

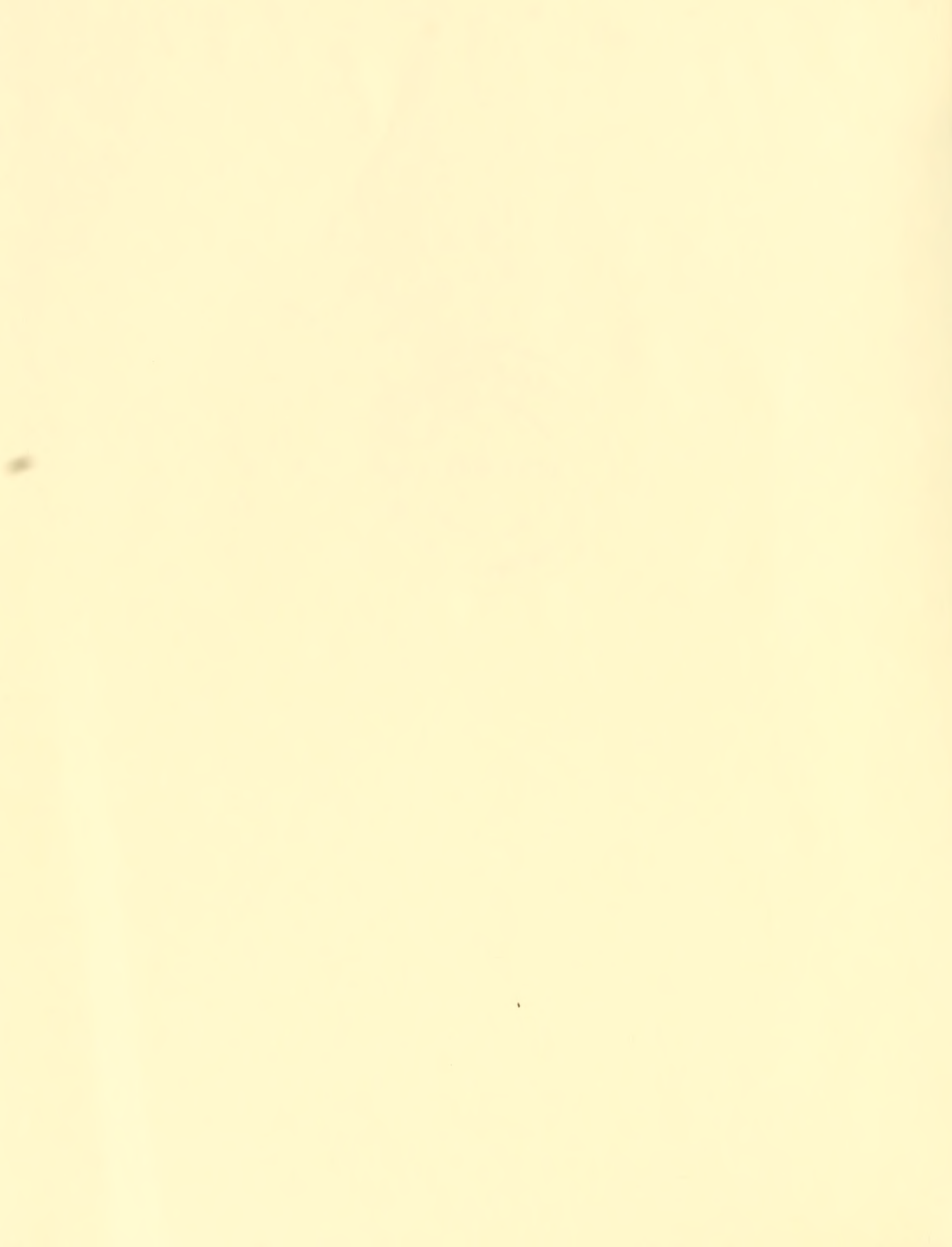
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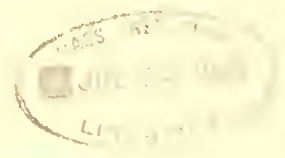
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CONTINUOUS TIME DIMENSION II:  
THE CASE OF UNCERTAINTY

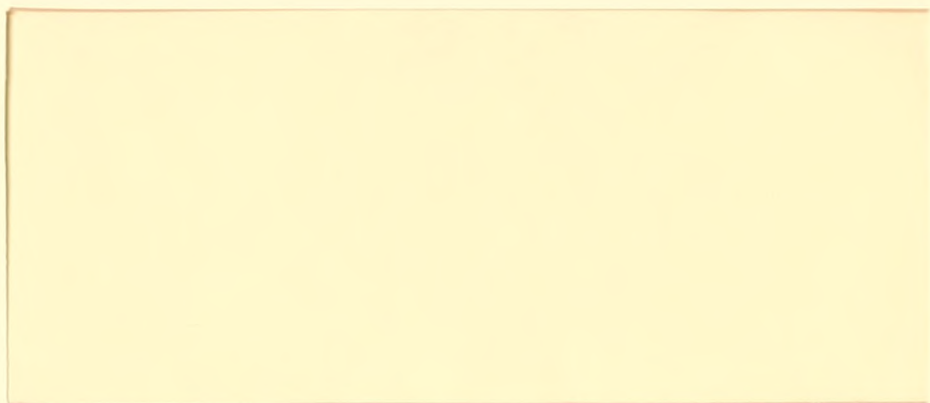
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Ayman Hindy and Chi-fu Huang

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March 1989

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# ON INTERTEMPORAL PREFERENCES WITH A CONTINUOUS TIME DIMENSION II: THE CASE OF UNCERTAINTY

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March 1989

## Abstract

This paper proposes a family of topologies on the space of consumption patterns in continuous time under uncertainty. Preferences continuous in any of the proposed topologies treat consumptions at nearly adjacent dates as almost perfect substitutes except possibly at surprises. The topological duals of the family of proposed topologies essentially contain processes that are the sums of processes of absolutely continuous paths and martingales. Thus if equilibrium prices for consumption come from the duals, consumptions at nearly adjacent dates have almost equal prices except possibly at surprises. In particular, if the information structure is generated by a Brownian motion, the duals are composed of Itô processes. We discuss the properties of prices of long-lived assets in economies populated with agents whose preferences are continuous in our topologies when there are no arbitrage opportunities. We also investigate some implications of our topologies on standard models of choice in continuous time as well as on recent models of non time-separable representations of preferences.

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*Mom, I can not eat any spinach now. I just had my dinner.*

*Anonymous*

## 1 Introduction and summary

No “standard” economic agent could use the above convincing argument to delay his consumption. “Standard” economic models assume that very recent consumption has no effect on one’s current appetite; an assumption that each of us repeatedly violates.

In addition, we frequently violate an important implication of the “standard models” concerning the prices of assets. If one knows for sure that the price of a certain asset tomorrow will be much higher than it is today then one would be willing to delay his consumption for one day and use the proceeds to buy such an asset. Alternatively, one could borrow from someone else who is willing to delay his consumption, pay him a relatively low rate of interest and use the funds to realize the gain in the price of the asset. Our behavior would force today’s price to increase and become very close to the price of the asset tomorrow. If no one were willing to delay consumption for one day for a small fee, then a situation of a big jump in prices over short periods in the absence of any new information might prevail in equilibrium. In fact, this is a prediction of “standard” models in which continuous changes in prices are obtained mainly by exogenously specifying continuously varying endowments. We ask: could one develop a model of preferences for consumption over time under uncertainty that agrees with our economic intuition in which past consumption has an effect on current utility and in which the continuity of prices is a phenomenon implied by continuity of preferences and independent of the nature of the endowments? This paper is an attempt to address these issues.

This is the second part of a series of two papers. The first part, Huang and Kreps (1987), which we henceforth abbreviate as H&K, addresses the following questions: How might one represent a consumption pattern in continuous time under certainty and what are the appropriate topologies on the space of consumption patterns that capture the idea that consumptions at nearly adjacent dates are almost perfect substitutes? Moreover, what form would the equilibrium prices take when individuals in the economy have preferences continuous in the appropriate topologies?

“Standard” answers exist for the questions raised by H&K: A consumption pattern on a time interval, say  $[0, 1]$ , is represented by a real-valued function  $c : [0, 1] \rightarrow \mathbb{R}_+$ , where  $c(t)$  is the consumption “rate” at time  $t$ . Two consumption patterns are close if they are close as functions in an  $L^p$  norm topology for some  $1 \leq p < \infty$ . An agent’s preferences are represented by

$\int_0^1 u(c(t), t) dt$ , a time-additive functional of consumption rates, where, using the terminology of Arrow and Kurz (1970),  $u(c, t)$  is a “felicity” function for consumption at time  $t$ . The equilibrium prices come from  $L^q$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , the topological dual of  $L^p$ ; that is, the price at time zero of a consumption pattern  $c$  can be represented as  $\int_0^1 c(t)y(t)dt$ , where  $y$  is from  $L^q$  and  $y(t)$  is interpreted as the time zero price of one unit of consumption per unit time at time  $t$ .

H&K argued that these standard answers are unsatisfactory for the following reasons. First, in modeling consumption over time, one should allow consumption at rates as well as in gulps – observed consumption behavior such as having meals is in gulps. Second and more important, the  $L^p$  topology on the space of consumption rates is so strong along the time dimension that consumptions at nearly adjacent dates are perfect nonsubstitutes! As a consequence of the strong topology, the space of equilibrium prices is too rich. It includes, for example, discontinuous functions of time and continuous functions of unbounded variation signifying that equilibrium prices for consumptions at nearly adjacent dates are either very different or, even though continuous, fluctuate in a nowhere differentiable manner.<sup>1</sup>

The most general way for representing a consumption pattern over time under certainty is by an increasing<sup>2</sup> and positive function on  $[0, 1]$ . Let  $x$  be such a function. Then  $x(t)$  denotes the accumulated consumption from time zero to time  $t$ . If  $x$  is an absolutely continuous function of time, its derivatives exist almost everywhere and can be interpreted as consumption rates. The discontinuities of  $x$  are the gulps of consumption. Letting  $X$  be the linear span of the space of these consumption patterns, H&K introduce a family of norm topologies on  $X$  so that, among other things, consumptions at nearly adjacent dates are almost perfect substitutes. An example of this family of norm topologies is the  $L^p$  topology on accumulated consumption, where the  $L^p$ -norm of a consumption pattern  $x$  is given by:

$$\left( \int_0^1 |x(t)|^p dt + |x(1)|^p \right)^{\frac{1}{p}}.$$

H&K also show that preferences continuous in any of the norm topologies can be represented by maximizing the time-additive functional of consumption rates in the “standard answers” only if the felicity functions are linear! The equilibrium prices basically come from absolutely continuous functions – prices of consumptions at nearly adjacent dates are almost equal and change over time in an almost differentiable manner.

<sup>1</sup>Note that one might expect equilibrium prices for consumption to fluctuate widely in an economy under uncertainty due to temporal resolution of uncertainty, however; see Huang (1985a, 1985b).

<sup>2</sup>Throughout this paper we will use weak relations: increasing means nondecreasing, positive means nonnegative, etc. When the relations are strict, we will say, for example, strictly increasing.

The purpose of this paper is to develop topologies on the space of consumption patterns under uncertainty that capture the notion that consumptions at adjacent dates are almost perfect substitutes. A crucial distinction between an economy under certainty and an economy under uncertainty is that in the latter the passage of time also reveals partially the true state of nature, or reveals *information*. In a world of uncertainty, the preferences of an individual over consumption patterns may be affected by the way uncertainty is resolved over time. A world in which any uncertainty is resolved gradually over time, with ample “warning” and “preparation” for new bits of information is certainly different from a world which is “sudden” and “full of surprises”. Thus it is unreasonable to require *a priori* that consumptions at nearly adjacent dates, which may be random, be almost perfect substitutes unless there are no “surprises” there. Similarly, we would not expect the equilibrium prices to be continuous except at (random) times of no surprise.

We introduce a family of norm topologies on the linear span of the set of consumption patterns under uncertainty, which is composed of processes with positive and increasing paths. This family of topologies are natural generalizations of the H&K topologies. In particular, they reduce to the H&K topologies when the true state of nature is revealed at time 0 and consumptions at nearly adjacent dates, where there is no discontinuity of information, are almost perfect substitutes. We then show that preferences continuous and uniformly proper (to be defined) in any one of the family of topologies exhibit intuitively appealing economic properties.

In general equilibrium theory, equilibrium prices come from the topological dual spaces. Thus we also characterize the topological duals of the suggested family of norm topologies and show that they are essentially composed of processes which are sums of processes of absolutely continuous paths and martingales. This is a very natural result. H&K have shown that in the case of certainty the shadow prices for consumption are essentially absolutely continuous functions. Hence preferences that treat consumption at nearly adjacent dates as almost perfect substitutes will give rise to nearly equal prices for consumption at nearly adjacent dates and these prices vary over time in an almost differentiable fashion; an intuitively attractive conclusion. In the case of uncertainty a new element, the pattern of information flow, affects the sample path properties of equilibrium prices. This effect is captured in the martingale component of the price process. It is known that a martingale can make discontinuous changes only at surprises and can fluctuate in a nowhere differentiable fashion. Thus equilibrium prices for consumption are continuous except possibly at surprises and can fluctuate in a nowhere differentiable manner (due to temporal resolution of uncertainty). This agrees with our economic

intuition.

We also investigate properties of arbitrage-free price processes in a dynamic securities market economy where an arbitrage opportunity is defined using concepts of continuity derived from our family of topologies. It is shown that, between lump-sum ex-dividend dates, price processes are continuous except possibly at surprises. In particular, if the information structure is generated by a Brownian motion and accumulated dividend process of a security is an absolutely continuous process, the ex-dividend price process for this security is an Itô process.

It is worthwhile to point out that earlier models of equilibrium using the standard representation of utility had to rely on exogenous factors in addition to preferences to characterize the sample path properties of equilibrium prices; see Duffie (1986) and Huang (1987). For example, in the case of a Brownian motion filtration, the price process for a security with absolutely continuous accumulated dividend process will not be an Itô process unless the aggregate endowment process is. In our setup, however, since consumption at nearly adjacent dates are almost perfect substitutes at times of no surprise, price processes of securities with absolutely continuous accumulated dividends will be Itô processes independently of the properties of the aggregate endowment process.

As for the existence of an Arrow-Debreu equilibrium in an economy populated with agents whose preferences are continuous in one of our topologies, we have little to report. Known sufficient conditions for the existence of an equilibrium are not satisfied by our topologies. This opens up a question about the existence of an equilibrium in economies of our type.

The rest of this paper is organized as follows. Section 2 formulates the stochastic environment under study with a time span  $[0, 1]$ . Taken as primitive is a probability space  $(\Omega, \mathcal{F}, P)$  and an information structure  $\mathbf{F} = \{\mathcal{F}_t; t \in [0, 1]\}$ , where each  $\omega \in \Omega$  denotes a state of nature,  $\mathcal{F}$ , a sigma-field, is the collection of distinguishable events at time 1,  $P$  is the probability beliefs about possible events held by the individuals we will consider, and  $\mathbf{F}$ , an increasing family of sub-sigma fields of  $\mathcal{F}$ , specifies how distinguishable events in  $\mathcal{F}$  are revealed from time 0 to time 1.

A consumption pattern  $\mathbf{x} = \{\mathbf{x}(t); t \in [0, 1]\}$  is a stochastic process having positive, increasing, and right-continuous sample paths which is consistent with  $\mathbf{F}$  or *adapted* to  $\mathbf{F}$ . The random variable  $\mathbf{x}(t)$  denotes the accumulated consumption from time 0 to time  $t$ . Let  $\mathbf{X}_+$  be the space of consumption patterns and  $\mathbf{X}$  be the linear span of  $\mathbf{X}_+$ . Our task is to define a topology  $\mathcal{T}$  on  $\mathbf{X}$  so that preferences continuous with respect to it exhibit economically desirable properties.

We put forth a wishlist for a topology  $\mathcal{T}$  and introduce a family of topologies to satisfy this wishlist in Section 3. The agenda on our wishlist is two-fold: First, an economy under certainty

is a special case of an economy under uncertainty where the true state of nature is revealed at time 0. In such event, we would like our topology to agree with the H&K topology. This necessitates that the norms of H&K be used path by path (state by state) on a consumption pattern. For the  $L^p$  example mentioned above, the path-wise distance between  $x \in \mathbf{X}_+$  and 0 is

$$\left( \int_0^1 |x(\omega, t)|^{p(\omega)} dt + |x(\omega, 1)|^{p(\omega)} \right)^{\frac{1}{p(\omega)}}, \quad (1)$$

where  $x(\omega, t)$  is the value of the random variable  $x(t)$  when the state of nature is  $\omega$ . Note that in (1)  $p$  is a function of the state of nature  $\omega$  – there is no *a priori* reason to expect that the trade-off of consumption across time is the same for all states.

Second, consider substitutability of consumption across states. One unit of consumption at a time in a particular event may be a close substitute to one unit of consumption at the same time in another event for an individual. But there is no economic reason to expect that *all* individuals with continuous preferences consider consumptions in different events to be close substitutes. Thus the topology on  $\mathbf{X}$  should not *a priori* build in substitutability of consumption across states in an arbitrary manner. On the other hand, it is quite intuitive that one unit of consumption in an event which is very unlikely to occur should be close to not consuming at all.

With all these considerations in mind, a natural way of aggregating the path-wise construction in the previous paragraph is to just take expectation. Taking expectations embodies the notion that consumptions at two disjoint events are perfect nonsubstitutes except when both events are negligible in probability and thus the two consumption patterns are almost indistinguishable from zero. Taking expectations also embodies the notion that, at any time, the differences of preferences for consumption in two events with the same probability is a decreasing function in the degree of overlap of the two events. We here remind the reader that defining such a strong topology and considering continuous preferences certainly does not rule out preferences that actually consider consumptions across states to be close substitutes. They will be special cases of preferences continuous in the “strong” topology. In the same  $L^p$  example above, except now that  $p$  is nonrandom, the norm of a consumption pattern  $x \in \mathbf{X}_+$  is then

$$\left( \mathbf{E} \left[ \int_0^1 |x(t)|^p dt + |x(1)|^p \right] \right)^{\frac{1}{p}}, \quad (2)$$

which is a standard  $L^p$  norm but on accumulated consumptions rather than on consumption rates.<sup>3</sup>

<sup>3</sup>The reader may have noticed that (2) is not finite for every  $x \in \mathbf{X}_+$ . Thus we will have to restrict our attention to a subset of  $\mathbf{X}_+$  depending on the topology.

In section 4 we develop some topological and uniform properties of the family of topologies defined in Section 3. The reader will see that the family of topologies inherit many properties of the H&K topologies when those properties are properly “aggregated” across states by taking expectations. Basically, the family of topologies exhibit the desired economic property: consumptions at nearly adjacent dates where there are no discontinuities of information are almost perfect substitutes. Since *uniformly proper* preferences (see Mas Colell (1986)) have been needed for the general equilibrium theory, we also investigate properties of preferences that are continuous and uniformly proper.

Section 5 characterizes the topological duals of the family of topologies under consideration. We show there that the topological dual spaces corresponding to our topology – spaces which contain the shadow price of consumption – consist of processes that can be decomposed into the sum of two components: an absolutely continuous part and a martingale part. We further show that a topological dual space corresponding to each one of our topologies may fail to be a sublattice in the order dual; a property known to be sufficient, together with other things, to guarantee existence of a Walrasian equilibrium in our model. In particular, in the case of Brownian motion information, the duals spaces are not sublattices of the order duals.

Section 6 examines how standard models with time-additive utility functions fare in our set up. Similar to the results of H&K, preferences represented by time-additive utility functions over consumption rates are continuous in any of our topologies if and only if they are linear. In addition, we examine some of the prevalent representations in the literature of “non time-additive” utility functions such as those in Bergman (1985), Constantinides (1988), Heaton (1988), and Sundaresan (1988). Although such representations have elements that capture the effect of past consumption on current utility, we find that most of them imply preferences that are *not* continuous in the sense that we advocate.

In section 7 we examine the implications of our topologies on the prices of long-lived securities in dynamic asset markets. Our discussion is along two lines. First, we examine asset prices under conditions of “no arbitrage.” We show that in this case the prices over time of long lived securities inherit the properties of the shadow prices of consumption that we discuss in section 5. In particular, we show that in the case of Brownian motion information structures, and absent any arbitrage opportunities, the prices of long lived securities with absolutely continuous accumulated dividend processes will be Itô processes. This result is purely driven by the continuity of preferences. Second, we discuss similar issues in the case of markets with continuous trading in dynamic equilibrium, under the assumption that such an equilibrium exists. Section 8 contains concluding remarks.



## 2 Formulation

Consider an economic agent who lives in a world of uncertainty from time 0 to time 1. There is a single consumption good available for consumption at any time  $t \in [0, 1]$ . Uncertainty is modeled by a probability space  $(\Omega, \mathcal{F}, P)$ . Each  $\omega \in \Omega$  represents a state of nature which is a complete description of one possible realization of all exogenous sources of uncertainty from time 0 to time 1. The sigma-field  $\mathcal{F}$  represents the collection of events which are distinguishable at time 1 and  $P$  is a probability measure on  $(\Omega, \mathcal{F})$ .

We take as given the time evolution of our agent's knowledge about the states of nature. This is modeled by a *filtration*  $\mathbf{F} = \{\mathcal{F}_t; t \in [0, 1]\}$ , which is an increasing family of sub sigma-fields of  $\mathcal{F}$ ; that is,  $\mathcal{F}_s \subseteq \mathcal{F}_t$  if  $s \leq t$ . Interpret  $\mathcal{F}_t$  as the information that the agent has at time  $t$ . We assume that  $\mathcal{F} = \mathcal{F}_1$ , that is, the true state of nature will be known at time 1, and the filtration  $\mathbf{F}$  satisfies the following *usual* conditions:

1.  $\mathbf{F}$  is *complete* in that  $\mathcal{F}_0$  contains all  $P$ -null sets;
2.  $\mathbf{F}$  is *right continuous* in that  $\mathcal{F}_t = \mathcal{F}_{t+}$ , where  $\mathcal{F}_{t+} = \bigwedge_{s > t} \mathcal{F}_s$ .

Besides these *usual* conditions, we will further suppose that the information structure satisfies a regularity condition, which almost always holds in applications. Before we do that, some definitions are in order.

**Definition 1** *The function  $T : \Omega \rightarrow [0, \infty]$  is an optional time with respect to  $\mathbf{F}$  if*

$$\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t \quad \forall t \in [0, 1].$$

An optional time can always be interpreted to be the first time a certain event happens. The condition  $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$  in the above definition then says that at any time  $t$ , it will be known whether a certain event happened or not.

**Definition 2** *An optional time  $T$  is said to be predictable if there exists a sequence of optional times  $\{T_n\}$  such that  $T_n \leq T$  a.s. and on the set  $\{T > 0\}$ , almost surely,  $T_n < T_{n+1} < T$  and  $T_n \nearrow T$ . The sequence  $\{T_n\}$  is said to announce  $T$ .*

Intuition suggests that if the first time an event happens is announced by the occurrence of a sequence of other events, then the event will not take one by surprise. This intuition turns out not to be correct for general information structures and is valid for a *quasi left-continuous* information structure:

**Definition 3** An information structure  $\mathbf{F}$  is quasi left-continuous if for a predictable time  $T$  and its announcing sequence  $\{T_n\}$  we have

$$\mathcal{F}_T = \bigvee_{n=1}^{\infty} \mathcal{F}_{T_n} \equiv \mathcal{F}_{T-},$$

where  $\mathcal{F}_T$  is a sigma-field of events prior to  $T$  consisting of all events  $A \in \mathcal{F}$  such that

$$A \cap \{T \leq t\} \in \mathcal{F}_t, \text{ for all } t \in [0, 1].$$

We assume throughout our analysis that  $\mathbf{F}$  is quasi left-continuous. This is without much loss of generality. For example, the filtration generated by a diffusion process, a Poisson process, or a combination of the two is quasi left-continuous once the filtration is completed.<sup>4</sup>

A measurable stochastic process  $Y$  is a mapping  $Y : \Omega \times [0, 1] \rightarrow \mathbb{R}$  that is measurable with respect to  $\mathcal{F} \otimes \mathcal{B}([0, 1])$ , the product sigma-field generated by  $\mathcal{F}$  and the Borel sigma-field of  $[0, 1]$ . For each  $\omega \in \Omega$ ,  $Y(\omega, \cdot) : [0, 1] \rightarrow \mathbb{R}$  is a *sample path* and for each  $t \in [0, 1]$ ,  $Y(\cdot, t) : \Omega \rightarrow \mathbb{R}$  is a random variable, which we will usually simply use  $Y(t)$  to denote. The process  $Y$  is said to be adapted to  $\mathbf{F}$  if for each  $t \in [0, 1]$ ,  $Y(t)$  is  $\mathcal{F}_t$ -measurable. This is a natural information constraint: the value of the process at time  $t$  cannot depend on the information yet to be revealed in the future. For brevity, all the processes to appear will be measurable and adapted to  $\mathbf{F}$  unless otherwise mentioned.

The life time consumption pattern of an agent is represented by a process  $x$  whose sample paths are positive, increasing, and right-continuous with  $x(\omega, t)$  denoting the accumulated consumption from time 0 to time  $t$  in state  $\omega$ . We denote the set of such processes by  $\mathbf{X}_+$ . The linear span of  $\mathbf{X}_+$ , the space of processes having right-continuous and bounded variation sample paths, will be denoted by  $\mathbf{X}$ .

For technical reasons, which will be made clear later, we will consider preferences defined only on a subset of  $\mathbf{X}_+$ . Denote this subset for now by  $\mathcal{L}$ .<sup>5</sup> The agent is assumed to have preferences over  $\mathcal{L}$  which are given by a complete and transitive binary relation  $\succeq$  that is “continuous.” Our task is to define a topology, say  $\mathcal{T}$ , on  $\mathcal{L}$  such that preferences continuous with respect to which exhibit desirable economic properties. This is the subject to which we now turn.

<sup>4</sup>It has been shown by Meyer (1963) that any information structure generated by a process that is continuous at predictable optional times (defined with respect to its natural filtration) and has the strong Markov property is quasi left-continuous.

<sup>5</sup>This subset in fact depends on the topology chosen. The reader can think of the the classical Banach space example:  $L^p(\Omega, \mathcal{F}, P)$  certainly does not include all the random variables and is “topology-dependent.”

### 3 A family of norm topologies

Before proceeding, we will first briefly review the family of H&K norm topologies on the linear span of the space of consumption patterns under certainty. This space, denoted by  $X$ , is a linear space composed of functions on  $[0, 1]$  that are right-continuous and of bounded variation. An element of the positive orthant of  $X$ , denoted by  $X_+$ , is an increasing, positive, and right-continuous function on  $[0, 1]$ . If  $x \in X_+$ ,  $x(t)$  denotes the accumulated consumption from time 0 to time  $t$ . A topology in the H&K family corresponds to a function  $\mu : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$  with  $\mu(0) = 0$  and  $\mu(\infty) = \infty$  that is strictly increasing, continuous, and concave. Defining  $\eta \equiv \mu^{-1}$ , the inverse of  $\mu$ ,  $\eta : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$  with  $\eta(0) = 0$  and  $\eta(\infty) = \infty$  is strictly increasing, continuous, and convex. Then

$$\inf\{a > 0 : \int_0^1 \eta(|x(t)|/a) dt + \eta(|x(1)|/a) \leq 1\}$$

defines a norm for  $x \in X$ . Let  $\succeq$  be a preference relation continuous in the topology defined by the above norm. H&K showed that the following properties are satisfied by  $\succeq$ .

(1) Two patterns of consumption that have almost equal accumulated consumption at every point in time are close: if  $\lim_{n \rightarrow \infty} \sup_{t \in [0, 1]} |x_n(t) - x(t)| = 0$ , then  $x_n \succeq y$  for all  $n$  implies  $x \succeq y$  and  $y \succeq x_n$  for all  $n$  implies  $y \succeq x$ .

(2) Sizable shifts in consumption across small amounts of time are regarded as insignificant: for  $x, y \in X_+$ , let

$$p(x, y) = \inf\{\epsilon > 0 : x(t + \epsilon) + \epsilon \geq y(t) \geq x(t - \epsilon) - \epsilon, t \in [0, 1]\}.$$

(This is the *Prohorov metric* on the space of increasing functions on  $[0, 1]$ .) If  $p(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $x_n \succeq y$  for all  $n$  implies  $x \succeq y$ , and  $y \succeq x_n$  for all  $n$  implies  $y \succeq x$ .

Moreover, for  $\succeq$  that is *uniformly proper* in the direction  $\chi_{[t, 1]}$ , the indicator function of  $[t, 1]$ , in the sense of Mas Colell (1986)<sup>6</sup>, shift of an increasingly large amount of consumption across a decreasingly small amount of time is regarded negligible. This is where the function  $\mu$  comes into the picture:

---

<sup>6</sup>The formal definition of *uniform properness* will be given later.

(3) Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences of consumption patterns with the properties that (i)  $\{x_n(1)\}$  and  $\{y_n(1)\}$  are both  $o(\mu(n))$ , and (ii)  $p(x_n, y_n) \leq 1/n$ , then for all sufficiently large  $n$ ,  $x_n + \chi_t \succ y_n$ , where  $\{x_n(1)\}$  is  $o(\mu(n))$  means  $\lim_{n \rightarrow \infty} x_n(1)/\mu(n) = 0$ .

Literally, (3) means if the amount of consumption shifted increases to infinity at a rate slower than the rate at which  $\mu$  increases to infinity and if the decreasingly small time interval across which consumption is shifted goes to zero faster than  $1/n$  goes to zero, then the shift is insignificant in terms of preferences.

Note that (1) and (2) are topological properties while (3) is a uniform property since the sequences in (3) diverge rather than converge.

With these above facts about the H&K topologies in mind, we now turn to two considerations that determine a family of norm topologies on subspaces of  $\mathbf{X}$ .

First, note that an economy under certainty is a special case of an economy under uncertainty. The former corresponds to a case of the latter where the true state of nature is revealed at time 0. Therefore, a candidate topology under uncertainty should have the property that it degenerates to an H&K topology when  $\mathcal{F}_0 = \mathcal{F}_1$ . That is, if the true state  $\omega \in \Omega$  were known at time zero, the topology on  $\mathbf{X}$  should be generated by the “norm:”

$$\|x\| = \inf\{a > 0 : \int_0^1 \varphi(\omega, |x(\omega, t)|/a) dt + \varphi(\omega, |x(\omega, 1)|/a) \leq 1\},$$

for some  $\varphi(\omega, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\varphi(\omega, 0) = 0$  and  $\varphi(\omega, \infty) = \infty$  that is continuous, strictly increasing, and convex. Note that  $\mu(\omega, \cdot) \equiv \varphi(\omega, \cdot)^{-1}$  is a  $\mu$  in H&K that measures how fast shifts of consumption across time increase to infinity in property (3) above. (Here, however,  $\mu$  may depend upon the state of nature since every state of nature can be viewed as a different economy under certainty and there is no *a priori* reason to expect that the trade-off of consumption over time will be identical in each one of these different economies.) This consideration necessitates that an H&K norm be used on a path of an  $x \in \mathbf{X}$ .

Second, we expect that one unit of consumption at any time and in an event which is negligible in terms of probability  $P$  should be “close” to not consuming at all. On the other hand, *a priori*, there is no reason to expect that one unit of consumption at time  $t$  in event  $A$  is a close substitute for one unit of consumption at the same time but in another event  $A'$  disjoint from  $A$  except when both  $A$  and  $A'$  are negligible in terms of probability  $P$ , since in which case, both consumption patterns are “close” to zero.

These two considerations suggest that we use H&K path-by-path and “paste” together these path-wise constructions by taking expectation according to  $P$ . This leads us to the notion of

a Generalized Orlicz space. (Musielak (1983) is a good reference.) The following is a brief introduction.

Consider the measure space  $(\Omega \times [0, 1], \mathcal{O}, P \times \lambda)$ , where  $\mathcal{O}$  is the *optional* sigma-field – the sigma-field generated by all the processes adapted to  $\mathbf{F}$  having right-continuous paths,  $\lambda$  is the Lebesgue measure on  $[0, 1]$ , and  $P \times \lambda$  is the product measure generated by  $P$  and  $\lambda$ . A mapping on  $\Omega \times [0, 1]$  that is measurable with respect to  $\mathcal{O}$  is an *optional process*. It is known that an optional process is adapted to  $\mathbf{F}$ ; see Chung and Williams (1983). Denote the space of optional processes with equivalence  $P \times \lambda$  almost everywhere by  $\mathbf{O}$ . Note that  $\mathbf{X}$  is a subspace of  $\mathbf{O}$ .

Let  $\Phi$  be the collection of functions  $\varphi : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  measurable with respect to  $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$  with, *almost surely*,

$$\varphi(\omega, 0) = 0 \quad \text{and} \quad \varphi(\omega, \infty) = \infty,$$

that are continuous, strictly increasing, convex, and integrable in that

$$\int_{\Omega} \varphi(\omega, z) P(d\omega) < \infty \quad \forall z > 0.$$

A *modular* is a function  $\xi : \mathbf{O} \rightarrow [0, \infty]$  such that

- (a)  $\xi(0) = 0$ ;
- (b)  $\xi(x) = 0$  implies  $x = 0$ ;
- (c)  $\xi(-x) = \xi(x)$ ;
- (d)  $\xi(\alpha x + \beta y) \leq \xi(x) + \xi(y)$  for  $x, y \in \mathbf{O}$  and for  $\alpha, \beta \geq 0$ ,  $\alpha + \beta = 1$ .

For  $\varphi \in \Phi$  and for all  $x \in \mathbf{O}$ ,  $\varphi(\omega, |x(\omega, t)|) : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is measurable with respect to  $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$  and

$$\begin{aligned} \xi(x) &\equiv \int_{\Omega} \int_0^1 \varphi(\omega, |x(\omega, t)|) dt P(d\omega) + \int_{\Omega} \varphi(\omega, |x(\omega, 1)|) P(d\omega) \\ &\equiv \mathbf{E} \left[ \int_0^1 \varphi(|x(t)|) dt + \varphi(|x(1)|) \right] \end{aligned} \tag{3}$$

is a *modular* on  $\mathbf{O}$ .

The modular  $\xi$  defines the so called *generalized Orlicz space*,  $L^\varphi$ , where

$$L^\varphi = \left\{ x \in \mathbf{O} : E \left[ \int_0^1 \varphi(\gamma |x(t)|) dt + \varphi(\gamma |x(1)|) \right] \rightarrow 0 \quad \text{as} \quad \gamma \rightarrow 0+ \right\}. \tag{4}$$

The set

$$L_0^\varphi = \left\{ x \in \mathbf{O} : E \left[ \int_0^1 \varphi(|x(t)|) dt + \varphi(|x(1)|) \right] < \infty \right\}, \tag{5}$$

will be called a *generalized Orlicz class*. It is known that  $L_0^\varphi$  is a convex subset of  $L^\varphi$  and  $L^\varphi$  is the smallest vector subspace of  $\mathbf{O}$  containing  $L_0^\varphi$ . The largest vector subspace of  $\mathbf{O}$  contained in  $L_0^\varphi$  is

$$E^\varphi = \left\{ x \in \mathbf{O} : E \left[ \int_0^1 \varphi(\gamma|x(t)|) dt + \varphi(\gamma|x(1)|) \right] < \infty \quad \forall \gamma > 0 \right\}. \quad (6)$$

An  $x \in E^\varphi$  is said to be a finite element of  $L^\varphi$ .

**Remark 1** If  $\varphi(\omega, x) = \varphi(x)$  is independent of  $\omega$ , we say that  $L^\varphi$  and  $L_0^\varphi$  are Orlicz space and Orlicz class, respectively. Also, in the case of certainty, there is no distinction between  $L^\varphi$ ,  $L_0^\varphi$ , and  $E^\varphi$ .

We will henceforth restrict our attention to  $E^\varphi$ , on which we define a norm:

**Definition 4** Given the function  $\varphi \in \Phi$ , we define a norm  $\|\cdot\|_\varphi$  on  $E^\varphi$  as follows:  $\forall x \in E^\varphi$ ,

$$\|x\|_\varphi = \inf \left\{ a > 0 : E \left[ \int_0^1 \varphi\left(\frac{|x(t)|}{a}\right) dt + \varphi\left(\frac{|x(1)|}{a}\right) \right] \leq 1 \right\}. \quad (7)$$

(For the fact that (7) defines a norm on  $E^\varphi$ , see Musielak (1989).)

Here it is worthwhile to remark that if  $\varphi(\omega, |z|) = |z|^p$  for some  $1 \leq p < \infty$ , then

$$L^\varphi = L_0^\varphi = E^\varphi = L^p(\Omega \times [0, 1], \mathcal{O}, P \times \lambda),$$

and the norm defined in Definition 4 is equivalent to the standard  $L^p$  norm.

As we mentioned earlier,  $\mathbf{X}$  is a proper subset of  $\mathbf{O}$ , therefore  $E^\varphi$  contains processes whose sample paths are not of bounded variation or right-continuous. Now putting  $\mathcal{E}^\varphi \equiv E^\varphi \cap \mathbf{X}$ , the following proposition shows that  $(\mathcal{E}^\varphi, \|\cdot\|_\varphi)$  is a normed space with compact order intervals, where  $\|\cdot\|$  is as defined in (7) but restricted to  $\mathcal{E}^\varphi$ .

**Proposition 1**  $(\mathcal{E}^\varphi, \|\cdot\|_\varphi)$  is a normed space with norm-compact order intervals.

PROOF. The first assertion follows from the fact that  $\|\cdot\|_\varphi$  is a norm on  $E^\varphi$  and  $\mathcal{E}^\varphi$  is a subspace of  $E^\varphi$ . A proof of the second assertion is provided in Appendix A. ■

Note that the norm  $\|\cdot\|_\varphi$  pastes together the path-wise H&K construction by taking expectation under  $P$  – as we envisioned earlier. Henceforth, denote the topology on  $\mathcal{E}^\varphi$  generated by  $\|\cdot\|_\varphi$  by  $\mathcal{T}_\varphi$ .

Denote the positive orthant of  $\mathcal{E}^\varphi$  by  $\mathcal{E}_+^\varphi \equiv \mathcal{E}^\varphi \cap \mathbf{X}_+$ , which is a cone. Each  $x \in \mathcal{E}_+^\varphi$  is then a consumption pattern. The cone  $\mathcal{E}_+^\varphi$  defines an order on  $\mathcal{E}^\varphi$  in the following way: Let  $x, y \in \mathcal{E}^\varphi$ . We say  $x$  is “greater” than  $y$ , denoted by  $x \geq y$ , if  $x - y \in \mathcal{E}_+^\varphi$ .

For the analysis to follow, we shall assume that our agent's preferences  $\succeq$  are defined on  $\mathcal{E}_+^\varphi$  and are continuous with respect to  $\mathcal{T}_\varphi$ . Using the terminology of general equilibrium theory,  $\mathcal{E}^\varphi$  and  $\mathcal{E}_+^\varphi$  are referred to as the *commodity space* and the *consumption set*, respectively.

A natural question that arises from the above definition of commodity spaces is how these spaces and their corresponding topologies compare. The answer to such a question is given in the following proposition.

**Proposition 2** *Let  $\varphi_1, \varphi_2 \in \Phi$  satisfy:*

$$\varphi_2(\omega, z) \leq k_1 \varphi_1(\omega, k_2 z) + h(\omega) \quad \forall z \geq 0 \quad \text{and} \quad P \times \lambda - a.e.$$

*where  $h$  is a positive integrable function on  $\Omega$  and  $k_1, k_2$  are strictly positive constants. Then  $\mathcal{E}^{\varphi_1} \subseteq \mathcal{E}^{\varphi_2}$ , and  $\mathcal{T}_{\varphi_1}$  is stronger than  $\mathcal{T}_{\varphi_2}$ .*

PROOF. See Musielak (1983, Theorem 8.5). ■

Among all the topologies  $\{\mathcal{T}_\varphi; \varphi \in \Phi\}$ , there is a weakest one – the one generated by  $\varphi(\omega, z) = |z|$  and denoted henceforth by  $\mathcal{T}$ . This is formally stated in the following corollary.

**Corollary 1**  *$\mathcal{T}$  is weaker than  $\mathcal{T}_\varphi$  for all  $\varphi \in \Phi$ .*

PROOF. It is easily verified that, for all  $\varphi \in \Phi$ ,

$$|z| \leq \varphi(\omega, |z|) + h(\omega) \quad \forall z \geq 0 \quad P \times \lambda - a.e.$$

for some positive integrable random variable  $h$  on  $(\Omega, \mathcal{F}, P)$ . The assertion then follows from Proposition 2. ■

The commodity space corresponding to  $\varphi(\omega, z) = |z|$  will be denoted by  $\mathcal{E}$ .

**Remark 2** *If  $\varphi_1$  and  $\varphi_2$  are independent of  $\omega$ , then the condition in Proposition 2 can be written as: There is  $k_1, k_2 > 0$  and  $z_0 > 0$  such that*

$$\varphi_2(z) \leq k_1 \varphi_1(k_2 z) \quad \forall z \geq z_0.$$

Now that we have defined a family of commodity spaces and their corresponding topologies, the next item on our agenda is to see whether continuous (and uniformly proper) preferences give rise to economically desirable behavior.

## 4 Topological and uniform properties of $\mathcal{T}_\varphi$

In this section we will investigate whether the topology  $\mathcal{T}_\varphi$  gives economically reasonable sense of “closeness” for consumption patterns in  $\mathcal{E}_+^\varphi$  for all  $\varphi \in \Phi$ . The reader will find out that  $\mathcal{T}_\varphi$  inherits many properties of a H&K topology when those properties are appropriately aggregated across states. In particular, consumptions at nearly adjacent dates where there is no discontinuity of information are almost perfect substitutes – a natural generalization of H&K.

We first record two facts about  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$ .

**Proposition 3** *Suppose that  $(\Omega, \mathcal{F}, P)$  is separable. Then  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  is a separable normed topological vector space.*

PROOF. Recall that  $(\Omega, \mathcal{F}, P)$  is separable if there exists a countable set  $\{B_n \in \mathcal{F}; n = 1, 2, \dots\}$  such that for every  $\epsilon > 0$  whatever small and  $A \in \mathcal{F}$  there exists  $B_n$  such that  $P(A \Delta B_n) < \epsilon$ , where  $\Delta$  denotes the symmetric difference of two sets. Then it is easy to see that a countable dense set of  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  is the set of right-continuous bounded simple processes that change their values at rational time points, take rational values, and vanish except at finite subsets of  $\{B_n\}$ . (A process  $Y$  is simple if there exists a *finite* subdivision  $0 = t_0 < t_1 < \dots < t_n = 1$  of  $[0, 1]$ , and random variables  $\alpha, \alpha_1, \dots, \alpha_{n-1}$ , where  $\alpha_i$  is  $\mathcal{F}_{t_i}$ -measurable and bounded such that:  $Y(\omega, t) = \alpha(\omega)\chi_{\{0\}}(t) + \sum_{i=1}^{n-1} \alpha_i(\omega)\chi_{(t_i, t_{i+1}]}(t)$ .) ■

**Remark 3** *In most applications,  $(\Omega, \mathcal{F}, P)$  is separable. For example, in many models of financial markets,  $\Omega = C[0, 1]$ , the space of continuous function on  $[0, 1]$ ,  $\mathcal{F}$  is the Borel sigma-field generated by open sets defined by the sup norm on  $C[0, 1]$ , and  $P$  is the Wiener measure. The fact that this probability space is separable can be found, for example, in Billingsley (1968).*

An immediate consequence of Proposition 3 is that any continuous preference relation  $\succeq$  on separable  $(\mathcal{E}_+^\varphi, \mathcal{T}_\varphi)$  has a numerical representation; see Debreu (1954).

Next we will examine in what alternative forms the three properties satisfied by a continuous preference relation in H&K as reviewed in the beginning of Section 3 continue to hold. In the process, we will also show that absolutely continuous consumption patterns are dense.

First, we give an example to show that two consumption patterns that have almost equal accumulated consumption at every point in time and in almost every state of nature may not be close.



**Example 1** Fix  $\varphi \in \Phi$ . Let  $B_n \in \mathcal{F}$  be such that  $B_{n+1} \subset B_n \forall n$ ,  $P(B_n) > 0$ , and  $P(B_n) \rightarrow 0$  as  $n \rightarrow \infty$ , and let the random variable  $Y_n \geq 0$  be such that  $\varphi(\omega, Y_n(\omega)) = n/P(B_n)$  a.s. Consider the sequence of consumption patterns

$$x_n(\omega, t) = \begin{cases} 0 & \text{if } t < 1; \\ Y_n(\omega)\chi_{B_n}(\omega) & \text{if } t = 1, \end{cases}$$

where  $\chi_{B_n}$  is the indicator function of the set  $B_n$ . That is,  $x_n$  provides for no consumption until time 1, at which time it provides  $Y_n$  units of consumption in the event  $B_n$  and nothing otherwise. Let  $x(t) = 0$  for all  $t$ . It is easy to see that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0,1]} |x_n(\omega, t)| = 0, \text{ a.s.}$$

But

$$\mathbf{E} \left[ \int_0^1 \varphi(x_n(t)) dt + \varphi(x_n(1)) \right] = \int_{B_n} \varphi(\omega, Y_n(\omega)) P(d\omega) = n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Note that  $\|x_n - x\|_\varphi \rightarrow 0$  if and only if for all  $\gamma > 0$ ,

$$\mathbf{E} \left[ \int_0^1 \varphi(\gamma(x_n(t) - x(t))) dt + \varphi(\gamma(x_n(1) - x(1))) \right] \rightarrow 0 \text{ as } n \rightarrow \infty;$$

see Musielak (1983, Theorem 1.6, p. 9). Thus  $\|x_n - x\|_\varphi \not\rightarrow 0$  as  $n \rightarrow \infty$  and we do not expect  $x_n \succeq y$  for all  $n$  to imply that  $x \succeq y$ .

In the above example,  $x_n$  converges uniformly to zero pointwise. Along the sets  $\{B_n\}$ , however,  $x_n(1)$  increases to infinity in such a way that  $\varphi(\omega, x_n(\omega, 1))$  grows to infinity faster than the probability of  $B_n$  decreases to zero. Thus  $x_n$  does not converge to zero in  $\mathcal{T}_\varphi$ . For  $x_n$  to converge to  $x$ ,  $\sup_{t \in [0,1]} |x_n(\omega, t) - x(\omega, t)|$  cannot grow too fast even on sets of gradually vanishing probability. The following proposition gives an alternative form of property (1) of H&K.

**Proposition 4** Let  $x, y, x_n \in \mathcal{E}^\varphi$  for all  $n$ . If  $\lim_{n \rightarrow \infty} \sup_{t \in [0,1]} |x_n(\omega, t) - x(\omega, t)| = 0$  a.s. and if there exists a random variable  $K$  with the property that  $\mathbf{E}[\varphi(K)] < \infty$ , such that  $\sup_{t \in [0,1]} |x_n(\omega, t) - x(\omega, t)| \leq K(\omega)$  a.s., then  $x_n \succeq y$  for all  $n$  implies  $x \succeq y$  and  $y \succeq x_n$  implies  $y \succeq x$ .

PROOF. By the continuity of  $\succeq$  with respect to  $\mathcal{T}_\varphi$ , we only have to show that  $\|x_n - x\|_\varphi \rightarrow 0$  as  $n \rightarrow \infty$ . Recall from Example 1 that it suffices to show for any  $\gamma > 0$

$$\mathbf{E} \left[ \int_0^1 \varphi(\gamma|x_n(t) - x(t)|) dt + \varphi(\gamma|x_n(1) - x(1)|) \right] \rightarrow 0.$$

The assertion then follows from the Lebesgue convergence theorem and the fact that  $\varphi(\omega, z)$  is continuous in  $z$  and  $\varphi(\omega, 0) = 0$ . ■

Second, we want to show that sizable shift of consumption within the “information constraint” over a small enough interval is regarded as insignificant. The information constraint here is absent in H&K. One unit of consumption at time  $t$  in an event  $A \in \mathcal{F}_t$  may not be advanced to time  $t - \epsilon$  in the same event for  $\epsilon > 0$  however small, since  $A$  may not be a distinguishable event in  $\mathcal{F}_{t-\epsilon}$  and doing so will violate the information constraint that consumption patterns be adapted to  $\mathbf{F}$ . On the other hand, one unit of consumption at time  $t$  in an event can always be delayed to any time  $s > t$  in the same event. Moreover, one unit of consumption at a “surprise” cannot be advanced to an instant before by the nature of a surprise.

The following proposition formalizes these discussions, whose proof is contained in Appendix A. A definition is needed.

**Definition 5** *Let  $T$  and  $S$  be two optional times. The stochastic interval  $[T, S]$  is the set*

$$\{(\omega, t) : T(\omega) \leq t \leq S(\omega)\}.$$

Let  $T \leq S$  be two optional times. It is known that  $[T, S] \in \mathcal{O}$  and thus the process

$$x(\omega, t) \equiv \chi_{[T, 1]}(\omega, t),$$

denoting a consumption pattern that provides for one unit of consumption at time  $T$ , is optional and therefore adapted (to  $\mathbf{F}$ ).

**Proposition 5** *Let  $T$  be an optional time. Then  $k\chi_{[T, 1]} \in \mathcal{E}_+^\varphi$  for all  $k > 0$  and  $T + \frac{1}{n}$  is also an optional time for all  $n$ . Fix  $k > 0$  and put*

$$x_n = k\chi_{[T + \frac{1}{n}, 1]} \text{ and } x = k\chi_{[T, 1]}.$$

*Suppose that  $P\{T = 1\} = 0$ . Then*

$$\|k\chi_{[T + \frac{1}{n}, 1]} - k\chi_{[T, 1]}\|_\varphi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Thus  $x_n \geq y$  for all  $n$  implies  $x \geq y$  and  $y \geq x_n$  for all  $n$  implies  $y \geq x$ . On the other hand, suppose that  $\{T_n\}$  is a sequence of optional times with  $T_n \leq T$ , and on the set  $\{T > 0\}$ ,  $T_n < T_{n+1} < T$ .*

*Putting  $x'_n = k\chi_{[T_n, 1]}$ ,*

$$\|x'_n - x\|_\varphi \rightarrow 0 \text{ as } n \rightarrow \infty$$

*if and only if  $T$  is predictable and  $\{T_n\}$  is an announcing sequence.*

PROOF. See Appendix A. ■

Proposition 5 states that delaying the consumption of  $k$  units, even for large  $k$ , for a small period of time will be regarded insignificant by an agent with preference relation continuous in  $\mathcal{T}_\varphi$ . However, advancing consumption over a small interval from an optional time to an earlier optional time is regarded insignificant if and only if the optional time is predictable; that is, consumptions at a random time and at random times instants before are almost perfect substitutes if and only if there is no left discontinuity of information at those random times instants before. It is worthwhile perhaps to recall that any deterministic time is predictable and thus consumptions at nearly adjacent deterministic times are always almost perfect substitutes! Discontinuity of information can only occur at random times. For example, the first time a Poisson process jumps is a surprise that happens at a random time.

**Remark 4** *Note that although  $T + \frac{1}{n}$  is an optional time whenever  $T$  is,  $T - \frac{1}{n}$  is in general not an optional time except when  $T$  is deterministic. Thus an announcing sequence of a predictable optional time  $T$  in general cannot be constructed by putting  $T_n = T - \frac{1}{n}$ .*

With the aid of Proposition 5, we now show in the following proposition that the set of absolutely continuous consumption patterns is dense in  $\mathcal{E}^\varphi$ . This is the counterpart of a result in H&K with one caveat. Under certainty, one unit of consumption at time  $t \in (0, 1)$  can be approximated arbitrarily closely by consuming at an average rate “around”  $t$ . For example,  $\chi_{[1/2, 1]}$  can be approximated arbitrarily closely by three sequences,

$$x_n(t) = n\left(t - \frac{1}{2} + \frac{1}{2n}\right)\chi_{[\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}]}(t) + \chi_{(\frac{1}{2} + \frac{1}{2n}, 1]}(t), \quad (8)$$

$$x_n(t) = n\left(t - \frac{1}{2} + \frac{1}{n}\right)\chi_{[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]}(t) + \chi_{(\frac{1}{2}, 1]}(t), \quad (9)$$

$$x_n(t) = n\left(t - \frac{1}{2}\right)\chi_{[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]}(t) + \chi_{(\frac{1}{2} + \frac{1}{n}, 1]}(t). \quad (10)$$

The sequences above shift consumption around, before, and after  $t = 1/2$ , respectively, at an average rate. They all converge to consuming one unit at time  $1/2$  in any of the H&K topologies. From Proposition 5, however, we know that consumption at nonpredictable optional times cannot be approximated by earlier consumption. Consumption at all times can nevertheless be approximated by *delayed* consumption except at the final date  $t = 1$ , which is not a random time and creates no problem. This observation is formalized in the following proposition, whose proof is contained in Appendix A.

**Proposition 6** *The set of absolutely continuous consumption patterns in  $\mathcal{E}^\varphi$  is dense with respect to the topology  $\mathcal{T}_\varphi$ .*

Proposition 5 also suggests a more general measure of “shifts” of consumption in the spirit of the Prohorov metric used by H&K.

Let  $x, y \in \mathcal{E}_+^\varphi$  and define a Prohorov metric path-by-path:

$$\rho_\omega(x, y) = \inf \{ \epsilon \geq 0 : x(t + \epsilon, \omega) + \epsilon \geq y(t, \omega) \geq x(t - \epsilon, \omega) - \epsilon, \forall t \in [0, 1] \}.$$

The  $\epsilon$  neighborhood of a path  $x(\omega, \cdot)$  is any other path  $y(\omega, \cdot)$  that lives entirely within a “sleeve” around  $x(\omega, \cdot)$  that is determined by moving up and to the left and down and to the right ( a distance  $\epsilon$  in each direction) from  $x(\omega, \cdot)$  at every point  $(t, x(\omega, t))$ . Hence if  $y(\omega, \cdot)$  is within  $\epsilon$  of  $x(\omega, \cdot)$  in the Prohorov sense, then at every time  $t$  in state  $\omega$ , the total consumption under  $x$  is within  $\epsilon$  of the total consumption under  $y$  at some time no more than  $\epsilon$  away from  $t$ .

The idea here is that if  $\rho_\omega(x, y)$  is uniformly smaller than  $\epsilon$  across states of nature, then sizable shifts of consumption at predictable optional times can only occur across a time interval less than  $\epsilon$  in width. Such shifts should not drastically change an agent’s utility for the original pattern  $x$ . At nonpredictable times, however, Proposition 5 implies that as  $\epsilon \rightarrow 0$ , the  $\epsilon$  neighborhood of  $x$  will not contain any  $y$  that advances consumption to earlier time since that will violate the information constraint. As it turns out,  $\rho_\omega(x, y)$  being uniformly small is too strong, and we can allow  $\rho_\omega(x, y)$  to be large on a set of small probability. (Note that  $\rho_\omega(x_n, x) \rightarrow 0$  a.s. for a sequence of consumption patterns  $(x_n)$  is too weak for  $\|x_n - x\|_\varphi \rightarrow 0$ . A counter example can be constructed along the lines of Example 1.)

**Proposition 7** *Let  $x, x_n \in \mathcal{E}_+^\varphi$  with the property that there exists an  $\mathcal{F}$ -measurable function, say  $K$ , such that*

$$\rho_\omega(x_n, x) \leq \frac{K(\omega)}{n} \quad P\text{-a.s.} \quad \text{and} \quad \mathbf{E}[\varphi(K)] < \infty.$$

*Then  $\|x_n - x\|_\varphi \rightarrow 0$  as  $n \rightarrow \infty$ . Thus  $x_n \succeq y$  for all  $n$  implies that  $x \succeq y$  and  $y \succeq x_n$  for all  $n$  implies that  $y \succeq x$ .*

PROOF. See Appendix A. ■

This proposition concludes our investigation into how continuous preferences behave with respect to “shifts” of consumption. With the aid of Proposition 7, the following proposition shows the fact that  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  is not a topological vector lattice.

**Proposition 8**  *$(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  is not a topological vector lattice.*

PROOF. The sequence of sure consumption patterns  $\chi_{[1/2,1]} - \chi_{[(n+1)/2n,1]}$  converges to zero by Proposition 7. But the positive part (or the negative part) obviously does not. ■

Third, we now consider how our topologies perform when an increasingly larger amount of consumption is shifted over a decreasingly smaller amount of time. Intuition suggests that as long as the increasingly larger amount being shifted goes to infinity slowly enough, the shift should be regarded as negligible. The measure of how slowly the amount of a shift increases to infinity is naturally related to  $\varphi$ , the state-by-state inverse of which measures the same thing in a corresponding economy under certainty.

We first put  $\mu \equiv \varphi^{-1}$ , the inverse of  $\varphi$  state-by-state. It is easily seen that  $\mu(\omega, \cdot)$  is strictly increasing, continuous, concave, and with  $\mu(\omega, 0) = 0$  and  $\mu(\omega, \infty) = \infty$ . The following proposition essentially shows that if the amount shifted increases to infinity slower than  $\mu$  grows to infinity state by state except possibly on a set of small probability, then the shift will be negligible.

**Proposition 9** *Let  $\mu = \varphi^{-1}$  and let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $\mathcal{E}_+^\varphi$ . Suppose that  $\{x_n(\omega, 1)\}$  and  $\{y_n(\omega, 1)\}$  are both  $o(\mu(\omega, n))$  and satisfy*

$$\rho_\omega(x_n, y_n) \leq \frac{K(\omega)}{n} \quad p - a.s.,$$

where  $K(\omega)$  is an  $\mathcal{F}$ -measurable function such that  $\mathbf{E}[\varphi(K)] < \infty$ . Then  $\|x_n - y_n\|_\varphi \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF. See Appendix A. ■

The sequences  $\{x_n\}$  and  $\{y_n\}$  in Proposition 9 are both divergent and thus we could not hope for the validity of statements such as

$$x^* + x_n \succ y_n, \quad \text{for } n \text{ sufficiently large} \tag{11}$$

when  $\succeq$  is continuous and strictly monotonically increasing in the direction of  $x^* \in \mathcal{E}_+^\varphi$  with  $x^* \neq 0$ , where  $\succ$  is the strict preference relation derived from  $\succeq$ . In Appendix B we give an example of a linear preference relation that violates (11). The reason for this is that although  $\|x_n - y_n\|_\varphi \rightarrow 0$  as  $n \rightarrow \infty$ ,  $x^* + x_n - y_n$  may lie outside of  $\mathcal{E}_+^\varphi$  and we cannot use the continuity and strict monotonicity of  $\succeq$  to conclude that  $x^* + x_n - y_n \succ 0$  for large  $n$  and thus  $x^* + x_n \succ y_n$ . However, (11) is a property of uniformly proper preferences defined formally below:

**Definition 6 (Mas Colell (1986))** *Preferences  $\succeq$  on  $\mathcal{E}_+^\varphi$  are said to be uniformly proper with respect to  $\mathcal{T}_\varphi$  in the direction  $x^* \in \mathcal{E}_+^\varphi$  with  $x^* \neq 0$  if there exists a  $\mathcal{T}_\varphi$  open neighborhood  $V$  of the origin such that, for every  $x \in \mathcal{E}_+^\varphi$ ,  $v \in V$ , and scalar  $a > 0$ ,  $x - ax^* + av \not\succeq x$ .*

The uniform properness is a form of strong monotonicity. The *uniform* part of properness comes from the fact that  $x^*$  and  $V$  must be chosen independently of  $x$ . It is important to note that  $x - ax^* + av \succeq x$  is allowed to fail by virtue of the fact that  $x - ax^* + av \notin \mathcal{E}_+^\varphi$ . (See Mas Colell (1986) for further details.)

**Proposition 10** *Let the sequences  $\{x_n\}$  and  $\{y_n\}$  be as in Proposition 9. Suppose that  $\succeq$  is uniformly proper in the direction  $x^*$ . Then*

$$x^* + x_n \succ y_n \quad \text{as } n \rightarrow \infty.$$

PROOF. Rather than mimicking the proof of H&K, we use a different argument. Richard and Zame (1986) have shown that if  $\succeq$  is uniformly proper, it can be extended to an open set containing  $\mathcal{E}_+^\varphi$ . By Proposition 9, for large  $n$ ,  $x^* + x_n - y_n$  lies inside this open set and  $x^* + x_n - y_n$  is strictly preferred to 0 for the extended preferences by continuity. Then it follows that  $x^* + x_n \succ y_n$  for large  $n$ . ■

In the positive orthant, (11) is not a topological property of continuous preferences – there exists continuous preferences that violate (11); it is a uniform property. On an open set containing  $\mathcal{E}_+^\varphi$ , however, (11) becomes a topological property for continuous preferences. Since continuous and uniformly proper preferences on  $\mathcal{E}_+^\varphi$  behave like continuous preferences on an open set containing  $\mathcal{E}_+^\varphi$ , they satisfy (11).

## 5 Duality

In standard general equilibrium theory of the Arrow-Debreu sort, equilibrium prices come from the space of linear functionals continuous in the topology with respect to which agents' preferences are assumed to be continuous. This space of continuous linear functionals is called the *topological dual space*.<sup>7</sup> Let  $\psi$  be a continuous linear functional that gives equilibrium prices. Then  $\psi(x)$  is the price at time 0 of the consumption claim  $x$ . We will find out soon that  $\psi$  can be represented in the form

$$\psi(x) = \mathbf{E} \left[ \int_{0-}^1 f(t) dx(t) \right] \quad x \in \mathcal{E}^\varphi,$$

where, roughly,  $f$  is a process that is decomposable into two parts: a process having absolutely continuous sample path and a martingale. We interpret  $f(\omega, t)$  to be the time 0 shadow price of one unit of consumption at time  $t$  in state  $\omega$ . Since a martingale can have discontinuities only

<sup>7</sup>More formally,  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  is a topological linear space and the space of continuous linear functionals on  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  is its topological dual.

at unpredictable optional times (see Meyer (1966, Theorem VI.14)), we thus conclude that consumption at nearly adjacent (random) dates have almost the same shadow prices except at surprises. It is known that a continuous martingale is either a constant or of unbounded variation. Thus when information is revealed continuously throughout, shadow prices for consumption can fluctuate widely in a nondifferentiable fashion. In particular, if the information structure is generated by a Brownian motion, the shadow price process for consumption over time is an Itô process.

We now proceed to characterize the topological dual of  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$ , denoted by  $\mathcal{E}^{\varphi^*}$ . We shall make use of the concept of complementary generalized Orlicz spaces for most of our analysis here. We provide some definitions.

**Definition 7** A function  $\varphi \in \Phi$  is said to be an *N-function* if the following conditions hold:

- $\lim_{x \rightarrow 0^+} \frac{\varphi(\omega, x)}{x} = 0 \quad \forall \omega \in \Omega$ ; and
- $\lim_{x \rightarrow \infty} \frac{\varphi(\omega, x)}{x} = \infty \quad \forall \omega \in \Omega$ .

**Remark 5** By convexity, for  $\varphi$  an *N-function*, we can write  $\varphi(\omega, |x|) = \int_0^{|x|} v(\omega, s) ds$ , where  $v(\omega, s)$  is the right-hand derivative of  $\varphi(\omega, s)$  for a fixed  $\omega$ .

**Definition 8** Let  $\varphi \in \Phi$  be an *N-function*, and let  $v(\omega, s)$  be as in the above remark. Let  $v^*(\omega, \sigma) = \sup\{s : v(\omega, s) \leq \sigma\}$ , then

$$\varphi^*(\omega, x) = \int_0^{|x|} v^*(\omega, \sigma) d\sigma$$

is said to be complementary to  $\varphi$ . The generalized Orlicz space  $L^{\varphi^*}$  is then said to be complementary to  $L^\varphi$ , where we recall the definition of a generalized Orlicz space in (4).

**Example 2** (1) If  $\varphi(\omega, x) = |x|^{p(\omega)}$ ,  $1 < p(\omega) < \infty$ , then  $\varphi^*(\omega, y) = |y|^{q(\omega)}$ , where  $\frac{1}{p(\omega)} + \frac{1}{q(\omega)} = 1$ . (2) If  $\varphi(\omega, x) = e^x - x - 1$ , then  $\varphi^*(\omega, y) = (1 + y)\ln(1 + y) - y$ .

The following proposition describes  $\mathcal{T}_\varphi$ -continuous linear functionals on the space  $\mathcal{E}^\varphi$  in terms of elements of  $L^{\varphi^*}$ .

**Proposition 11** *Let  $\varphi \in \Phi$  be an  $N$ -function and with a complementary function  $\varphi^*$ . Then  $\psi : \mathcal{E}^\varphi \rightarrow \mathbb{R}$  is a  $\mathcal{T}_\varphi$ -continuous linear functional if and only if there exists an adapted process  $f$  with absolutely continuous sample paths, and with  $f' \in L^{\varphi^*}$ , so that*

$$\psi(x) = \mathbf{E} \left[ \int_{0-}^1 g(t) dx(t) \right] \quad \forall x \in \mathcal{E}^\varphi, \quad (12)$$

where  $g(t) = f(t) + m(t)$ , and

$$m(t) = \mathbf{E}[-f'(1) - f(1)|\mathcal{F}_t] \quad \forall t \in [0, 1] \quad (13)$$

is a martingale, or equivalently,  $g(t) = f(t) - f'(1) - f(1)$ , which is a process with absolutely continuous paths but may not be adapted.

PROOF. See Appendix A. ■

An important special case of Proposition 11 is when  $\varphi(\omega, z) = |z|^p$  with  $1 < p < \infty$ . The case when  $p = 1$  is not covered in Proposition 11, however, since  $|z|$  is not an  $N$  function. The following proposition gives the duality result for this  $L^p$  family, whose proof is left for the reader.

**Proposition 12** *Let  $\varphi(\omega, z) = |z|^p$ , for all  $\omega$ , with  $1 \leq p < \infty$ . Let  $q$  be such that  $1/p + 1/q = 1$ . Then  $\psi : \mathcal{E}^\varphi \rightarrow \mathbb{R}$  is a  $\mathcal{T}_\varphi$ -continuous linear functional if and only if there exists an adapted process  $f$  with absolutely continuous sample paths so that*

$$\psi(x) = \mathbf{E} \left[ \int_{0-}^1 g(t) dx(t) \right] \quad \forall x \in \mathcal{E}^\varphi, \quad (14)$$

where

$$(f', f'(1)) \in L^q(\Omega \times [0, 1], \mathcal{O}, P \times \lambda) \times L^q(\Omega, \mathcal{F}, P),$$

$g(t) = f(t) + m(t)$ , and

$$m(t) = \mathbf{E}[-f'(1) - f(1)|\mathcal{F}_t] \quad \forall t \in [0, 1] \quad (15)$$

is a martingale, or equivalently,  $g(t) = f(t) - f'(1) - f(1)$ , which is a process with absolutely continuous paths but is not necessarily adapted.

Note that in the above proposition, when  $p = 1$ , the dual space is composed of processes that are sums of processes with Lipschitz continuous paths and bounded martingales.

Another important case of the function  $\varphi$  which is not covered in the above propositions is when  $\varphi$  is asymptotically linear, in that for some  $\omega \in \Omega$ , we have  $\lim_{x \rightarrow \infty} \frac{\varphi(x)}{x} = \alpha > 0$ . In the



case of certainty, H&K have shown that the space of shadow prices in this case is the space of Lipschitz continuous functions. We obtain a similar characterization for the case of uncertainty under the assumption that the function  $\varphi$  is “integrably asymptotically linear.” This result is recorded in the following proposition.

**Proposition 13** *Let the function  $\varphi$  be integrably asymptotically linear, in that for any  $\epsilon > 0$ , there exists a random variable  $K$ , with the property that  $\mathbf{E}[\varphi(K)] < \infty$ , such that*

$$\varphi(\omega, x) \leq \alpha x + \epsilon \quad \forall x \geq K(\omega) \text{ a.s.} \quad (16)$$

*In this case, we have the equivalence  $\mathcal{E}^\varphi = \mathcal{E}$ , and  $\mathcal{T}_\varphi$  is the same as  $\mathcal{T}$ , where we recall that  $\mathcal{E}$  and  $\mathcal{T}$  are the consumption space, and the topology constructed using  $\varphi(x) = |x|$  for all  $\omega \in \Omega$ . In particular, this shows that an element of the dual space to  $\mathcal{E}^\varphi$  is the sum of a process with Lipschitz continuous sample paths and a bounded martingale.*

PROOF. See Appendix A. ■

In H&K, a continuous linear functional can be represented essentially by an absolutely continuous function. Here, however, a continuous linear functional is represented by a process that is the sum of an absolutely continuous process and a martingale. It is known that the sample path properties of a martingale are determined by the way information is revealed over time: A martingale can have a discontinuity only at nonpredictable optional times; see, for example, Meyer (1966, Theorem VI.14) It then follows that prices for consumptions at nearly random adjacent dates are almost equal when there are no “surprises.” An information structure  $\mathbf{F}$  is said to be continuous if  $P(A|\mathcal{F}_t) = \mathbf{E}[\chi_A|\mathcal{F}_t]$  is a continuous process for all  $A \in \mathcal{F}$ ; that is, the posterior probability of any event evolves continuously. It is known that an information structure is continuous if and only if all optional times are predictable; see Huang (1985a). Thus when  $\mathbf{F}$  is continuous, the dual of  $\mathcal{E}^\varphi$  consists only of continuous processes and prices of consumption at nearly adjacent random dates are almost equal. It is also known that if a martingale is continuous on a stochastic interval, it is either a constant or of unbounded variation there (Fisk (1965)). Hence there are cases where prices for consumption at nearly adjacent dates are almost equal but fluctuate widely in a nowhere differentiable fashion. This comes about because of uncertainty. When there is no uncertainty, the martingale part of the prices for consumption disappears and prices degenerate to absolutely continuous functions of time.

An important special case of continuous information structure is when  $\mathbf{F}$  is generated by a Brownian motion. In this case, the dual space only contains Itô processes. This characterization is given in the following proposition:

**Proposition 14** *Let  $\mathbf{F}$  be generated by a Brownian motion and let  $g$  be an element of any of the dual spaces characterized in Propositions 11, 12 and 13, then  $g$  is an Itô process:*

$$g(t) = \int_0^t f'(s) ds + \int_0^t \theta(s)^\top dw(s),$$

where  $f$  is a process with absolutely continuous paths and  $f'$  denotes its derivative with respect to time, where  $w$  is a Brownian motion and where  $\theta$  is adapted to  $\mathbf{F}$  and satisfies:

$$\int_0^1 |\theta(t)|^2 dt < \infty \quad \text{a.s.}$$

PROOF. From Propositions 11, 12, and 13, we know that  $g(t) = f(t) + m(t)$ , where  $f$  is an absolutely continuous process and  $m(t)$  is a martingale. The assertion then follows from the fact that a martingale adapted to a Brownian motion filtration can always be represented as an Itô integral (see Clark (1970, Theorems 3 and 4)). ■

An example of a discontinuous information structure is when information is obtained by observing the realizations of a simple Poisson process. In this case, a martingale may have a discontinuity only when the Poisson process jumps. That is, the shadow price process can have a discontinuity only at a “surprise” which occurs when the Poisson process jumps.

Before we leave this section, we show below that the dual spaces characterized above in general fail to be sub-lattices of their order duals. We demonstrate this by showing two examples. The first example uses a very specialized information structure, while the second is in the context of a Brownian motion information structure, which is the case of prevalent applications.

**Example 3** *Let  $\Omega = \{\omega_1, \omega_2\}$  and the filtration be composed of*

$$\mathcal{F}_t = \begin{cases} \{\Omega, \emptyset\} & \forall t \in [0, 1/2), \\ \{\{\omega_1\}, \{\omega_2\}, \Omega, \emptyset\} & \forall t \in [1/2, 1]. \end{cases}$$

*Assume that each of the two possible states has a strictly positive probability of occurrence.*

*Define two processes with absolutely continuous sample paths as follows:*

$$f_1(\omega, t) = \begin{cases} 0 & \forall \omega \in \Omega \quad t \in [0, 1/2) \\ 4(t - 1/2) & \text{if } \omega = \omega_1 \quad \forall t \in [\frac{1}{2}, \frac{3}{4}), \\ -4(t - 1/2) & \text{if } \omega = \omega_2 \quad \forall t \in [\frac{1}{2}, \frac{3}{4}), \\ 1 & \text{if } \omega = \omega_1 \quad \forall t \in [\frac{3}{4}, 1], \\ -1 & \text{if } \omega = \omega_2 \quad \forall t \in [\frac{3}{4}, 1]. \end{cases} \quad f_2(\omega, t) = \begin{cases} 2t & \forall \omega \in \Omega \quad t \in [0, 1/2) \\ 1 - 4(t - 1/2) & \text{if } \omega = \omega_1 \quad \forall t \in [\frac{1}{2}, 1], \\ 1 & \text{if } \omega = \omega_2 \quad \forall t \in [\frac{1}{2}, 1]. \end{cases}$$

*Note that*

$$-f_1(\omega, 1) - f_1'(\omega, 1) = \begin{cases} -1 & \text{if } \omega = \omega_1, \\ 1 & \text{if } \omega = \omega_2; \end{cases} \quad -f_2(\omega, 1) - f_2'(\omega, 1) = \begin{cases} 5 & \text{if } \omega = \omega_1, \\ -1 & \text{if } \omega = \omega_2. \end{cases}$$

It is easily verified that  $g_i(t) = f_i(t) - \mathbf{E}[f_i(1) + f'_i(1)|\mathcal{F}_t]$  lies in the dual spaces characterized in Propositions 11, 12, and 13 for  $i = 1, 2$ . Note that the order dual of  $\mathcal{E}^\varphi$  is composed of positive processes  $g$  that are measurable with respect to  $\mathcal{F} \otimes \mathcal{B}([0, 1])$  so that the integral

$$\mathbf{E} \left[ \int_{0-}^1 g(t) dx(t) \right]$$

is well-defined for  $x \in \mathcal{E}_+^\varphi$ . The order generated by the order dual is then the pointwise order; that is,  $g_1 \leq g_2$  if  $g_1(\omega, t) \leq g_2(\omega, t)$  a.e. From Proposition 11, we can think of the topological dual of  $\mathcal{E}^\varphi$  as composed of processes not necessarily adapted but with absolutely continuous sample paths. In this example, we can think of  $g_i$  as

$$g_i(\omega, t) = f_i(\omega, t) - f_i(\omega, 1) - f'_i(\omega, 1).$$

Then the pointwise minimum of  $g_1$  and  $g_2$  is

$$g_1(\omega, t) \wedge g_2(\omega, t) = \begin{cases} -1 & \text{if } \omega = \omega_1 \text{ } t \in [0, \frac{1}{2}), \\ 4(t - 1/2) - 1 & \text{if } \omega = \omega_1 \text{ } t \in [\frac{1}{2}, \frac{3}{4}), \\ 0 & \text{if } \omega = \omega_1 \text{ } t \in [\frac{3}{4}, 1], \\ -1 + 2t & \text{if } \omega = \omega_2 \text{ } t \in [0, \frac{1}{2}), \\ 0 & \text{if } \omega = \omega_2 \text{ } t \in [\frac{1}{2}, 1]. \end{cases}$$

On the interval  $[0, \frac{1}{2})$ ,  $g_1 \wedge g_2$  has different derivatives along  $\omega_1$  and  $\omega_2$ . Since a martingale on  $[0, \frac{1}{2})$  must be a constant, it is impossible to write  $g_1 \wedge g_2$  as a sum of an adapted process with absolutely continuous paths and a martingale and thus it cannot be an element of the topological dual of  $\mathcal{E}^\varphi$ .

The next example is in the context of a Brownian motion filtration.

**Example 4** Let the filtration be a Brownian motion filtration. From Proposition 14 we know that the dual contains only Itô processes. It is well known that the minimum of two Itô processes is a generalized Itô process in that it can be decomposed into two parts: a continuous process with bounded variation sample paths and an Itô integral (see Harrison (1985), §6). That is, by taking the pointwise minimum of two Itô processes, one creates a process whose time trend part may have a singular component. Thus the space of Itô processes is not a lattice. We note, in contrast, that the space of generalized Itô processes<sup>8</sup> is indeed a lattice.

<sup>8</sup>A generalized Itô process is a continuous process with bounded variation sample paths plus an Itô integral.

## 6 Comparison with standard models

In this section we will see how the standard models fare in the context of our current model. Using a standard model, one assumes that an agent maximizes his expected utility of the following form

$$U(x) = \mathbf{E} \left[ \int_0^1 u(x'(t), t) dt \right], \quad (17)$$

for absolutely continuous consumption pattern, where  $u(z, t)$  is a time-additive “felicity” function. This model does not permit gulps of consumption. This creates no major problem. Recall from Proposition 6 that the set of absolutely continuous consumption patterns is dense in  $\mathcal{E}^\varphi$ . We can therefore follow H&K in defining the expected utility of a consumption pattern involving gulps to be the limit of the expected utilities of a sequence of approximating absolutely continuous consumption patterns.

The above procedure will work provided that  $U(x)$  is  $\mathcal{T}_\varphi$ -continuous. We show, however, in what follows that if  $u(z, t)$  is jointly continuous then  $U(x)$  is  $\mathcal{T}_\varphi$  continuous only if  $u(z, t)$  is linear. Our proof is identical to that of H&K in the case of certainty by using Proposition 7 and the fact that a nonrandom consumption is certainly feasible in a world of uncertainty.

**Proposition 15** *Let  $u(z, t) : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$  be jointly continuous. The utility function  $U : \mathcal{E}_+^\varphi \rightarrow \mathbb{R}$  is continuous in  $\mathcal{T}_\varphi$  only if there exists continuous functions  $\alpha : [0, 1] \rightarrow \mathbb{R}$  and  $\beta : [0, 1] \rightarrow \mathbb{R}$  such that  $u(z, t) = \alpha(t)z + \beta(t)$ .*

PROOF. See Appendix A. ■

The converse of the above proposition is a direct application of the duality results in Section 5.

**Proposition 16** *Let  $u(z, t) = \alpha(t)z + \beta(t)$ . Then  $U$  of (17) is continuous in  $\mathcal{T}_\varphi$  if  $\alpha' \in L^\varphi$ , where  $\alpha'(t)$  is the derivative of  $\alpha(t)$ .*

PROOF. Observe that

$$U(x) = \mathbf{E} \left[ \int_0^1 \alpha(t)x'(t)dt + \int_0^1 \beta(t)dt \right] = \mathbf{E} \left[ \int_0^1 \alpha(t)dx(t) + \int_0^1 \beta(t)dt \right].$$

Thus  $U$  is a linear functional plus a constant. Then the assertion follows directly from Proposition 11. ■

The above analysis reveals that the standard representation of utility as the expectation of the integral of felicity of current consumption fails (except in a very special case) to produce

preferences continuous in our topologies. The problem is that the additive form of felicity of current consumption implies that consumption at an earlier date, no matter how recent it is, has no effect on current satisfaction. This is in direct conflict with our concept of preference continuity in which consumptions at nearby dates are close substitutes if there is no discontinuity of information.

The standard model also fails to capture our economic intuition in situations different from the single perishable consumption good case that we deal with here. Take for example the case of durable goods. One could interpret the consumption plan  $x(t)$  as the level of accumulated purchase of durable good up to time  $t$ . A durable good is most often acquired in single units, and only a few times over the horizon  $[0, 1]$ . However, the owner of the good receives a continuous flow of services from the earlier acquired durable goods. His level of satisfaction from owning the good at any time should reflect his enjoyment from the services provided by the good, in spite of the fact that the good was purchased in the past.

We would like to propose an alternative representation of utility, which keeps the spirit of the standard model, is mathematically tractable, and gives rise to preferences continuous in our topologies. To achieve this objective, we will express utility as an integral of felicity as in the standard model. However, the felicity at time  $t$  depends not only on consumption at time  $t$ , but also on consumption in the “recent past.” Our construction goes as follows: let  $x(\omega, t)$  be a consumption plan. Consider an adapted process  $\theta(\omega, t)$ , which is uniformly bounded across the states of nature  $\omega$ , which is differentiable in  $t$ , and whose first derivatives are uniformly bounded over  $\omega$ . An example of such a process is the Gauss-kernel given by:

$$\theta(\omega, t) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{t^2}{2\sigma^2}}. \quad (18)$$

Now using the process  $\theta$ , construct from  $x$  another process  $\hat{x}$  which gives a path-by-path weighted average of recent past consumption. In other words, put

$$\hat{x}(\omega, t) = \int_0^k \theta(\omega, s) dx(t-s). \quad (19)$$

The new process  $\hat{x}$  will be the element over which preferences are expressed using the standard model. Let  $u(\hat{x}, t)$  be a felicity function which is state-independent, continuous and concave in  $\hat{x}$ . Let

$$U(x) = \mathbf{E}\left[\int_0^1 u(\hat{x}, t) dt\right]. \quad (20)$$

The fact that under these conditions, the utility function given in (20) is continuous in any of our topologies is recorded in the following proposition.

**Proposition 17** *Let the averaging process  $\theta$  and the felicity function  $u$  satisfy the conditions described above. Let  $U(x)$  be constructed as in (20). Preferences represented by  $U$  are continuous in any  $\mathcal{T}_\varphi$ .*

PROOF. See Appendix A ■

It is important to point out that many authors have attempted to capture the effects of the time complementarity of consumption using functional forms that appear to be similar to (20). It turns out, however, that there is a crucial difference. In our proposed representation of utilities, the felicity function  $u$  depends only on the “smoothed” consumption process  $\hat{x}$  and time. In contrast, most “non time-additive” formulations in the literature posit a felicity function, say  $v$ , which takes as arguments the current consumption “rate”  $x'(t)$  together with the “smoothed” earlier consumption  $\hat{x}(t)$ . Thus one represents preferences by:

$$V(x) = \mathbf{E}\left[\int_0^1 v(x', \hat{x}, t) dt\right]. \quad (21)$$

Notable examples of such representations include Bergman (1985), Constantinides (1988), Heaton (1988), and Sundaresan (1988). Inclusion of the “smoothed” recent consumption in the instantaneous utility function  $v$  captures the effect of past consumption on one’s current satisfaction. However, including current consumption in the felicity function destroys the continuity of preferences in the sense we argue for in this paper except possibly for uninteresting special cases. We record this fact in the following proposition.

**Proposition 18** *Let  $v(c, z, t): \mathbb{R}_+^2 \times [0, 1]$  be jointly continuous. The utility function  $V: \mathcal{E}_+^\varphi \rightarrow \mathbb{R}$  defined in (21) is continuous in  $\mathcal{T}_\varphi$  only if there exists jointly continuous functions  $\alpha: \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$  and  $\beta: \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ , and a subset  $\Lambda$  of  $\mathbb{R}_+$  with a strictly positive Lebesgue measure such that  $v(c, z, t) = \alpha(z, t)c + \beta(z, t)$  for all  $z \in \Lambda$ .*

PROOF. See Appendix A. ■

## 7 Issues related to arbitrage and general equilibrium

One of the central topics in modern financial theories concerns properties of security prices when there are no arbitrage opportunities. Using a similar commodity space and a Mackey topology generated by the space of bounded and continuous processes and barring arbitrage opportunities, Huang (1985b) has shown that security prices *over time* are continuous at predictable optional times when there are no lump-sum dividends. In particular, price processes will be

generalized Itô processes between lump-sum ex-dividend dates if the information structure is generated by a Brownian motion. However, requiring the dual space of the commodity space to have only continuous processes is a bit too strong. It implies that consumption at nearly adjacent dates are almost perfect substitutes even at times of surprise. Using the family of economically more reasonable norm topologies discussed in previous sections, we get essentially all the results of Huang (1985b) in the following manner.

Take the commodity space to be  $(\mathcal{E}^\varphi, \tau_\varphi)$ . Assume that  $(\mathcal{E}^\varphi, \tau_\varphi)$  is separable. There are  $N$  securities which are available for trading any time in  $[0, 1]$ . Security  $n$  is denoted by its accumulated dividend process  $x_n \in \mathcal{E}^\varphi$ . Let  $S_n$  denote the ex-dividend price process for security  $n$ . Since securities are traded ex-dividends, we assume that  $x_n(0) = 0$  for all  $n$  and  $S_n(1) = 0$ ; that is, there is no accumulated dividend at time 0 and the ex-dividend price of a security at time 1 is zero. A trading strategy  $\theta$  is an  $N$ -dimensional process that prescribes the portfolio strategy for the traded securities. Without getting into technical details, we shall only allow agents to use *simple trading strategies* – strategies that are adapted, bounded, left-continuous, and change their values at a finite number of non-random time points.<sup>9</sup> shall say  $m \in \mathcal{E}^\varphi$  is *marketed* if there is a strategy  $\theta$  so that

$$m(t) - m(0) = \theta(0)^\top S(0) + \int_0^{t-} \theta(s)^\top dS(s) + \int_0^{t-} \theta(s)^\top dx(s),$$

where  $^\top$  denotes “transpose,”  $x(t) = (x_1(t), \dots, x_N(t))^\top$ , and the integrals are defined path-by-path; alternatively, we say that  $m$  is *financed* by  $\theta$ . Let  $M$  denote the space of marketed consumption patterns. It is clear that  $M$  is a linear subspace of  $\mathcal{E}^\varphi$ . An element  $m \in M$  financed by  $\theta$  is a *simple free lunch* if  $\theta(0)^\top S(0) \leq 0$ ,  $m \in \mathcal{E}_+^\varphi$ , and  $m \neq 0$ . Barring simple free lunches, each  $m \in M$  has a unique price at time 0,  $m(0) + \theta(0)^\top S(0)$ , where  $\theta$  finances  $m$ , and we can define a linear functional on  $M$  by  $\pi(m) = m(0) + \theta(0)^\top S(0)$ .

Now we shall show that the linear functional  $\pi$  has a nice representation if the securities market does not admit *free lunches*, a concept due to Kreps (1981). A *free lunch* is a sequence  $\{(m_n, x_n) \in M \times \mathcal{E}^\varphi; n = 1, 2, \dots\}$  and a bundle  $k \in \mathcal{E}_+^\varphi$  with  $k \neq 0$  such that

$$m_n - x_n \in \mathcal{E}_+^\varphi \quad \forall n, \quad \|x_n - k\|_\varphi \rightarrow 0,$$

and

$$\liminf_n \pi(m_n) \leq 0.$$

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<sup>9</sup>To consider strategies more general than simple strategies, we will first need to discuss how general stochastic integrals are defined, which we do not want to do here. Interested readers should consult Huang (1985b) for a discussion in a context similar to our setup here.

Suppose that there are no free lunches. Theorem A.1 of Duffie and Huang (1986) shows that there exists an extension of  $\pi$  to all of  $\mathcal{E}^\varphi$ ,  $\psi$ , that is strictly positive in that  $\psi(x) > 0$  if  $x \in \mathcal{E}_+^\varphi$  and  $x \neq 0$ .

Let  $\varphi$  satisfy conditions of Propositions 11, 12, or 13, then

$$\psi(x) = \mathbf{E} \left[ \int_{0-}^1 g(t) dx(t) \right] \quad \forall x \in \mathcal{E}^\varphi,$$

for some strictly positive process  $g$ , which is the sum of a martingale and a process of absolutely continuous sample paths. Let  $S_m(t)$  be the ex-dividend price at time  $t$  of  $m \in M$ . It follows from Proposition 5.1 of Huang (1985b) that

$$S_m(t) = \frac{\mathbf{E} \left[ \int_0^1 g(s) dm(s) | \mathcal{F}_t \right] - \int_0^t g(s) dm(s)}{g(t)}. \quad (22)$$

Note that the first term of the numerator of (22) is a martingale and thus the properties of  $g$  translate into those of  $S_m$  in a natural way. For example,  $S_m$  must be continuous at predictable optional times if  $m$  is also continuous there and, between discontinuities of  $m$ ,  $S_m$  can have discontinuities only at surprises. In particular, suppose that  $\mathbf{F}$  is generated by a Brownian motion and  $m$  is an absolutely continuous process. Since a martingale then can be represented by an Itô integral and the second term in the numerator of (22) is an absolutely continuous process, the numerator of (22) is an Itô process. The denominator is a martingale plus a process of absolutely continuous paths and thus is an Itô process. Since  $g$  is strictly positive, Itô's lemma implies that  $S_m$  is an Itô process. These are all economically appealing properties of price processes for securities over time.

As for the existence of an Arrow-Debreu equilibrium in an economy with a commodity space  $\mathcal{E}^\varphi$  equipped with the topology  $\mathcal{T}_\varphi$ , however, we have little to offer. Since  $(\mathcal{E}_+^\varphi, \mathcal{T}_\varphi)$  has an empty interior, existing general equilibrium theories offer two possibilities. First, if  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  were a topological vector lattice, then an equilibrium is ensured if agents' preferences are uniformly proper, among other things; see Mas Colell (1986). Second, if, among other things, the topological dual of  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  were a sublattice of its order dual and if preferences are uniformly proper, there exists an equilibrium; see Mas Colell and Richard (1987). Unfortunately, neither  $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$  is a topological vector lattice nor its dual is a sublattice of its order dual in general; for the former recall Proposition 8; for the latter see Example 3 and 4. Whether there exists an Arrow-Debreu equilibrium in our economy is an open question.

However, provided that there exists an Arrow-Debreu equilibrium, one can implement the Arrow-Debreu equilibrium by continuously trading a few long-lived securities as in Duffie (1986),



Duffie and Huang (1985), and Huang (1987) and thus establish the existence of a dynamic equilibrium with dynamically complete markets. In Duffie (1986) and Huang (1987), the commodity space is the space of consumption rates and the sense of closeness between two consumption patterns is defined by the standard  $L^p$  norm on consumption rates as functions of  $(\omega, t)$ . Thus consumption at nearly adjacent dates are perfect nonsubstitutes even at times of no surprise. As a consequence, the properties of shadow prices for consumption over time in a pure exchange equilibrium depend crucially on the properties of the aggregate endowment process. For example, in the case of a Brownian motion filtration, the price process for a security with absolutely continuous accumulated dividend process will not be an Itô process unless the aggregate endowment process is. This does not conform with our intuition. In our setup, however, since consumption at nearly adjacent dates are almost perfect substitutes at times of no surprise, price processes of securities with absolutely continuous accumulated dividends will be Itô processes independently of the properties of the aggregate endowment process.

## 8 Concluding remarks

In this paper we have advanced a family of topologies defining closeness between consumption patterns over time under uncertainty that capture the intuitive idea that consumptions at nearly adjacent dates are almost perfect substitutes if there are no surprises there. The intuitive idea we have tried to formalize is certainly not new. Our contribution is topologizing this idea and, more important, characterizing the topological dual spaces. We feel that our choice of topologies is the natural one that conceptualizes the aforementioned idea for the following two reasons. First, in the degenerate case where the true state of nature is revealed at time 0, we reach the conclusion that the shadow prices of consumption are absolutely continuous functions. Hence, preferences continuous in our topologies give rise to equilibrium prices in which the price of consumption at adjacent dates are almost equal and in which prices change smoothly in a differentiable manner. We believe that this is an intuitively attractive result.

Second, in the nondegenerate case, the topological duals are the natural generalizations of those under certainty. In this case, prices are processes that can be decomposed as the sum of two components: a process of absolutely continuous sample path and a martingale. In the case of uncertainty a new element, the pattern of information flow, affects the sample path properties of equilibrium prices. This effect is captured in the martingale component of the price process. It is known that a martingale can make discontinuous changes only at surprises. Thus equilibrium prices for consumption are continuous except possibly at surprises. This is

also an intuitively appealing result which holds independently of the nature of the endowment process.

The proposed family of natural topologies, however, does not give rise to certain mathematical properties known to be sufficient for the existence of an Arrow-Debreu equilibrium. The resolution of this issue should be of high priority.

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## Appendix A Proofs

### PROOF OF PROPOSITION 1:

PROOF. Let  $x^* \geq x_*$  be elements of  $\mathcal{E}^\varphi$  and let  $I$  denote the order interval  $[x_*, x^*]$ . Assume without loss of generality that  $x_* = 0$ , since our topologies are linear. We will show that  $I$  is totally bounded and closed and therefore compact; see Schaefer (1980, p.25).

To show that  $I$  is totally bounded it suffices to show that for any  $\epsilon > 0$  there exists a finite number of elements of  $I$ ,  $\{x_n; n = 1, 2, \dots, N(\epsilon)\}$  such that

$$I \subset \bigcup_{n=1}^{N(\epsilon)} O_{x_n}^\epsilon,$$

where  $O_{x_i}^\epsilon$  is a  $\mathcal{T}_\varphi$ -neighborhood of  $x_i$  with  $\|\cdot\|_\varphi$  diameter less than  $\epsilon$ . Fix  $\epsilon > 0$ . From Proposition 7, we know there exists  $N(\epsilon) > 0$  such that  $\rho_\omega(x, y) \leq x^*(\omega, 1)/N(\epsilon)$  implies  $\|x - y\|_\varphi < \epsilon/2$  for  $x, y \in \mathcal{E}_+^\varphi$ . Construct  $N(\epsilon)$  processes with right-continuous, nonnegative, and nondecreasing sample paths in the following fashion. For each state  $\omega$ , begin first by building a Prohorov sleeve around  $x^*(\omega, \cdot)$  with Prohorov distance  $x^*(\omega, 1)/N(\epsilon)$ . Let  $\hat{x}_1(\omega, \cdot)$  be the function prescribed by the lower boundary of that sleeve. Formally, let

$$\hat{x}_1(\omega, t + x^*(\omega, 1)/N(\epsilon)) \equiv \max\{x^*(\omega, t) - x^*(\omega, 1)/N(\epsilon), 0\} \quad \forall t \in \mathfrak{R},$$

where we have used the convention that  $x^*(\omega, t) = 0$  if  $t < 0$ . Next we define  $\hat{x}_n$  recursively as follows:

$$\hat{x}_{n+1}(\omega, t + x^*(\omega, 1)/N(\epsilon)) \equiv \max\{\hat{x}_n(\omega, t) - x^*(\omega, 1)/N(\epsilon), 0\} \quad \forall t \in \mathfrak{R}.$$

Note better that the processes  $\hat{x}_n$  constructed may not be adapted. By construction,  $\rho_\omega(\hat{x}_n, \hat{x}_{n+1}) \leq x^*(\omega, 1)/N(\epsilon)$  and thus  $\|\hat{x}_n - \hat{x}_{n+1}\|_\varphi < \epsilon/2 \forall n = 1, 2, \dots, N(\epsilon) - 1$ .

Suppose for the time being that the processes  $\hat{x}_n$  are adapted. Let

$$O_{\hat{x}_n}^\epsilon = \{x \in \mathcal{E}_+^\varphi : \|x - \hat{x}_n\|_\varphi < \epsilon/2\}.$$

It is then clear that

$$I \subset \bigcup_{n=1}^{N(\epsilon)} O_{\hat{x}_n}^\epsilon$$

and thus  $I$  is totally bounded. Now if  $\hat{x}_n$  are not adapted, pick any element  $x_n \in O_{\hat{x}_n}^\epsilon$  and put

$$O_{x_n}^\epsilon = \{x \in \mathcal{E}_+^\varphi : \|x - x_n\|_\varphi < \epsilon\},$$

we claim that  $O_{\hat{x}_n}^\epsilon \subset O_{x_n}^\epsilon$ . To see this, let  $x \in O_{\hat{x}_n}^\epsilon$ . Then

$$\|x - x_n\|_\varphi \leq \|x - \hat{x}_n\|_\varphi + \|\hat{x}_n - x_n\|_\varphi < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

by triangle inequality. We thus conclude that  $I$  is totally bounded.

Next we want to show that  $I$  is closed. Let  $x_n \in I$  and  $\|x_n - x\|_\varphi \rightarrow 0$  as  $n \rightarrow \infty$  for some  $x \in \mathcal{E}_+^\varphi$ . Then there exists a subsequence of  $(x_n)$  that converges  $P \times \lambda$ -a.e. to  $x$ . The fact that  $x \in I$  follows from arguments similar to those in the proof of Proposition 1 of H&K. ■

## PROOF OF PROPOSITION 5:

PROOF. First, we show that  $k\chi_{[T,1]} \in \mathcal{E}_+^\varphi$  for all optional time  $T$  and all  $k > 0$ . Note that

$$\mathbf{E} \left[ \int_0^1 \varphi(\gamma k \chi_{[T,1]}(t)) dt \right] \leq \mathbf{E} [\varphi(\gamma k)] < \infty,$$

where the first inequality follows from the fact that  $\varphi$  is strictly increasing and the second inequality follows from the hypothesis that  $\varphi$  is integrable.

Second, the fact that  $T + \frac{1}{n}$  is an optional time whenever  $T$  is can be seen by directly checking the definition:

$$\{\omega \in \Omega : T + \frac{1}{n} \leq t\} = \{\omega \in \Omega : T \leq t - \frac{1}{n}\} \in \mathcal{F}_{t-\frac{1}{n}} \subset \mathcal{F}_t \quad \forall t \in [0, 1].$$

Third, the hypothesis that  $\varphi$  is integrable and Lebesgue convergence theorem imply that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{E} \left[ \int_0^1 \varphi(\gamma k (\chi_{[T,1]}(t) - \chi_{[T+\frac{1}{n},1]}(t))) dt \right] &= \mathbf{E} \left[ \int_0^1 \lim_{n \rightarrow \infty} \varphi(\gamma k (\chi_{[T,1]}(t) - \chi_{[T+\frac{1}{n},1]}(t))) dt \right] \\ &= \mathbf{E} \left[ \int_0^1 \varphi(\gamma k \lim_{n \rightarrow \infty} (\chi_{[T,1]}(t) - \chi_{[T+\frac{1}{n},1]}(t))) dt \right] = 0, \end{aligned}$$

where the second and the third equalities follow from the fact that  $\varphi(\omega, z)$  is continuous in  $z$  and is zero at  $z = 0$ , respectively. Thus, by the continuity of  $\succeq$ ,  $x_n \succeq y$  for all  $n$  implies  $x \succeq y$  and  $y \succeq x_n$  for all  $n$  implies that  $y \succeq x$ .

Fourth, suppose that  $\{T_n\}$  is a sequence of optional times with  $T_n \leq T$ , and  $T_n < T$  and  $T_n < T_{n+1}$  on the set  $\{T > 0\}$  and that  $k\chi_{[T_n,1]} \rightarrow k\chi_{[T,1]}$  in  $\|\cdot\|_\varphi$ . This implies that

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[ \int_0^1 \varphi(\gamma k (\chi_{[T,1]}(t) - \chi_{[T_n,1]}(t))) dt \right] \rightarrow 0$$

for all  $\gamma > 0$ . This implies that  $k\chi_{[T_n,1]} \rightarrow k\chi_{[T,1]}$  in  $P \times \lambda$ -measure, and hence there exists a subsequence  $\{T_{n_i}\}$  such that

$$\lim_{n_i \rightarrow \infty} \chi_{[T_{n_i},1]}(\omega, t) - \chi_{[T,1]}(\omega, t) = 0 \quad P \times \lambda - a.e.,$$

which implies that  $T_{n_i} \rightarrow T$  a.s. and thus  $T$  is predictable.

The proof for the last assertion is just the reverse of the above paragraph. ■

## PROOF OF PROPOSITION 6:

PROOF. The proof of Proposition 3 essentially implies that the set of right-continuous, and bounded processes that change their values at a finite number of optional times is dense in  $(\mathcal{E}^\varphi, \tau_\varphi)$ . It then suffices to show that one unit of consumption at an optional time can be approximated arbitrarily closely by a sequence of absolutely consumption patterns. We will show here that consumption at a predictable time can be approximated by consuming at “rate” instants before. The rest of the proof is left for the reader.

Let  $T$  be predictable and  $\{T_n\}$  be its announcing sequence. Assume without loss of generality that  $T > 0$  a.s. (On the set  $\{T = 0\}$ , we can shift consumption to the right at rates.) Putting

$$S_n \equiv \mathbf{E}[T | \mathcal{F}_{T_n}],$$

it can be verified that  $S_n$  is an optional time by checking the definition. Since  $T > T_n$  a.s., we know  $S_n > T_n$  a.s. Define

$$\begin{aligned} x(t) &= \chi_{[T,1]}(t) \\ x_n(t) &= \frac{t - T_n}{S_n - T_n} \chi_{[T_n, S_n]}(t) + \chi_{(S_n, 1]}(t). \end{aligned}$$

Note that  $x_n$  is an absolutely continuous consumption pattern. One can verify that  $x_n \rightarrow x$  in  $\overline{T}_\varphi$ . ■

PROOF OF PROPOSITION 7:

PROOF. Note again that  $\|x_n - x\|_\varphi \rightarrow 0$ , if and only if for any  $\gamma > 0$

$$E \left[ \int_0^1 \varphi(\gamma |x_n(t) - x(t)|) dt + \varphi(\gamma |x_n(1) - x(1)|) \right] \rightarrow 0;$$

see Musielak (1983, Theorem 1.6). Note that if  $\mathbf{E}[\rho(x_n, x)] \rightarrow 0$ , then

$$P\left(\left\{\rho(x_n, x) \geq \frac{1}{m}\right\}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For any integer  $m$ , let  $N(m)$  be such that  $P\left(\left\{\rho(x_n, x) \geq \frac{1}{m}\right\}\right) < \frac{1}{m}$ , for all  $n > N(m)$ .

Next write:

$$E \left[ \int_0^1 \varphi(\gamma |x_n(t) - x(t)|) dt + \varphi(\gamma |x_n(1) - x(1)|) \right] = I_1^{m,n} + I_2^{m,n},$$

for  $n > N(m)$ , where

$$\begin{aligned} I_1^{m,n} &= \int_{\{\rho(x_n, x) \geq \frac{1}{m}\}} \left[ \int_0^1 \varphi(\gamma |x_n(t) - x(t)|) dt + \varphi(\gamma |x_n(1) - x(1)|) \right] P(d\omega), \\ I_2^{m,n} &= \int_{\{\rho(x_n, x) < \frac{1}{m}\}} \left[ \int_0^1 \varphi(\gamma |x_n(t) - x(t)|) dt + \varphi(\gamma |x_n(1) - x(1)|) \right] P(d\omega). \end{aligned}$$

We will show that both integrals converge to zero, as  $m$  (and hence  $n$ )  $\rightarrow \infty$ .

First consider  $I_1^{m,n}$ . Since

$$\int_0^1 \varphi(\gamma |x_n(t) - x(t)|) dt \leq \int_0^1 \varphi(\gamma (x_n(1) + x(t))) dt \leq \varphi(\gamma (x_n(1) + x(1))) \quad \forall \omega \in \Omega,$$

we have

$$\begin{aligned} \int_0^1 \varphi(\gamma|x_n(t) - x(t)|) dt + \varphi(\gamma|x_n(1) - x(1)|) &\leq 2\varphi(\gamma(x_n(1) + x(1))) \\ &\leq 2\varphi(\gamma(K + 2x(1))) \end{aligned}$$

But since  $\mathbf{E}[\varphi(K)] < \infty$  and  $x \in \mathcal{L}^\varphi$ , it then follows that  $I_1^{m,n} \rightarrow 0$  as  $m, n \rightarrow \infty$ , by Lebesgue dominated convergence theorem.

Now consider  $I_2^{m,n}$ . Note first that for  $\omega \in \{\rho_\omega(x_n, x) < \frac{1}{m}\}$ , we have

$$|x_n(1) - x(1)| < \frac{1}{m} \quad \text{or} \quad \varphi(\gamma|x_n(1) - x(1)|) < \varphi\left(\frac{\gamma}{m}\right).$$

Thus

$$\int_{\{\rho(x_n, x) < \frac{1}{m}\}} \varphi(\gamma|x_n(1) - x(1)|) P(d\omega) < \mathbf{E}\left[\varphi\left(\frac{\gamma}{m}\right)\right].$$

Next consider the integral part. For any  $\omega \in \{\rho(x_n, x) < \frac{1}{m}\}$ , we can bound the inside integral by:

$$\frac{1}{m} \sum_{k=1}^m \varphi\left(\gamma\left(\max\left\{x\left(\omega, \frac{k}{m}\right), x_n\left(\omega, \frac{k}{m}\right)\right\} - \min\left\{x\left(\omega, \frac{(k-1)}{m}\right), x_n\left(\omega, \frac{(k-1)}{m}\right)\right\}\right)\right).$$

But since for precisely those sample functions  $\rho(x_n, x) < \frac{1}{m}$  we have

$$\max\{x(\omega, t), x_n(\omega, t)\} \leq x\left(\omega, t + \frac{1}{m}\right) + \frac{1}{m}$$

and

$$\min\{x(\omega, t), x_n(\omega, t)\} \geq x\left(\omega, t - \frac{1}{m}\right) - \frac{1}{m},$$

so the summation above is bounded by:

$$\frac{1}{m} \sum_{k=1}^m \varphi\left(\gamma\left(x\left(\omega, \frac{(k+1)}{m}\right) - x\left(\omega, \frac{(k-2)}{m}\right) + \frac{2}{m}\right)\right).$$

Finally, as  $\varphi$  is convex, equal to zero at zero and increasing, this in turn is bounded by:

$$\frac{1}{m} \varphi\left(\gamma \sum_{k=1}^m \left[x\left(\omega, \frac{(k+1)}{m}\right) - x\left(\omega, \frac{(k-2)}{m}\right) + \frac{2}{m}\right]\right) \leq \frac{1}{m} \varphi(\gamma(3x(\omega, 1) + 2)).$$

Therefore,

$$\int_{\{\rho(x_n, x) < \frac{1}{m}\}} \int_0^1 \varphi(\gamma|x_n(t) - x(t)|) dt P(d\omega) \leq \frac{1}{m} \mathbf{E}\left[\varphi(\gamma(3x(1) + 2))\right],$$

$$I_2^{m,n} < \frac{1}{m} \mathbf{E}\left[\varphi(\gamma(3x(1) + 2))\right] + \mathbf{E}\left[\varphi\left(\frac{\gamma}{m}\right)\right],$$

and  $I_2^{m,n} \rightarrow 0$  as  $m, n \rightarrow \infty$ , proving our result. ■



PROOF OF PROPOSITION 9:

PROOF. For any  $\omega \in \Omega$ , and for any  $\gamma > 0$ , using the same arguments as in the proof of Proposition 7, we get:

$$\begin{aligned} & \int_0^1 \varphi(\omega, \gamma|x_n(\omega, t) - y_n(\omega, t)|) dt + \varphi(\omega, \gamma|x_n(\omega, 1) - y_n(\omega, 1)|) \\ & \leq \frac{K_\omega}{n} \varphi(\omega, \gamma(3x_n(\omega, 1) + 2)) + 2\varphi(\omega, \gamma|x_n(\omega, 1) - y_n(\omega, 1)|). \end{aligned}$$

The second term on the right-side of the inequality vanishes for any  $\gamma$ , since

$$|x_n(\omega, 1) - y_n(\omega, 1)| \leq \rho_\omega(x_n, y_n) \leq \frac{K_\omega}{n}.$$

It then follows from Lebesgue convergence theorem, continuity of  $\varphi(\omega, z)$  in  $z$ ,  $\varphi(\omega, 0) = 0$ , and the hypothesis that  $\mathbf{E}[\varphi(K)] < \infty$  that

$$\mathbf{E} \left[ 2\varphi(\gamma|x_n(1) - y_n(1)|) \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The first term vanishes since,

$$x_n(\omega, 1)/\mu(\omega, n) \rightarrow 0 \quad \text{thus} \quad \gamma(3x_n(\omega, 1) + 2)/\mu(\omega, n) \rightarrow 0.$$

Hence, for any  $\delta$  however small, there is  $N_\delta$ , such that:

$$\forall n > N_\delta: \quad \gamma(3x_n(\omega, 1) + 2) < \delta\mu(\omega, n).$$

Applying  $\varphi(\omega, \cdot)$  to both sides of the above relation, and using the shape of  $\varphi$ , for all  $\delta < 1$ , we conclude that:

$$\frac{K_\omega}{n} \varphi_\omega(\gamma(3x_n(\omega, 1) + 2)) < \frac{K_\omega}{n} \varphi_\omega(\delta\mu(\omega, n)) < \frac{K_\omega}{n} \delta \varphi_\omega(\mu(\omega, n)) \leq K_\omega \delta$$

for all  $n > N_\delta$ .

Noting that by Jensen's inequality,  $\varphi(\mathbf{E}[K]) \leq \mathbf{E}[\varphi(K)] < \infty$  and since  $\mathbf{E}[\varphi(K)] < \infty$ , we conclude that  $\|x_n - y_n\|_\varphi \rightarrow 0$ , by Lebesgue convergence theorem. ■

PROOF FOR PROPOSITION 11:

PROOF. Integration by parts path-by-path gives

$$\begin{aligned} \psi(x) &= \mathbf{E} \left[ \int_{0-}^1 g(t) dx(t) \right] \\ &= \mathbf{E} \left[ \int_{0-}^1 f(t) dx(t) + \int_{0-}^1 m(t) dx(t) \right] \\ &= \mathbf{E} \left[ f(1)x(1) - \int_0^1 x(t) f'(t) dt + m(1)x(1) \right] \\ &= \mathbf{E} \left[ - \int_0^1 x(t) f'(t) dt - x(1)f'(1) \right], \end{aligned} \tag{23}$$

where  $f'$  denotes the derivative of  $f$  and we have used the fact that

$$\mathbf{E}\left[\int_{0-}^1 m(t)dx(t)\right] = \mathbf{E}\left[\int_{0-}^1 m(1)dx(t)\right] = \mathbf{E}[m(1)x(1)] \quad (24)$$

in the third equality (see Dellacherie and Meyer (1982, VI.57)).

Musiela (1983, Corollary 13.14, p.87) shows that (23) is a linear functional continuous in  $\mathcal{T}_\varphi$  if  $f' \in L^\varphi$ . This is the sufficiency part.

Now consider the necessity part. Let  $\psi : \mathcal{E}^\varphi$  be a  $\mathcal{T}_\varphi$ -continuous linear functional. By the Hahn-Banach theorem,  $\psi$  can be extended to be a continuous linear functional on the whole of  $E^\varphi$ , of which we recall the definition from (6). Let  $\Psi$  denote this extension. We first show that  $\Psi : E^\varphi \rightarrow \mathfrak{R}$  can be represented in the form:

$$\Psi(x) = \mathbf{E}\left[\int_0^1 x(t)y(t)dt + x(1)y(1)\right] \quad \forall x \in E^\varphi \quad (25)$$

for some  $y \in L^\varphi$ . For this, we first consider the notion of *modular convergence*: A sequence  $\{x_n\} \in L^\varphi$  is said to converge in modular to  $x$ , if

$$\mathbf{E}\left[\int_0^1 \varphi(\gamma|x_n(t) - x(t)|) dt + \varphi(\gamma|x_n(1) - x(1)|)\right] \rightarrow 0 \quad \text{for some } \gamma > 0.$$

The sequence  $\{x_n\} \in L^\varphi$  converges in norm to  $x$ , if and only if

$$\mathbf{E}\left[\int_0^1 \varphi(\gamma|x_n(t) - x(t)|) dt + \varphi(\gamma|x_n(1) - x(1)|)\right] \rightarrow 0 \quad \text{for all } \gamma > 0;$$

see Musiela (1983, Theorem 1.6, p.3). Norm convergence of  $x_n$  clearly implies modular convergence.

Musiela (1983, Theorem 13.17, p.88) shows that (25) is true under the additional restriction on  $\varphi$  that

$$\forall x_0 > 0 \quad \text{there exists } c > 0 \quad \text{such that } \frac{\varphi(\omega, x)}{x} \geq c \quad \text{for } x \geq x_0 \quad \forall \omega \in \Omega.$$

Musiela (1983, Theorem 13.15, p.87), however, shows that this restriction is required to guarantee that a norm continuous linear functional on  $L^\varphi$  is also modular continuous. In our formulation, we do not need this restriction, since we do not deal with the generalized Orlicz space  $L^\varphi$ . We only consider the subspace  $E^\varphi$ , and on this subspace norm convergence and modular convergence are equivalent; Musiela (1983, 5.2.B, p.18). We therefore conclude that restricting our attention to  $E^\varphi$  allows us to relax the above described condition. The interested reader can easily verify this by consulting Musiela (1983, Theorems 13.15 and 13.17).

Now define

$$f(\omega, t) \equiv - \int_0^t y(\omega, s)ds \quad \forall t \in [0, 1] \omega \in \Omega,$$

which is clearly adapted and path-wise absolutely continuous. Denoting the derivative of  $f(\omega, t)$  with respect to  $t$  by  $f'(\omega, t)$ , it is clear that  $f'(t) = -y(t)$ . Reversing the arguments in deriving (23), we prove the necessity part. ■

## PROOF OF PROPOSITION 13:

PROOF. Let  $\varphi$  be integrably asymptotically linear. We will first show that if  $x$  is an element of  $\mathcal{E}$ , then it is also an element of  $\mathcal{E}^\varphi$ . Note the following:

$$\begin{aligned} \|x\|_\varphi &= \mathbf{E}\left[\int_0^1 \varphi(|x(t)|)\chi_{\{|x(t)|\leq K\}} dt + \varphi(|x(1)|)\chi_{\{x(1)\leq K\}}\right] \\ &+ \mathbf{E}\left[\int_0^1 \varphi(|x(t)|)\chi_{\{|x(t)|\geq K\}} dt + \varphi(|x(1)|)\chi_{\{x(1)\geq K\}}\right] \\ &\leq 2\mathbf{E}[\varphi(K)] + \alpha\mathbf{E}\left[\int_0^1 |x(t)|dt + |x(1)|\right] + 2\epsilon < \infty. \end{aligned}$$

Hence  $x$  is an element of  $\mathcal{E}^\varphi$ .

Since  $\mathcal{T}$  is weaker than  $\mathcal{T}_\varphi$ , we only need to show that convergence in  $\mathcal{T}$  implies convergence in  $\mathcal{T}_\varphi$ . Consider a sequence of elements  $\{x_n\}$  that converges to  $x$  in  $\mathcal{T}$ . This implies that  $\{x_n\}$  converges in the product measure generated by  $P$  and Lebesgue measure to  $x$ . Consider now

$$\begin{aligned} \|x_n - x\|_\varphi &= \mathbf{E}\left[\int_0^1 \varphi(|x_n(t) - x(t)|)\chi_{\{|x_n(t) - x(t)|\leq K\}} dt + \varphi(|x_n(1) - x(1)|)\chi_{\{x_n(1) - x(1)\leq K\}}\right] \\ &+ \mathbf{E}\left[\int_0^1 \varphi(|x_n(t) - x(t)|)\chi_{\{|x_n(t) - x(t)|\geq K\}} dt + \varphi(|x_n(1) - x(1)|)\chi_{\{x_n(1) - x(1)\geq K\}}\right] \\ &\leq \mathbf{E}\left[\int_0^1 \varphi(|x_n(t) - x(t)|)\chi_{\{|x_n(t) - x(t)|\leq K\}} dt + \varphi(|x_n(1) - x(1)|)\chi_{\{x_n(1) - x(1)\leq K\}}\right] \\ &+ \alpha\mathbf{E}\left[\int_0^1 |x_n(t) - x(t)|dt + |x_n(1) - x(1)|\right] + 2\epsilon. \end{aligned}$$

The first term on the right-hand side of the inequality converges to zero by Lebesgue convergence theorem. The second term goes to zero as  $n \rightarrow \infty$  by hypothesis. Since  $\epsilon$  is arbitrarily small,  $\|x_n - x\|_\varphi \rightarrow 0$  as  $n \rightarrow \infty$ . ■

## PROOF OF PROPOSITION 15:

PROOF. Suppose that  $u$  is not linear. Then there exists two scalars  $r, \hat{r}$  and  $t \in [0, 1]$  such that  $u((r + \hat{r})/2, t) \neq u(r, t)/2 + u(\hat{r}, t)/2$ . Without loss of generality, assume that  $u((r + \hat{r})/2, t) > u(r, t)/2 + u(\hat{r}, t)/2$ . By joint continuity, there exists  $\epsilon > 0$  and some interval  $I$  containing  $t$  such that  $u((r + \hat{r})/2, s) - \epsilon > u(r, s)/2 + u(\hat{r}, s)/2$  for all  $s \in I$ . Consider a sequence of nonrandom absolutely continuous consumption patterns constructed as follows: Off of  $I$ , consume at rate 1 in each  $x_n$ . On  $I$ , subdivide  $I$  into  $2n$  equal sized intervals, and consume at rate  $r$  on the even subintervals and  $\hat{r}$  on the odd. This sequence of consumption patterns converges in the Prohorov metric to the consumption pattern  $x$  that has  $x'(s) = 1$  off  $I$  and  $x'(s) = (r + \hat{r})/2$  on  $I$ . By Proposition 7,  $x_n \rightarrow x$  in  $\mathcal{T}_\varphi$ . But  $U(x) > U(x_n) + \epsilon\lambda(I)$ , where  $\lambda(I)$  is the Lebesgue measure of  $I$ . Thus  $U$  is not continuous in  $\mathcal{T}_\varphi$ . ■

## PROOF OF PROPOSITION 17:

PROOF. Fix a topology  $\mathcal{T}_\varphi$ . Let  $x_n$  be a sequence of consumption plans that converge to  $x$  in  $\mathcal{T}_\varphi$ . Assume, without loss of generality, that  $\sup_n \|x_n\|_\varphi < \infty$ . We will first show that  $u(\hat{x}_n, t)$  converge to  $u(\hat{x}, t)$  in the product measure  $\nu = P \times \lambda$ , where  $P$  is the probability measure on  $(\Omega, \mathcal{F})$ , and  $\lambda$  is the Lebesgue measure on the Borel sigma-field on  $[0, 1]$ .

First observe that if  $x_n \rightarrow x$  in  $\mathcal{T}_\varphi$ , then  $x_n \rightarrow x$  in  $\mathcal{T}$ . Hence,

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[ \int_0^1 |x_n(t) - x(t)| dt + |x_n(1) - x(1)| \right] = 0. \quad (26)$$

This implies that  $x_n$  converge to  $x$  in the product measure  $\nu$ , and that  $\int_0^1 |x_n(t) - x(t)| dt$  converges to zero in  $P$ -measure. Therefore, for any  $m$ , we can find  $N(m)$  such that  $\nu \left( \{(\omega, t) : |x_n(\omega, t) - x(\omega, t)| > \frac{1}{m}\} \right) < \frac{1}{m}$ , and  $P \left[ \int_0^1 |x_n(t) - x(t)| dt > \frac{1}{m} \right] < \frac{1}{m}$ , for all  $n > N(m)$ .

Now consider  $\hat{x}_n(\omega, t) - \hat{x}(\omega, t) = \int_0^k \theta(\omega, s) dx_n(\omega, t-s) - dx(\omega, t-s)$ . Integrating by parts, and using the assumption that  $\theta(\omega, k)$  vanishes, we get:

$$\hat{x}_n(\omega, t) - \hat{x}(\omega, t) = - \left( x_n(\omega, t) - x(\omega, t) \right) \theta(\omega, 0) - \int_0^k \left( x_n(\omega, t-s) - x(\omega, t-s) \right) \theta_s(\omega, s) ds, \quad (27)$$

where  $\theta_s(\omega, s)$  denotes the partial derivative of  $\theta(\omega, s)$  with respect to  $s$ . But since both  $\theta$  and its derivative are uniformly bounded over  $\omega$ , the left-hand side in (27) is bounded by  $M_1 |x_n(\omega, t) - x(\omega, t)| + M_2 \left[ \int_0^1 |x_n(\omega, t) - x(\omega, t)| dt \right]$ , where  $M_1$  and  $M_2$  are two constants.

From this, we can easily conclude that

$$\nu \left( \{(\omega, t) : |\hat{x}_n(\omega, t) - \hat{x}(\omega, t)| > \frac{1}{m} (M_1 + M_2)\} \right) < \frac{2}{m} \quad \forall n > N(m). \quad (28)$$

Hence  $\hat{x}_n$  converge in  $\nu$ -measure to  $\hat{x}$ . Now consider the convergence of  $u(\hat{x}_n)$ . First, assume that  $u$  is uniformly continuous in  $\hat{x}$ . Since the felicity function  $u$  is state-independent, and uniformly continuous in  $\hat{x}$ , it then follows that if  $|\hat{x}_n - \hat{x}| < c_1$  then  $|u(\hat{x}_n) - u(\hat{x})| < \alpha c_1$ , where  $\alpha$  is independent of  $\omega$ , and  $\hat{x}$ . We use this assumption of uniform continuity, together with (28) to conclude that  $u(\hat{x}_n)$  converges in  $\nu$ -measure to  $u(\hat{x})$ . By Jensen's inequality, we have  $\sup_n \mathbf{E} \left[ \int_0^1 u(\hat{x}_n) dt \right] \leq \sup_n u(\mathbf{E} \left[ \int_0^1 \hat{x}_n dt \right]) < \infty$ . It then follows by Lebesgue dominated convergence theorem that  $U(x_n)$  converge to  $U(x)$ .

Next consider the case when  $u$  is merely continuous in  $\hat{x}$ . For every integer  $m$ , define  $u_m(\hat{x}) = u(\hat{x}) \chi_{[\frac{1}{m} \leq \hat{x} \leq m]}$ . For a fixed  $m$ , the function  $u_m$  is uniformly continuous in  $\hat{x}$ , and hence the arguments in the previous paragraph show that  $U_m(x_n)$  converge to  $U_m(x)$ , where  $U_m(x)$  is defined exactly as  $U(x)$ , except that  $u$  is substituted by  $u_m$ . But  $u_m(\hat{x})$  converge monotonically to  $u(\hat{x})$  as  $m \rightarrow \infty$ . Applying the monotone convergence theorem, we conclude that

$$\lim_{n \rightarrow \infty} U(x_n) = \lim_{m, n \rightarrow \infty} U_m(x_n) = \lim_{m \rightarrow \infty} U_m(x) = U(x).$$

This shows that the utility definition given by (20) gives preferences continuous in  $\mathcal{T}_\varphi$ .  $\blacksquare$

## PROOF OF PROPOSITION 18:

PROOF. Suppose not. By joint continuity, there exists two scalars  $r, \hat{r}$  and  $t \in [0, 1]$  such that  $v((r + \hat{r})/2, z, t) \neq v(r, z, t)/2 + v(\hat{r}, z, t)/2$  for all values of  $z$ . Without loss of generality, assume that  $v((r + \hat{r})/2, z, t) > v(r, z, t)/2 + v(\hat{r}, z, t)/2$ . By joint continuity again, there exists  $\epsilon > 0$  and some interval  $I$  containing  $t$  such that  $v((r + \hat{r})/2, z, s) - \epsilon > v(r, z, s)/2 + v(\hat{r}, z, s)/2$  for all  $s \in I$ . Consider a sequence of nonrandom absolutely continuous consumption patterns constructed as follows: Off of  $I$ , consume at rate 1 in each  $x_n$ . On  $I$ , subdivide  $I$  into  $2n$  equal sized intervals, and consume at rate  $r$  on the even subintervals and  $\hat{r}$  on the odd. This sequence of consumption patterns converges in the Prohorov metric to the consumption pattern  $x$  that has  $x'(s) = 1$  off  $I$  and  $x'(s) = (r + \hat{r})/2$  on  $I$ . By Proposition 7,  $x_n \rightarrow x$  in  $\mathcal{T}_\varphi$ . Proposition 17 shows that  $\lim_{n \rightarrow \infty} \int_0^1 v(x', \hat{x}_n, t) dt = \int_0^1 v(x', \hat{x}, t) dt$ . But for any  $\hat{x}_n$ , we have:

$$\int_0^1 v(x', \hat{x}_n, t) dt > \int_0^1 v(x', \hat{x}, t) dt + \epsilon \lambda(I).$$

Taking limits of both sides as  $n \rightarrow \infty$ , we get:

$$\int_0^1 v(x', \hat{x}, t) dt = \lim_{n \rightarrow \infty} \int_0^1 v(x', \hat{x}_n, t) dt \geq \lim_{n \rightarrow \infty} \int_0^1 v(x', \hat{x}_n, t) dt + \epsilon \lambda(I).$$

Hence  $V(x) \geq \lim_{n \rightarrow \infty} V(x_n) + \epsilon \lambda(I)$ , where  $\lambda(I)$  is the Lebesgue measure of  $I$ . Thus  $V$  is not continuous in  $\mathcal{T}_\varphi$ . ■

## Appendix B Example of a $\mathcal{T}_\varphi$ continuous preference that violates (11)

We first show that any linear preferences with the marginal utility being the *optional projection* of a continuous and uniformly bounded process are continuous in all of  $\mathcal{T}_\varphi$  for all  $\varphi \in \Phi$ .

**Proposition 19** *Let  $f: \Omega \times [0, 1] \rightarrow \mathbb{R}$  be an  $\mathcal{F} \times \mathcal{B}(\mathbb{R})$ -measurable process, not necessarily adapted, with the the properties that*

$$f(\omega, \cdot) \in C[0, 1] \quad \text{and} \quad \text{esssup}_\omega \left[ \max_t f(\omega, t) \right] < \infty,$$

where  $C[0, 1]$  denotes the space of continuous functions on  $[0, 1]$ . Define the optional projection of  $f$ , denoted  $f^*$  by:

$$f^*(\omega, t) = \mathbf{E} \left[ f(t) \middle| \mathcal{F}_t \right].$$

Note that by construction, the process  $f^*$  is adapted. Preference relations which are represented by:

$$x \succeq y \quad \text{if} \quad \mathbf{E} \left[ \int_0^1 f^*(t) dx(t) \right] \geq \mathbf{E} \left[ \int_0^1 f^*(t) dy(t) \right], \quad \forall x, y \in \mathcal{E}_+^\varphi$$

are continuous in  $\mathcal{T}_\varphi$ .

PROOF. Given  $x_n, x \in \mathcal{E}_+^\varphi$ , and  $f^*$  as in the hypothesis, we show that if:

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[ \int_0^1 |x_n(t) - x(t)| dt + |x_n(1) - x(1)| \right] = 0, \quad (29)$$

then

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[ \int_0^1 f^*(t) dx_n(t) - \int_0^1 f^*(t) dx(t) \right] = 0. \quad (30)$$

Then  $\succeq$  is continuous in  $\mathcal{T}$  and thus in  $\mathcal{T}_\varphi$  for all  $\varphi \in \Phi$  since, by Corollary 1,  $\mathcal{T}$  is weaker than any  $\mathcal{T}_\varphi$ . To prove this, we will use the fact that since  $f^*$  is the optional projection of  $f$  then

$$\mathbf{E} \left[ \int_0^1 f^*(t) dx_n(t) \right] = \mathbf{E} \left[ \int_0^1 f(t) dx_n(t) \right] \quad \forall x_n \in \mathcal{E}^\varphi;$$

Dellacherie and Meyer (1982, Theorem VI.57). Now (29) implies that:

$$\lim_{n \rightarrow \infty} P \left( \left\{ \left[ \int_0^1 |x_n(t) - x(t)| dt + |x_n(1) - x(1)| \right] > \frac{1}{m} \right\} \right) = 0. \quad (31)$$

Therefore, for any  $m > 0$ , choose  $N(m)$  such that for all  $n > N(m)$ :

$$P \left( \left\{ \left[ \int_0^1 |x_n(t) - x(t)| dt + |x_n(1) - x(1)| \right] > \frac{1}{m} \right\} \right) < \frac{1}{m}.$$

Write (30) as:

$$\mathbf{E} \left[ \int_0^1 f(t) d(x_n(t) - x(t)) \right] = I_1^{m,n} + I_2^{m,n},$$

where

$$\begin{aligned} I_1^{m,n} &= \int_{\Omega_1^{m,n}} \int_0^1 f(\omega, t) d(x_n(\omega, t) - x(\omega, t)) P(d\omega) \\ I_2^{m,n} &= \int_{\Omega_2^{m,n}} \int_0^1 f(\omega, t) d(x_n(\omega, t) - x(\omega, t)) P(d\omega), \end{aligned}$$

and where

$$\begin{aligned} \Omega_1^{m,n} &= \left\{ \omega \in \Omega: \left[ \int_0^1 |x_n(t) - x(t)| dt + |x_n(1) - x(1)| \right] < \frac{1}{m} \right\}, n \geq N(m), \\ \Omega_2^{m,n} &= \left\{ \omega \in \Omega: \left[ \int_0^1 |x_n(t) - x(t)| dt + |x_n(1) - x(1)| \right] \geq \frac{1}{m} \right\}, n \geq N(m). \end{aligned}$$

We now show that both  $I_1^{m,n}$  and  $I_2^{m,n}$  converge to zero as  $m$  (and hence  $n$ )  $\rightarrow \infty$ .

First consider  $I_1^{m,n}$ . Assume to begin with that  $f(\omega, \cdot) \in C^1[0, 1]$ , the space of continuously differentiable functions on  $[0, 1]$ . This implies that  $f_t(\omega, t) < K_\omega < \infty$  for  $t \in [0, 1]$  for some  $\mathcal{F}$ -measurable function  $K$ , where we use  $f_t$  to denote  $\partial f(\omega, t)/\partial t$ .

For any  $\omega \in \Omega_1^{m,n}$ , consider  $\int_0^1 f(\omega, t) d(x_n(\omega, t) - x(\omega, t))$ . Integration by parts gives

$$\begin{aligned} & \int_0^1 f(\omega, t) d(x_n(\omega, t) - x(\omega, t)) \\ &= (x_n(\omega, 1) - x(\omega, 1))f(\omega, 1) - (x_n(\omega, 0) - x(\omega, 0))f(\omega, 0) - \int_0^1 [x_n(\omega, t) - x(\omega, t)]f_t(\omega, t) dt \end{aligned}$$

Hence

$$\left| \int_0^1 f(\omega, t) d(x_n(\omega, t) - x(\omega, t)) \right| \leq \frac{1}{m} |f(\omega, 1)| + \left| (x_n(0) - x(0))f(\omega, 0) \right| + \frac{K_\omega}{m}.$$

By the right continuity of the sample functions of  $\{x_n\}$  and  $x$  and the fact that  $x_n(0)$  and  $x(0)$  are constants, we have

$$\lim_{m,n \rightarrow \infty} |x_n(0) - x(0)| = 0.$$

Thus, as  $m \rightarrow \infty$ , we get

$$\lim_{m,n \rightarrow \infty} \int_0^1 f(t) d(x_n(t) - x(t)) = 0 \quad \forall \omega \in \bigcap_m \bigcup_{n > N(m)} \Omega_1^{m,n}.$$

But if

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 f(\omega, t) d(x_n(\omega, t) - x(\omega, t)) &= 0 \quad \forall f(\omega, \cdot) \in C^1[0, 1] \quad \forall \omega \in \bigcap_m \bigcup_{n > N(m)} \Omega_1^{m,n}, \\ \lim_{n \rightarrow \infty} \int_0^1 f(\omega, t) d(x_n(\omega, t) - x(\omega, t)) &= 0 \quad \forall f(\omega, \cdot) \in C[0, 1] \quad \forall \omega \in \bigcap_m \bigcup_{n > N(m)} \Omega_1^{m,n}; \end{aligned}$$

see Billingsley (1968, Theorem 7.1, p. 42).

In addition, since for all  $\omega \in \Omega_1^{m,n}$  we have:

$$\int_0^1 f(\omega, t) d(x_n(\omega, t) - x(\omega, t)) \leq A \left[ \int_0^1 d((x_n(t) - x(t))) \right] \leq A(x_n(1) - x(1)),$$

where

$$A \equiv \sup_{\omega \in \Omega} \sup_{t \in [0,1]} |f(\omega, t)|,$$

and since from (29) we have that

$$\mathbb{E} \left[ |x_n(1) - x(1)| \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

it then follows from Strook (1987, Exercise III.3.19) that  $I_1^{m,n} \rightarrow 0$  as  $n \rightarrow \infty$ .

Now consider  $I_2^{m,n}$ . Since  $P(\Omega_2^{m,n}) \rightarrow 0$  from (31), and since

$$\int_0^1 f(t) d(x_n(t) - x(t)) \leq A(x_n(1) - x(1))$$

therefore  $I_2^{m,n} \rightarrow 0$  as  $n, m \rightarrow \infty$  by Lebesgue convergence theorem. ■

**Proposition 20** *There exist preferences  $\succeq$  on  $\mathcal{E}_+^\varphi$  continuous in  $\mathcal{E}_\varphi$ , that do not satisfy (11).*

PROOF. It suffices to provide an example. Let  $\mu \equiv \varphi^{-1}$  and let  $x^* \in \mathcal{E}_+^\varphi$  be such that  $x + x^* \succ x$  for all  $x \in \mathcal{E}_+^\varphi$ . Define the continuous function:

$$f(\omega, t) = \alpha \sqrt{1/\mu(1/t)} + 1 \quad \text{where} \quad \alpha > \alpha^* = \mathbf{E} \left[ \int_0^1 f(t) dx^*(t) \right].$$

Let

$$f^*(\omega, t) = \mathbf{E} \left[ \alpha \sqrt{1/\mu(1/t)} + 1 \middle| \mathcal{F}_t \right]$$

and define preferences on  $\mathcal{E}_+^\varphi$  by

$$x \succeq y \quad \text{if} \quad \mathbf{E} \left[ \int_0^1 f^* dx \right] \geq \mathbf{E} \left[ \int_0^1 f^* dy \right].$$

By construction and by Proposition 19,  $\succeq$  is continuous in  $\tau_\varphi$ . Next consider the two sequences

$$x_n(\omega, t) = \mathbf{E} \left[ \sqrt{\mu(n)} \right] \chi_{[0,1]}, \quad \text{and} \quad y_n(\omega, t) = \mathbf{E} \left[ \sqrt{\mu(n)} \right] \chi_{[1/n, 1]}.$$

It is clear that both sequences satisfy the requirements of Proposition 9. However, for every  $n$  and  $\omega$ :

$$\mathbf{E} \left[ \int_0^1 f(t) d(x_n(t) + x^*(t)) \right] = \alpha^* + \mathbf{E} \left[ \sqrt{\mu(n)} \right]$$

and

$$\int_0^1 f(\omega, t) dy_n(\omega, t) = \left( 1 + \alpha \sqrt{1/\mu(\omega, n)} \right) \mathbf{E} \left[ \sqrt{\mu(n)} \right].$$

Hence

$$\begin{aligned} \mathbf{E} \left[ \int_0^1 f^*(t) dy_n \right] &= \mathbf{E} \left[ \sqrt{\mu(n)} \right] + \alpha \mathbf{E} \left[ \sqrt{\mu(n)} \right] \mathbf{E} \left[ \sqrt{1/\mu(n)} \right] \\ &\geq \alpha + \mathbf{E} \left[ \sqrt{\mu(n)} \right], \end{aligned}$$

where we have used Jensen's Inequality and the fact that

$$\mathbf{E} \left[ \int_0^1 f^*(t) dx_n(t) \right] = \mathbf{E} \left[ \int_0^1 f(t) dx_n(t) \right] \quad \forall x_n \in \mathcal{E}^\varphi;$$

Dellacherie and Meyer (1982, Theorem VI.57). Hence for any  $n$ :

$$\mathbf{E} \left[ \int_0^1 f^*(t) d(x_n(t) + x^*(t)) \right] < \mathbf{E} \left[ \int_0^1 f^*(t) dy_n(t) \right]$$

and therefore:

$$\lim_{n \rightarrow \infty} x_n - x^* \not\geq \lim_{n \rightarrow \infty} y_n,$$

which proves that  $\succeq$  violates (11). ■









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