

WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

ON SOME COMPUTATIONAL ASPECTS OF
REAL BUSINESS CYCLE THEORY

Jean-Pierre Danthine, Universite de Lausanne
John B. Donaldson, Columbia University and
Rajnish Mehra, Sloan School of Management, M.I.T. and
University of California, Santa Barbara

WP# 1984-88

Revised February 1988

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

ON SOME COMPUTATIONAL ASPECTS OF
REAL BUSINESS CYCLE THEORY

Jean-Pierre Danthine, Universite de Lausanne
John B. Donaldson, Columbia University and
Rajnish Mehra, Sloan School of Management, M.I.T. and
University of California, Santa Barbara

WP# 1984-88

Revised February 1988

We gratefully acknowledge the invaluable programming assistance of
Yu-Hua Chu. Thanks are also due to Gary Hansen for helpful comments.

MIT LIBRARIES
MAY 15 1988
RECEIVED

1. Introduction

It has been common practice in the recent Real Business Cycle (RBC) literature to approximate, quadratically, the return function about the steady state and then to use this approximate return function as the basis for generating the economy's equilibrium time series. This is done for well known reasons of analytic and computational simplicity: with a quadratic return function the decision rules are linear and may be easily determined. To compute the optimal decision rules numerically via standard value iteration procedures is simply too intensive when the number of decision and state variables is large. Nevertheless, it is legitimate to question the extent to which accuracy is compromised using such approximate procedures.

The answer to this question may very well be entirely model specific. Indeed, if it were feasible to undertake such a comparative evaluation for all model contexts, there would be no need to resort to such approximations in the first place. In this paper we undertake such a comparison for the basic one good stochastic growth model as analyzed in Hansen (1985) (his 'divisible labor economy'). That is, we solve for the optimal decision rules and equilibrium time series using both standard value iterative and quadratic approximation techniques and compare results. Some familiarity with the basic methodology is assumed.

2. The Basic Model

The model we investigate is a familiar one

$$\max E\left(\sum_{t=0}^{\infty} \beta^t u(c_t, 1-N_t)\right)$$

$$\text{s.t.} \quad c_t + z_t \leq f(k_t, N_t) \tilde{\lambda}_t, \quad k_0 \text{ given}$$

$$k_{t+1} = z_t + (1 - \Omega)k_t, \quad 0 \leq N_t \leq 1, \quad \text{where}$$

c_t , k_t , N_t and z_t are, respectively, per capita consumption, capital stock, labor service supplied, and investment in period t , E is the expectations operator, β the period discount factor, Ω the depreciation factor, $U(\cdot)$ the representative agent's period utility function, $f(\cdot)$ the production function, and $\tilde{\lambda}_t$ the random period shock to technology. The precise specification of the functional forms employed here is as follows: $U(c_t, 1-N_t) = \frac{1}{\delta} [c_t^\gamma (1-N_t)^{1-\gamma}]^\delta$, and $f(k_t, N_t) = \frac{L}{v} [k_t^\alpha N_t^{1-\alpha}]^v \tilde{\lambda}_t$. For $v = L = 1$ and $\alpha = .36$, the above technology specification is the same as Hansen (1985); the preference specification is taken from Kydland and Prescott (1982).

The choice of shock process requires a somewhat more extensive discussion. In Hansen's (1985) work, the shock to technology is described by first order Markov process of the form

$$(2) \quad \lambda_{t+1} = \rho \lambda_t + \tilde{\varepsilon}_t$$

with $\rho = .95$ and $\tilde{\varepsilon}_t$ lognormally distributed with $E\tilde{\varepsilon}_t = 1 - \rho = .05$ and $\sigma_{\tilde{\varepsilon}} = .00712$. While such a shock process is perfectly natural for the quadratic return, linear decision rule setting, it does not immediately translate into the standard value iterative context. Under these latter procedures, it is first necessary to define, precisely, the region in which the state variables -- in this case, capital stock and the shock to technology -- lie over the problem's infinite horizon. For Hansen's specification of the production technology, the range of possible capital stock values will be bounded above by some k_{\max} provided the shock to technology is itself bounded.

Given upper and lower bounds on the shock, it is then necessary (see, e.g., Danthine and Donaldson (1981), Mehra and Prescott (1985), and Greenwood and Huffman (1986)) to select a discrete set of N possible values of λ , typically with mean one, the probability of relative occurrence of which is governed by a prespecified N dimensional probability transition matrix.

Hansen's (1985) shock process, however, is not uniformly bounded above over the infinite horizon and thus has no direct representation in the transition matrix setting. In order to undertake our comparisons, we therefore specify a two state Markov process on the shock to technology which has the same mean and variance as the process chosen by Hansen (1985).¹ Thus, for both the value iterative and quadratic approximate procedures we assume that the Markov process on the technology shock assumes values $\lambda_t = .98$ and $\lambda_t = 1.02$ with transition density

$$\lambda_{t+1} = \begin{matrix} .98 & 1.02 \\ \left[\begin{array}{cc} .97 & .03 \\ .03 & .97 \end{array} \right] \end{matrix}$$

We denote the conditional distribution of next period's shock given today's shock of λ by $dF(\lambda_{t+1}; \lambda)$.

3. The Standard Solution

The sequence of approximating value functions is described by the recursive equation

(3)

$$V_n(k, \lambda) = \max\{U(f(k, N_n)\lambda - k_n + (1-\Omega)k, 1-N_n) + \beta \sum_{t=1}^2 V_{n-1}(k_n, \lambda_t) dF(\lambda_t; \lambda)\},$$

$$\text{s.t.} \quad 0 \leq N_n \leq 1$$

$$(1-\Omega)k \leq k_n \leq f(k, N_n)\lambda + (1-\Omega)k$$

where the subscript n denotes the n th iteration. Experience has shown us that optimizing over next period's capital, rather than investment, substantially reduces round off errors.²

The state and decision variables are constrained to assume values in a discrete set or "grid" which reasonably approximates their actual domain of definition. A simple serial search procedure for determining the optimal (k_n^*, N_n^*) by evaluating the above expression for all possible (k_n, N_n) is, however, computationally very slow. Assuming as many as 250 distinct possible values for the capital stock, two values of the technology shock and 100 possible levels of labor supply, each stage of the iteration process would require $(250 \times 2) \times (250 \times 100)$ (number of possible state variable combinations \times number of possible pairs of decision levels) = 12,500,000 independent evaluations of the above expression to determine the maximum. Although this number can be substantially reduced by applying known properties of the decision variables (such as the fact that k_n is increasing in k), in light of the hundred plus iterations that may be necessary for policy convergence, the total computation time is frustratingly large.

A natural device for reducing these computations by a factor of 100 is to devise a simple and rapid procedure for determining directly the optimal N_n associated with each possible choice of k_n . This is what we do. Noting that, for a given (k, λ) :

(i) searching over the region of ordered pairs (k_n, N_n) for which $0 \leq N_n \leq 1$ and $(1 - \Omega)k \leq k_n \leq f(k, N_n) + (1 - \Omega)k$ is equivalent to searching over the region $(1 - \Omega)k \leq k_n \leq f(k, 1_n) + (1 - \Omega)k$ and

$$N_0 \leq N_n \leq 1 \text{ where } N_0 = N_0(k_n, k, \lambda) \equiv \left[\frac{v[k_n - (1 - \Omega)k]}{Lk^{\alpha v} \lambda} \right]^{\frac{1}{(1-\alpha)v}}, \text{ and}$$

(ii) maximizing $U(\cdot) + V_{n-1}(\cdot)$ with respect to N_n is equivalent to maximizing $U(\cdot)$ alone with respect to N_n , since $V_{n-1}(\cdot)$ is independent of N_n ,

we can solve for the optimal N_n associated with each (k_n, k, λ) triple by maximizing

$$(4) \quad (aN_n^{(1-\alpha)v} - b)\gamma^\delta (1 - N_n)^{(1-\gamma)\delta} c$$

with respect to N_n , where $a = a(k, \lambda) = \frac{Lk^{\alpha v} \lambda}{v} \geq 0$, $b = b(k_n, k) =$

$k_n - (1 - \Omega)k \geq 0$, and $c = \frac{1}{\delta}$. Since equation (4) is a concave function of N_n we may differentiate and set the resultant expression equal to zero to determine the optimal N_n . This yields the expression

$$(5) \quad \frac{\gamma a (1 - \alpha) v N_n^{(1-\alpha)v-1}}{aN_n^{(1-\alpha)v} - b} = (1 - \gamma) \frac{1}{1 - N_n}.$$

Rearranging terms this may be rewritten as

$$(6) \quad AN_n^{(1-\alpha)v} - BN_n^{(1-\alpha)v-1} - C = 0$$

where $A = (1 - \gamma)a + \gamma a(1 - \alpha)v \geq 0$, $B = \gamma a(1 - \alpha)v \geq 0$ and $C = (1 - \gamma)b$. Rearranging equation (6) again gives

$$(7) \quad AN_n - B - CN_n^{1-(1-\alpha)v} = 0, \text{ or}$$

$$(8) \quad N_n = \frac{B}{A} + \frac{C}{A} N_n^{1-(1-\alpha)v}.$$

Equation (8) is amenable to standard fixed point procedures.

Graphing each side of (8), as per Figure (1) below, it is

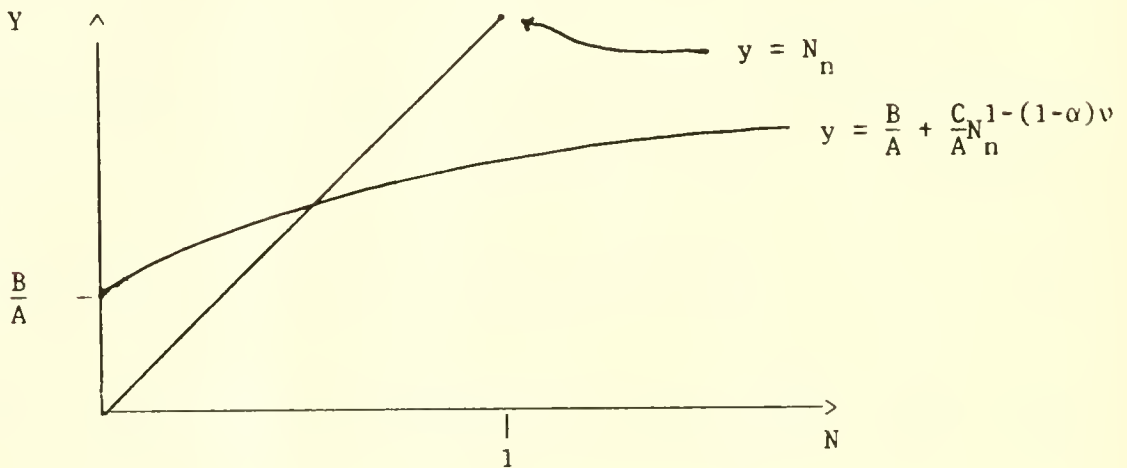


Figure (1)

apparent that the recursive iteration procedure defined by

$$N_n^{(0)} \equiv 1$$

$$N_n^{(1)} = \frac{B}{A} + \frac{C}{A} (N_n^{(0)})^{1-(1-\alpha)v}$$

$$(9) \quad N_n^{(2)} = \frac{B}{A} + \frac{C}{A} (N_n^{(1)})^{1-(1-\alpha)v}$$

⋮

⋮

$$N_n^{(k)} = \frac{B}{A} + \frac{C}{A} (N_n^{(k-1)})^{1-(1-\alpha)v}$$

will approximate the solution. The remaining issue is to establish a convergence criterion for the procedure. Suppose the solution to

equation (8) is an $N_n = \frac{B}{A} + \frac{C}{A} N_n^{1-(1-\alpha)v}$, then for an $N_n^{(i)}$,

$$\begin{aligned}
 (10) \quad N_n^{(i+1)} &= \frac{B}{A} + \frac{C}{A} (N_n^{(i)})^{1-(1-\alpha)v} = \frac{B}{A} + \frac{C}{A} (N_n)^{1-(1-\alpha)v} \\
 &\quad + \frac{C}{A} (1-(1-\alpha)v) N_n^{-(1-\alpha)v} (N_n^{(i)} - N_n) \\
 &= N_n + \frac{C}{A} (1-(1-\alpha)v) N_n^{-(1-\alpha)v} (N_n^{(i)} - N_n).
 \end{aligned}$$

If we desire a solution for which $N_n^{(i)} - N_n < .5\|K\|$, where $\|K\|$ denotes the width of the partition of possible capital stock levels, then the convergence requirement becomes:

$$\begin{aligned}
 \left| N_n^{(i+1)} - N_n^{(i)} \right| &\cong \left| N_n + \frac{C}{A} (1-(1-\alpha)v) N_n^{-(1-\alpha)v} (N_n^{(i)} - N_n) - N_n^{(i)} \right| \\
 &= \left| \left[\frac{C}{A} (1-(1-\alpha)v) N_n^{-(1-\alpha)v} - 1 \right] (N_n^{(i)} - N_n) \right| \\
 &\leq \left| \frac{C}{A} (1-(1-\alpha)v) N_n^{-(1-\alpha)v} - 1 \right| (.5 \|K\|), \text{ where}
 \end{aligned}$$

$\frac{C}{A}$, and N_n can be estimated from the steady state certainty levels of

capital and labor; that is $\frac{C}{A} \cong \frac{(1-\gamma)[k^{SS} - (1-\Omega)k^{SS}]v}{[(1-\gamma) + \gamma(1-\alpha)v]Lk^{\alpha v}\lambda}$, $N_n \cong .3 = N^{SS}$,

and $(1 - (1 - \gamma)v) = .36$ (in the case of Hansen's (1985) choice of param-

eters). Given the determination of the optimal $N_n = N_n(k_n, k, \lambda)$, the

related optimal k_n was determined via a modified exhaustive search

procedure. The time series output of the model was constructed by first

generating a sequence of three thousand shocks to the technology with

respect to the chosen transition matrix and then by allowing the economy to evolve from an arbitrarily determined initial state via repeated application of the optimal policy rules. We have not subjected the data to any detrending procedure as we thought it worthwhile first to assess the model's direct output without the modifications a smoothing procedure would introduce.³ Limiting ourselves for the moment to the statistics reported in Hansen (1985), we find the time series properties of this optimal economy to be summarized below:

Table (1)

$$\beta = .96, \alpha = .36, \gamma = .33, \Omega = .1, \lambda_1 = 1.02, \lambda_2 = .98,$$

$$\text{Persistence} = .97, L = 1$$

$$\text{Capital Stock Partition} = .005$$

	(a) standard deviation (percent)	(b) correlation with output
output:	2.43	1.00
consumption:	2.09	.898
investment:	4.49	.881
capital stock:	2.22	.868
hours:	.69	.99
productivity:	2.23	.276

We note that for a state space of 250 possible capital stock levels, and 100 possible labor supply levels, the entire computational procedure required 31.33 minutes of VAX-11-780 CPU time.

3. An Approximate Solution

Our procedure is similar to that of Section (2) except that we undertake a quadratic approximation of the return function about the economy's certainty steady state. As before, we first outline our iterative procedure and then present our comparative time series results.

Let the superscript ss denote the steady state certainty ($\lambda_t \equiv 1$) value of the relevant variable.⁴ Following Kydland and Prescott (1982) and Hansen (1985), we approximate the agent's utility function about the steady state by the following expression:

$$(11) \quad \hat{U}^*(k, N, \lambda, z) = \bar{U} + BX + X'QX, \text{ where}$$

$$(12) \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k - k^{ss} \\ N - N^{ss} \\ \lambda - \lambda^{ss} \\ z - z^{ss} \end{bmatrix};$$

$$(13) \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} [\hat{u}(k^{ss} + \Delta_k, N^{ss}, \lambda^{ss}, z^{ss}) - \hat{u}(k^{ss} - \Delta_k, N^{ss}, \lambda^{ss}, z^{ss})] \frac{1}{2\Delta_k} \\ [\hat{u}(k^{ss}, N^{ss} + \Delta_n, \lambda^{ss}, z^{ss}) - \hat{u}(k^{ss}, N^{ss} - \Delta_n, \lambda^{ss}, z^{ss})] \frac{1}{2\Delta_k} \\ [\hat{u}(k^{ss}, N^{ss}, \lambda^{ss} + \Delta_\lambda, z^{ss}) - \hat{u}(k^{ss}, N^{ss}, \lambda^{ss} - \Delta_\lambda, z^{ss})] \frac{1}{2\Delta_k} \\ [\hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss} + \Delta_z) - \hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss} - \Delta_z)] \frac{1}{2\Delta_k} \end{bmatrix},$$

where $\hat{u}(k, N, \lambda, z) = u(f(k, N)\lambda - z, 1 - N)$;

$$(14) \quad \bar{U} = \hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss}); \text{ and}$$

Q is a 4×4 matrix with diagonal entries of the form:

$$\begin{aligned}
 q_{11} = & [\hat{u}(k^{SS} + \Delta_k, N^{SS}, \lambda^{SS}, z^{SS}) - \hat{u}(k^{SS}, N^{SS}, \lambda^{SS}, z^{SS}) \\
 & + \hat{u}(k^{SS} - \Delta_k, N^{SS}, \lambda^{SS}, z^{SS})] \frac{1}{2\Delta_k^2} \\
 & \vdots \\
 & \vdots \\
 & \vdots \\
 q_{44} = & [\hat{u}(k^{SS}, N^{SS}, \lambda^{SS}, z^{SS} + \Delta_z) - \hat{u}(k^{SS}, N^{SS}, \lambda^{SS}, z^{SS}) \\
 & + \hat{u}(k^{SS}, N^{SS}, \lambda^{SS}, z^{SS} - \Delta_z)] \frac{1}{2\Delta_z^2}
 \end{aligned}
 \tag{15}$$

and off diagonal elements of the form, e.g.:

$$\begin{aligned}
 q_{13} = & [\hat{u}(k^{SS} + \Delta_k, N^{SS}, \lambda^{SS} + \Delta_\lambda, z^{SS}) - \hat{u}(k^{SS} - \Delta_k, N^{SS}, \lambda^{SS} + \Delta_\lambda, z^{SS}) \\
 & - \hat{u}(k^{SS} + \Delta_k, N^{SS}, \lambda^{SS} - \Delta_\lambda, z^{SS}) + \hat{u}(k^{SS} - \Delta_k, N^{SS}, \lambda^{SS} - \Delta_\lambda, z^{SS})] \frac{1}{(\Delta_k \cdot \Delta_\lambda)^2}
 \end{aligned}
 \tag{16}$$

In the above expressions, the symbol Δ indicates the deviation from the steady state while \bar{U} represents the certainty steady state level of period utility. The deviations themselves were chosen proportional to the steady state levels with the same proportionately constant across all state variables.

Turning to the iterative procedure itself, it is evident that for fixed (k, λ) , $V_0^A(k, \lambda)$ can be expressed as

$$(17) \quad V_0^A(k, \lambda) = \text{CONSTANT} + b_2 x_2 + \sum (q_{2j} + q_{j2}) x_j x_2 + q_{22} x_2^2$$

where the constant has the value

$$(18) \quad \text{CONSTANT} = \bar{u} + \sum_{i \neq 2} b_i x_i + \sum_{\substack{i \neq 2 \\ j \neq 2}} q_{ij} x_i x_j, \text{ and } X_4 = (1 - \Omega)k - z^{\text{SS}}.$$

The superscript A is intended to remind us that these are approximate value functions. We note also that the precise representations of the x_i 's, b_i 's and q_{ij} 's have been suppressed for ease of presentation.

Noting that equation (17) is of the form $V = C + Bx_2 + Ax_2^2$, it is clear that the maximum will occur at

$$(19) \quad x_2 = -\frac{B}{2A} = \frac{-(\sum_{j \neq 2} 2q_{j2} x_j + b_2)}{2q_{22}}.$$

from which the optimal $\tilde{N}_0 = N^{\text{SS}} + x_2$ is easily computed. It follows that

$$(20) \quad V_0^A(k, \lambda) = u(f(k, N_0)\lambda, 1 - N_0) \text{ for all } (k, \lambda).$$

More generally,

$$(21) \quad V_n^A(k, \lambda) = \max_{0 < N_n < 1} \{V^*(k, N_n, \lambda, k_n) + \beta \sum_{t=1}^2 V_{n-1}^A(k_n, \lambda_t) dF(\lambda_t, \lambda)\}.$$

$$(1 - \Omega)k \leq k_n \leq f(k, N_n)\lambda - (1 - \Omega)k$$

We point out that in this expression we optimize over next period's capital stock, rather than this period's investment. This requires some modification of $U^*(\cdot)$, but again seems to substantially reduce errors resulting from rounding off.⁴

An examination of the region defining the set of possible choices of the state variables allows us to express (21) equivalently as

$$(22) \quad V_n^A(k, \lambda) = \max \left\{ u^*(k, N_n, \lambda, k_n) + \beta \sum_{t=1}^2 V_{n-1}^A(k_n, \lambda_t) dF(\lambda_t, \lambda) \right\}$$

$$\text{s.t.} \quad (1 - \Omega)k \leq k_n \leq f(k, 1) + (1 - \Omega)k$$

$$N_0(k_n, k, \lambda) \leq N_n \leq 1$$

Substituting and rearranging terms gives:

$$(23) \quad V_n^A(k, \lambda) = \max_{(1-\Omega)k \leq k_n \leq f(k, N_n) + (1-\Omega)k} \left\{ \max_{N_0(k_n, k, \lambda) \leq N_n \leq 1} \{ C(k, \lambda, k_n) \right.$$

$$\left. + b_2 x_2 + \sum_{j \neq 2} (q_{j2} + q_{2j}) x_j x_2 + q_{22} x_2^2 \} \right\} \text{ where}$$

$$(24) \quad C(k, \lambda, k_n) = \hat{u}(k^{SS}, N^{SS}, \lambda^{SS}, z^{SS}) + \sum_{i \neq 2} b_i x_i + \sum_{\substack{i \neq 2 \\ j \neq 2}} q_{ij} x_i x_j$$

$$+ \beta \sum_{t=1}^2 V_{n-1}^A(k_n, \lambda_t) dF(\lambda_t, \lambda).$$

As before the optimum N_n can be computed directly by differentiating the expression of line (24) to give

$$(25) \quad x_2 = - \frac{B}{2A} = \frac{-(b_2 + \sum_{j \neq 2} 2q_{j2} x_j)}{2q_{22}} \equiv \tilde{N}_n - N^{SS}.$$

We note also that if $\tilde{N}_n > 1$, then $N_n = 1$; if $\tilde{N}_n < N_0$, $N_n = N_0$ and if $N_0 < \tilde{N}_n < 1$, then $N_n = \tilde{N}_n$. Given this information, the computation of the optimal k_n reduces to solving

$$(26) \quad V_n^A(k, \lambda) = \max \left\{ u^*(k, \tilde{N}_n, \lambda, k_n) + \beta \sum_{t=1}^2 V_{n-1}^A(k_n, \lambda_t) dF(\lambda_t, \lambda) \right\}$$

$$(1 - \Omega)k \leq k_n \leq f(k, 1) + (1 - \Omega)k$$

This latter maximization was solved using a modified standard grid search procedures. The resulting savings policy function is linear and is the same as if this expression had also been solved directly. Although we could as well directly solve for the capital next period, the application of the linear decision rule so obtained would not necessarily confine the economy to the prechosen state space. In effect, the discretizing of the state space precludes us from using the Kydland and Prescott (1982) procedure. The results of the simulation for the same parameters as in Table (1) are reported below:

Table (2)

$$\beta = .96, \alpha = .36, \gamma = .33, \Omega = .1, \lambda_1 = 1.02, \lambda_2 = .98,$$

$$\text{Persistence} = .97, \Delta = .001$$

$$\text{Capital Stock Partition} = .005$$

	(a) standard deviation (percent)	(b) correlation with output
output:	2.18	1.00
consumption:	2.27	.992
investment:	2.64	.678
capital stock:	.59	.899
hours:	.31	.99
productivity:	2.02	.062

Comparing Tables (1) and (2) we note that the standard deviations of the output, consumption, hours and productivity series are more or less the same. The exact solution has substantially greater standard deviation of investment and capital stock, however. It should be noted that each of the time series from which these statistics were calculated was (1) subjected to identical shocks (randomly generated) and (2) had its origin in the same initial capital stock level. It is also worth noting that the levels of all series (expected) were nearly identical.

4. Sensitivity Analysis

There are a number of related issues which ought to be addressed if we are to assess fully the appropriateness of this approximation. In particular, we examine (1) the accuracy of the approximation with regard to the magnitude of the deviation from steady state levels, (2) the comparative time paths to the steady state of the approximate and exact economies and (3) the sensitivity of the approximation to changes in parameter values. Each is considered in turn:

4.1 Magnitude of Deviations: As noted in Section 4, for the approximate results reported thus far, all deviation from the steady state (the Δ 's) were chosen proportional to the corresponding steady state level itself with the proportionality factor set at .001. In this subsection we recompute the statistics of Table (2) for the case in which this proportionality factor is .01 and .02. These results are reported below:

Table (3)

$$\Delta = .01$$

$$\alpha = .36, \beta = .96, \gamma = .33, \Omega = .1, \lambda_1 = 1.02, \lambda_2 = .98$$

$$\text{Persistence} = .97, L = 1$$

$$\text{Capital Stock Partition} = .005$$

	(a) standard deviations (percent)	(b) correlation with output
output:	2.87	1.00
consumption:	2.51	.968
investment:	7.43	.879
capital stock:	3.36	.895
hours:	1.12	.986
productivity:	2.43	.350

$$\Delta = .02$$

(other parameters unchanged)

	(a) standard deviations	(b) correlations with output
output:	3.40	1.00
consumption:	3.05	.956
investment:	9.16	.898
capital stock:	5.41	.910
hours:	1.33	.966
productivity:	2.87	.523

These tables reveal the tendency of the standard deviation in all series to increase as the proportional deviation Δ increases; capital stock variation appears to increase at the greatest rate. This is not surprising as the quadratic approximation is effectively reducing the level of concavity in the utility function which has the consequence of making increased variation in all the series optimal. With regard to the investment function in particular, its linearity under the quadratic approximate regime contrasts with its concavity under the 'exact' formulation. For any given variation in output, greater variation in investment will be observed under the linear decision rule.

We also note that the correlation of both consumption and hours with output declines slightly with the increased Δ , while wage (productivity) correlation with output increases very dramatically. The correlation of output with the remaining other series is substantially unaffected. We are unable to provide intuition as to why this should be so.

4.2 Convergence to Steady State: As noted earlier, our program is designed so that all time series are generated by subjecting the technology to the exact same sequence of random shocks. The initial capital shock level is identical across all cases as well. This set-up has the advantage of allowing us to study relative convergence rates under both procedures.

In general, the exact economy converges to the steady state capital stock level much less rapidly than does the approximate economy. The exact behavior is as follows: both economies linger at the boundary of the steady state without actually entering it for many periods. This may simply reflect a long sequence of the poor random shocks. However, the approximate economy reaches the boundary in approximately one-fourth the time required for the exact economy. This is all the more striking when we realize that the range of the stationary distribution on capital stock for the exact economy fully contains the range for the approximate economy. These figures are, respectively, [1.2, 1.26] and [1.22, 1.235]. Thus, not only does the exact economy require more time to converge, but also the amount by which capital must increase to enter the steady state ($k_0 < 1.2$) is less. These reported range values are also consistent with the greater relative variability of capital stock for the exact economy.

4.3 Sensitivity to Parameters: Of all the parameters necessary to define this class of problems, the choice of the preference parameter γ is perhaps most open to question and we examine the sensitivity of our approximation to changes in that value. This has the added advantage of leaving the steady state basis of the approximation unchanged. We find that with the $\Delta = .01$ deviation, the discrepancies between the exact and approximate cases can vary substantially with the choice of preference parameters. This is illustrated below where we first allow for changes in γ (Table 4).

Table 4

$\beta = .96$, $\delta = -.5$, $\Omega = .1$, $\Delta = .01$
 $\lambda_1 = 1.02$, $\lambda_2 = .98$, Persistence = .97, $L = 1$
 Capital Stock Partition = .005
 $\gamma = .25$

	<u>Exact Economy</u>		<u>Approximate Economy</u>	
	(a) standard deviation	(b)	(a) correlation with output	(b)
output:	2.56	1.00	3.63	1
consumption:	2.12	.971	4.16	.908
investment:	6.32	.875	14.52	.823
capital stock:	2.49	.857	6.23	.931
hours:	1.13	.991	1.64	.971
productivity:	2.33	.286	3.16	.597

$\gamma = .40$

	<u>Exact Economy</u>		<u>Approximate Economy</u>	
	(a)	(b)	(a)	(b)
output:	2.48	1.00	2.89	1.00
consumption:	1.99	.976	2.36	.981
investment:	5.80	.888	5.54	.909
capital stock:	2.47	.870	2.89	.879
hours:	.82	.994	.96	.988
productivity:	2.28	.307	2.32	.297

5. Concluding Comments

The figures reported here suggest that the quadratic approximation will provide a very reasonable approximation to its exact counterpart when the shocks are low and the deviations which form the basis of the approximation are small. It is in the capital stock and investment series that the greatest discrepancy arises.

All of this is a bit unsatisfying, however, as there are many sources of approximation error in a numerical exercise as complex as this and it seems impossible to assign any discrepancies in the 'exact'

and 'approximate' solutions to a particular source. Thus the use of such methods in increasingly complex numerical exercises will remain, invariably, partially an act of faith.

FOOTNOTES

1. This was accomplished using the method of moments.
2. If we optimize directly over next period's capital stock then investment z and consumption are computed as residuals. If we optimize over z , the z is discretized, $(1-\Omega)k$ is discretized and the sum $z + (1-\Omega)k$ discretized -- which gives three sources of round off errors in the measurement of the critical next period's capital stock level.
3. Since the output data of this model does not, by construction, exhibit any trend, it is not entirely clear that such a procedure would be appropriate anyway.
4. These steady state values were computed as in Kydland and Prescott (1982).

REFERENCES

- Danthine, J. P. and J. B. Donaldson, "Stochastic Properties of Fast vs. Slow Growing Economies," Econometrica 49(1981), 1007-1033.
- Greenwood, J. and G. Huffman, "A Dynamic Equilibrium Model of Inflation and Unemployment," Working Paper, University of Western Ontario (1986).
- Hansen, G. D., "Indivisible Labor and the Business Cycle," Journal of Monetary Economics 16 (1985), 309-327.
- Kydland, F. E. and E. C. Prescott, "Time to Build and Aggregate Fluctuations," Econometrica 50 (1982), 1345-1370.
- Mehra, R. and E. C. Prescott, "The Equity Premium: A Puzzle," Journal of Monetary Economics, 15 (1985), 145-161.

Date Due

Date Due	

MIT LIBRARIES



3 9080 005 239 378

