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CORRELATOR ERRORS DUE TO FINITE OBSERVATION INTERVALS

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Abstract

A measurement signal-to-noise ratio is defined as the reciprocal of a normalized rms measurement error. This ratio is evaluated for the general continuous correlator and for the general sampling correlator. The limiting behavior of the signal-to-noise ratio is studied for several specific cases. The continuous correlator cases considered are those where the averaging filter is: (a) a perfect integrator, and (b) an RC low-pass filter. The sampling correlator special case considered is that where the sampling is done periodically.

CORRELATOR ERRORS DUE TO FINITE OBSERVATION INTERVALS

I. Introduction

In studies dealing with stochastic functions of time, one continually meets statistical functions of the type

$$\begin{aligned} R_{12}(\tau) &\equiv \langle f_1(t) f_2(t - \tau) \rangle_{av} \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} f_1(t) f_2(t - \tau) dt . \end{aligned} \quad (1)$$

Such a function provides a measure of the statistical correlation between the time functions f_1 and f_2 . This function is well known as the "autocorrelation function" if f_1 and f_2 are the same, and as the "crosscorrelation function" if f_1 and f_2 are different (ref. 1, 2).

It is frequently of interest to be able to determine experimentally such a function as (1). From the definition of the correlation function, we see that the operations of time delay, multiplication and averaging over all time are involved. The operations of delay and multiplication are physically feasible, but the requirement of experimentally averaging over an infinite time interval obviously cannot be met. It is the purpose of this paper to investigate the errors incurred by confining one's measurements to a finite observation interval in time.*

Practical correlators are of two types: the "continuous correlators," where all operations are performed continuously in time throughout the observation interval (ref. 3, 4), and the "sampling correlators," where the operations are performed upon samples representing the values of the input functions at specific instants of time (ref. 5, 6). We shall discuss the two types of correlators separately.

II. The Continuous Correlator

a. General Relations

The continuous type of correlator is shown schematically by Fig. 1. One of the correlator input functions, f_2 , is delayed in time by an amount τ_0 , and then multiplied by the other input function, f_1 , to form a product function, x

$$x(t) = f_1(t) f_2(t - \tau_0) . \quad (2)$$

The infinite-time average of the product function is thus the desired correlation function evaluated for the specific delay τ_0

$$\langle x(t) \rangle_{av} = R_{12}(\tau_0) . \quad (3)$$

*After the writing of this report it was learned that Johnson and Middleton (9) were making a somewhat similar but more general study.

The operation of averaging is performed by passing the product function, x , through an averaging filter. The output, y , of the averaging filter may be related to its input by means of a convolution integral involving the filter unit impulse response, or weighting function, $h(t)$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau . \quad (4)$$

Throughout this paper we will assume that the averaging filter is either passive or absolutely stable, thus requiring that the filter impulse response be zero for negative values of its argument.

We may now place into evidence the effect of an observation interval of finite length, T_o , by defining a new impulse response function, $h(t, T_o)$, which is equal to the filter unit impulse response throughout the observation interval and equal to zero thereafter

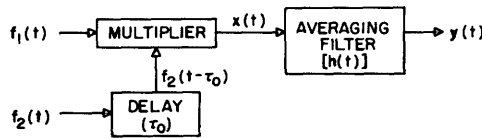


Fig. 1 The continuous correlator.

$$h(t, T_o) \equiv \begin{cases} h(t) & \text{for } t \leq T_o , \\ 0 & \text{for } T_o < t . \end{cases} \quad (5)$$

Thus, our filter input-output relation may be written as

$$y(t) = \int_{-\infty}^{+\infty} h(\tau, T_o) x(t - \tau) d\tau . \quad (6)$$

The continuous correlator output may then be expressed in terms of the correlator inputs

$$y(t) = \int_{-\infty}^{+\infty} h(\tau, T_o) f_1(t - \tau) f_2(t - \tau_o - \tau) d\tau . \quad (7)$$

Let us now assume that the correlator inputs are statistically stationary functions of time. The correlator output will then also be a statistically stationary function of time whose average value is directly related to the desired correlation function. The variations of the correlator output about this average value will be considered as measurement errors.

The average value of the correlator output is by definition

$$\langle y(t) \rangle_{av} \equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} y(t) dt . \quad (8)$$

From the filter input-output relation, (6), we then have

$$\begin{aligned}
\langle y(t) \rangle_{av} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} dt \int_{-\infty}^{+\infty} h(\tau, T_0) x(t - \tau) d\tau \\
&= \int_{-\infty}^{+\infty} h(\tau, T_0) d\tau \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} x(t - \tau) dt \quad (9)
\end{aligned}$$

We are dealing with statistically stationary time functions, hence a translation in time does not change the value of an average. Thus, we may rewrite (9) as

$$\langle y(t) \rangle_{av} = \int_{-\infty}^{+\infty} h(\tau, T_0) d\tau \langle x(t) \rangle_{av} \quad (10)$$

Consequently, using (3), we see that the average value of the correlator output is equal to the desired correlation function multiplied by a constant

$$\langle y(t) \rangle_{av} = R_{12}(\tau_0) \int_{-\infty}^{+\infty} h(\tau, T_0) d\tau \quad (11)$$

At this point in our discussion, it is convenient to define a system function, $H(\omega, T_0)$, as the Fourier transform of the impulse response, $h(t, T_0)$

$$H(\omega, T_0) \equiv \int_{-\infty}^{+\infty} h(t, T_0) e^{-i\omega t} dt \quad (12)$$

This function is thus the system function of the averaging filter when we take into account the effect of a finite observation interval. Setting $\omega = 0$ in (12) gives

$$H(0, T_0) = \int_{-\infty}^{+\infty} h(t, T_0) dt \quad (13)$$

Then, using (13) in (11), we see that we may also express the average value of the correlator output as

$$\langle y(t) \rangle_{av} = R_{12}(\tau_0) H(0, T_0) \quad (14)$$

The mean-square measurement error is defined as

$$\begin{aligned}
\sigma_y^2 &\equiv \langle [y(t) - \langle y(t) \rangle_{av}]^2 \rangle_{av} \\
&= \langle y^2(t) \rangle_{av} - \langle y(t) \rangle_{av}^2 \quad (15)
\end{aligned}$$

The mean square of the correlator output may be obtained from the autocorrelation function, $R_y(\tau)$, of the correlator output, or from its Fourier transform, $G_y(\omega)$, the spectral density of the correlator output

$$\langle y^2(t) \rangle_{av} = R_y(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_y(\omega) d\omega \quad (16)$$

The spectral density of the correlator output may be expressed in terms of the spectral density, $G_x(\omega)$, of the product function, x

$$G_y(\omega) = G_x(\omega) |H(\omega, T_0)|^2 \quad (17)$$

Thus the mean square of the correlator output may be expressed as

$$\langle y^2(t) \rangle_{av} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_x(\omega) |H(\omega, T_0)|^2 d\omega \quad (18)$$

Let us now define a new variable, $\xi(t)$, as the variation of the product function, $x(t)$, about its mean

$$\begin{aligned} \xi(t) &\equiv x(t) - \langle x(t) \rangle_{av} \\ &= x(t) - R_{12}(\tau_0) \end{aligned} \quad (19)$$

The autocorrelation function, $R_x(\tau)$, of the product function may then be expressed in terms of the autocorrelation function, $R_\xi(\tau)$, of $\xi(t)$ as

$$R_x(\tau) = R_\xi(\tau) + R_{12}^2(\tau_0) \quad (20)$$

Similarly, the spectral density, $G_x(\omega)$, of the product function may be expressed in terms of the spectral density, $G_\xi(\omega)$ of $\xi(t)$ as

$$G_x(\omega) = G_\xi(\omega) + 2\pi R_{12}^2(\tau_0) \delta(\omega) \quad (21)$$

where $\delta(\omega)$ is a Dirac delta function. Then substituting (21) into the expression (18) for the mean square of the correlator output, we obtain

$$\begin{aligned} \langle y^2(t) \rangle_{av} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_\xi(\omega) |H(\omega, T_0)|^2 d\omega \\ &+ R_{12}^2(\tau_0) |H(0, T_0)|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_\xi(\omega) |H(\omega, T_0)|^2 d\omega \\ &+ \langle y(t) \rangle_{av}^2 \end{aligned} \quad (22)$$

Thus we finally see that the mean-square measurement error may be expressed as

$$\sigma_y^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_\xi(\omega) |H(\omega, T_0)|^2 d\omega \quad (23)$$

Let us now define $R_h(\tau)$ as the Fourier transform of $|H(\omega, T_0)|^2$

$$R_h(\tau) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} |H(\omega, T_o)|^2 e^{i\omega\tau} d\omega . \quad (24)$$

Then, from (17) we see that $R_h(\tau)$ may be thought of as the autocorrelation function of the output of the averaging filter when the filter input is "white" noise with a spectral density of unity. An application of Parseval's Theorem then enables us to express $R_h(\tau)$ directly in terms of the impulse response $h(t, T_o)$

$$R_h(\tau) = \int_{-\infty}^{+\infty} h(\lambda, T_o) h(\tau + \lambda, T_o) d\lambda . \quad (25)$$

If we apply Parseval's Theorem to the expression for the mean-square correlator output error, (23), we obtain

$$\sigma_y^2 = \int_{-\infty}^{+\infty} R_\xi(\tau) R_h(\tau) d\tau . \quad (26)$$

In the preceding development, we have derived expressions for the average value of the correlator output and for the mean square of the variations of the correlator output about its mean. It is now desirable to consider some quantity providing a measure of the accuracy of the continuous correlator. For this purpose, it is convenient to define a measurement, rms, signal-to-noise ratio as the ratio of the average value of the correlator output to the rms value of the measurement error.

$$\left(\frac{S}{N}\right)_{\text{meas.}} \equiv \frac{\langle y(t) \rangle_{\text{av}}}{\sigma_y} . \quad (27)$$

This quantity is thus the reciprocal of the normalized rms measurement error. We may then express this measurement signal-to-noise ratio either as

$$\left(\frac{S}{N}\right)_{\text{meas.}} = \frac{R_{12}(\tau_o) H(0, T_o)}{\left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} G_\xi(\omega) |H(\omega, T_o)|^2 d\omega \right]^{1/2}} \quad (28)$$

from (14) and (23), or as

$$\left(\frac{S}{N}\right)_{\text{meas.}} = \frac{R_{12}(\tau_o) \int_{-\infty}^{+\infty} h(\tau, T_o) d\tau}{\left[\int_{-\infty}^{+\infty} R_\xi(\tau) R_h(\tau) d\tau \right]^{1/2}} \quad (29)$$

from (11) and (26). Now, remembering the relation between the impulse response, $h(t, T_o)$, and the averaging filter impulse response, $h(t)$, we see that we may change

the limits on the numerator integral to $(0, T_o)$. Further, the definition of $R_h(\tau)$, (25), shows that this quantity is zero outside the interval $(-T_o, +T_o)$. Thus, we may rewrite (29) as

$$\left(\frac{S}{N}\right)_{\text{meas.}} = \frac{R_{12}(\tau_o) \int_0^{T_o} h(\tau) d\tau}{\left[2 \int_0^{T_o} R_{\xi}(\tau) R_h(\tau) d\tau\right]^{1/2}} \quad (30)$$

since both $R_{\xi}(T)$ and $R_h(T)$ are even functions of T .

A study of the expressions for the measurement signal-to-noise ratio shows that this ratio is a function not only of the properties of the averaging filter but also of the statistical properties of the correlator input functions. Further, the dependence of this ratio on the statistics of the correlator inputs is through the autocorrelation function of their product function, $x(t)$

$$\begin{aligned} R_x(\tau) &= \langle x(t) x(t - \tau) \rangle_{\text{av}} \\ &= \langle f_1(t) f_2(t - \tau_o) f_1(t - \tau) f_2(t - \tau_o - \tau) \rangle_{\text{av}} \end{aligned} \quad (31)$$

This autocorrelation function is determined by the fourth order joint distribution of the correlator input functions, $f_1(t)$ and $f_2(t)$. Thus, the measurement signal-to-noise ratio can be determined from the second order distributions (or from derived functions such as the spectral densities) of the correlator inputs only in the trivial case where the correlator input functions are statistically independent.

b. Perfect Integrator Averaging Filter

Let us consider now the special case where the averaging filter of the continuous correlator is a perfect integrator, but where only a finite duration, T_o , observation interval is permitted. In this case

$$h(t, T_o) = \begin{cases} 1 & \text{for } 0 \leq t \leq T_o \\ 0 & \text{otherwise,} \end{cases} \quad (32)$$

for the filter input-output relation (6) then becomes

$$y(t_1) = \int_{t_1 - T_o}^{t_1} x(t) dt \quad (33)$$

Let us now evaluate the expression (30) for the measurement signal-to-noise ratio. From (32) we see that the numerator integral of (30) becomes just T_o . To evaluate the denominator of (30) we must first determine $R_h(\tau)$ for this case. From (25) and (32),

we see that

$$R_h(\tau) = \begin{cases} T_o - \tau & \text{for } 0 \leq \tau \leq T_o \\ 0 & \text{for } \tau > T_o \end{cases} \quad (34)$$

The denominator integral of (30) then becomes

$$2 \int_0^{T_o} R_\xi(\tau) R_h(\tau) d\tau = 2T_o \int_0^{T_o} \left(1 - \frac{\tau}{T_o}\right) R_\xi(\tau) d\tau \quad (35)$$

Thus, finally, the measurement signal-to-noise ratio may be expressed as

$$\left(\frac{S}{N}\right)_{\text{meas.}} = \frac{R_{12}(\tau_o) \sqrt{T_o}}{\left[2 \int_0^{T_o} \left(1 - \frac{\tau}{T_o}\right) R_\xi(\tau) d\tau\right]^{1/2}} \quad (36)$$

when the averaging filter is a perfect integrator and a finite duration observation interval is used.

It is of interest to consider the behavior of the ratio (36) for very small and very large values of T_o .

For very small values of T_o , the autocorrelation function $R_\xi(\tau)$ becomes nearly equal to its value for τ equal to zero. Hence

$$2 \int_0^{T_o} \left(1 - \frac{\tau}{T_o}\right) R_\xi(\tau) d\tau \rightarrow 2R_\xi(0) \int_0^{T_o} \left(1 - \frac{\tau}{T_o}\right) d\tau = R_\xi(0) T_o \quad (37)$$

and (36) becomes

$$\left(\frac{S}{N}\right)_{\text{meas.}} \rightarrow \frac{R_{12}(\tau_o)}{[R_\xi(0)]^{1/2}} \text{ as } T_o \rightarrow 0 \quad (38)$$

which is nothing, more or less, than the signal-to-noise ratio at the output of the multiplier. Thus we see that for very small duration observation intervals, the measurement signal-to-noise ratio becomes independent of the duration of the observation interval. From (37), we see that the specification of "small T_o " in (38) corresponds to the requirement that the autocorrelation function $R_\xi(\tau)$ be substantially equal to the value of $R_\xi(0)$ for values of τ in the interval $(0, T_o)$.

For very large values of T_o we have

$$\lim_{T_o \rightarrow \infty} 2 \int_0^{T_o} \left(1 - \frac{\tau}{T_o}\right) R_\xi(\tau) d\tau = 2 \int_0^{\infty} R_\xi(\tau) d\tau = G_\xi(0) \quad (39)$$

In this case (36) becomes

$$\left(\frac{S}{N}\right)_{\text{meas.}} \rightarrow \frac{R_{12}(\tau_o) \sqrt{T_o}}{[G_{\xi}(0)]^{1/2}} \text{ as } T_o \rightarrow \infty . \quad (40)$$

Thus, for very large values of T_o , the measurement signal-to-noise ratio is directly proportional to the square root of the duration of the observation interval.

c. RC Low-Pass Averaging Filter

A second special case of interest is that where the averaging filter has the form of an RC low-pass filter. Such a filter is characterized by a system function of the form

$$H(\omega) = \frac{1}{1 + \frac{i\omega}{a}} \quad (41)$$

where a is the angular half-power frequency of the filter. The corresponding impulse response is

$$h(t) = \begin{cases} a e^{-at} & 0 \leq t \\ 0 & t < 0 \end{cases} . \quad (42)$$

The filter impulse response then becomes

$$h(t, T_o) = \begin{cases} a e^{-at} & 0 \leq t \leq T_o \\ 0 & \text{otherwise} \end{cases} . \quad (43)$$

when we take into account the effect of an observation interval of finite duration.

Let us evaluate the expression (30) for the measurement signal-to-noise ratio.

From (43), we see that the numerator integral of (30) becomes

$$\int_0^{T_o} h(t, T_o) dt = 1 - e^{-aT_o} \quad (44)$$

From (25) and (43), we see that

$$\begin{aligned} R_h(\tau) &= a^2 \int_0^{T_o - \tau} e^{-aT} e^{-a(T+\tau)} dT \\ &= \begin{cases} \frac{a}{2} e^{-a\tau} \left[1 - e^{-2a(T_o - \tau)} \right] & \text{for } 0 \leq \tau \leq T_o \\ 0 & \text{otherwise} \end{cases} . \end{aligned} \quad (45)$$

Thus, the measurement signal-to-noise ratio may be expressed as

$$\left(\frac{S}{N}\right)_{\text{meas.}} = \frac{R_{12}(\tau_0) \left(1 - e^{-\alpha T_0}\right)}{\left[\alpha \int_0^{T_0} e^{-\alpha \tau} \left[1 - e^{-2\alpha(T_0 - \tau)}\right] R_{\xi}(\tau) d\tau \right]^{1/2}} \quad (46)$$

when the averaging filter has the form of an RC low-pass filter, and a finite duration observation interval is used.

Again, let us consider the behavior of the measurement signal-to-noise ratio for very small and very large values of T_0 .

For very small values of T_0

$$1 - e^{-\alpha T_0} \rightarrow \alpha T_0 \text{ as } \alpha T_0 \rightarrow 0. \quad (47)$$

Then, substituting (47) into (46) gives

$$\left(\frac{S}{N}\right)_{\text{meas.}} \rightarrow \frac{R_{12}(\tau_0) \sqrt{T_0}}{\left[2 \int_0^{T_0} \left(1 - \frac{\tau}{T_0}\right) R_{\xi}(\tau) d\tau \right]^{1/2}} \text{ as } \alpha T_0 \rightarrow 0. \quad (48)$$

Thus, for very small values of αT_0 , the RC low-pass filter acts as a perfect integrator. The specification of "small αT_0 " in (48) corresponds to the requirement that the duration, T_0 , of the observation interval be small compared to the reciprocal of the half-power angular frequency, α , of the low-pass filter (i.e. small compared to the filter time constant). The limiting expression, (38), thus holds for values of T_0 small compared to those required by (48). For very large values of T_0 , the exponential terms containing T_0 vanish in (46). In this case we have

$$\left(\frac{S}{N}\right)_{\text{meas.}} \rightarrow \frac{R_{12}(\tau_0) \sqrt{\frac{1}{\alpha}}}{\left[\int_0^{\infty} e^{-\alpha \tau} R_{\xi}(\tau) d\tau \right]^{1/2}} \text{ as } T_0 \rightarrow \infty. \quad (49)$$

Thus for very large values of T_0 , the measurement signal-to-noise ratio becomes independent of the duration of the observation interval. This limiting case has been considered in some detail by Fano, (ref. 7).

Let us further investigate the behavior of the measurement signal-to-noise ratio for large values of T_0 . While $R_{\xi}(\tau)$ has no constant terms, it may well have both periodic and random terms which are also the periodic and random terms in $R_x(\tau)$.

Let us write

$$R_{\xi}(\tau) = \sum_{p=1}^P R_{px}(0) \cos \omega_p \tau + R_{rx}(\tau) \quad (50)$$

where $R_{rx}(\tau)$ is the random term and we have assumed that there are P periodic terms.

Let us now use (50) in (49). First of all, for the periodic terms

$$R_{px}(0) \int_0^{\infty} e^{-a\tau} \cos \omega_p \tau d\tau = R_{px}(0) \frac{a}{a^2 + \omega_p^2} \quad (51)$$

Consider now the $R_{rx}(\tau)$ term. Because of the very purpose of the low-pass averaging filter, we commonly meet the condition that the bandwidth of the averaging filter is very small compared to the bandwidth of the random part of the product function, $x(t)$. Hence

$$\begin{aligned} \int_0^{\infty} R_{rx}(\tau) e^{-a\tau} d\tau &\doteq \int_0^{\infty} R_{rx}(\tau) d\tau \\ &= \frac{G_{rx}(0)}{2} \end{aligned} \quad (52)$$

since the stated bandwidth conditions imply that $e^{-a\tau}$ is substantially constant, and equal to unity throughout the range of values of τ for which $R_{rx}(\tau)$ has significantly nonzero values.

We may finally say that

$$\left(\frac{S}{N}\right)_{\text{meas.}} \doteq \frac{R_{12}(\tau_0) \sqrt{\frac{1}{a}}}{\left[\sum_{p=1}^P \frac{R_{px}(0) a}{a^2 + \omega_p^2} + \frac{G_{rx}(0)}{2} \right]^{1/2}} \quad (53)$$

for the case of an RC low-pass averaging filter when used with very long observation intervals.

III. The Sampling Correlator

a. General Relations

The sampling type of correlator is shown schematically by Fig. 2.

The output of a sampler corresponds to the value of its input at the instant of sampling. The sampler outputs are multiplied to form a product sample

$$x(t_s) = f_1(t_s) f_2(t_s - \tau_0) \quad (54)$$

where t_s is the instant of sampling f_1 .

The process of averaging in the sampling correlator is performed by adding a finite number, M , of the product samples. When M product samples have been added, their sum is presented as the correlator output

$$y = \sum_{s=1}^M f_1(t_s) f_2(t_s - \tau_0) \quad (55)$$

The correlator output, y , is thus a random variable dependent upon the values of the correlator inputs at the sampling instants, t_s , and upon the number of samples, M . The mean value of the correlator output is directly related to the desired correlation function, and the variations of the correlator output about its mean will be considered as measurement errors.

The mean, or statistical average, of the correlator output is

$$\bar{y} = \sum_{s=1}^M \overline{f_1(t_s) f_2(t_s - \tau_0)} \quad (56)$$

since the mean of a sum is the sum of the means. Under the assumption that we are dealing with ergodic input functions, we may replace the operation of statistical averaging by that of time averaging. Thus

$$\overline{f_1(t_s) f_2(t_s - \tau_0)} = \langle f_1(t) f_2(t - \tau_0) \rangle_{av} \equiv R_{12}(\tau_0) \quad (57)$$

irrespective of the value of t_s . The mean value of the correlator output is then given by

$$\bar{y} = M R_{12}(\tau_0) \quad (58)$$

and is M times the desired correlation function evaluated for the specific delay of τ_0 .

The mean measurement error is by definition

$$\sigma_y^2 = \overline{[y - \bar{y}]^2} = \overline{y^2} - \bar{y}^2 \quad (59)$$

The mean square of the correlator output is given by

$$\begin{aligned} \overline{y^2} &= \overline{\left[\sum_{s=1}^M f_1(t_s) f_2(t_s - \tau_0) \right]^2} \\ &= \sum_{j=1}^M \sum_{k=1}^M \overline{f_1(t_j) f_2(t_j - \tau_0) f_1(t_k) f_2(t_k - \tau_0)} \quad (60) \end{aligned}$$

In this double summation, there are M terms of the form ($j=k$), and for each of these terms we have

$$\overline{f_1^2(t_j) f_2^2(t_j - \tau_0)} = \langle f_1^2(t) f_2^2(t - \tau_0) \rangle_{av}$$

since we are dealing with statistically stationary input functions. The mean-square correlator output may then be expressed as

$$\overline{y^2} = M \langle f_1^2(t) f_2^2(t - \tau_0) \rangle_{av} + \sum_{j=1}^M \sum_{k=1}^M \overline{f_1(t_j) f_2(t_j - \tau_0) f_1(t_k) f_2(t_k - \tau_0)} \quad (61)$$

(k ≠ j)

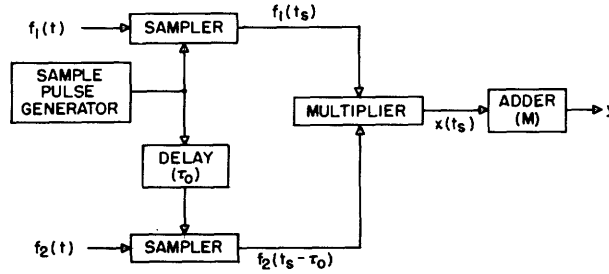


Fig. 2 The sampling correlator.

Now, as in the continuous correlator case, we will define a measurement rms signal-to-noise ratio as the ratio of the average value of the correlator output to the rms variation of the correlator output about its mean

$$\left(\frac{S}{N}\right)_{\text{meas.}} \equiv \frac{\overline{y}}{\sigma_y} \quad (62)$$

Thus, for the general case of sampling at arbitrary instants of time, the measurement signal-to-noise ratio for the sampling correlator is given by

$$\left(\frac{S}{N}\right)_{\text{meas.}} = \left[\frac{M R_{12}(\tau_0)}{M \langle f_1^2(t) f_2^2(t - \tau_0) \rangle_{av} - M^2 R_{12}^2(\tau_0)} + \sum_{j=1}^M \sum_{k=1}^M \overline{f_1(t_j) f_2(t_j - \tau_0) f_1(t_k) f_2(t_k - \tau_0)} \right]^{1/2} \quad (63)$$

(k ≠ j)

We note that this ratio is a function of the number of samples, M ; the instants of sampling, t_s ; and of the statistical properties of the correlator input functions, including the fourth order probability density required to determine the mean of the product: $f_1(t_j) f_2(t_j - \tau_0) f_1(t_k) f_2(t_k - \tau_0)$.

b. Periodic Sampling

Perhaps the case of greatest practical interest is that where the sampling is done periodically. In this case, the M samples are spaced equally throughout the observation interval of duration T_0 . The sampling period is thus

$$t_o = \frac{T_o}{M} \quad (64)$$

If then the observation interval starts at $t=0$ (a matter of convenience only) the k^{th} sampling instant is kt_o . We may rewrite our expression (61) for the mean square of the correlator output as

$$\begin{aligned} \overline{y^2} &= M \langle f_1^2(t) f_2^2(t - \tau_o) \rangle_{av} \\ &+ \sum_{j=1}^M \sum_{k=1}^M \frac{f_1(jt_o) f_2(jt_o - \tau_o) f_1(kt_o) f_2(kt_o - \tau_o)}{(k \neq j)} \end{aligned} \quad (65)$$

Let us consider the double summation terms in (65). For this purpose, it is convenient to visualize an M by M array of the elements (j, k) making up the sum. First of all, the principal diagonal terms (j, j) are missing because of the restriction $(k \neq j)$. Second, each element (j, k) is equal to the element (k, j) . Thus every term above the principal diagonal has an equal counterpart below that diagonal. Further, each element (j, k) is equal to each element of the form $(j + i, k + i)$, since we are dealing with statistically stationary input functions. All of the $(M-m)$ elements in the m^{th} diagonal above the principal diagonal are equal. Applying these relations, (65) becomes

$$\begin{aligned} \overline{y^2} &= M \langle f_1^2(t) f_2^2(t - \tau_o) \rangle_{av} \\ &+ 2 \sum_{m=1}^{M-1} (M-m) \langle f_1(t) f_2(t - \tau_o) f_1(t+mt_o) f_2(t+mt_o - \tau_o) \rangle_{av} \end{aligned} \quad (66)$$

and our measurement signal-to-noise ratio may be expressed as

$$\left(\frac{S}{N} \right)_{\text{meas.}} = \left[\frac{M R_{12}(\tau_o)}{M \langle f_1^2(t) f_2^2(t - \tau_o) \rangle_{av} - M^2 R_{12}^2(\tau_o)} + 2 \sum_{m=1}^{M-1} (M-m) \langle f_1(t) f_2(t - \tau_o) f_1(t+mt_o) f_2(t+mt_o - \tau_o) \rangle_{av} \right]^{1/2} \quad (67)$$

To visualize the variation of the measurement signal-to-noise ratio as a function of the duration of the observation interval, it is convenient to define a new variable, ξ , equal to the variation of the product function, x , about its mean

$$\begin{aligned}\xi(t) &\equiv x(t) - \langle x(t) \rangle_{av} \\ &= f_1(t) f_2(t - \tau_o) - R_{12}(\tau_o)\end{aligned}\quad (68)$$

Using this variable, we may rewrite our expression (66) for the mean square of the correlator output

$$\begin{aligned}\overline{y^2} &= M \left[\langle \xi^2(t) \rangle_{av} + R_{12}^2(\tau_o) \right] \\ &+ 2 \sum_{m=1}^{M-1} (M-m) \left[\langle \xi(t) \xi(t + m\tau_o) \rangle_{av} + R_{12}(\tau_o) \right]\end{aligned}\quad (69)$$

and hence

$$\overline{y^2} = M R_{\xi}(0) + M^2 R_{12}^2(\tau_o) + 2 \sum_{m=1}^{M-1} (M-m) R_{\xi}(m\tau_o)\quad (70)$$

a result which will be recognized as being essentially the same as that obtained by Costas (ref. 8) in his study of a single random variable.

If we then use (70), we may finally express the measurement signal-to-noise ratio for the sampling correlator as

$$\left(\frac{S}{N}\right)_{\text{meas.}} = \frac{R_{12}(\tau_o) \sqrt{M}}{\left[R_{\xi}(0) + 2 \sum_{m=1}^{M-1} \left(1 - \frac{m}{M}\right) R_{\xi}\left(\frac{mT_o}{M}\right) \right]^{1/2}}\quad (71)$$

in the case of periodic sampling.

If the duration, T_o , of the observation interval is held fixed, and the number of samples, M , in that interval is allowed to increase without limit, then (71) becomes in the limit the same as the expression (36) obtained for the case of the continuous correlator with a perfect integrator as an averaging filter. The results obtained in II(b) then apply to the case of the sampling correlator satisfying the above conditions,

Let us now consider the limiting case where, for a given number M of samples, the duration of the observation interval approaches zero. In this case

$$R_{\xi}\left(\frac{mT_o}{M}\right) \rightarrow R_{\xi}(0)\quad (72)$$

for all m . Then

$$2 \sum_{m=1}^{M-1} \left(1 - \frac{m}{M}\right) R_{\xi}\left(\frac{mT_o}{M}\right) \rightarrow (M-1) R_{\xi}(0)\quad (73)$$

and the measurement signal-to-noise ratio becomes

$$\left(\frac{S}{N}\right)_{\text{meas.}} \rightarrow \frac{R_{12}(\tau_o)}{[R_{\xi}(0)]^{1/2}} \text{ as } T_o \rightarrow 0 . \quad (74)$$

Thus we see that, for very short duration observation intervals, the measurement signal-to-noise ratio is independent of the number of samples and is dependent solely upon the properties of the correlator input functions. The specification of "very small T_o " in (74) corresponds to the requirement that the correlation function $R_{\xi}(\tau)$ be essentially constant throughout the interval $(0, T_o)$ in τ , as shown by (72).

Let us finally consider the case where the duration of the observation interval becomes very large. In this region, entirely different behaviors of the measurement signal-to-noise ratio may be obtained, depending upon the relation between the method of sampling and the statistics of the correlator input functions. The most favorable case is that where $R_{\xi}(\tau)$ contains no periodic components. In this case $R_{\xi}(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. Therefore, for a given value of M , $R(mT_o/M) \rightarrow 0$, for all m , as $T_o \rightarrow \infty$. The measurement signal-to-noise ratio then becomes

$$\left(\frac{S}{N}\right)_{\text{meas.}} \rightarrow \frac{R_{12}(\tau_o)}{[R_{\xi}(0)]^{1/2}} \sqrt{M} \text{ as } T_o \rightarrow \infty \quad (75)$$

which corresponds to the situation where successive product samples are statistically independent.

On the other hand, the most unfavorable situation arises when $R_{\xi}(\tau)$ contains a periodic component whose period is that of the sampling process. In this case $R_{\xi}(\tau)$ contains a component

$$R_{px}(0) \cos \omega_p \tau = R_{px}(0) \cos \left(\frac{2\pi M}{T_o} \tau \right) \quad (76)$$

and the summation in the denominator of (71) contains a set of terms

$$2 \sum_{m=1}^{M-1} \left(1 - \frac{m}{M}\right) R_{px}(0) \cos \left(\frac{2\pi M}{T_o} \cdot \frac{mT_o}{M} \right) = (M-1) R_{px}(0) . \quad (77)$$

In this case the measurement signal-to-noise ratio becomes

$$\left(\frac{S}{N}\right)_{\text{meas.}} \rightarrow \frac{R_{12}(\tau_o)}{[R_{px}(0)]^{1/2}} \text{ as } T_o \rightarrow \infty \quad (78)$$

which is independent of the duration T_o of the observation interval, and of the number of samples, M !

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