not appear directly in the felicity function. This feature, which is absent in almost all non–time–additive preferences in the literature, is the key to representing the notion of local substitution. For the details, we refer the reader to Hindy, Huang, and Kreps (1992) and Hindy and Huang (1992).

Preferences of representative consumers have been used, in general equilibrium models, to explain the behavior of the returns on financial assets. Recent empirical studies of Eichenbaum and Hansen (1990), Gallant and Tauchen (1989), and Heaton (1993) suggest that there is substitution over short periods and habit formation over long periods. In particular, Heaton (1993) showed, after correcting for the problem of temporal aggregation, that habit formation alone does not provide significant explanation power for asset pricing over the time–additive models; while local substitution, or durability, does. In addition, given local substitution, or durability, habit formation over long horizons becomes more significant in its explanatory power. These results imply that a model which combines local substitution and habit formation may better explain the behavior of security returns. The analysis of the optimization problem for a single consumer is an important step in that direction.

In the durable good version of the model, we identify $C(t)$ with the total purchases of a durable good till time t. Purchases of the good are irreversible either because there is no secondary market for the good, or because of prohibitively high selling costs. We invite the reader to think of such durables as grocery, cloth, small appliances, and, even, vacations. The good provides a flow of services proportional to the stock $z(t)$ of the good at time t. Absent any new purchases, the service flow declines over time because of the deterioration in the stock of the durable good. The standard of living of the agent is given by $y(t)$ and reflects past consumption experience. We take it that the standard of living deteriorates at a much slower rate than the stock of the durable. This assumption is reflected in the restriction that $\beta > \lambda$. Discrete time versions of this model have been studied by Dunn and Singleton (1986), Eichenbaum, Hansen and Singleton (1988) and Hotz, Kydland and Sedlacek (1988), among others.

In the third interpretation of the model, we take $C(t)$ to be the total purchases, till time t, of a composite commodity. This commodity provides the agent with two sources of utility. For example, consider food that provides "calories" and "vitamins", or cloth that provides "shelter" and "style". The two components of the dual purpose commodity are captured by $z(t)$ and $y(t)$. The half-lives of the two components are different. Furthermore, the two sources of satisfaction are complementary. For example, a higher level of energy increases the marginal utility of vitamins. This feature is captured by the assumption that $u_{12} > 0$. In the balance of the paper we will use the terminology "consumption" and "purchases of the durable good" interchangeably.

The specification of preferences in (1) leads to very interesting behavior as well as a technically challenging optimization problem. From the analysis in Hindy and Huang (1993), who study a model without habit formation, it is known that the optimization problem specified here belongs to the class of free boundary singular control problems. The main feature of the problem is that consumption occurs periodically. The agent consumes only when the marginal utility of wealth is equal to a linear combination of the marginal utility of the stock of the durable good and the marginal utility of the standard of living. This condition is satisfied only for a particular combination of wealth, stock of the durable, and standard of living. Searching for this combination, or free boundary, is the essence of solving the utility maximization problem.

We first discuss, formally, necessary and sufficient conditions for a consumption–investment policy to be optimal. In particular, we provide a verification theorem that shows that the value function — the maximum attainable utility starting from any initial position — satisfies a differential inequality. Such differential inequalities are the hallmark of free boundary control problems. The verification theorems we provide require that the value function be twice continuously differentiable in W and once continuously differentiable in z and y.

In general, it is difficult to ascertain the smoothness of the value function. In Hindy and Huang (1993) such smoothness could be established because there is a closed–form expression for the value function. In the problem we study in this paper, such a closed–form solution is not currently available. For this reason, our numerical analysis relies on a weaker notion for the solution of the dynamic programming equation. Such notion, known as viscosity solution, is described in details in a companion paper by Hindy, Huang, and Zhu (1993). We record the verification theorems for smooth value functions for two reasons. First, the discussion is useful for conveying the economic intuition of the optimal solution. Second, the verification theorems can be utilized in the future if a closed–form solution is obtained.

We describe a numerical approach for solving the utility maximization problem. The idea of the numerical approach is to approximate the controlled processes, which are continuous-time diffusions, by appropriately chosen Markov chains on a finite state space. In addition, we approximate the utility function in (1) by one which is appropriate for the Markov chain. The approximating Markov chains are chosen to satisfy a "local consistency" condition. This condition is, roughly, that the conditional expected change in the Markov chain and its conditional variance locally match the drift and variance of the original controlled process.

We approximate the decision problem we study here by a sequence of Markov chain control problems. Each Markov chain control problem in the sequence is readily solvable numerically. We then show that the value functions in the sequence of approximating control problems converge to the value function of the original controlled diffusion problem. We remark that existence of the value function and the feasibility of computing its values with high accuracy do not guarantee existence of optimal policies that achieve the value function. Contrast this with classical feedback control case. In that case, optimal controls can be expressed as explicit functions of the value function and its derivatives. As a consequence, existence of a smooth value function immediately implies existence of optimal controls. In our case, with a free–boundary problem, there is no direct relationship between the value function and the optimal controls, which, if exist, are not feedback.

We, hence, show that the sequence of optimal investment policies and optimal consumption regions in the approximating Markov chain problems converge to the optimal solutions in the original problem, if the latter exist. In this paper, we only sketch without proof the numerical technique and the convergence results. The companion paper, Hindy, Huang, and Zhu (1993) provides the details and proofs.

We provide numerical solutions for the infinite horizon maximization problem when the price of the risky assets follow geometric Brownian Motion. We analyze a felicity function $u(z, y, t)$ of the multiplicative form $e^{-\delta t} z^{\alpha_1} y^{\alpha_2}$ for some constants δ, α_1, and α_2. Given the convergence results, all the numerical solutions we discuss here can be made arbitrarily close to the optimal solution of the original problem. In particular, the regions of optimal consumption and the optimal investment policies we report are very good approximations of the optimal policies in the continuous time problem, if the latter exist. For ease of exposition, we will call the solutions we report here "optimal". The reader should remember that such solutions are rather ϵ–optimal in the sense that they are very close approximations to the optimal solution.

We compute the optimal consumption boundary $W^*(z, y)$. An agent with wealth W, stock of the durable good z and standard of living y such that $W < W^*(z, y)$ optimally refrains from consumption. Such an agent waits as wealth increases, on average, and both z and y decline till the first time that the trajectory of the "state variables" (W, z, y) hits the consumption boundary. An agent with state variables such that $W > W^*(z, y)$ optimally consumes an amount of consumption that reduces wealth and increases z and y instantly to bring the state variables to the consumption boundary. Once on the consumption boundary, an agent consumes the minimum amount required to keep the state variables from crossing the boundary.

The striking feature of the solution is that the consumption boundary $W^*(z, y)$ is cyclical as a function of z and y. For a fixed standard of living y, the critical wealth required for an agent to start consuming is not a monotone function in the stock of the durable good z. Instead, it increases for a while with increases in z, then declines with further increases in z, and then reverses its trend and increases again with increases in z. Similarly, $W^*(z, y)$ is a cyclical function of the standard of living y for fixed z.

The optimal investment policy $A^*(W, z, y)$— the proportion of wealth invested in the risky assets— follows a similar cyclical pattern as both the stock of the durable good and the standard of living change. In particular, the partial derivatives $\frac{\partial A^*}{\partial z}$ and $\frac{\partial A^*}{\partial y}$ change their sign periodically as z and y, respectively, change. The cyclical behavior of the critical wealth level and the investment policy contrasts with the behavior of an agent in the presence of local substitution, or durability, but without habit formation. In this case, studied in Hindy and Huang (1993), the critical wealth required to begin consumption is linear in the stock of the durable good. Moreover, the proportion optimally invested in the risky asset is constant.

The cyclical pattern in the consumption boundary and the investment policy results from the interaction of the durability and habit formation effects. An additional unit of the durable good has two conflicting effects. The first is a direct satiating effect. The agent's appetite for more of the durable good is reduced. The second is an indirect stimulating effect. As the stock of the durable good increases, the agent's appetite for a higher standard of living increases. This is the effect of complementarity or habit formation. As a result, the agent would like to consume more of the good to increase the standard of living. The satiating and stimulating effects also influence the investment behavior of the agent. When the satiating effect dominates, the agent invests a high fraction of wealth in the risky assets since he can afford to tolerate high losses in wealth. On the other hand, when the stimulating effect is dominant, the agent invests a smaller fraction of wealth in the risky asset. The agent in this case behaves in a more risk averse manner to protect the standard of living.

The interesting feature of the solution is that the relative strength of the satiating and stimulating effects alternates as z and y change. We document the conflict between the two effects of consumption in a series of graphs that display the variations of marginal utilities of the stock of the durable and the standard of living as the state variables vary. We also discuss the long term behavior of optimally invested wealth and optimally followed standard of living and stocks of durable goods in a population of agents. As a consequence of the cyclical nature of the consumption boundary, initial differences in life style— the ratios z/W and y/W— between agents who are identical in preferences persist indefinitely. In essence, agents in the population are divided into distinct life-style classes. Members of the same class eventually adopt the same life-style. However, a member of one class, behaving optimally, does not migrate to another class. This contrasts with the analysis of Constantinides (1990) and Hindy and Huang (1993) in which the ratio of the stock of the durable to wealth converges to a steady state distribution regardless of the initial endowments.

We also study the implications of our model on the determination of the risk premium. We compute the risk premium in an economy with constant stochastic returns to scale. We show that the interaction between the satiating and stimulating effects of consumption leads to cyclical movement in the risk premium. In the stationary investment environment we study, the risk premium changes cyclically because the attitudes of the representative consumer towards risk change in cycles. The culprit, of course, is the interaction between durability, or local substitution, with habit formation.

Finally, we remark that our analysis is confined to the decision of an individual to purchase a durable good. Constructing aggregate time series of purchases in an economy populated with many agents of the type we study is important for understanding the relationships between aggregate consumption and asset returns. Such an interesting study, however, is beyond the scope of this paper.

The rest of the paper is organized as follows. Section 2 describes the setup of the optimization problem. The necessary and sufficient conditions for optimality are discussed in section 3. The numerical technique and its implementation are discussed in sections 4 and 5, respectively. Section 6 reports an example of the numerical solution of an infinite horizon program when the price of the risky asset follows a geometric Brownian motion. The implications of the solution on the determination of the risk premium on risky assets in equilibrium is discussed in section 7. Section 8 contains some concluding remarks.

2 Formulation

Consider an agent who lives from time $t = 0$ to $t = \infty$ in a world of uncertainty where there is a single good available for consumption at any time. The agent has the opportunity to invest in a frictionless securities market with $N + 1$ long lived securities, continuously traded, and indexed by $n = 0, 1, 2, \ldots, N$. Security n, where $n = 1, 2, \ldots, N$, is risky, pays dividends at rate

$\rho_n(t)$, and sells for $S_n(t)$ at time t. We assume that $\rho_n(t)$ can be written as $\rho_n(S(t))$, where we have used $S(t)$ to denote the column vector $[S_1(t), S_2(t), \ldots, S_N(t)]^{\top}$[1]. Security 0 is locally riskless, does not pay dividend, and sells for $B(t) = \exp\{\int_0^t r(S(s), s)ds\}$ at time t, where $r(S)$ is the instantaneous riskless interest rate at time t and $r(\cdot)$ is Borel measurable.

The price process for the risky securities follows a diffusion process given by:

$$(4) \quad S(t) + \int_0^t \rho(S(s))\, ds = S(0) + \int_0^t \mu(S(s))\, ds + \int_0^t \sigma(S(s))\, dB(s) \quad \forall t \in [0, \infty) \quad a.s.,$$

where B is an M-dimensional, $M \geq N$, standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$.[2] Assume that this diffusion process is strictly positive with probability one and that for each integer $m > 0$, there exists a constant c_m so that $E[|S(t)|^{2m}] \leq e^{c_m t}$.[3]

We assume that the agent has only access to the information contained in the historical prices of the risky securities. We denote this information structure by $\mathbf{F} = \{\mathcal{F}_t; t \in [0, \infty)\}$, where $\mathcal{F}_t$ is the smallest sub–sigma–field of $\mathcal{F}$ with respect to which $\{S(s); 0 \leq s \leq t\}$ is measurable. We assume that $\mathcal{F}_t$ contains all the probability zero sets of $\mathcal{F}$, or $\mathbf{F}$ is *complete*. All processes to be discussed will be adapted to $\mathbf{F}$.[4]

The agent can consume the single good at "gulps" at any moment, and can consume at finite rates over intervals. The agent can also refrain from consumption altogether for some time. Moreover, the sample path of cumulative consumption at any time t can have a singular component, that is a continuous nontrivial increasing function whose derivative is zero for almost all t.

Let $\mathbf{X}_+$ be the space of all processes x whose sample paths are positive[5], increasing and right–continuous. Recall that an increasing function $x(\omega, .)$ has a finite left–limit at any $t \in$

[1]The superscript $^{\top}$ denotes transpose.

[2]A process Y is a mapping $Y: \Omega \times [0, \infty) \to \Re$ that is measurable with respect to $\mathcal{F} \otimes \mathcal{B}([0, \infty))$, the product sigma–field generated by $\mathcal{F}$ and the Borel sigma-field of $[0, \infty)$. For each $\omega \in \Omega, Y(\omega, .): [0, \infty) \to \Re$ is a *sample path* and for each $t \in [0, \infty), Y(., t): \Omega \to \Re$ is a random variable. We will use the following notation: If μ is a vector in $\Re^N$, let $|\mu|$ be the Euclidean norm of μ. In addition, if σ is a matrix, let $|\sigma|^2$ denote $\operatorname{tr}(\sigma\sigma^{\top})$, where tr is the *trace* of a square matrix. The notation *a.s.* denotes statements which are true with probability one. For brevity, we will sometimes use $\mu(t)$, $\rho(t)$, $\sigma(t)$, and $r(t)$ to denote $\mu(S(t))$, $\rho(S(t))$, $\sigma(S(t))$, and $r(S(t))$, respectively.

[3]The latter can be ensured by a growth condition on $(\mu - \rho)$ and on σ; see Friedman (1975, theorem 5.2.3).

[4]The process Y is said to be adapted to $\mathbf{F}$ if for each $t \in [0, \infty)$, $Y(t)$ is $\mathcal{F}_t$-measurable. This is a natural information constraint: the value of the process at time t cannot depend on information yet to be revealed.

[5]We use weak relations. Positive means nonnegative and increasing means nondecreasing.

$(0, \infty)$ denoted by $x(\omega, t^-)$. We will use the convention that $x(\omega, 0^-) = 0$ *a.s.* Since left limits exist for the sample paths of any $x \in \mathbf{X}_+$, a jump of $x(\omega, .)$ at τ is $\Delta x(\omega, \tau) \equiv x(\omega, \tau) - x(\omega, \tau^-)$.

The stochastic process $C \in \mathbf{X}_+$ is a consumption pattern available to the agent with $C(\omega, t)$ denoting the *cumulative* consumption from time 0 to time t in state ω. For any $\omega \in \Omega$, the points of discontinuity of $C(\omega, t)$ are the moments when the agent consumes a "gulp". Moreover, $C(\omega, t)$ has an absolutely continuous component over the intervals during which the agent is consuming at rates $C'(\omega, t)$, where $C'(\omega, t)$ denotes the consumption rate at time t in state ω. Finally, $C(\omega, t)$ may have a singular part.

An investment strategy is an N-dimensional process $A = \{A(t) \equiv (A_1(t), \dots, A_N(t)); t \in [0, \infty)\}$, where $A_n(t)$ denotes the proportion of wealth invested in the n-th risky security at time t before consumption and trading. The proportion invested in the riskless security is $1 - A(t)^\top \mathbf{1}$, where $\mathbf{1}$ is a vector of 1's. A consumption plan $C \in X_+$ is said to be *financed* by an investment strategy A if

$$
(5) \quad \begin{aligned} W(t) &= W(0) + \int_0^t \Big(W(s) r(s) + W(s) A^\top(s) I_{S^{-1}}(s)(\mu(s) - r(s)S(s)) \Big)\, ds - C(t^-) \\ &\quad + \int_0^t W(s) A^\top(s) I_{S^{-1}}(s) \sigma(s)\, dB(s), \qquad \forall t \in [0, \infty) \quad a.s. \end{aligned}
$$

where $W(t)$ is wealth at time t before consumption and $I_{S^{-1}}(t)$ is an $N \times N$ diagonal matrix with the n–th element on the diagonal equal to $S(t)^{-1}$. Note that the wealth process has left–continuous sample paths and that $W(t^+) = W(t) - \Delta C(t)$.

The agent derives satisfaction from past consumption and from final wealth and utility is given by (1). The felicity function at time t is defined over the stock of durable, z, and the standard of living, y, defined, respectively, as

$$
(6) \quad z(t) = z(0^-) e^{-\beta t} + \beta \int_{0^-}^t e^{-\beta(t-s)}\, dC(s) \quad a.s.,
$$

$$
(7) \quad y(t) = y(0^-) e^{-\lambda t} + \lambda \int_{0^-}^t e^{-\lambda(t-s)}\, dC(s) \quad a.s.,
$$

where $z(0^-) \geq 0$ and $z(0^-) \geq 0$ are constants, β and λ, with $\beta > \lambda$, are weighting factors, and the integrals in (6) and (7) are defined path by path in the Lebesgue-Stieltjes sense. Note that the lower limit of the integrals in (6) and (7) is 0^-, to account for the possible jump of C at $t = 0$. Also note that z and y are right continuous processes which jump whenever C does and have singular component whenever C does.

A consumption plan C and the investment strategy A that finances it are said to be *admissible* if (5) is well-defined[6], $E[\int_0^\infty e^{-\delta t}|u(z(t), y(t))|dt] < \infty$, where z and y are associated with C, and for all integers $m > 0$ and $T \in \Re_+$, there exists k_m^T so that, for all $t \leq T$,

(8) $E[|C(t)|^{2m}] \leq e^{k_m^T t}$ and $E[|W(t)|^{2m}] \leq e^{k_m^T t}$.

Denote by $\mathcal{C}$ and $\mathcal{A}$ the space of admissible consumption plans and trading strategies, respectively. Formally, the agent manages wealth dynamically to solve the following program:

$$(9) \quad \begin{array}{ll} \sup_{C\in\mathcal{C}} & E\left[\int_0^\infty e^{-\delta t}u(z(t), y(t))dt\right] \\ \text{s.t.} & C \text{ is financed by } A \in \mathcal{A} \text{ with } W(0) = W_0, \\ & \text{and } W(t) - \Delta C(t) \geq 0 \quad \forall t \in [0, \infty), \end{array}$$

where W_0 is the initial wealth of the agent and u is the felicity function at time t. Note that the second constraint of (9) is a positive wealth constraint — wealth after consumption at any time must be positive.

3 Necessary and Sufficient Conditions for Optimality

In this section we discuss, formally, necessary and sufficient conditions for a consumption–investment policy to be optimal. In particular, we provide a verification theorem that shows that the value function satisfies a differential inequality. Such differential inequalities are the hallmark of free boundary control problems. The verification theorem requires that the value function be twice continuously differentiable in W and once continuously differentiable in z and y.

In general, it is difficult to ascertain the smoothness of the value function. In Hindy and Huang (1993) such smoothness could be established because there is a closed–form expression for the value function. In the problem we study in this paper, such a closed–form solution is not currently available. For this reason, the numerical analysis relies on the notion of viscosity solution as described in details in Hindy, Huang, and Zhu (1993). We record the verification

[6] For this we mean both the Lebesgue integral and the Itô integral are well-defined. When $A(t)$ is a feedback control depending on $(W(t), z(t^-), S(t), t)$ and $C(t)$ depends on the history of (W, S), we mean there exists a solution W to the stochastic differential equation (5) for every pair of controls.

theorems for smooth value functions for two reasons. First, the discussion is useful for conveying the economic intuition of the optimal solution. Second, the verification theorems can be utilized in the future if a closed-form solution is obtained.

3.1 Necessary Conditions

We use Bellman's optimality principle to derive necessary conditions about J assuming that it is continuously differentiable, twice in W, and once in y and z. One difficulty that arises is that the usual Bellman equation in dynamic programming is derived when the admissible consumption plans are purely at rates and are feedback controls. Here, cumulative consumption can be in gulps and may have singular parts. Moreover, if cumulative consumption has singular components, it cannot be expressed in feedback form. Thus, our derivation of the necessary conditions will be heuristic in nature. We refer the reader to Zhu (1991) and the references therein for recent work on the class of singular control problems.

First, we observe that

$$J(W, z, y, S) = J(W - \Delta, z + \beta\Delta, y + \lambda\Delta, S). \tag{10}$$

if a consumption gulp of size Δ is prescribed at the state (W, z, y, S). This is so because both quantities are equal to $E[\int_0^\infty e^{-\delta t} u(z(t), y(t))dt]$, where $\{z(t)\,,\, y(t); s \in [0, \infty)\}$ are defined along the optimal path on $[0, \infty)$. Moreover, the size of the gulp should be chosen to maximize J. Thus, we must have

$$\begin{aligned} & J_W(W - \Delta, z + \beta\Delta, y + \lambda\Delta, S) \\ = \; & \beta J_z(W - \Delta, z + \beta\Delta, y + \lambda\Delta, S) + \lambda J_y(W - \Delta, z + \beta\Delta, y + \lambda\Delta, S), \end{aligned} \tag{11}$$

where J_W, J_z and J_y denote the first partial derivatives of J with respect to its first, second, and third arguments, respectively.

We now show that (10) and (11) imply that J_W must be equal to $\beta J_z + \lambda J_y$ at any (W, z, y, S) where a gulp of consumption is prescribed. Differentiating (10) with respect to W and noticing that Δ is an implicit function of W, y, and z defined through (11) gives

$$\begin{aligned} & J_W(W, z, y, S) \\ = \; & [-J_W(W - \Delta, z + \beta\Delta, y + \lambda\Delta, S) + \beta J_z(W - \Delta, z + \beta\Delta, y + \lambda\Delta, S) \end{aligned}$$

$$
\begin{aligned}
&+\lambda J_y(W-\Delta, z+\beta\Delta, y+\lambda\Delta, S)]\frac{\partial\Delta}{\partial W} + J_W(W-\Delta, z+\beta\Delta, S) \\
= \;& J_W(W-\Delta, z+\beta\Delta, y+\lambda\Delta, S),
\end{aligned}
$$

where we used (11) to get the second equality. Similarly, we have $J_z(W,z,y,S) = J_z(W-\Delta, z+\beta\Delta, y+\lambda\Delta, S)$ and $J_y(W,z,y,S) = J_y(W-\Delta, z+\beta\Delta, y+\lambda\Delta, S)$. Given (11), we then have $J_W(W,z,y,S) = \beta J_z(W,z,y,S) + \lambda J_y(W,z,y,S)$ at (W,z,y,S) where a consumption gulp is prescribed.

Second, assume that J is sufficiently smooth for the generalized Itô's lemma to apply.[7] For any time t, the principle of optimality in dynamic programming and the generalized Itô's lemma imply that

$$
\begin{aligned}
0 = \;& \max_{dC,A} E_t\Big[\int_t^{t+\Delta t} e^{-\delta s} u(z(s),y(s))ds + \int_t^{t+\Delta t} e^{-\delta s}[\mathcal{D}^A J(s) - \delta J(s)]\, ds \\
(12) \quad & + \int_{t-}^{t+\Delta t} e^{-\delta s}[\beta J_z(s) + \lambda J_y(s) - J_W(s)]\, dC(s) + \textstyle\sum_i [J(\tau_i^+) - J(\tau_i)] - \sum_i [J_W(\tau_i)\Delta W(\tau_i) \\
& + J_z(\tau_i)\Delta z(\tau_i) + J_y(\tau_i)\Delta y(\tau_i)\Big] \quad a.s.,
\end{aligned}
$$

where dC denotes the consumption plan on the time interval $[t, t+\Delta t)$, τ_i is the i–th jump point prescribed by dC on $[t, t+\Delta t)$, $\mathcal{D}^A J$ is the differential operator associated with A (see Hindy and Huang (1993, footnote 8), $J(s)$ and its derivatives are evaluated at $(W(s), z(s^-), y(s^-), S(s))$, and we have assumed, without loss of generality, that Itô integral appearing in Itô's lemma is a martingale and thus vanishes when the conditional expectation is taken.

Since no consumption is always feasible, putting $dC = 0$ in (12) and letting Δt decrease to zero gives

$$(13) \quad 0 \geq u(z,y) + \max_A [\mathcal{D}^A J] - \delta J\,.$$

This relation must hold for all values of W, y, z,and S.

Suppose t is not a point for consumption gulps. Then for small enough Δ, by right–continuity of C, there will not be any gulps on $[t, t+\Delta t)$. For a nontrivial consumption dC, with no discontinuities on $[t, t+\Delta t)$, to maximize the right-hand side of (12) for small enough Δt, it must be the case that $J_W(t) = \beta J_z(t) + \lambda J_y(t)$. This, together with earlier discussion

[7] The generalized Itô lemma we will use throughout can be found in Krylov (1980, theorem 2.10.1) and Dellacherie and Meyer (1982, VIII.27).

when there is a consumption gulp at t, implies that nontrivial consumption, independent of its form, can only occur at time t when

$$\begin{aligned} & \beta J_z(W(t), z(t^-), y(t^-), S(t)) + \lambda J_y(W(t), z(t^-), y(t^-), S(t)) \\ = \; & J_W(W(t), z(t^-), y(t^-), S(t)). \end{aligned}$$

Moreover, when optimal cumulative consumption is continuous on $[t, t + \Delta t)$, standard arguments in dynamic programming show that (13) holds as an equality at t when Δt decreases to zero.

Next suppose that the optimal consumption plan prescribes zero consumption on $[t, t+\Delta t)$. For zero dC to maximize the right-hand side of (12) for small enough Δt, it must be the case that

$$\begin{aligned} & \beta J_z(W(t), z(t^-), y(t^-), S(t)) + \lambda \beta J_z(W(t), z(t^-), y(t^-), S(t)) \\ & \leq J_W(W(t), z(t^-), y(t^-), S(t)). \end{aligned}$$

Moreover, given that $dC = 0$ is the optimal consumption plan at t, (13) holds as an equality.

In summary, we have derived the following necessary conditions for optimality:

- consumption gulp at t:

$$\beta J_z(t) + \lambda J_y(t) - J_W(t) = 0 \qquad u(z(t), y(t)) + \max_A [\mathcal{D}^A J(t)] - \delta J(t) \leq 0$$

- continuous consumption at t:

$$\beta J_z(t) + \lambda J_y(t) - J_W(t) = 0 \qquad u(z(t), y(t)) + \max_A [\mathcal{D}^A J(t)] - \delta J(t) = 0$$

- no consumption at t:

$$\beta J_z(t) + \lambda J_y(t) - J_W(t) \leq 0 \qquad u(z(t), y(t)) + \max_A [\mathcal{D}^A J(t)] - \delta J(t) = 0$$

These conditions can be written compactly as a *differential inequality*:

$$\text{(14)} \quad \max\left\{ \max_A [u(z, y) + \mathcal{D}^A J - \delta J], \; \beta J_z + \lambda J_y - J_W \right\} = 0,$$

plus the condition that nontrivial consumption occurs only at times t where $\beta J_z(t) + \lambda J_y(t) = J_W(t)$.

In addition to the above necessary conditions, it is clear that J must satisfy the following *boundary condition*:

$$(15)\quad \lim_{W \downarrow 0} J(W, z, y, S) = \int_0^\infty e^{-\delta t} u(ze^{-\beta t}, ye^{-\lambda t}) dt\,.$$

Condition (15) is implied by the constraint that wealth at any time cannot become negative. Thus, whenever wealth is zero, the only feasible policy afterwards is no consumption.

3.2 Sufficient Conditions

In this section we provide a verification theorem to check the optimality of investment–consumption plans. We show that if there exists a solution $\hat{J}$ to the differential inequality (14) with the boundary condition (15), and $\hat{J}$ satisfies some regularity conditions, then $\hat{J}(W, z, y, S) \geq J(W, z, y, S)$ for all (W, z, y, S). We then give conditions so that $\hat{J}(W, z, y, S)$ is attained by a candidate feasible investment and consumption policy. It then follows that $J(W, z, y, S) = \hat{J}(W, z, y, S)$ and the candidate investment and consumption policy is the optimal policy.

Proposition 1 *Let $\hat{J} : \Re_+^{N+3} \to \Re_+$ be positive, concave in its first three arguments, continuously differentiable over $\Re_+^{N+3}$ in all of its arguments, and twice continuously differentiable over $\Re_+^{N+3}$ in its first $N+3$ arguments, except possibly on a smooth manifold $\mathcal{M}$,[8] satisfying the differential inequality of (14) with the boundary condition*

$$\lim_{W \downarrow 0} \hat{J}(W, z, y, S) = \int_0^\infty e^{-\delta t} u(ze^{-\beta t}, ye^{-\lambda t}) dt < \infty.$$

In addition, $\hat{J}$ satisfies the growth condition: for every $T \in \Re_+$, there exists $K_1^T > 0$ and $K_2^T > 0$ so that

$$(16)\quad |\hat{J}(X)| \leq K_1^T (1 + |X|)^{K_2^T} \quad \forall X \equiv (W, z, y, S) \in R_+^{N+3}.$$

Assume furthermore that there exists a consumption policy $C^* \in \mathcal{C}$ financed by $A^* \in \mathcal{A}$, with the associated state variables W^*, y^*, and z^*, such that, putting $\varrho = \inf\{t \geq 0 : W^*(t) = 0\}$,

[8] For the definition of smooth manifolds see Hindy and Huang (1993, footnote 8).

$\forall\, t \in (0, \varrho]$, $P-a.s.$

$$-\delta \hat{J} + \mathcal{D}^{A^*} \hat{J} + u(z^*, y^*) = 0 \tag{17}$$

$$\int_0^t [\hat{J}_W(W^*, z^*, y^*, S) - \beta \hat{J}_z(W^*, z^*, y^*, S) - \lambda \hat{J}_y(W^*, z^*, y^*, S)]\, dC^*(s) = 0 \tag{18}$$

$$\lim_{t \to \infty} E[e^{-\delta t} \hat{J}(W^*(t), z^*(t^-), y^*(t^-), S^*(t))] = 0, \tag{19}$$

and, almost surely,

$$\begin{aligned} &\hat{J}(W(t_i), z(t_i^-), y(t_i^-), S(t_i)) \\ &= \hat{J}(W(t_i) - \Delta C^*(t_i), z(t_i^-) + \beta \Delta C^*(t_i), y(t_i^-) + \lambda \Delta C^*(t_i), S(t_i)), \end{aligned} \tag{20}$$

where t_i's are the times of gulps prescribed by C^* on $[0, \varrho)$. Then $\hat{J} = J$ and (C^*, A^*) is an optimal consumption and investment policy.

Proof. The proof is similar to that of Proposition 1 and Theorem 1 in Hindy and Huang (1993) ∎

Note that Proposition 1 requires two conditions. First, $\hat{J}$ is positive. Second, the expected value of $\hat{J}(t)$ along the optimal path must converge to zero as t increases to infinity. The latter condition ensures that the agent exhibits enough impatience so that accumulating wealth indefinitely without consumption is not optimal. We imposed the former condition for technical convenience and we can replace it by a stronger condition which requires that (19) holds not just for the optimal policy but for all feasible plans.

As a recipe for decision making, Proposition 1 outlines general principles that the agent should follow as long as his wealth is strictly positive. In particular, the theorem instructs the agent to use a control policy that keeps the triple (W, z, y) in the region where the differential equation (17) is satisfied. In addition, the agent may consume only when marginal utility of wealth is equal to the sum of (β times) the marginal utility of the service flow and (λ times) the marginal utility of the standard of living. Finally, in the contingency that the optimal consumption policy calls for a "gulp", the size of the gulp should be chosen such that the value function immediately before is equal to the value function immediately after the gulp.

The optimal consumption and portfolio policy characterized in Proposition 1 of course satisfies all the necessary conditions of Section 3.1. Applying Proposition 1 requires that one know *a priori* that the value function is twice continuously differentiable. Absent this knowledge, we can use a weaker notion of solution to the differential inequality (14). In Hindy, Huang, and Zhu (1993), we use the notion of viscosity solutions, Crandall and Lions (1982), and show that such a solution of (14) is indeed the value function of the optimal consumption problem.

4 Numerical Approach

In this section we describe the numerical approach utilized to solve the infinite horizon utility maximization problem. We replace the continuous time processes (W, z, y) by a sequence of approximating discrete parameter Markov chains. The original optimization problem is then approximated by a sequence of discrete parameter control problems. We show that the value function of each Markov chain control problem satisfies an iterative, and hence easily programmable, discrete Bellman equation. Bellman's equation of the discrete control problem is a natural finite-difference approximation to the continuous Bellman's 'nonlinear' partial differential equation. We also show that the method of Markov chain approximation produces a consistent, stable, and convergent numerical scheme. Furthermore, the special monotonicity and concavity features of the problem allow us to prove convergence not only of the discrete approximations of the value function $J(W, y, z)$, but also of the corresponding discrete approximations of the optimal policies, when the latter exist.

For numerical implementation, we need to restrict the domain of definition of the state variables to a finite cubic region $[0, M_W] \times [0, M_y] \times [0, M_z]$. For this purpose, we introduce the reflection processes L_t, R_t and D_t at the boundaries of the cubic region. We hence consider the following modified state variables:

$$\begin{aligned} d\tilde{y}_t &= -\lambda \tilde{y}_t dt + \lambda dC_t - dL_t\,, \\ d\tilde{z}_t &= -\beta \tilde{z}_t dt + \beta dC_t - dR_t\,, \quad \text{and} \\ d\tilde{W}_t &= \tilde{W}_t(r + A_t(\mu - r))dt - dC_t + \tilde{W}_t A_t \sigma dB_t - dD_t\,, \end{aligned}$$

where L_t, R_t and D_t are nondecreasing and increase only when the state processes (W, z, y) hit the cutoff boundaries $W = M_W$, $z = M_z$, and $y = M_y$, respectively.

We will assume that the riskless rate r is constant and that there is a single risky asset whose price process is a geometric Brownian Motion given as

$$S(t) + \int_0^t \rho(s)\, ds = S(0) + \int_0^t \mu S(s)\, ds + \int_0^t \sigma S(s)\, dB(s)\,,$$

where $\mu > r$ and σ are positive constants and where $B(t)$ is a standard Brownian Motion. Since the investment environment is stationary, the value function J depends only on the level of wealth W, the stock of the durable good z, the standard of living y, and time t. It is also easy to see that $J(W, z, y, t) = e^{-\delta t} J(W, z, y, 0)$ since the felicity function is time separable with a constant impatience parameter. We will focus our analysis on the function $J(W, z, y, 0)$ and denote it henceforth by $J(W, z, y)$.

We will solve the following infinite horizon program on the restricted domain:

$$\text{(21)}\quad \text{Max} \qquad E\int_0^\infty e^{-\delta t} u(\tilde{y}_t, \tilde{z}_t) dt \equiv J^M(W, y, z)$$

subject to the dynamics of $\tilde{z}$ and $\tilde{y}$ and the dynamic budget constraint. Bellman's equation for this dynamic program takes the form:

$$\text{(22)}\quad \max\Big\{-\delta J^M + \max_A\{\mathcal{L}^A J^M\} + u(y, z), \lambda J_y^M + \beta J_z^M - J_W^M\Big\} = 0,$$

on $[0, M_W] \times [0, M_y] \times [0, M_z]$, together with the appropriate boundary conditions, where the operator $\mathcal{L}^A$ is given by $\mathcal{L}^A = \frac{1}{2} W^2 A^2 \sigma^2 \frac{\partial^2}{\partial W^2} + W[r + A(\mu - r)]\frac{\partial}{\partial W} - \lambda y \frac{\partial}{\partial y} - \beta z \frac{\partial}{\partial z}$.

We follow the lead of Soner (1986) and add to (22) the boundary constraints

$$\frac{\partial J^M}{\partial W} = 0, \qquad \frac{\partial J^M}{\partial y} = 0, \quad \text{and} \qquad \frac{\partial J^M}{\partial z} = 0,$$

respectively, at the cutoff boundaries $W = M_W$, $y = M_y$, and $z = M_z$. These constraints reflect the asymptotic behavior of the value function on the original unbounded domain. It is worthwhile to remark that in the theoretical analysis of convergence of the numerical scheme, the value functions J^M converge pointwise to J as the sequence of restricted domains Q_M increases to Q, *regardless of* the specification of the behavior of J^M at the boundaries. The boundary conditions we chose, however, affect the quality of the actual numerical solution.

4.1 The Markov Chain Scheme

We introduce a sequence of grids

$$G_h = \{(k,i,j) : W = k \times h, y = i \times h, z = j \times h; 0 \le k \le N_W, 0 \le i \le N_y, 0 \le j \le N_z\},$$

where h is the step size, and where $N_y = M_y/h$, $N_z = M_z/h$, and $N_W = M_W/h$ are integers. Each grid point $(k,i,j) \in G_h$ corresponds to a state (W,y,z) with $W = k \times h$, $y = i \times h$, and $z = j \times h$. For simplicity, we take the same step size in all three directions. For a fixed grid, we introduce the space of discrete strategies

$$\mathcal{AC}_N = \{(A,C); A = l \times \Delta, \Delta C = 0 \text{ or } h;\ 0 \le l \le N_A\}$$

where Δ is the step size of control, $N_A = \bar{A}/\Delta$ is total number of control steps with $\bar{A}$ an artificial bound to be relaxed in the limit as $\Delta \downarrow 0$ and $N_A \times \Delta \uparrow \infty$.

We approximate the continuous time process $(\tilde{W}, \tilde{z}, \tilde{y})$ by a sequence of discrete parameter Markov chains $\{(W_n^h, y_n^h, z_n^h); n = 1, 2, \cdots\}$ with h denoting the granularity of the grid and n denoting the steps of the Markov chain. For every h, the chain has the property that, at each step n, there is a choice between investment and consumption. At any time, a chain can either make an instantaneous jump, ($\Delta C = h$), or follow a "random walk", ($\Delta C = 0$), to the neighboring states on the grid. At the cutoff boundaries the chain is reflected to the interior in a manner consistent with its dynamics in the interior of the domain. Fix a chain and its corresponding grid size h. The transition probabilities for this Markov chain are specified as follows:

1. The case of no consumption—$(\Delta C = 0)$:

The chain can possibly move from the current state (k,i,j) only to one of the four neighboring states: $(k+1,i,j), (k-1,i,j), (k,i-1,j)$, and $(k,i,j-1)$. For any investment policy $A = l \times \Delta$, the transition probabilities in this case are defined as

$$\begin{aligned} P_h^A[(k,i,j),(k+1,i,j)] &= \frac{k^2h^2A^2\sigma^2/2 + kh^2(r+\mu A)}{Q^h(k,i,j)}, \\ P_h^A[(k,i,j),(k-1,i,j)] &= \frac{k^2h^2A^2\sigma^2/2 + kh^2(rA)}{Q^h(k,i,j)}, \end{aligned}$$

$$\begin{aligned}
P_h^A[(k,i,j),(k,i-1,j)] &= \frac{\lambda i h^2}{Q^h(k,i,j)}, \\
P_h^A[(k,i,j),(k,i,j-1)] &= \frac{\beta j h^2}{Q^h(k,i,j)}, \quad \text{and} \\
P_h^A[(k,i,j),(k,i,j)] &= 1 - P_h^A[(k,i,j),(k+1,i,j)] - P_h^A[(k,i,j),(k-1,i,j)] \\
&\quad - P_h^A[(k,i,j),(k,i-1,j)] - P_h^A[(k,i,j),(k,i,j-1)]
\end{aligned}$$

where the normalization factor $Q^h(k,i,j)$ is given by

$$Q^h(k,i,j) = \max_{0 \le l \le N_A} \left\{ k^2h^2(l \times \Delta)^2\sigma^2 + kh^2[r + l \times \Delta(\mu + r)] + (\lambda i + \beta j)h^2 \right\}.$$

The recipe for these transition probabilities is a slightly modified version of those suggested by Kushner (1977).

Furthermore, we define the incremental difference $\Delta y_n^h \equiv y_{n+1}^h - y_n^h$, $\Delta z_n^h \equiv z_{n+1}^h - z_n^h$, and $\Delta W_n^h \equiv W_{n+1}^h - W_n^h$. At step n of the chain, with the previously defined time scale $\Delta_n t^h$, one can verify that

$$\begin{aligned}
E_n^h[\Delta y_n^h] &= -\lambda y \Delta t_n^h - \Delta L_n^h, \\
E_n^h[\Delta z_n^h] &= -\beta z \Delta t_n^h - \Delta R_n^h, \\
E_n^h[\Delta W_n^h] &= W[r + A(\mu - r)]\Delta t_n^h - \Delta D_n^h, \quad \text{and} \\
E_n^h[\Delta W_n^h - E_n^h[\Delta W_n^h]]^2 &= W^2A^2\sigma^2\Delta t_n^h + O(\Delta t_n^h),
\end{aligned} \tag{23}$$

where E_n^h denotes expectation conditional on the nth-time state (W_n^h, y_n^h, z_n^h), and where the reflecting processes ΔL_n^h, ΔR_n^h, and ΔD_n^h equal to the positive value h only when y_n^h, z_n^h, and W_n^h reach their respective boundaries. This implies that the first and second moments of the Markov chain approximate those of the continuous process $(\tilde{W}_t, \tilde{y}_s, \tilde{z}_s)$. We call this property "local" consistency of the Markov chain.

2. The case of consumption— $(\Delta C = h)$:

The chain jumps along the direction $(-1, \lambda, \beta)$ from the current state (k,i,j) to the state $(k-1, i+\lambda, j+\beta)$. However, the later state, $(k-1, i+\lambda, j+\beta)$, is usually not on the grid except in the trivial case $\lambda = 1 = \beta$. For ease of programming, we take the intersection of the direction vector $(-1, \lambda, \beta)$ with the corresponding surface boundary and randomize between three grid points adjacent to the intersection point. We randomize in such a way that the expected random increment will be along the direction $(-1, \lambda, \beta)$ and of length equal to the distance from the starting state to the point of intersection.

Without loss of generality, we assume hereafter that $\lambda < 1 < \beta$. In this case, the intersection occurs inside a triangle spanned by the three grid points $(k, i, j+1)$, $(k-1, i, j+1)$, and $(k-1, i+1, j+1)$. The transition probabilities can be defined as:

$$\begin{aligned} P_h^C[(k,i,j),(k,i,j+1)] &= \frac{\beta-1}{\beta}, \\ P_h^C[(k,i,j),(k-1,i,j+1)] &= \frac{1-\lambda}{\beta}, \text{ and} \\ P_h^C[(k,i,j),(k-1,i+1,j+1)] &= \frac{\lambda}{\beta}, \end{aligned}$$

In this case, also, we can verify the property of "local" consistency:

$$(24) \quad E_n^h[\Delta y_n^h] = \lambda \Delta C, \qquad E_n^h[\Delta z_n^h] = \beta \Delta C, \qquad \text{and} \qquad E_n^h[\Delta W_n^h] = -\Delta C$$

where $\Delta C = \frac{h}{\beta}$ is the increment of consumption. These quantities correspond to the respective changes in the continuous time process for a consumption increment of the same magnitude ΔC. The movements of the Markov chain are depicted in Figure 1.

4.2 The Markov–Chain Decision Problem

A policy (A, C) is admissible if it preserves the Markov property in that

$$\text{Prob}\left(\begin{array}{c} (W_{n+1}^h, y_{n+1}^h, z_{n+1}^h) = (W', y', z') \\ \text{conditional on } \left\{ \begin{array}{l} (W_n^h, y_n^h, z_n^h) = (W, y, z) \\ (W_k^h, y_k^h, z_k^h), \ k \le n. \end{array} \right. \end{array} \right) = \left\{ \begin{array}{ll} P_h^A[(k,i,j),(k',i',j')], & \text{if } \Delta C = 0, \\ P_h^C[(k,i,j),(k',i',j')], & \text{if } \Delta C = h, \end{array} \right.$$

where $W = kh$ and $W' = k'h$, $y = ih$ and $y' = i'h$, $z = jh$ and $z' = j'h$. The control problem for the discrete parameter Markov chain is then to solve the program:

$$(25) \quad J^h(k,i,j) \equiv \max_{A,C} \ E_{k,i,j}^h \sum_{n=0}^{\infty} e^{-\lambda t_n^h} u(y_n^h, z_n^h) \Delta t_n^h,$$

where $t_n^h = \sum_{0 \le l \le n} \Delta t_l^h$ and $\Delta t^h(k,i,j) = h^2/Q^h(k,i,j)$. Note, this is analogous to (21) in the sense that the sum in (25) approximates the expected integral in (21).

The discrete dynamic programming equation now takes the iterative form

$$
\begin{aligned}
J^h(k,i,j) \; := \;& \max\Big\{ \textstyle\sum_{k',i',j'} P_h^C[(k,i,j),(k',i',j')]J^h(k',i',j'), \\
(26) \qquad & \max_{0\le l\le N_A}\{e^{-\delta\Delta t^h(k,i,j)} \textstyle\sum_{k',i',j'} P_h^A[(k,i,j),(k',i',j')]J^h(k',i',j')\} \\
& +U(i,j)\Delta t^h(k,i,j)\Big\}
\end{aligned}
$$

for $(k,i,j) \in G_n$, with iterative reflection at the artificial boundaries, where we recall that $P_h^A[*,*]$ and $P_h^C[*,*]$ are the transition probabilities of the chain. Now, let us denote

$$
\begin{aligned}
D_i^- J(k,i,j) &= \frac{J(k,i,j)-J(k,i-1,j)}{h}, \qquad & D_j^- J(k,i,j) &= \frac{J(k,i,j)-J(k,i,j-1)}{h}, \\
D_i^+ J(k,i,j) &= \frac{J(k,i+1,j)-J(k,i,j)}{h}, \qquad & D_j^+ J(k,i,j) &= \frac{J(k,i,j+1)-J(k,i,j)}{h}, \\
D_k^+ J(k,i,j) &= \frac{J(k+1,i,j)-J(k,i,j)}{h}, \qquad & D_k^- J(k,i,j) &= \frac{J(k,i,j)-J(k-1,i,j)}{h}, \text{ and}
\end{aligned}
$$

$$
D_k^2\, J(k,i,j) = \frac{J(k+1,i,j)-2J(k,i,j)+J(k-1,i,j)}{h^2}.
$$

Using this notation, we can express the discrete Bellman's equation (26) in the form:

$$
\begin{aligned}
-\Big[\frac{1-e^{-\delta\Delta t^h(k,i,j)}}{\Delta t^h(k,i,j)}\Big]J^h + \max_A\{\mathcal{L}_h^A J^h(k,i,j)\} + U(i,j) \;&\le\; 0, \\
\lambda D_i^+ J^h(k-1,i,j+1) + \beta D_j^+ J^h(k,i,j) - D_k^- J^h(k,i,j+1) \;&\le\; 0, \text{ with} \\
\mathcal{L}_h^A = \frac{1}{2}W^2A^2\sigma^2 D_k^2 + W(r+A\mu)D_k^+ - WArD_k^- - \lambda y D_i^- - \beta z D_j^-.&
\end{aligned}
$$

Furthermore, one of these two inequalities must hold as an equality at each $(k,i,j) \in G_h^+$.

Clearly, $(1-e^{-\lambda\Delta t^h})/\Delta t^h$ approximates the discount factor λ as $h \to 0$. As a result, the above discrete differential inequalities are a finite–difference approximation of Bellman's equation (22) for the maximization problem on the finite domain $[0,M_W]\times[0,M_y]\times[0,M_z]$. At the cutoff boundaries, we specify a slightly different finite–difference approximation. We refer the reader to Hindy, Huang, and Zhu (1993) for the details.

4.3 Convergence of the Markov chain Approximation

In this section, we state, without proofs, the convergence results of the Markov chain approximations. Specifically, we state that the value functions in the sequence of Markov chain control problems converge to the value function of the continuous time problem on the bounded domain. Furthermore, the optimal portfolio policies and the optimal consumption regions in the sequence of the Markov chain decision problems converge to those of the continuous time problem, if the latter exist. For an extensive discussion of the sense of convergence and for the technical details and proofs, we refer the reader to Hindy, Huang, and Zhu (1993).

Theorem 1 *The discrete Markov chain approximation yields a consistent and stable finite-difference scheme for Bellman's equation (22). Consider a sequence of grids and let the grid size $h \downarrow 0$ such that $(hk, hi, hj) \to (W, z, y)$. The sequence of solutions $J^h(k,i,j)$ of the discrete Bellman's equation converges to the value function $J(W,y,z)$ of continuous-time problem as $h \downarrow 0$. Furthermore, if there exists an optimal policy (A^*, C^*) for the original problem, then as $h \downarrow 0$, the sequence of approximate policies $A^h(k,i,j)$ converges to the optimal investment policy $A^*(W,z,y)$. Finally, if the point (W,z,y) is in the optimal consumption (abstinence) region in the original problem, then almost all grid points that converge to (W,z,y) will be in the consumption (abstinence) regions of the corresponding Markov-chain problems.*

5 Numerical Implementation

In this section, we describe briefly the implementation of the numerical scheme for solving the discrete Markov chain program. We designed the numerical scheme to reflect the iterative nature of the discrete Bellman's equation:

$$\begin{aligned} J^h(k,i,j) &= \max\Big\{ \textstyle\sum_{k',i',j'} P_h^C[(k,i,j),(k',i',j')]J^h(k',i',j') \\ &+ \max_{0\le l\le N_A}\{e^{-\lambda\Delta t^h} \textstyle\sum_{k',i',j'} P_h^A[(k,i,j),(k',i',j')]J^h(k,i,j)\} + U(i,j)\Delta t\Big\}. \end{aligned}$$

At each state, we first need to choose between consumption ($\Delta C = h$) and no consumption ($\Delta C = 0$). If consumption is chosen, we use the consumption scheme

$$J^h(k,i,j) = \sum_{k',i',j'} P_h^C[(k,i,j),(k',i',j')]J^h(k',i',j').$$

Otherwise, we use the investment scheme

$$J^h(k,i,j) = \max_{0\le l\le N_A} e^{-\lambda\Delta t} \sum_{k',i',j'} P_h^A[(k,i,j),(k',i',j')]J^h(k,i,j) + U(i,j)\Delta t$$

where $P_h^A[*,*]$ and $P_h^C[*,*]$ are given in section 4.

The numerical algorithm is a combination of two iterative schemes: approximation in policy space and approximation in value space. The first method can be viewed as a "descent" method in the space of control policies, while the second method calculates the $(n+1)$-step value function with the updated policies from the previous iteration.

Specifically, we implement the following steps.

1– Guess an initial value function $J_0^h(k,i,j) = K \times U(i,j)$ and set the initial policies as $A_0^h(k,i,j) = 0 = \Delta C_0^h(k,i,j)$ for $(k,i,j) \in G_h$ where $K > 0$ is a constant, and $U(i,j) = u(y,z)$ is simply the utility function.

2– Given the n–step value function $\{J_n^h(k,i,j) : (k,i,j) \in G_h\}$ and the n-step policies $\{A_n^h(k,i,j), \Delta C_n^h(k,i,j)\}$, compute at each state $(k,i,j) \in G_h$ the updated policies $\{A_{n+1}^h(k,i,j), \Delta C_{n+1}^h(k,i,j)\}$. At each $(k,i,j) \in G_h$, compute the maximum attainable utility from investment as

$$\begin{aligned}\max_A e^{-\lambda\Delta t}\Big\{&P_h^A[(k,i,j),(k+1,i,j)]J_n^h(k+1,i,j) + P_h^A[(k,i,j),(k-1,i,j)]J_n^h(k-1,i,j)\\ &+P_h^A[(k,i,j),(k,i-1,j)]J_n^h(k,i-1,j) + P_h^A[(k,i,j),(k,i,j-1)]J_n^h(k,i,j-1)\\ &+P_h^A[(k,i,j),(k,i,j)]J_n^h(k,i,j)\Big\} + U(i,j)\Delta t^h,\end{aligned}$$

where Δt^h is the interpolation time and $P^A[*,*]$'s are the transition probabilities of the chain. Meanwhile, we also compute the utility for consumption by

$$\frac{\beta-1}{\beta}J_n^h(k,i,j+1) + \frac{1-\lambda}{\beta}J_n^h(k-1,i,j+1) + \frac{\lambda}{\beta}J_n^h(k-1,i+1,j+1)$$

which equates the marginal utility of wealth to a combination of the marginal utility of consumption and the marginal utility of living standard. Then, update the $(n+1)$–step policies by choosing

$$A_{n+1}^h(k,i,j) \in \operatorname{argmax}_{A=l\times\Delta,\ 0\le l\le N_A}\Big\{e^{-\lambda\Delta t^h}\sum_{k',i',j'} P^A[(k,i,j),(k',i',j')]J_n^h(k',i',j')\Big\},$$

and $\Delta C_{n+1}^h(k,i,j) = 0$ if the utility from investing is higher than that from consuming. If the reverse is true, we choose $\Delta C_{n+1}^h(k,i,j) = h$ and $A_{n+1}^h(k,i,j)$ is set to some arbitrary number.

3– With the updated policy $(A_{n+1}^h, \Delta C_{n+1}^h)$, we now evaluate the utility functional to obtain the $(n+1)$–step value function J_{n+1}^h. In this step, first update the transition probabilities $P^A[*,*]$

according to the new policies. Then at each $(k,i,j) \in G_h^0$, compute $J_{n+1}^h(k,i,j)$ by solving a linear system:

$$\begin{aligned} J_{n+1}^h(k,i,j) &= e^{-\lambda\Delta t^h}\Big\{P_h^A[(k,i,j),(k+1,i,j)]J_{n+1}^h(k+1,i,j) \\ &+P_h^A[(k,i,j),(k-1,i,j)]J_{n+1}^h(k-1,i,j) \\ &+P_h^A[(k,i,j),(k,i-1,j)]J_{n+1}^h(k,i-1,j) \\ &+P_h^A[(k,i,j),(k,i,j-1)]J_{n+1}^h(k,i,j-1) \\ &+P_h^A[(k,i,j),(k,i,j)]J_{n+1}^h(k,i,j)\Big\} \\ &+U(i,j)\Delta t^h, \end{aligned}$$

where $\Delta t^h = h^2/Q^{n+1}(k,i,j)$, and $Q^{n+1}(k,i,j)$ is the normalizing constant as in the previous step but with the $(n+1)$-step updated policies $A_{n+1}^h(k,i,j)$. On the other hand, if consumption is prescribed at $(k,i,j) \in G_h^0$ (i.e. $\Delta C_{n+1}^h(k,i,j) = h$), compute $J_{n+1}^h(k,i,j)$ by solving the following different linear system:

$$\begin{aligned} J_{n+1}^h(k,i,j) &= \frac{\beta-1}{\beta}J_{n+1}^h(k,i,j+1) + \frac{1-\lambda}{\beta})J_{n+1}^h(k-1,i,j+1) \\ &+\frac{\lambda}{\beta}J_{n+1}^h(k-1,i+1,j+1) \end{aligned}$$

Finally, the evaluation at the cutoff-boundary can be obtained according to the reflection rules specified in Hindy, Huang, and Zhu (1993, §5.4). Throughout the entire procedure, the value at $k=0$ is chosen to satisfy the boundary condition at $W=0$.

4– Compute the roundoff error

$$ERR(n) = \max_{1\le k\le N_W} \frac{1}{(N_y+1)(N_z+1)}\sqrt{\sum_{i,j}[J_{n+1}^h(k,i,j) - J_n^h(k,i,j)]^2}.$$

If $ERR(n) \le \epsilon$, the desired precision is achieved and the program is terminated with the results taken as the value $J^h(k,i,j)$ and the optimal policies $\{A^h(k,i,j), C^h(k,i,j)\}$. Otherwise, increase n by 1 and return to steps 2 and 3. Here, $\epsilon > 0$ is the tolerance error prescribed in the program.

Computing the value function at every iteration requires solving a large linear system of equations. We use a relaxation method to solve this system rather than a method that requires computing the inverse of a matrix. The large size of the involved three dimensional matrices renders the latter algorithm numerically very expensive. The usage of the relaxation technique is guaranteed to produce accurate results because of the fact that the matrices describing the linear system to be solved define a contraction mapping as explained in Hindy, Huang, and Zhu (1993, §6.1). The numerical results are discussed in the following section.

6 Numerical Solution

In this section, we report the optimal consumption and portfolio rules in an infinite horizon program when the utility function is $u(z,y,t) = e^{-\delta t} z^{\alpha_1} y^{\alpha_2}$, with $0 < \alpha_i < 1$, $i = 1,2$. The discount factor δ expresses the impatience of the agent. The differential inequality (14) simplifies to

$$(27)\quad \max\left\{ z^{\alpha_1} y^{\alpha_2} + \max_{A}[\mathcal{D}^A J] - \delta J,\ \beta J_z + \lambda J_y - J_W \right\} = 0.$$

The boundary condition when $W = 0$ is

$$(28)\quad J(0,z,y) = \frac{z^{\alpha_1} y^{\alpha_2}}{\alpha_1 \beta + \alpha_2 \lambda + \delta}.$$

We solve this problem numerically and restrict the domain of the state variables (W,z,y) to the cube $[0,1] \times [0,1] \times [0,1]$. Observe that the multiplicative specification of the utility function is well defined for all strictly positive values of z and y. Furthermore, the utility function is increasing in both z and y. A similar multiplicative specification was introduced by Abel (1990). The life time satisfaction of the agent increases as the standard of living increases, ceteris paribus. This contrasts with the "difference specification" studied by Constantinides (1990) and Sundaresan (1989) who analyze the form $u(z,y,t) = e^{-\delta t}(z-y)^{\gamma}$, for some $\gamma < 1$. This utility function is only well defined when $z \geq y$. For $z < y$, we can take $u(z,y,t) = -\infty$. This extension maintains the concavity of the utility function.

Constantinides (1990) and Sundaresan (1989) study the special case when $\beta = \infty$ and hence $z(t)$ is the consumption rate $c(t)$ at time t. The difference utility function implies that the life time satisfaction of an agent is lower, the higher is his standard of living. Furthermore, this specification requires that initial wealth W_0 be at least equal to z_0/r for the life time utility maximization problem to have a feasible policy that keeps the consumption rate higher than the standard of living at all future dates. An agent who starts with $W_0 < z_0/r$ is certain to fail to maintain the initial standard of living. As a result, the maximum life time utility of such an agent is $-\infty$. For interest rates in the range 2% – 20%, an agent requires 5 – 50 times the initial standard of living in financial wealth to avoid the extremely painful consequences of the difference specification. This requirement restricts the domain of the state space where the life time utility optimization problem is meaningful.

Before we proceed, we observe that the felicity function $u(z, y, t)$ is homogeneous of degree $\alpha_1 + \alpha_2$ in z and y. In addition, the dynamics of the state variables W, z, and y, given in (5), (6), and (7), respectively, are linear. As a result, the value function J is homogeneous of degree $\alpha_1 + \alpha_2$. In other words, $J(kW, kz, ky) = k^{\alpha_1+\alpha_2} J(W, z, y)$ for all positive k. In principle, we could use the homogeneity of the value function to reduce the number of state variables from three to two. For example, we could divide by the stock level z and write $J(W, z, y) = z^{\alpha_1+\alpha_2} J(W/z, 1, y/z)$. We could then deal with the new state variables $x_1 \equiv \frac{W}{z}$ and $x_2 \equiv \frac{y}{z}$ and rewrite Bellman's equation in terms of x_1 and x_2.

A reformulated problem with two state variables, however, is not easier to analyze computationally. Let the value function in the reformulated problem be $z^{\alpha_1+\alpha_2} V(x_1, x_2)$. The optimality condition $J_W - \beta J_z - \lambda J_y$ in the original formulation corresponds to $(1 + \beta x_1)V_{x_1} + (\beta x_2 - \lambda)V_{x_2} - \beta(\alpha_1 + \alpha_2)V$ in the new formulation. Note that in the new formulation, this condition is expressed as a linear partial differential equation with zero-order term $\beta(\alpha_1 + \alpha_2)V$. As a result, the numerical solution of this equation is susceptible to numerical instabilities that require very small grid size or the use of implicit techniques. For more on the stability of numerical schemes, we refer the reader to Celia and Gray (1992, §4.2). The same condition in the original formulation does not contain any zero-order terms and hence is immune to numerical instability.

There is, hence, a tradeoff in choosing one of the two theoretically equivalent formulations of the problem. On the one hand, the three dimensional formulation requires more computer memory to process the three dimensional grid. The numerical algorithm, however, is simple and free of numerical instability. On the other hand, the two dimensional formulation demands less storage because of the reduction in the number of state variables. This formulation, however, requires special techniques to handle numerical instability. Such techniques reduce the speed of convergence of the algorithm. We elected to solve the problem numerically in its three dimensional formulation after our initial experiments suggested that the three dimensional numerical procedure converges faster, for the same accuracy, than the corresponding two dimensional one.

We also tested the numerical procedure by solving the special case in which $\lambda = 0$. This is the case in which the standard of living is constant and does not change with the level of consumption. Hindy and Huang (1993) report the closed form solution for this problem. The free boundary that determines the region of consumption in the state space (W, z) is given by a straight line $W = k^* z$. The critical ratio k^* depends on all the parameters of the decision problem and ranges from 10 to 20 for typical parameter values. The numerical procedure produces straight line free boundaries whose slope agree with the analytically computed k^* to an accuracy of 10^{-5}. Such computations were performed on the original two dimensional formulation, rather than on the equivalent one dimensional reformulation, of the decision problem.

6.1 Optimal Consumption Rules

We present examples of the numerical solutions. In these examples, we chose the following parameter values

$$r = 6\% \quad \mu = 12\% \quad \sigma = 23.1\% \qquad \lambda = 0.4 \quad \beta = 7.0 \quad \text{and} \quad \delta = 0.5\,.$$

Given theorem 1, all numerical solutions we discuss here can be made arbitrarily close to the optimal solution of the original problem. In particular, the regions of optimal consumption and the optimal investment policies we report are very good approximations of the optimal policies in the continuous time problem. For ease of exposition, we will call the solutions we report here "optimal". The reader should remember that such solutions are rather ϵ–optimal in the sense that they are very close approximations to the optimal solution.

Figures 2 and 3 display the optimal free boundary in the case of $\alpha_1 = 0.8\,, \alpha_2 = 0.2$ and $\alpha_1 = 0.2\,, \alpha_2 = 0.8$, respectively. The striking feature in both figures is the cyclic shape of the free boundary. For a stock of the durable good z and a standard of living y, let $W^*(z, y)$ be the corresponding critical level of wealth on the consumption boundary. If wealth W is such that $W > W^*$, the optimal behavior of the agent is to consume immediately the amount of wealth that brings the state variables to the consumption boundary. If, on the other hand, $W < W^*$, the optimal behavior is to consume nothing and to wait for wealth to grow and for z and y to decline until the first time when the state variables reach the consumption boundary. Subsequently, the optimal policy is to consume the minimum amount required to keep the level of wealth from exceeding the critical value W^* corresponding to the durable stock and the standard of living.

For a fixed level of the standard of living y, the critical value of wealth W^* is neither monotonically increasing nor monotonically decreasing in z. Instead, the critical value $W^*(z,y)$ is cyclical in z for a fixed y. Similarly, the figures show that $W^*(z,y)$ is cyclical in y for fixed values of z. This contrasts with the case of constant standard of living, $\lambda = 0$, analyzed in Hindy and Huang (1993). In that case, the critical wealth level that determines the states of consumption is monotonically increasing in the stock of the durable good. The higher the level of the stock is, the higher is the level of wealth required for the agent to start consumption.

Monotonicity of the critical level W^* in the case of non-changing standard of living is a consequence of the tradeoffs that the agent faces. In that case, a decision to consume would increase the level of the durable good and simultaneously reduce the level of financial wealth. If the marginal utility of wealth exceeds the marginal utility of the stock of the durable good, the optimal decision is to refrain from consumption to increase the expected level of wealth. Otherwise, the optimal choice is to consume to increase the level of the durable good and equalize the marginal utilities of W and z.

Fix the level of wealth. By concavity of the indirect utility function, the higher the level of the durable good stock is, the *lower* is the marginal value of an additional unit of the good relative to the marginal value of one additional unit of wealth. In other words, the higher the value of the stock is, the more useful it is to increase expected wealth rather than the level of the durable stock. As a result, higher levels of the stock of the durable good require higher levels of wealth before consuming is optimal.

Such is not the case with the introduction of the standard of living that increases with consumption. In that case, consumption increases both the stock of the durable and the standard of living and reduces financial wealth. Recall that whether the agent optimally consumes or not is determined by the quantity $J_W - (\beta J_z + \lambda J_y)$. The agent consumes only when this quantity is equal to zero. When J_W exceeds the sum of βJ_z and λJ_y, it is optimal to refrain from consumption to increase the expected level of wealth and reduce the levels of both z and y. The complementarity between the standard of living and the stock of the durable good introduces new effects. Recall that the second partial derivatives u_{zy} is strictly positive. Furthermore, our numerical results show that J_{zy} is also strictly positive. As a result, as the level of the stock of the durable good increases, so does the marginal value of an increase in the standard of living. This complementarity effect is the source of the cyclical form of the consumption boundary.

Fix the level of wealth and the standard of living and consider how the tradeoffs facing the agent change as the stock of the durable good increases. An increase in the stock of the durable good z, ceteris paribus, has two opposing effects. The first is a satiating direct effect. An increase in z reduces the marginal utility J_z. The second is a stimulating indirect effect. An increase in z increases the marginal utility J_y because of the complementarity and habit formation effects. The total effect of an increase in z depends on the size of the stock of the durable good relative to W and y. Sometimes, the relative values of W, z, and y are such that the satiating effect dominates and an increase in z leads to a reduction in the sum $\beta J_z + \lambda J_y$ relative to J_W. As a consequence, refraining from consumption is the optimal choice. Other times, the relative values of W, z, and y are such that the stimulating, complementarity–induced, effect dominates and an increase in z leads to an increase in the sum $\beta J_z + \lambda J_y$ relative to J_W. In this case, consuming the amount required to equalize J_W with $\beta J_z + \lambda J_y$ is the optimal choice.

The important feature of the interaction between the stimulating and the satiating effects of increasing the stock of the durable good is that relative dominance alternates between them as z increases. For low levels of z, the satiating effect dominates and an increase in z decreases the sum $\beta J_z + \lambda J_y$. For higher values of z, ceteris paribus, the stimulating effect dominates and further increases in z increase the sum $\beta J_z + \lambda J_y$. For still higher values of z, the satiating effect regains dominance and increases in z reduce the sum $\beta J_z + \lambda J_y$. We present samples of the alternating behavior of the sum $\beta J_z + \lambda J_y$ in Figures 4 and 5.

An interesting consequence of the cyclical form of the consumption boundary is that a cross section of agents with identical levels of wealth and standard of living will exhibit alternating patterns in their consumption behavior. Agents with relatively low levels of the stock of the durable good will consume because a unit of consumption is a valuable addition to their stock of the durable good. In their case the value of J_z is high enough to encourage consumption. Agents with relatively high levels of the stock of the durable good will also consume but for a different reason. In their case, the marginal value of J_y is high enough to induce consumption. In essence, a unit of consumption is a value increment to their standard of living. Agents with intermediate levels of the stock of the durable good have relatively low levels of both J_z and J_y. Their optimal policy is to refrain from consumption on the grounds that increasing financial wealth is more valuable than increasing either the stock of durable or the standard of living. Since agents consume because of different reasons, it is impossible to infer which of two agents, with identical levels of W and y, has a higher level of z.

Figures 2, 3, 4, and 5 reveal, and a detailed examination of the numerical results confirms, that the sum $\beta J_z + \lambda J_y$ is cyclical in the value of the standard of living y for fixed W and z. The economic reasoning is the same. An increase in the standard of living y, ceteris paribus, has two opposing effects. The satiating direct effect is due to the natural decline in J_y. The stimulating indirect effect is due to the habit formation effect that increases J_z, the marginal utility of the durable good, as the standard of living increases. The relative strength of these two opposing effects alternates as y increases and hence the sum $\beta J_z + \lambda J_y$ changes in cycles as the standard of living increases. The consequences are also the same as those from the cyclical effect of z. In particular, a cross section of agents with identical wealth and stock of the durable good will reflect this cyclical pattern in its consumption behavior. Specifically, agents with relatively high and low standards of living consume, albeit for different reasons, whereas agents of intermediate standards of living optimally invest in the financial asset and are content to derive utility from the current stock of the durable and the current standard of living.

It is easy to see from the figures, and the numerical results substantiate, that the same cyclical effect is present in the equivalent two state variable formulation of the problem. Define $x_1 \equiv \frac{W}{z}$ and $x_2 \equiv \frac{y}{z}$. Define the consumption boundary corresponding to x_1, $x_2^*(x_1)$, as follows:

if the state variables x_1 and x_2 are such that $x_2 > x_2^*(x_1)$, the optimal policy is to consume the amount required to bring x_1 and x_2 to the consumption boundary. Otherwise, it is optimal not to consume and wait till the trajectory of (x_1, x_2) reaches the consumption boundary. The numerical results show that the consumption boundary x_2^* is a cyclic function of x_1.

It is also interesting to consider the long run behavior of the optimally controlled state variables in a population of agents. In the case of constant standard of living, $\lambda = 0$, Hindy and Huang (1993, §7) show that the ratio of the optimally controlled wealth, W^*, and stock of the durable good, z^*, reaches a steady state distribution. After long enough time, the distribution of the ratio W^*/z^* is independent of the initial conditions W_0/z_0. In other words, after long enough time, all agents would have probabilistically the same level of the durable good, relative to wealth, regardless of the discrepancies in their initial starting condition.

This result of eventual similarity of the optimal ratio of the stock of durable to the level of wealth does not obtain in the presence of a standard of living that changes with consumption. Refer again to figures 2 and 3 and observe that the state space below the consumption boundary $W^*(z, y)$ is divided into different distinct regions. In the restricted state space $[0,1]\times[0,1]\times[0,1]$, there are three distinct regions which we label as regions I, II, and III. Recall that these are the regions where it is optimal not to consume. If an agent starts inside region I, for example, then the optimal policy is to abstain from consumption and wait for wealth to grow, on average, and for both z and y to decline till the trajectory of the state variables (W, z, y) reaches the consumption boundary. At that time, the agent consumes the amount required to keep the state variables (W, z, y) from moving across the consumption boundary.

From the geometry of the consumption regions, I, II, and III, and from the dynamics of the state variables, it is clear that if an agent starts inside one of those regions, the trajectory of the state variables will remain inside that region during the period of no consumption. Furthermore, once the trajectory of (W, z, y) reaches the boundary, consumption will reflect that trajectory to the *inside* of the region from which it originated. As a result, once the agent starts inside one of the regions, the optimal behavior confines the values of all future state variables to be inside that particular region.

The cyclical shape of the consumption boundary divides the state space into distinct regions.

As a result, the population of agents is divided into classes that depend on the initial starting levels of W, z, and y. Once in a particular class, an agent can not migrate to another class because of the shape of the consumption boundary. An agent that starts in the consumption region with wealth higher than the critical level corresponding to his endowment of z and y takes an immediate gulp of consumption and moves to a point on the consumption boundary. The entry point into the region of no consumption determines the class of the agent to which he will be confined forever.

Although agents are identical in their preferences, initial variations in endowments of wealth, stock of the durable good, and standard of living persist indefinitely. The population is divided into distinct classes. The members of each class may become eventually similar in the distribution of stocks of the durable good and standard of living relative to financial wealth. However, the eventual distributions of the optimal ratios z/W and y/W vary among classes. Furthermore, a member of one class may not emigrate to another class even after a very long time. It is worth mentioning that this eternal confinement of an agent to the class in which he was born is a result of our assumptions on the available investment opportunity. The price of the risky asset changes in small amounts over short periods of time because we assumed it to be a diffusion. We conjecture that if the investment opportunity admits large jumps in wealth, then an agent has a chance to migrate to another class. For tractability, we limited our analysis to asset prices of the diffusion family. Introducing a price process of the jump-diffusion family is an interesting further direction of this research.

6.2 Optimal Investment Rules

In this section, we present the optimal proportion of wealth A^* invested in the risky asset. From Bellman's equation, direct computation shows that

$$(29) \quad A^*(W,z,y) = -\frac{J_W}{WJ_{WW}}\frac{\mu - r}{\sigma^2}.$$

In the case of constant standard of living studied in Hindy and Huang (1993), the optimal fraction of wealth invested in the risky asset is a constant that does not vary with the level of wealth or the stock of the durable good. The constant fraction invested in the risky asset is higher for higher durability of the good (lower β), for lower interest rate r, and for lower rate of discount δ. Furthermore, as the durability effect weakens ($\beta \uparrow \infty$) the optimal fraction of wealth invested in the risky asset converges to the constant fraction used by an investor with time additive utility over consumption rates as given in Merton (1971).

In the current model with the standard of living that changes with the level of consumption, the fraction invested in the risky asset is not a constant. Furthermore, the optimal fraction of wealth invested in the risky asset exhibits cyclical behavior as the value of the stock of the durable and the standard of living change. Specifically, the partial derivatives $\frac{\partial A^*}{\partial z}$ and $\frac{\partial A^*}{\partial y}$ are cyclical functions in z and y, respectively. Fixing the level of wealth W and the stock of the durable good z, the numerical solution reveals that, for small values of y relative to W, $\frac{\partial A^*}{\partial y} < 0$ and hence the optimal fraction A^* *declines* as y increases. In this low range of y, an agent with a higher standard of living invests more defensively, ceteris paribus, than another agent with a lower standard of living. In other words, the agent with the higher standard of living behaves in a *more* risk averse manner to protect his standard of living.

In a higher range of y, $\frac{\partial A^*}{\partial y} > 0$ and hence as the standard of living increases, the optimal investment proportion A^* increases, ceteris paribus. For two agents with standard of living in this range, the agent with the higher standard of living acts in a *less* risk averse manner and invests more aggressively in the risky asset. For still higher values of y, the investment behavior changes and $\frac{\partial A^*}{\partial y} < 0$ reversing the trend for investment in the preceeding range of living standard. The same cyclical behavior of the optimal investment policy is exhibited as the stock of the durable good varies. A^* initially increases as z increases, ceteris paribus. As z reaches a critical level, A^* declines with further increases in z. For still higher levels of z, $\frac{\partial A^*}{\partial z} > 0$. The variation of A^* with wealth, however, does not exhibit this cyclical behavior. As W increases, ceteris paribus, A^* *declines.* A richer individual would invest a lower fraction of wealth in the risky asset than, an otherwise identical, poorer individual.

Note that, over the long run, each individual goes through cycles of investment behavior. During the periods of no consumption, wealth increases, on average, while z and y decline. The investment policy of the agent adjusts continuously as the state variables (W, z, y) change. The result is frequent upward and downward revision of the fraction of wealth invested in the risky asset. This cyclical investment behavior occurs, even in the stationary environment we analyze, due to the interaction of habit formation and durability effects. This behavior contrasts sharply with the constant fraction policies analyzed by Merton (1971) in the case of time additive utility, by Grossman and Laroque (1990) in the case of transaction costs, and by Hindy and Huang (1993) in the case of durable goods. This behavior is also different from the monotone behavior reported by Constantinides (1990) and Sundaresan (1989). In both studies the fraction of wealth invested in the risky asset *declines* as the standard of living increases relative to wealth.

To understand the sources of the cyclical investment behavior, consider the sign of $\frac{\partial A^*(W,z,y)}{\partial y}$, where $A^*(W, z, y)$ is given in (29). The sign of $\frac{\partial A^*(W,z,y)}{\partial y}$ is the same as the sign of $[-J_{WW}J_{Wy} + J_W J_{WWy}]$. However, from the properties of the value function J, we know that $J_{WW} < 0$, and $J_W > 0$. Furthermore, the term $J_W J_{WWy}$ is numerically much smaller than the term $-J_{WW}J_{Wy}$. Hence, the sign of $\frac{\partial A^*(W,z,y)}{\partial y}$ is, approximately, the same as the sign of J_{Wy}. Figures 6 and 7 display typical behavior of the marginal value of wealth as the state variables (W, z, y) vary.

Consider the variations of the marginal value of wealth J_W as the standard of living y changes. Figures 6 and 7 show that J_W is a cyclical function of y. This is a result of the habit formation effect. An increase in the standard of living y has two conflicting effects on the marginal utility of wealth. The direct satiating effect tends to reduce the marginal value of wealth. Due to this effect, increasing the standard of living reduces the appetite of the agent for further improvements in the living standard. Hence, the agent discounts additional units of wealth. There is also an indirect stimulating effect. As the standard of living increases, the agent's appetite for the durable good increases. This is the habit formation effect. Due to that increased appetite, the agent places high value on additional units of wealth.

The relative strength of the direct satiating effect and the indirect stimulating effect change as the state variables (W, z, y). At some values of the state variables, the direct satiating effect dominates. As a result, $\frac{\partial J_W}{\partial y} < 0$ and $\frac{\partial A^*}{\partial y} < 0$ in that range. For other values of the state variables, the indirect stimulating effect dominates. Hence, $\frac{\partial J_W}{\partial y} > 0$ and $\frac{\partial A^*}{\partial y} > 0$ in that range. The relative dominance of the direct and indirect effect alternates as the standard of living, relative to W and z, changes. The same intuition explains the variations of A^* with z. The cyclical variation of J_W with z is displayed in figures 6 and 7.

7 Equilibrium Risk Premium

In this section, we discuss the implications of the consumption and investment behavior we studied in this paper on the determination of the risk premium. We use the representative agent framework of Cox, Ingersoll, and Ross (1985) and presume that there is a risky production technology whose rates of returns are given by a (μ, σ)–Brownian Motion with μ and σ strictly positive scalars.[9] Using arguments similar to those of Cox, Ingersoll, and Ross (1985), we conclude that the equilibrium riskless rate, r, is given by:

$$(30) \quad r = \mu - \frac{-W\hat{J}_{WW}}{\hat{J}_W}\sigma^2,$$

where $\hat{J}$ is the indirect utility function for an agent who maximizes life-time utility under the constraint that all investment is in the risky production technology.

We computed numerically the equilibrium risk premium $\mu - r$ in an economy with a representative agent whose preferences were analyzed in section 6. Specifically, we studied the case with utility function $u(z, y, t) = e^{-0.5t}z^{0.2}y^{0.8}$. Furthermore, we chose, somewhat arbitrarily, the parameter values $\mu = 12\%, \sigma = 23.1\%, \lambda = 0.4$ and $\beta = 7.0$. We report samples of the equilibrium risk premium in Figures 8, 9, and 10. Table 1 compares the results with those in economies with a representative agent with time additive utility as in Merton (1973) and with durability effects as in Hindy and Huang (1993).

As the figures show, the equilibrium risk premium is not constant. This result contrasts with the case without the habit formation effect analyzed by Hindy and Huang (1993) in which the risk premium is a constant. Furthermore, in that case, the risk premium increases as the durability effect, as captured by β, decreases. Durability and habit formation effects combined lead to cyclical behavior of the risk premium. As our discussion in section 6.2 reveals, the attitudes of the investor towards risk change with changes in the level of wealth relative to both

[9] More specifically, the level of risky capital, $V(t)$, at time t starting from one unit continuously reinvested evolves according to the equation: $dV(t) = \mu V(t)\,dt + \sigma V(t)\,dB(t)$, where B is one dimensional standard Brownian motion.

Model	Utility Specification	β	λ	$\mu - r$
Time Additive Preferences	$z^{0.2}y^{0.8}$	∞	zero	4.27%
Durability	$z^{0.2}y^{0.8}$	7.00	zero	3.87%
Durability and Habit Formation	$z^{0.2}y^{0.8}$	7.00	0.40	1.40%—6.25%

Table 1: Comparison of Risk Premia Among Different Models

the stock of the durable good and the standard of living. Sometimes the habit formation effect dominates and the agent displays more risk averse behavior to protect the standard of living from adverse investment shocks. Other times, the durability effect dominates and the agent displays less risk averse behavior and tolerates higher probabilities of a decline in wealth.

The cyclical risk aversion translates into cyclical behavior of the risk premium. Naturally, at the times when the representative investor is more risk averse, a high risk premium is required to achieve equilibrium in the financial market. Similarly, at the times when the investor is less risk averse, a lower risk premium is sufficient to clear the market.

As table 1 shows, the risk premium at states when the habit formation effect dominates is higher than that in an, otherwise identical, economy with time–additive preferences. On the other hand, the risk premium at states when the durability effect dominates is lower than that in the corresponding time–additive economy. These results contrast with those obtained by Constantinides (1990) who studies habit formation *without* durability. Using a difference specification of the form $u(c, z) = (c - z)^{\alpha}$, for some $\alpha < 1$, and where c is the consumption *rate*, Constantinides (1990) shows that the equilibrium risk premium is *always* higher than that in a corresponding time–additive economy. In the economy we study, the variations in the risk premium above and below that in a corresponding time-additive economy are due to the changes in the relative importance of durability versus habit formation as the state of the economy (W, z, y) varies.

8 Concluding Remarks

In this paper we provide a complete analysis of the problem of optimal consumption and portfolio choice for an agent with utility functions that admit three different interpretations. In one interpretation, the preferences of the agent exhibit both local substitution and habit formation. In a second interpretations, the model represents habit formation over the service flows from irreversible purchases of a durable good. In a third version, the model represents preferences for consumption of a dual purpose commodity that provides two sources of utility.

We provide numerical techniques for solving this optimization program which is from the class of free boundary singular control problems. Our technique is based on approximating the control problem by a sequence of controlled Markov chains. A companion paper, Hindy, Huang, and Zhu (1993), provides all the technical details. Free boundary control problems appear in many areas in economics. We hope that the technique we presented finds use in many other applications.

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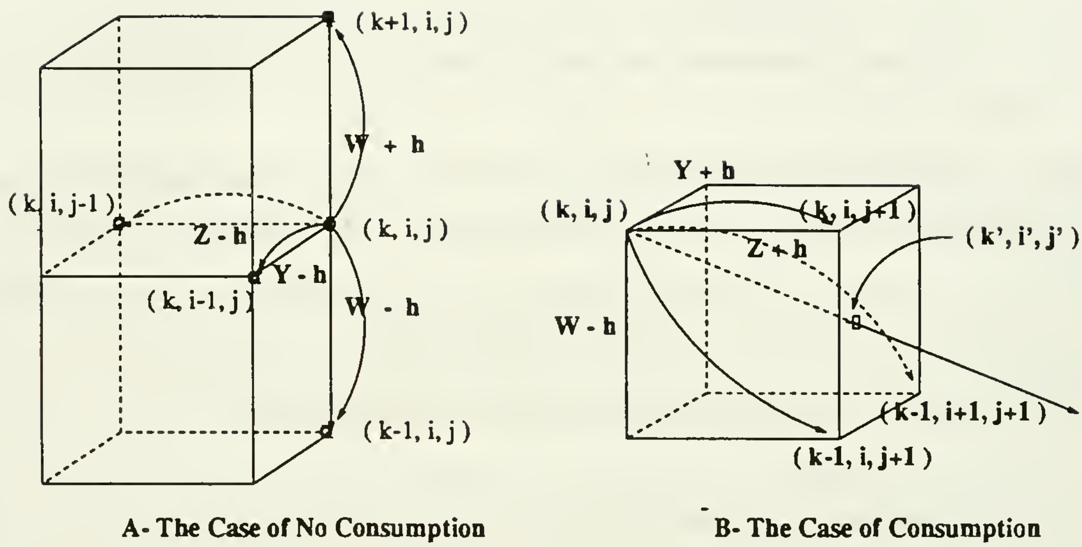

- <u>Case A</u> At state (k, i, j), when $\Delta C = 0$, the Markov chain moves to one of the four neighboring states: $(k+1, i, j)$, $(k-1, i, j)$, $(k, i-1, j)$, $(k, i, j-1)$ with the appropriate probabilities.

- <u>Case B</u> At state (k, i, j), when $\Delta C = h$, the Markov chain jumps to the intersection (k', i', j'). This is implemented by randomizing among the three neighboring states: $(k, i, j+1)$, $(k-1, i, j+1)$ and $(k-1, i+1, j+1)$.

Figure 1: **Local Transitions of The Controlled Markov Chain**

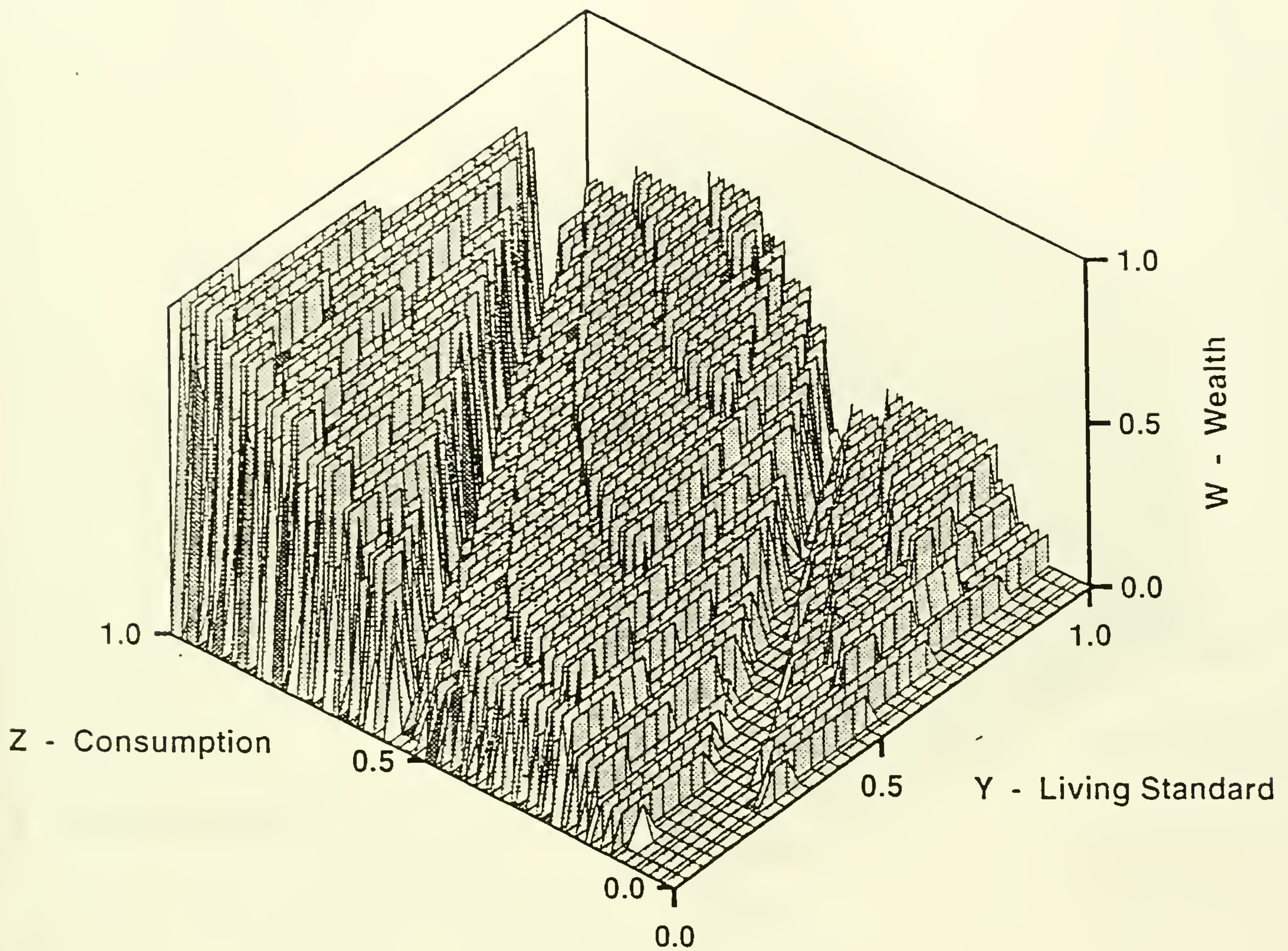

Figure 2: The optimal Free Boundary for $u(z, y) = z^{0.8}y^{0.2}$

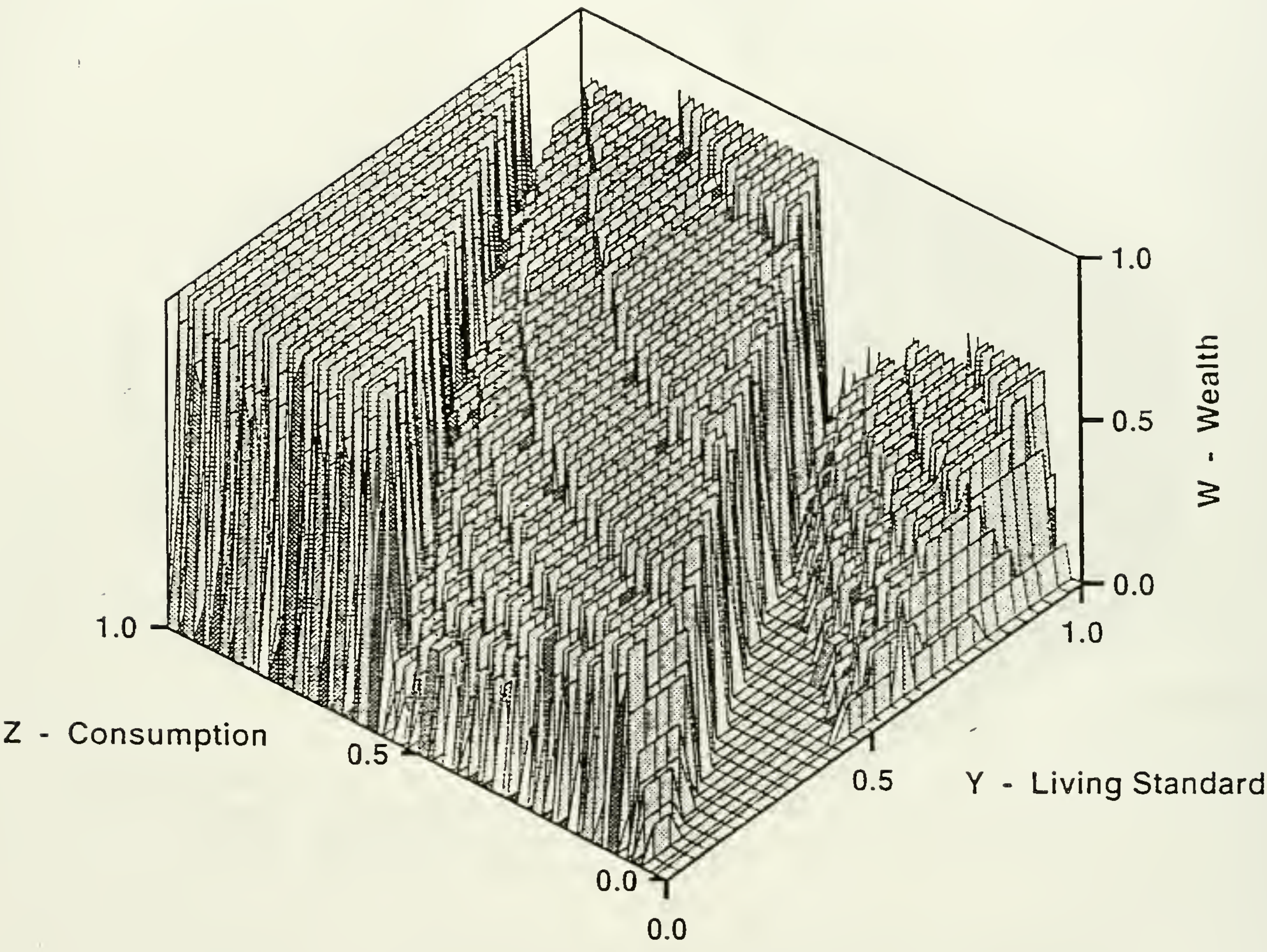

Figure 3: The optimal Free Boundary for $u(z, y) = z^{0.2}y^{0.8}$

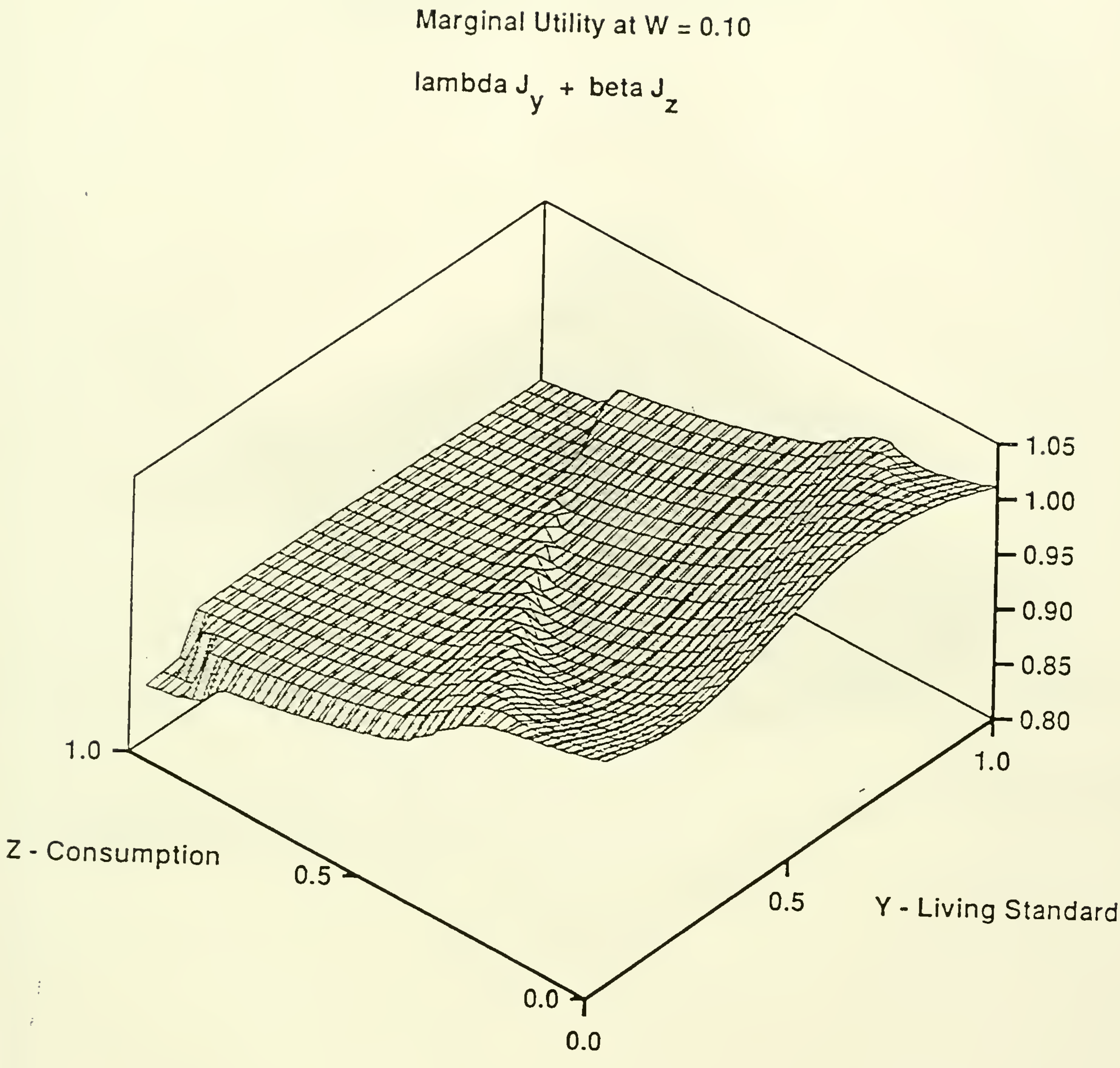

Figure 4: Samples of The Alternating Behavior of The Sum $\beta J_z + \lambda J_y$—I.

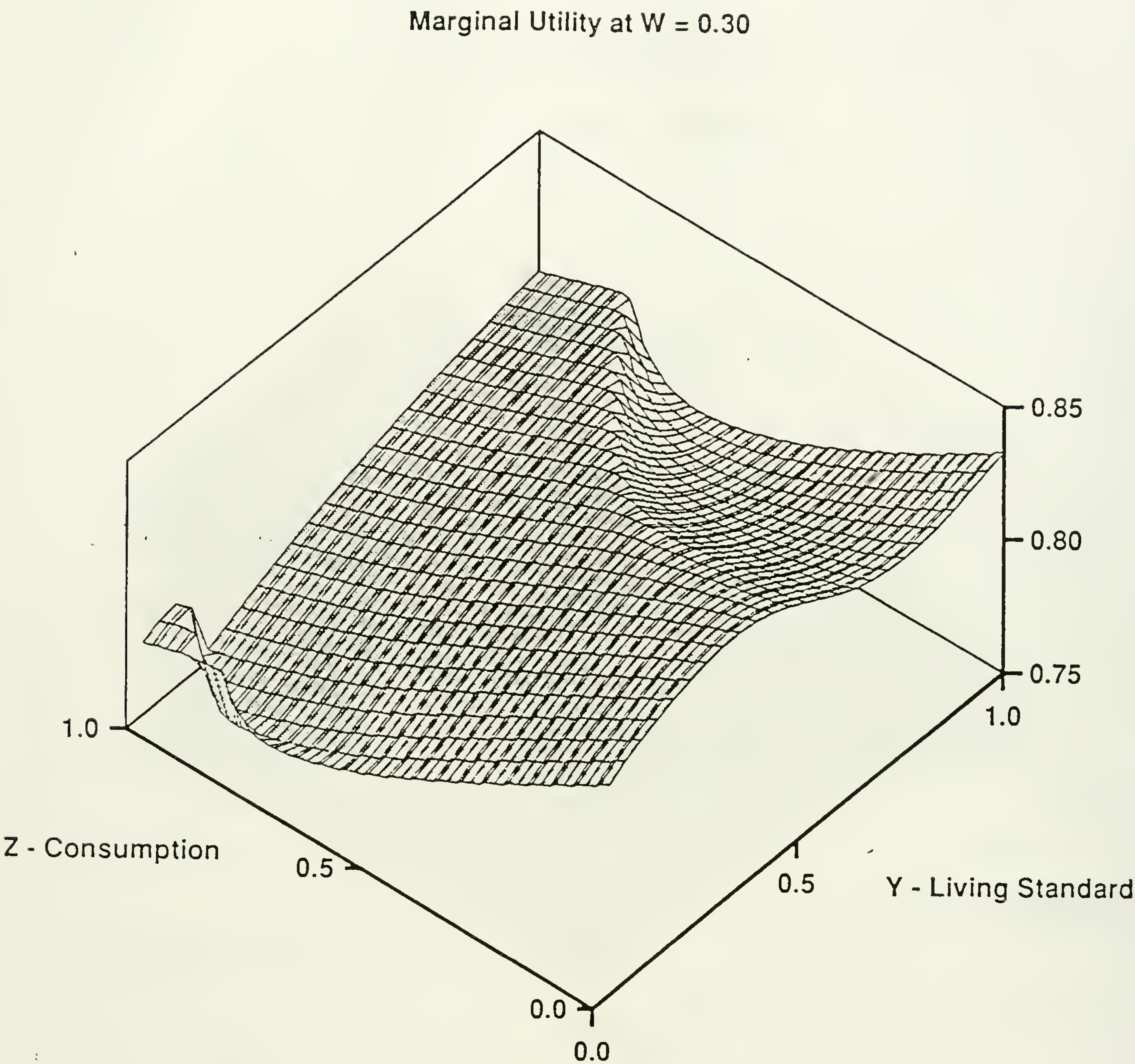

Figure 5: Samples of The Alternating Behavior of The Sum $\beta J_z + \lambda J_y$—II.

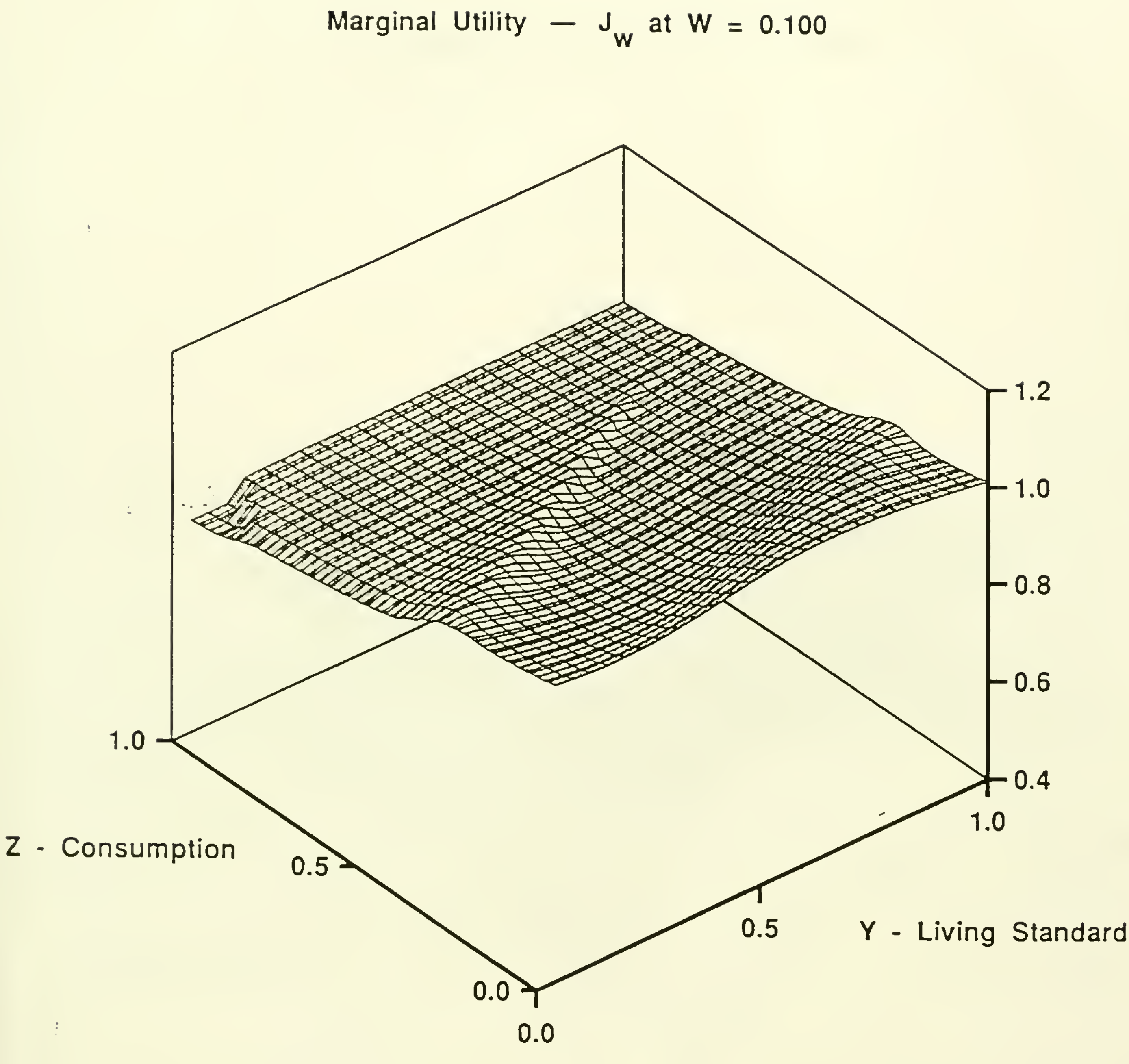

Figure 6: Samples of The Variations in Marginal Value of Wealth—I.

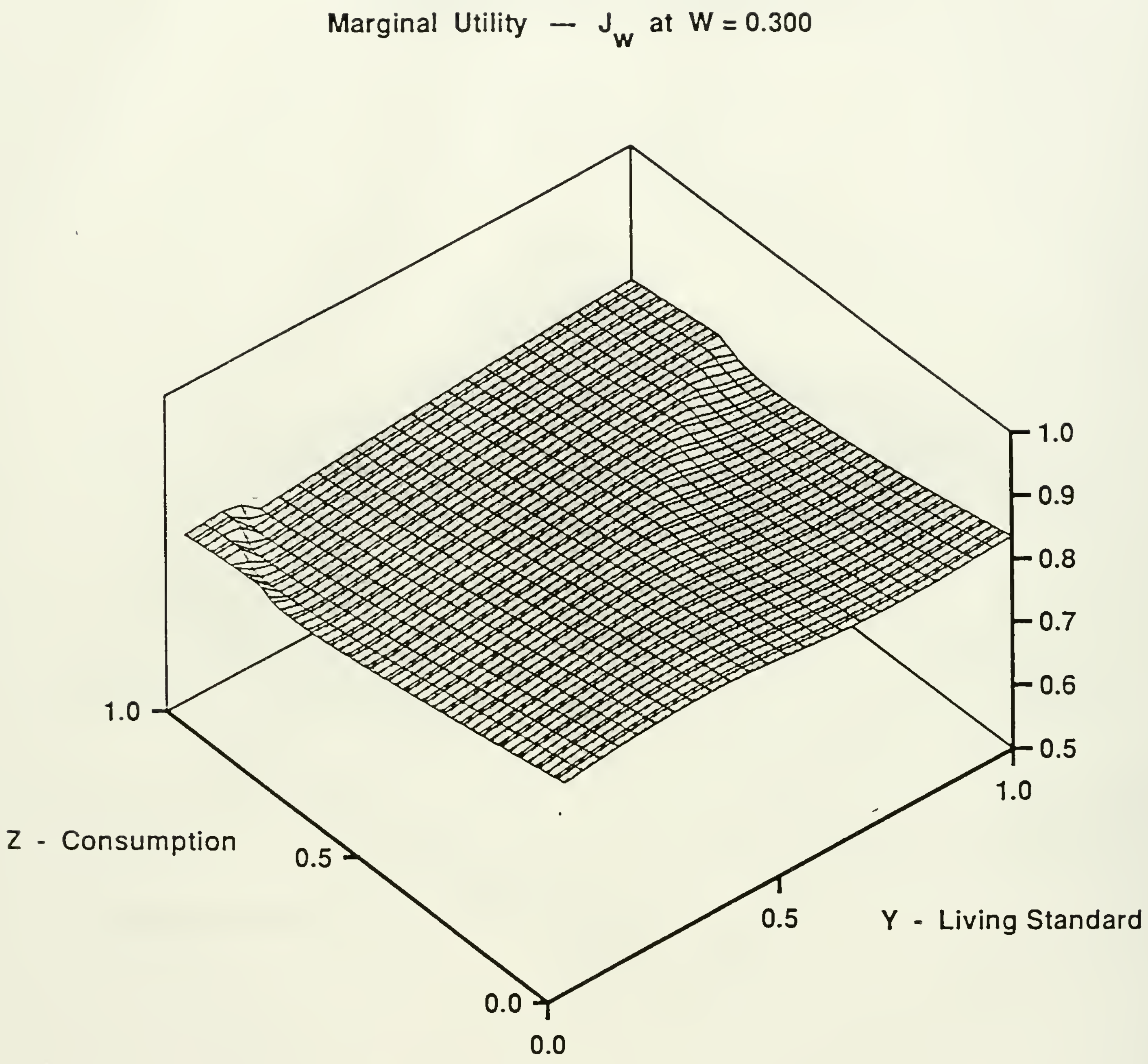

Figure 7: Samples of The Variations in Marginal Value of Wealth—II.

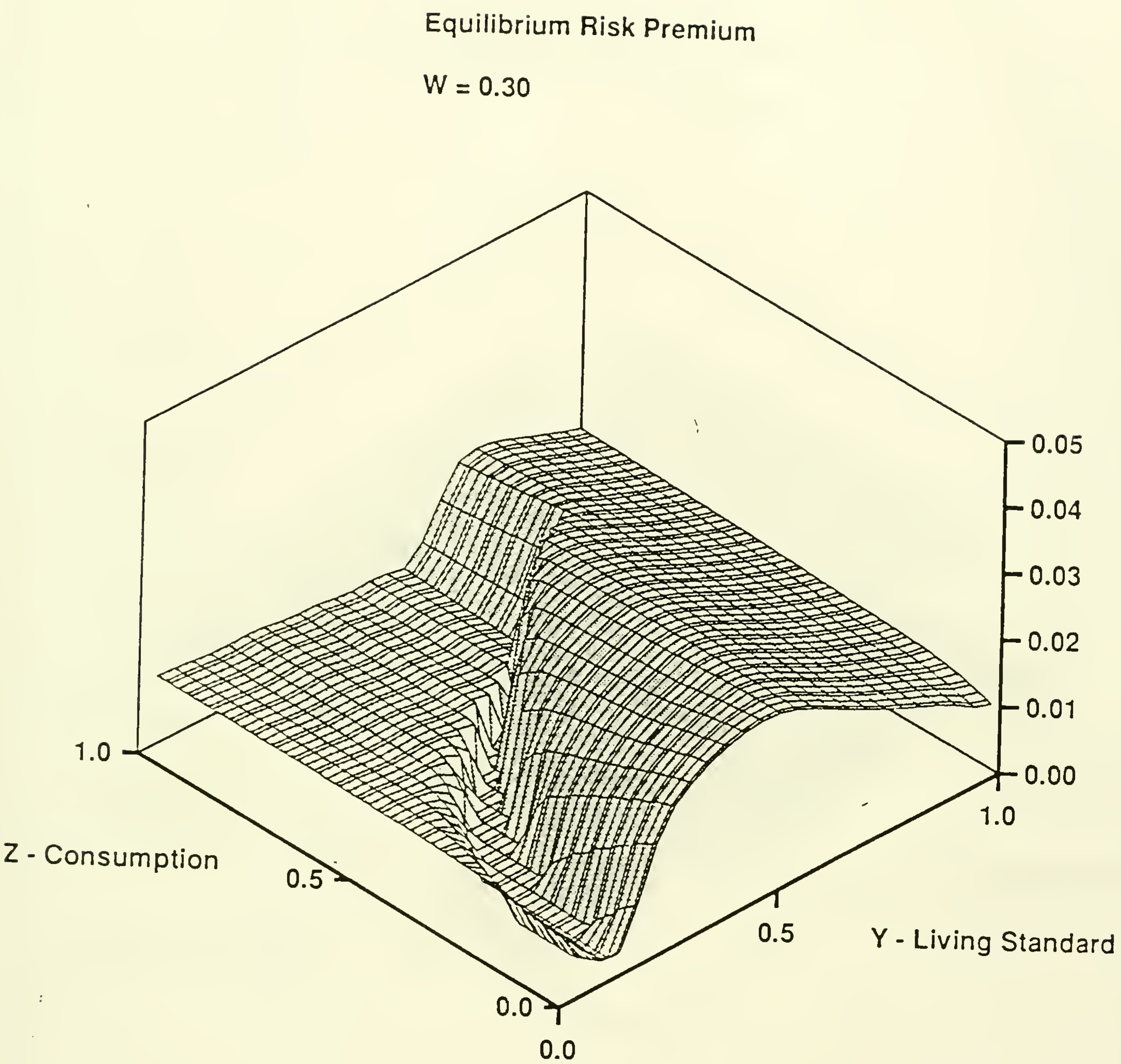

Figure 8: Samples of The Equilibrium Risk Premium—I.

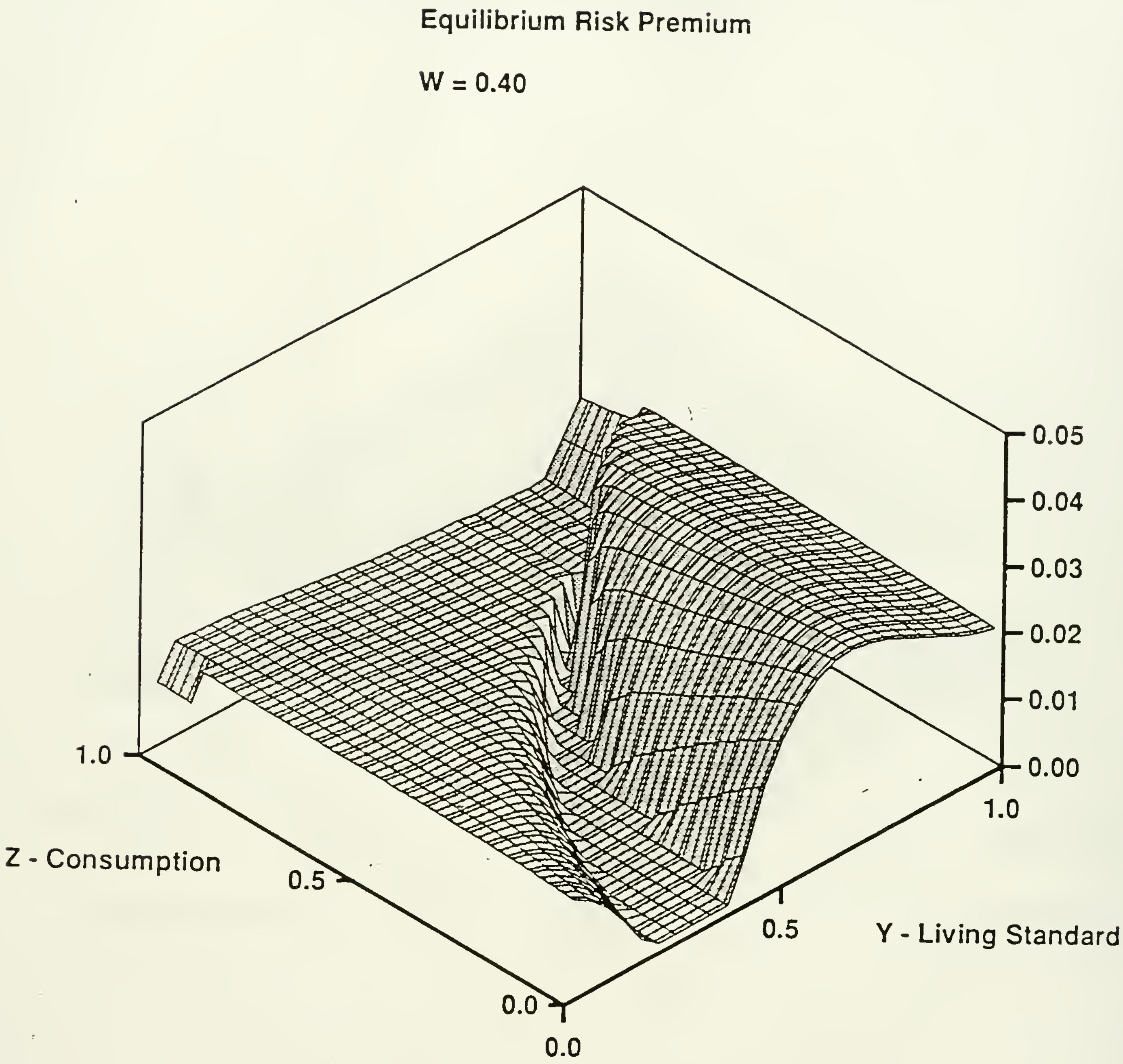

Figure 9: Samples of The Equilibrium Risk Premium—II.

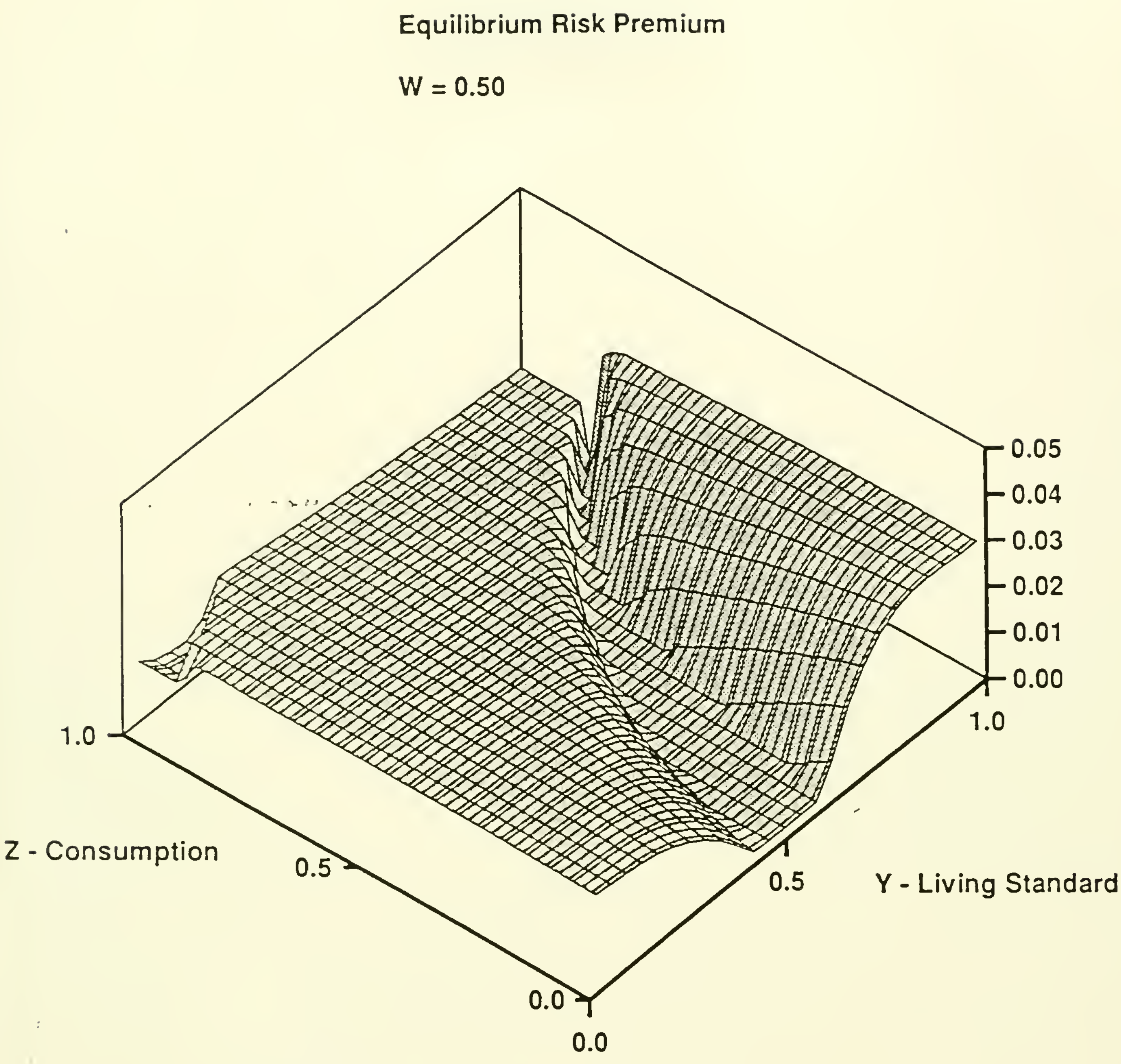

Figure 10: Samples of The Equilibrium Risk Premium—III.