SIGNAL-TO-NOISE RATIO IN CORRELATION DETECTORS

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Abstract

This paper discusses the operation in the presence of noise of a correlation detector consisting of a multiplier followed by a low-pass filter. It is shown that in the most favorable cases the output signal-to-noise power ratio is proportional to the corresponding input ratio and to the ratio of the signal bandwidth to the bandwidth of the low-pass filter.

ON THE SIGNAL-TO-NOISE RATIO IN CORRELATION DETECTORS

1. Introduction

Lee, Wiesner and others (1, 2, 3, 4) have suggested a number of important physical applications of correlation functions. Notable among these are the detection of small periodic signals buried in noise and the determination of the transfer characteristics of linear systems in the presence of noise generated within the systems.

The limits of performance of such schemes depend upon our ability to approximate experimentally the mathematical definition of the desired correlation function. More precisely, the crosscorrelation function of $f_1(t)$ and $f_2(t)$ is defined as

$$\Phi_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t)\, f_2(t-\tau)\, dt \quad . \tag{1}$$

That is, $\Phi_{12}(\tau)$ is the average of the product $f_1(t)\, f_2(t-\tau)$ over all values of t. The autocorrelation function $\Phi(\tau)$ of f(t) is defined in the same manner by letting $f_1(t) = f_2(t) = f(t)$. It is clear that the main limitation to the experimental determination of any correlation function lies in the fact that we cannot average over all values of t. The resulting random error leads to a finite signal-to-noise ratio at the output of the system, which clearly depends upon the length of the time interval over which the average is performed. The situation is very similar to that encountered in connection with conventional filtering, of which the correlation techniques may be considered as an extension. Narrowing the band of a filter corresponds to increasing the averaging time of a correlator.

Two types of correlators appear to be of practical importance. In one type (5, 6) the two functions of time are sampled periodically over a time interval T; corresponding samples are then multiplied and added. The other type of correlator performs a continuous multiplication of the two time functions; the resulting product function is passed through a low-pass filter which, in effect, performs a weighted average (7, 8).

The noise reduction characteristics of correlators of the first type have been studied extensively by Lee (3). The effect of the sampling frequency, including the limiting case of infinite sampling frequency, has been analyzed more recently by Costas (9). The purpose of the present paper is to determine the corresponding characteristics for a correlator of the second type. A more complete analysis of this problem will be presented in a forthcoming report by W. B. Davenport, Jr. (11).

2. Method of Analysis

The correlator considered in this paper is defined by the following operations (as indicated in Fig. 1):

1. One of the input functions is delayed by a time τ to obtain $f_2(t-\tau)$.
2. The other input function $f_1(t)$ is multiplied by $f_2(t-\tau)$ to yield the product function

$\phi_\tau(t) = f_1(t)\, f_2(t - \tau)$.

3. The product function $\phi_\tau(t)$ is passed through an RC filter with an amplitude response

$$|Z(\omega)| = \frac{1}{\sqrt{1 + (\frac{\omega}{a})^2}} \qquad (2)$$

and a corresponding impulse response

$$z(t) = ae^{-at} \quad . \qquad (3)$$

The output from the filter may be expressed as

$$\psi_\tau(t) = a \int_{-\infty}^{t} F(x, \tau) e^{-a(t-x)}\, dx \qquad (4)$$

indicating that the filter performs a weighted average of $\Phi_\tau(t)$ over the past, using the impulse response of the filter as a weighting function.

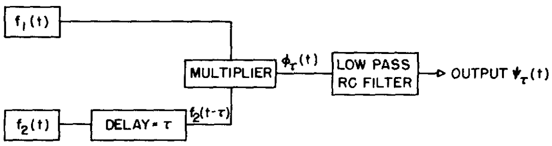

Fig. 1 Schematic diagram of a correlator.

Our problem is to determine the power ratio of the d-c component of $\psi_\tau(t)$, which is the desired output, to the a-c component, which represents the noise. The method of solution* consists of determining first the autocorrelation function of the product function $\phi_\tau(t)$. Then we shall be able to compute, by well-known methods, the autocorrelation function of the a-c output of the filter, whose value for $\tau = 0$ represents the noise power.

3. Autocorrelation Detection of a Sinusoid Mixed with Noise

We shall consider first the case in which

$$f(t) = A \cos \omega_o t + n(t) \qquad (5)$$

where A is a constant and n(t) is a random noise with a power spectrum

$$G_n(\omega) = \frac{N_o}{1 + (\frac{\omega}{a_n})^2} \quad . \qquad (6)$$

Such a noise may be obtained by passing a white noise through a low-pass filter such as that of Eq. 2 with a cut-off frequency equal to a_n.

The corresponding autocorrelation function of n(t) is given by

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*A bibliography on correlation functions may be found in Ref. 3.

$$\Phi_n(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N_o}{1 + (\frac{\omega}{\alpha_n})^2} \cos \omega\theta \, d\omega = N_o \frac{\alpha_n}{2} e^{-\alpha_n|\theta|} \quad . \tag{7}$$

The same function f(t) is fed to both inputs of the correlator. The product function $\phi_\tau(t)$ becomes in this case

$$\begin{aligned} \phi_\tau(t) &= \left[A \cos \omega_o t + n(t)\right] \left[A \cos \omega_o(t - \tau) + n(t - \tau)\right] \\ &= A^2 \cos \omega_o t \cos \omega_o(t - \tau) + A \cos \omega_o t \, n(t - \tau) \\ &\quad + A \cos \omega_o(t - \tau) \, n(t) + n(t) \, n(t - \tau) \quad . \end{aligned} \tag{8}$$

The constant and periodic components of $\phi_\tau(t)$ are

$$\left[\phi_\tau(t)\right]_p = \frac{1}{2} A^2 \left[\cos \omega_o(2t - \tau) + \cos \omega_o \tau\right] + \Phi_n(\tau) \quad . \tag{9}$$

The random component of $\phi_\tau(t)$ is

$$\left[\phi_\tau(t)\right]_r = A \cos \omega_o t \, n(t - \tau) + A \cos \omega_o(t - \tau) \, n(t) + \left[n(t) \, n(t - \tau) - \Phi_n(\tau)\right] \quad . \tag{10}$$

The autocorrelation function of $\left[\phi_\tau(t)\right]_r$ is, by definition

$$\psi_{1,\tau}(\theta) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\phi_\tau(t)\right]_r \left[\phi_\tau(t - \theta)\right]_r dt \tag{11}$$

and is then equal to the sum of the autocorrelation functions of the three terms of Eq. 9 and of the crosscorrelation functions of each pair of terms, in all possible combinations and permutations. All but one of the terms of $\psi_{1,\tau}(\theta)$ are readily computed, and $\psi_{1,\tau}(\theta)$ may be written in the form

$$\begin{aligned} \psi_{1,\tau}(\theta) &= \frac{1}{2} A^2 \left[2 \cos \omega_o \theta \, \Phi_n(\theta) + \cos \omega_o(\theta + \tau) \, \Phi_n(\theta - \tau) + \cos \omega_o(\theta - \tau) \, \Phi_n(\theta + \tau)\right] \\ &\quad - \Phi_n^2(\tau) + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[n(t) \, n(t - \theta) \, n(t - \theta) \, n(t - \theta - \tau)\right] dt \quad . \end{aligned} \tag{12}$$

Use has been made of the well-known fact that the average value of the product of two statistically independent time functions is equal to the product of their average values.

The computation of the autocorrelation function of $n(t) \, n(t - \tau)$, that is, of the integral in Eq. 12, presents some difficulty. The desired function depends, in general, on statistical characteristics of n(t) other than $\Phi_n(\theta)$. However, if the first four probability densities of n(t) are gaussian, the desired autocorrelation function is found to depend only upon $\Phi_n(\theta)$. Shot noise can be shown (10) to meet these requirements if certain reasonable assumptions are made about its physical nature. On the basis of the same assumptions, the desired autocorrelation function may be computed directly by means of simple extensions of two methods used by S. O. Rice (10) in secs. 2.6 and 4.5 of his paper, "Mathematical Analysis of Random Noise". Both procedures lead to the same result, the one corresponding to sec. 2.6 being presented in Appendix I. The final expression for $\psi_{1,\tau}(\theta)$ is found to be

$$\psi_{1,\tau}(\theta) = \frac{1}{2}A^2\Big[2\cos\omega_o\theta\,\phi_n(\theta) + \cos\omega_o(\theta+\tau)\,\Phi_n(\theta-\tau) + \cos\omega_o(\theta-\tau)\,\Phi(\theta+\tau)\Big] + \Phi_n^2(\theta) + \Phi_n(\theta-\tau)\,\Phi_n(\theta+\tau) \quad . \qquad (13)$$

The autocorrelation function $\psi_{2\tau}(\theta)$ of the random component of the output from the RC filter is easily determined as the convolution integral of $\psi_{1\tau}(\theta)$ and the Fourier transform of $|Z(\omega)|^2$. As a matter of fact, we need to compute only the value of $\psi_{2\tau}(\theta)$ for $\theta = 0$, which represents the mean square value of the random component of the output. We then obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\left(\frac{\omega}{a}\right)^2}\cos\omega\theta\, d\omega = \frac{a}{2}e^{-a|\theta|} \qquad (14)$$

and

$$\psi_{2\tau}(0) = \frac{a}{2}\int_{-\infty}^{\infty}\psi_{1,\tau}(\theta)\,e^{-a|\theta|}d\theta \quad . \qquad (15)$$

The integration is readily carried out, with the help of Eqs. 7 and 13, and yields

$$\psi_{2\tau}(0) = N_o A^2 \frac{a a_n}{4\omega_o}\left\{2\frac{\frac{a+a_n}{\omega_o}}{1+\left(\frac{a+a_n}{\omega_o}\right)^2} + \frac{e^{-a_n\tau}\left[\frac{a+a_n}{\omega_o}\cos\omega_o\tau + \sin\omega_o\tau\right] + e^{-a\tau}\left[\frac{a+a_n}{\omega_o}\cos 2\omega_o\tau - \sin 2\omega_o\tau\right]}{1+\left(\frac{a+a_n}{\omega_o}\right)^2}\right.$$

$$\left. + \frac{e^{-a\tau}\left[\frac{a_n - a}{\omega_o}\cos 2\omega_o\tau + \sin 2\omega_o\tau\right] - e^{-a_n\tau}\left[\frac{a_n - a}{\omega_o}\cos\omega_o\tau + \sin\omega_o\tau\right]}{1+\left(\frac{a_n - a}{\omega_o}\right)^2}\right\}$$

$$+ N_o^2\frac{a_n^2\, a}{4}\left\{\frac{1+e^{-(2a_n+a)\tau}}{2a_n + a} + e^{-2a_n\tau}\,\frac{1-e^{-a\tau}}{a}\right\} \quad . \qquad (16)$$

In all practical cases $a \ll a_n$ and $a_n\tau \gg 1$ because of the very purpose of the correlation measurement. It follows that most of the terms in Eq. 15 may be neglected, and $\psi_{2\tau}(0)$ is given, to a good approximation, by

$$\psi_{2\tau}(0)\, N_o = \frac{\alpha}{4}\left\{\frac{2A^2}{1+(\frac{\omega_o}{\alpha_n})^2}\,(1 + e^{-\alpha\tau}\cos 2\omega_o\tau) + \frac{N_o\alpha_n}{2}\right\} \quad . \tag{17}$$

The input $\phi_\tau(t)$ to the averaging network contains, in addition to the random component considered above, two d-c components and a periodic component of frequency $2\omega_o$, as indicated in Eq. 9. For all practical purposes the periodic component is filtered out. The d-c component resulting from the signal has a magnitude

$$\frac{1}{2} A^2 \cos\omega_o\tau \quad . \tag{18}$$

The d-c component resulting from the noise has a magnitude

$$\Phi_n(\tau) = N_o \frac{\alpha_n}{2} e^{-\alpha_n|\tau|} \tag{19}$$

and is therefore negligible for sufficiently large values of τ. Thus the ratio of the signal power to the noise power at the output of the averaging network is

$$\left(\frac{S}{N}\right)_{out} = \frac{2P^2\cos^2\omega_o\tau}{N_\alpha\left[N_{\alpha_n} + \frac{4P}{1+(\frac{\omega_o}{\alpha_n})^2}\,(1 + e^{-\alpha\tau}\cos 2\omega_o\tau)\right]} \tag{20}$$

where

$$P = \frac{1}{2} A^2 \tag{21}$$

$$N_{\alpha_n} = N_o \frac{\alpha_n}{2} \tag{22}$$

are the signal power and the noise power input to the correlator, and

$$N_\alpha = N_o \frac{\alpha}{2} \tag{23}$$

is the noise power that would be passed by the averaging network if fed by the original white noise of density N_o. If we use an optimum value of τ, for which $\cos\omega_o\tau = \cos 2\omega_o\tau = 1$, Eq. 19 reduces further to

$$\left(\frac{S}{N}\right)_{out} = \frac{2P^2}{N_\alpha\left[N_{\alpha_n} + \frac{4P}{1+(\frac{\omega_o}{\alpha_n})^2}\,(1 + e^{-\alpha\tau})\right]} \quad . \tag{24}$$

The significance of this result is best understood by considering the two extreme cases of very large and very small input signal-to-noise ratios. We obtain then

$$\left(\frac{S}{N}\right)_{out} = \begin{cases} \dfrac{P}{2N_a} \dfrac{1 + \left(\frac{\omega_o}{a_n}\right)^2}{1 + e^{-a\tau}} & \text{for } \dfrac{4P}{N_{a_n}} \gg \dfrac{1 + \left(\frac{\omega_o}{a_n}\right)^2}{1 + e^{-a\tau}} \\ \dfrac{2P^2}{N_a N_{a_n}} & \text{for } \dfrac{4P}{N_{a_n}} \ll \dfrac{1 + \left(\frac{\omega_o}{a_n}\right)^2}{1 + e^{-a\tau}} \end{cases} \quad . \tag{25}$$

In addition, we should take into proper account the fact that the RC network at the input to the correlator must affect the signal amplitude as well as the noise which determines the power spectrum of noise. Thus it would be more appropriate to express $(S/N)_{out}$ in terms of the actual signal power input

$$P_o = P\left[1 + \left(\frac{\omega_o}{a_n}\right)^2\right] \quad . \tag{26}$$

We have, then, for the two extreme cases

$$\left(\frac{S}{N}\right)_{out} = \begin{cases} \dfrac{P_o}{2N_a(1 + e^{-a\tau})} \\ \left(\dfrac{P_o}{N_a}\right)^2 \dfrac{2a}{a_n\left[1 + \left(\frac{\omega_o}{a_n}\right)^2\right]} \end{cases} \quad . \tag{27}$$

In the first case the $(S/N)_{out}$ is independent of the input bandwidth a_n of the correlator. In the second case $(S/N)_{out}$ is roughly proportional to $1/a_n$ as long as $a_n > \omega_o$.

4. Crosscorrelation Detection of a Sinusiod Mixed with Noise

In this case, the input to the correlator consists of the two functions

$$f_1(t) = A \cos \omega_o t + n(t)$$

$$f_2(t) = B \cos \omega_o t \tag{28}$$

where n(t) is a random noise. The product function input to the averaging network is, therefore

$$\phi_\tau(t) = f_1(t)\, f_2(t - \tau) = AB \cos \omega_o t \cos \omega_o(t - \tau) + Bn(t) \cos \omega_o(t - \tau)$$

$$= \frac{1}{2} AB(\cos \omega_o \tau + \cos \omega_o(2t - \tau)) + Bn(t) \cos \omega_o(t - \tau) \quad . \tag{29}$$

We shall see that in this case the output signal from the correlator is directly proportional to the first power of A rather than to A^2, as in the autocorrelation case. In other words, crosscorrelation may be used for the purpose of linear detection. In view of this fact, we wish to allow A to vary with time as

$$A = A_o(1 + F(t)) \quad . \tag{30}$$

Then, $\phi_\tau(t)$ will consist of the following components

1. A d-c component: $1/2\ A_o B \cos\omega_o\tau$
2. An amplitude modulated component of frequency $2\omega_o$:

$$\frac{1}{2}\ A_o B(1 + F(t)) \cos\omega_o(2t - \tau)$$

which will be eliminated in the averaging process

3. A signal component: $1/2\ A_o BF(t) \cos\omega_o\tau$
4. A noise component: $Bn(t) \cos\omega_o(t - \tau)$.

Let $\Phi_s(\theta)$ and $\Phi_n(\theta)$ be the autocorrelation functions of F(t) and n(t) respectively. If F(t) and n(t) are independent random functions, the autocorrelation function $\psi_{1,\tau}(\theta)$ of the random part of $\phi_\tau(t)$ is simply the sum of the autocorrelation functions of the signal component and of the noise component

$$\psi_{1,\tau}(\theta) = \frac{1}{4}\ A_o^2\ B^2 \cos^2\omega_o\tau\,\Phi_s(\theta) + \frac{1}{2}\ B^2 \cos\omega_o\theta\ \Phi_n(\theta) \quad . \tag{31}$$

The problem of separating the signal from the noise becomes at this point a special case of optimum filter design. If we assume, for simplicity, that an RC network is used for this purpose, as in sec. 3, we obtain for the output noise power

$$N_{out} = N_o B^2 \frac{\alpha\alpha_n}{8} \int_{-\infty}^{\infty} e^{-\alpha_n|\theta|}\, e^{-\alpha|\theta|} \cos\omega_o\theta\, d\theta = \frac{1}{2}\ B^2 N_\alpha \frac{1}{1 + \left(\frac{\omega_o}{\alpha_n + \alpha}\right)^2} \frac{\alpha_n}{\alpha_n + \alpha} \quad . \tag{32}$$

Since in all practical cases $\alpha_n \gg \alpha$ this equation reduces to

$$N_{out} = \frac{1}{2}\ B^2 N_\alpha \frac{1}{1 + \left(\frac{\omega_o}{\alpha_n}\right)^2} \quad . \tag{33}$$

The quantities α, α_n, N_o and N_α have the same meaning as in sec 3.

The output signal power is given by

$$S_{out} = \frac{1}{4}\ A_o^2\ B^2 \cos^2\omega_o\tau \int_{-\infty}^{\infty} \frac{\alpha}{2}\ \Phi_s(\theta)\, e^{-\alpha|\theta|} d\theta \quad . \tag{34}$$

However, since α must be sufficiently large to permit the signal to pass through the RC network without appreciable distortion, the integral in Eq. 34 may be considered equal to $\Phi_s(0)$. Thus, setting $\cos \omega_o \tau = 1$ we obtain

$$\left(\frac{S}{N}\right)_{out} = \frac{P\left[1 + \left(\frac{\omega_o}{\alpha_n}\right)^2\right] \Phi_s(0)}{N_\alpha} \tag{35}$$

where $P = 1/2\, A_o^2$ is the input carrier power. Again, we should take into account the fact that the signal must have gone through the same input RC network as the noise, in which case the carrier power input to the whole system would be

$$P_o = P\left[1 + \left(\frac{\omega_o}{\alpha_n}\right)^2\right] \quad . \tag{36}$$

Then, noting that $\Phi_s(0)$, mean square value of F(t), represents the ratio of the side-band power P_s to the carrier power P_o, we obtain finally

$$\left(\frac{S}{N}\right)_{out} = \frac{P_s}{N_\alpha} \quad . \tag{37}$$

If F(t) were equal to zero, that is, if the signal were unmodulated, and the d-c component were considered as the output signal, $(S/N)_{out}$ would still be given by Eq. 37, with P_o substituted for P_s.

5. Comparison of Correlation and Filtering

It is interesting to compare the results of secs. 3 and 4 with the signal-to-noise ratio that would be obtained by means of a tuned filter followed by a linear detector. For this purpose, we assume that the filter consists of an RLC network with the same damping factor α as the averaging RC network of the correlator. The noise power admitted by such a filter with a bandwidth equal to 2α is

$$N = \alpha N_o = 2N_\alpha \tag{38}$$

where N_o is the power density of the input noise. The output signal-to-noise ratio for an unmodulated carrier of power P_o and of frequency equal to the mean frequency of the filter has been computed by many investigators (10). The exact expression involves Bessel functions of imaginary argument, but for large values of P_o/N it reduces to

$$\left(\frac{S}{N}\right)_{out} = \frac{P_o}{N_\alpha} \tag{39}$$

which is the result obtained in the case of crosscorrelation. If the side-band power P_s is substituted for P_o, the same equation applies to the case of a modulated carrier, as long as $P_s/N_\alpha >> 1$. In conclusion, for large signal-to-noise ratios the crosscorrelation method of detection is not more effective than the linear rectification method. For small signal-to-noise ratios, however, the crosscorrelation method is superior, being free from the additional overmodulation noise.

The autocorrelation method is inferior to the crosscorrelation method by at least the factor $2(1 + e^{-\alpha\tau})$. It becomes considerably inferior for small values of input signal-to-noise ratios.

6. Detection of a Known Random Signal Mixed with Noise

Suppose the two functions to be correlated are

$$f_1(t) = s(t) + n_1(t) \tag{40}$$

$$f_2(t) = as(t) + n_2(t) \tag{41}$$

where $s(t)$, $n_1(t)$ and $n_2(t)$ are independent random functions and a is a constant. Let us consider first the case in which the three functions are obtained by passing white noise through appropriate low-pass filters such as that of Eq. 2. We may take as autocorrelation functions of $s(t)$, $n_1(t)$ and $n_2(t)$, respectively,

$$\Phi_s(\theta) = S\frac{\alpha_s}{2} e^{-\alpha_s|\theta|} \tag{42}$$

$$\Phi_1(\theta) = N_1\frac{\alpha_1}{2} e^{-\alpha_1|\theta|} \tag{43}$$

$$\Phi_2(\theta) = N_2\frac{\alpha_2}{2} e^{-\alpha_2|\theta|} \quad . \tag{44}$$

The $s(t)$ component of $f_2(t)$ might be delayed relative to the $s(t)$ component of $f_1(t)$; no loss of generality results from neglecting such a delay because its effect is only to change the origin of τ in the crosscorrelation function.

The product function becomes in this case

$$\phi_\tau(t) = as(t)\,s(t-\tau) + s(t)\,n_2(t-\tau) + as(t-\tau)\,n_1(t) + n_1(t)\,n_2(t-\tau) \quad . \tag{45}$$

The autocorrelation function of $\phi_\tau(t)$ is

$$\begin{aligned}\psi_{1,\tau}(\theta) = {} & a^2\left[\Phi_s^2(\theta) + \Phi_s^2(\tau) + \Phi_s(\theta-\tau)\,\Phi_s(\theta+\tau)\right] \\ & + \Phi_s(\theta)\,\Phi_2(\theta) + a^2\Phi_s(\theta)\,\Phi_1(\theta) + \Phi_1(\theta)\,\Phi_2(\theta) \quad .\end{aligned} \tag{46}$$

Use has been made of the results of App. I and of the fact that the average of the product of two independent random functions is equal to the product of their averages. The desired output of the correlator, that is, the d-c component of $\phi_\tau(t)$, is

$$a\,\Phi_s(\tau) = aS\frac{\alpha_s}{2} e^{-\alpha_s|\tau|} \quad . \tag{47}$$

The mean-square value of the random component of the output from the filter is

$$\psi_{2,\tau}(0) = \frac{\alpha}{2} \int_{-\infty}^{\infty} \left[\psi_{1,\tau}(x) - a^2 \Phi_s(\tau)\right] e^{-\alpha|x|} dx \quad . \tag{48}$$

If $\alpha << \alpha_s$, this expression becomes, to a good approximation,

$$\psi_{2,\tau}(0) = \frac{\alpha}{2}\left\{a^2 S^2 \frac{\alpha_s}{4}\left[1 + (1 + 2\alpha_s\tau) e^{-2\alpha_s\tau}\right] + S\frac{\alpha_s}{2}\left[\frac{a^2 N_1 \alpha_1}{\alpha_s + \alpha_1} + \frac{N_2\alpha_2}{\alpha_s + \alpha_2}\right] + \frac{N_1 N_2}{2}\frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2}\right\} \quad . \tag{49}$$

The output noise-to-signal power ratio is then, for $\tau = 0$

$$R_o = \frac{\alpha}{\alpha_s} + \frac{N_1\alpha_1}{S\alpha_s}\frac{\alpha}{(\alpha_s + \alpha_1)} + \frac{N_2\alpha_2}{a^2 S\alpha_s}\frac{\alpha}{\alpha_s + \alpha_2} + \frac{N_1\alpha_1}{S\alpha_s}\frac{N_2\alpha_2}{a^2 S\alpha_s}\frac{\alpha}{\alpha_1 + \alpha_2} \quad . \tag{50}$$

Let $R_1 = N_1\alpha_1/S\alpha_s$ be the noise-to-signal power ratio for $f_1(t)$, and $R_2 = N_2\alpha_2/a^2 S\alpha_s$ be the similar ratio for $f_2(t)$. Equation 50 becomes

$$R_o = \frac{\alpha}{\alpha_s} + R_1\frac{\alpha}{\alpha_s + \alpha_1} + R_2\frac{\alpha}{\alpha_s + \alpha_2} + R_1 R_2\frac{\alpha}{\alpha_1 + \alpha_2} \quad . \tag{51}$$

Let us consider now the band-pass case in which the three autocorrelation functions are

$$\Phi_s(\theta) = S\alpha_s e^{-\alpha_s|\theta|} \cos\omega_o\theta \tag{52}$$

$$\Phi_1(\theta) = N_1\alpha_1 e^{-\alpha_1|\theta|} \cos\omega_o\theta \tag{53}$$

$$\Phi_2(\theta) = N_2\alpha_2 e^{-\alpha_2|\theta|} \cos\omega_o\theta \quad . \tag{54}$$

Equations 45, 46 and 48 are still valid. The desired output of the correlator is in this case

$$a\,\Phi_s(\tau) = aS\alpha_s e^{-\alpha_s|\tau|} \cos\omega_o\tau \quad .$$

Substitution of Eqs. 52, 53 and 54 in Eq. 46 yields, after some elementary algebraic manipulations,

$$\psi_{1,\tau}(\theta) = a^2 S^2 \alpha_s^2 e^{-2\alpha_s|\tau|}\cos^2\omega_o\tau + \frac{1}{2}\left[a^2 S^2\alpha_s^2 e^{-2\alpha_s|\theta|} + a^2 SN_1\alpha_s\alpha_1 e^{-(\alpha_s+\alpha_1)|\theta|} + SN_2\alpha_s\alpha_2 e^{-(\alpha_s+\alpha_2)|\theta|} + N_1N_2\,\alpha_1\alpha_2 e^{-(\alpha_1+\alpha_2)|\theta|}\right](1 + \cos 2\omega_o|\theta|) + \frac{1}{2}a^2 S^2\alpha_s^2 e^{-\alpha_s|\theta-\tau|+|\theta+\tau|}(\cos\omega_o\tau + \cos\omega_o\theta) \quad . \tag{55}$$

The terms in $\cos\omega_o\theta$ and $\cos 2\omega_o\theta$ correspond to power spectra concentrated about the frequencies ω_o and $2\omega_o$. If the bandwidth α of the averaging network is $\alpha \ll \omega_o$ and also $\alpha_s \ll \omega_o$, these terms may be disregarded. Then the computation of the output noise, $\psi_{2,\tau}(0)$, hardly differs from that of the previous case. The result is, for $\alpha \ll \alpha_s$,

$$\psi_{2,\tau}(0) = \frac{\alpha}{2}\left\{a^2 S^2 \frac{\alpha_s}{2}\left[1 + (1 + 2\alpha_s\tau)e^{-2\alpha_s\tau}\cos 2\omega_o\tau\right] + S\alpha_s\left[\frac{a^2 N_1 \alpha_1}{\alpha_s + \alpha_1} + \frac{N_2\alpha_2}{\alpha_s+\alpha_2}\right] + N_1 N_2 \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right\} \quad . \tag{56}$$

The noise-to-signal power ratio is then, for $\tau = 0$, one-half of the value given by Eqs. 50 and 51. R_1 and R_2 still represent the power ratios for $f_1(t)$ and $f_2(t)$.

It should be noted that the power spectrum of the noise at the input to the low-pass filter is essentially flat up to a radian frequency of the order of magnitude of α_s. Thus, if an RLC band-pass filter of bandwidth α and mean frequency $p \ll \alpha_s$ were used instead of a low-pass filter, the output noise power would still be given approximately by Eq. 56. This result would be of importance if τ were changing at a rate $d\tau/dt = r$, in which case the desired output would have to be detected by means of a filter tuned to the frequency $r\omega_o$.

APPENDIX I

Determination of the Integral

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[n(t)\, n(t-\tau)\, n(t-\theta)\, n(t-\theta-\tau)\right] dt \quad . \tag{I-1}$$

The noise may be considered for our purposes as the linear superposition of identical time functions $F(t - t_k)$, where t_k is the time at which the random event k (physically, the arrival of an electron) takes place. The events producing the $F(t - t_k)$ are distributed individually and collectively at random, with a probability

$$p(K) = \frac{(\nu T)^K}{K!}\, e^{-\theta T} \tag{I-2}$$

that exactly K events occur in a time interval T. The parameter ν is the average number of events per unit time. Let us divide the time scale into intervals T, much larger than the interval over which F(t) is appreciably different from zero. In any interval in which exactly K events occur, the noise is

$$n_K(t) = \sum_{k=1}^{K} F(t - t_k) \quad . \tag{I-3}$$

Averaging the product

$$N_K(t) = r_K(t)\, n_K(t-\tau)\, n_K(t-\theta)\, n_K(t-\theta-\tau)$$

over all intervals T in which exactly K events occur, that is, with respect to the time of occurrence of the K events $t_1, t_2, \ldots t_k$, yields

$$\left[N_K(t)\right]_{av} =$$

$$\sum_{k=1}^{K} \sum_{\ell=1}^{K} \sum_{p=1}^{K} \sum_{q=1}^{K} \int_0^T \frac{dt_1}{T} \cdots \int_0^T \frac{dt_K}{T}\, F(t-t_k)\, F(t-\tau-t_\ell)\, F(t-\theta-t_p)\, F(t-\theta-\tau-t_q) .$$

The K terms in the above expression, for which $k = \ell = p = q$, yield a contribution

$$\frac{K}{T} \int_0^T F(t)\, F(t-\tau)\, F(t-\theta)\, F(t-\theta-\tau)\, dt \quad . \tag{I-4}$$

The $3K(K-1)$ terms, in which the parameters k, ℓ, p, q are equal in pairs, yield a contribution

$$\frac{K(K-1)}{T^2} \left\{ \left[\int_0^T F(t)\, F(t-\tau)\, dt\right]^2 + \left[\int_0^T F(t)\, F(t-\theta)\, dt\right]^2 + \left[\int_0^T F(t)\, F(t-\theta-\tau)\, dt\right] \left[\int_0^T F(t-\theta)\, F(t-\tau)\, dt\right] \right\} . \tag{I-5}$$

All other terms involve $\int_0^T F(t)\, dt$ as a multiplier and, therefore, must vanish if the average value of n(t) is to be zero.

The next step is the averaging of $\left[N_k(t)\right]_{av}$ with respect to K, which yields the average of $N_k(t)$ over all the intervals of length T. Using Eqs. I-1, I-4, and I-5, we obtain for this over-all average

$$\sum_{K=1}^{\infty} \frac{(\nu T)^K}{K!} e^{-\nu T} \left[N_K(t)\right]_{av} = \theta \int_{-\infty}^{\infty} F(t)\, F(t-\tau)\, F(t-\theta)\, F(t-\theta-\tau)\, dt$$

$$+ \nu^2 \left\{ \left[\int_{-\infty}^{\infty} F(t)\, F(t-\tau)\, dt\right]^2 + \left[\int_{-\infty}^{\infty} F(t)\, F(t-\theta)\, dt\right]^2 + \left[\int_{-\infty}^{\infty} F(t)\, F(t-\theta-\tau)\right] \left[\int_{-\infty}^{\infty} F(t-\theta)\, F(t-\tau)\, dt\right] \right\} . \tag{I-6}$$

The limits of integration have been changed to $-\infty$ and ∞, in view of the fact that T was selected sufficiently large to cover the region over which the integrands are appreciably different from zero. If the number of events per unit time is sufficiently large, the term in Eq. I-6 proportional to ν may be neglected in comparison with the term proportional to ν^2. It should be noted, in this regard, that the assumption that ν is very large is entirely equivalent to the assumption that n(t) has a Gaussian probability distribution. The terms proportional to ν^2 in Eq. I-6 are readily recognized as squares or products of values of the autocorrelation function $\Phi_n(\theta$, of n(t)). Thus, using the Ergodic theorem, we finally obtain

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} n(t)\, n(t-\tau)\, n(t-\theta)\, n(t-\theta-\tau)\, dt = \Phi_n^2(\tau) + \Phi_n^2(\theta) + \Phi_n(\theta-\tau)\, \Phi_n(\theta+\tau). \tag{I-7}$$

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