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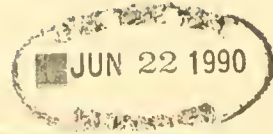
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SUBSTITUTION I: THE CASE OF CERTAINTY

Ayman Hindy and Chi-fu Huang

March 1990

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# OPTIMAL CONSUMPTION WITH INTERTEMPORAL SUBSTITUTION I: THE CASE OF CERTAINTY

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## Abstract

We study the problem of optimal consumption choice in continuous time under certainty for a class of utility functions that capture the notion that consumptions at nearby dates are almost perfect substitutes. The class we consider excludes all time-additive and almost all the non time-additive utility functions used in the literature. We provide necessary and sufficient conditions for a consumption policy to be optimal. Furthermore, we demonstrate our general theory by solving in a closed form the optimal consumption policy for a particular felicity function. The optimal policy in our solution consists of a (possible) initial "gulp" of consumption, or an initial period of no consumption, followed by consumption at the rate that maintains a constant ratio of wealth to average past consumption.

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# 1 Introduction

The choice of an optimal consumption plan for an individual under certainty is a classical problem in economics; see, for example, Ramsey (1928) and Modigliani and Brumberg (1954). In a large part of the literature that addresses this problem, the preferences of an individual are represented by a time-additive utility function. It has long been recognized that a time-additive utility function does not incorporate the intuitively appealing notion that past consumption can contribute to an individual's future satisfaction. Nevertheless, this class of preferences has been utilized because of its tractability. In discrete time models, one can argue that if the length of time periods is chosen appropriately, then past consumption would have negligible effect on current satisfaction. Such arguments, however, no longer apply when we consider a continuous time model.

Recently, a class of non-time-additive utility functions, due originally to Koopmans (1960) and Uzawa (1968), has gained increasing significance in the literature of continuous time equilibrium models; see, for example, Epstein (1987). Also, non-time-additive utility functions that exhibit habit formation are objects of current research interest in continuous time financial asset pricing models; see, for example, Constantinides (1988), Heaton (1988), and Sundaresan (1989). A common feature of these utility functions is that the felicity, or the index of instantaneous satisfaction at any time, is a function of the current consumption and a certain functional of the past consumption. A high level of past consumption can serve either to depress current appetite as in the case of the Uzawa type utility functions or to increase current appetite as in the case of the habit formation models.

These non-time-additive utility functions offer intuitively appealing notions of interactions between past and current consumption. They are also mathematically tractable. Optimal consumption policies can be characterized using dynamic programming. For particular functional forms of the felicity function, one can even have closed form expressions for these policies. Unfortunately, these utility functions fail to exhibit one key economic property that underlies much of our intuition about the behavior of asset prices over time in the absence of arbitrage possibilities.

A common hypothesis maintained by many financial economists is that asset prices cannot have predictable jumps. To support this hypothesis, the argument goes as follows: suppose agents knew that the price of an asset were to increase by a discrete amount at a specific moment. If an agent could borrow funds for a very short period of time over which the interest

he would have to pay is negligible, then he would be able to realize an “arbitrage profit”. Having access to a frictionless market, he would simply borrow some funds just before the jump in the price were supposed to occur, buy the asset, wait for its price to increase, and then liquidate his position. The demand of agents exploiting this arbitrage opportunity would push the price of the asset up before the time when the jump were supposed to appear and hence the jump would be eliminated. The validity of this argument depends crucially on the hidden assumption that prices for consumption at nearly adjacent dates are almost equal and thus the interest that the agent pays on funds borrowed over a short time interval is negligible.

There are several economic forces that will bring about the closeness of prices for consumptions at nearly adjacent dates. The most primitive among these forces is the hypothesis that agents treat consumptions at nearly adjacent dates as almost perfect substitutes and thus ensure that almost equal prices are established for them. Huang and Kreps (1989), in a model under certainty, and Hindy and Huang (1989), in a model under uncertainty, study preferences that exhibit this property. They show in particular that if the felicity function depends explicitly on current consumption and is strictly concave in it, then consumptions at nearly adjacent dates *cannot* be treated as almost perfect substitutes. Since the functional forms of the felicity function used both in the Uzawa type and in the habit formation type preferences are strictly concave, agents with those preferences do not view consumptions at nearby dates as close substitutes.<sup>1</sup> Therefore, such models will reach economic conclusions that violate some of our basic intuition about price behavior over time.

The purpose of this paper is to investigate the optimal consumption problem under certainty for a class of utility functions that treat consumptions at nearby dates as close substitutes. The key feature of this class of utility functions is that the felicity function at any time depends only upon an exponentially weighted average of past consumption – one derives satisfaction only from past consumption. We will demonstrate the tractability of this family by showing how dynamic programming can be used to give sufficient conditions for optimality. In particular, for a specific functional form of the felicity function, we solve in closed form the optimal consumption policy.

Intuition suggests that when one derives current satisfaction only from past consumption, the optimal consumption policy may specify periods of consumption mixed with periods of no consumption. Our closed form solution exhibits this property. In particular, the optimal consumption policy in our solution calls for a (possible) initial “gulp” of consumption, or an

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<sup>1</sup>Heaton (1990) studies preferences in discrete time whose continuous time limits satisfy the notion of local substitution studied by Huang and Kreps (1989) and Hindy and Huang (1989).

initial period of no consumption, followed by consumption at the rate that maintains a constant ratio of wealth to average past consumption. The consumption rate is determined by the interplay between the effect of past consumption, the current return on investment and the impatience of the agent.

The rest of this paper is organized as follows. Section 2 sets up the consumption problem under certainty in continuous time. Section 3 heuristically derives necessary conditions for a consumption policy to be optimal. Section 4 provides sufficient conditions for optimality. In section 5, we solve the optimal consumption problem in closed form for a particular felicity function. Section 6 contains concluding remarks.

## 2 The Setup

Consider an economic agent who lives from time  $t = 0$  to  $t = T$  in a certain world where there is a single consumption good available at any time between 0 and  $T$ . The agent can consume “gulps” of the good at any moment, and can consume at finite rates over intervals. He can also refrain from consumption for any period as he sees fit. We represent the agent’s consumption pattern over his life span by a positive<sup>2</sup>, increasing, right continuous function  $C: [0, T] \rightarrow \mathfrak{R}_+$ , with  $C(t)$  denoting the *accumulated* consumption from time zero to time  $t$ . Note that the only possible kind of a discontinuity of  $C$  is a jump. Note also that we allow  $C$  to have a “singular” component, that is a continuous increasing function whose derivative is zero for almost all  $t$ . The famous Cantor function is an example; see, for example, Royden (1968, p.48). The consumption set of the agent,  $X_+$ , is the space of all such functions and the commodity space,  $X$ , is the linear span of  $X_+$ . Note that  $X$  is the space of right-continuous and finite variation functions. Note also that a finite variation function  $x$  on  $[0, T]$  has a finite left-limit at any  $t \in (0, T]$  denoted by  $x(t^-)$  and a finite right-limit at any  $t \in [0, T)$  denoted by  $x(t^+)$ . By right-continuity, for any  $x \in X$  we have  $x(t^+) = x(t)$ ,  $t \in [0, T)$ . For convenience, we will use the convention that  $x(0^-) = 0$ . Since left-limits exist for any  $x \in X$ , a jump of  $x$  at  $t$  is  $\Delta x(t) \equiv x(t) - x(t^-)$ .

The points of discontinuity of a consumption pattern  $C$  are the moments when the agent decides to consume a “gulp”, and the size of the jump is the amount of that “gulp”. Note that there can be only at most a countable number of times of “gulps” since  $C$  is an increasing right continuous function of time. Let these moments be  $\tau_1, \tau_2, \dots$ , and let the jump at time  $\tau_i$  be

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<sup>2</sup>We use weak relations in our discussion. Hence *increasing* is equivalent to *nondecreasing*, for example. When a relation is strict, we will explicitly state so.

$\Delta C(\tau_i)$ . Moreover,  $C$  is absolutely continuous on the intervals on which the agent consumes at rates and the derivative of  $C$  on these intervals, denoted  $c(t)$ , is the “rate” of consumption there. When this derivative is zero,  $C$  is constant on the interval, and the agent is not consuming at all.

The agent starts at time 0 with endowment  $W(0)$ , and at any time  $t$  he can invest part of or all his wealth in a riskless asset with instantaneous rate of return  $r(t)$ . We assume that the interest rate  $r(t)$  is a continuous function of time. The agent’s wealth *before* consumption at time  $t$ , denoted  $W(t)$ , changes according to the following dynamics:

$$\begin{aligned} dW(t) &= W(t)r(t)dt - dC(t) && \text{for } t \in (\tau_i, \tau_{i+1}) \text{ for all } i, \\ W(\tau_i^+) &= W(\tau_i) - \Delta C(\tau_i) && \text{for all } \tau_i. \end{aligned} \tag{1}$$

Note that  $W(t)$  is a left continuous function and (1) is just the budget constraint over time. We impose the condition that  $W(t) \geq 0$  for all  $t \in [0, T]$ .

The agent has preferences for consumption over the period  $[0, T]$  and for final wealth  $W(T)$ .<sup>3</sup> He feels, however, that consumption at one time is a substitute for consumption at other, nearby times. Therefore, delaying or advancing consumption for a very small period of time has a very small effect on his total satisfaction. Huang and Kreps (1989), for the case of certainty, and Hindy and Huang (1989), for the case of uncertainty, formalize the idea that consumption preferences can be similar to those expressed by the agent. In particular, the authors advance a family of topologies on the space of consumption patterns over time that capture the notion that consumptions at nearly adjacent dates are almost perfect substitutes. In this paper, we use one member of this family given as follows. Fix  $p \geq 1$ . Define the topology  $\mathcal{T}$  on the commodity space  $X$  to be the topology generated by the norm:<sup>4</sup>

$$\|C\| = \left[ \int_0^T |C(t)|^p dt + |C(T)|^p \right]^{1/p}. \tag{2}$$

Moreover, Huang and Kreps (1989) and Hindy and Huang (1989) show that there are intuitively appealing functional forms of utility which express preferences for consumption with intertemporal substitution properties. The agent has one of them. Fix any consumption pattern  $C$  and a corresponding terminal wealth  $W(T)$ . At any time  $t$ , define the exponentially weighted

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<sup>3</sup>The astute reader might have observed that the final wealth should be  $W(T^+)$ , since wealth is a left continuous function of time. To simplify notation, we will use the slightly inaccurate  $W(T)$  for final wealth. Our justification is that consumption gulps at  $T$  add nothing to the satisfaction of the agent. Hence, at the optimal solution  $W(T) = W(T^+)$ .

<sup>4</sup>The special treatment of  $C(T)$  in the norm accounts for the fact that two consumption patterns which agree on  $[0, T)$  but disagree on  $T$  could not be distinguished if the norm were without the term involving  $C(T)$ .

average of past consumption,  $z(t)$ , by:

$$z(t) = z(0^-)e^{-\beta t} + \beta \int_{0^-}^t e^{-\beta(t-s)} dC(s), \quad (3)$$

where  $z(0^-) \geq 0$  is a constant and  $\beta$  is a weighting factor. Note that the lower limit of the integral in (3) is  $0^-$ , to account for the possible jump of  $C$  at  $t = 0$ , and that  $z(t)$  is a right continuous function. Moreover,  $z$  has a jump exactly when  $C$  does and  $z$  has a singular component when  $C$  does. Also observe that higher values of  $\beta$  imply higher emphasis on the recent past and less emphasis on consumption in the distant past. Note also that the average past consumption  $z(t)$  changes according to the following dynamics:

$$\begin{aligned} dz(t) &= \beta[dC(t) - z(t)dt] && \text{for } t \in (\tau_i, \tau_{i+1}) \text{ for all } i, \\ z(\tau_i) &= z(\tau_i^-) + \beta\Delta C(\tau_i) && \text{for all } \tau_i. \end{aligned} \quad (4)$$

Given  $z(0^-)$ , the agent's utility for the consumption pattern  $C$  and final wealth  $W(T)$  is given by:

$$U(z(0^-), 0; (C, W(T))) = \int_0^T u(z(t), t) dt + V(W(T)), \quad (5)$$

where both  $u: \mathfrak{R}_+ \times [0, T] \rightarrow \mathfrak{R} \cup \{-\infty\}$  and  $V: \mathfrak{R}_+ \rightarrow \mathfrak{R} \cup \{-\infty\}$  are continuous and increasing functions. Furthermore,  $V$  is strictly concave and  $u$  is strictly concave in its first argument. Note that we have allowed the possibility that  $u$  or  $V$  can take the value  $-\infty$  at zero. For example,  $V(W) = \frac{1}{\alpha}W^\alpha$ , where  $\alpha < 0$ . The function  $u$  is, in the terminology of Arrow and Kurz (1970), the felicity function that assigns the level of satisfaction derived from past consumption and  $V$  is the bequeath function. Preferences given by (3) and (5) are continuous in the product topology generated by  $\mathcal{T}$  and the Euclidean topology on  $\mathfrak{R}_+$ ; see Hindy and Huang (1989), proposition 17.

Starting from a given level of wealth, each consumption plan  $C$  determines, via the budget constraint, the value of terminal wealth  $W(T)$ . It also determines for every time  $t$ , the wealth  $W(t)$ , and the average past consumption  $z(t^-)$ . Note that at any  $t$ , the value  $z(t^-)$  is the average past consumption up till  $t$ , excluding the possible consumption at  $t$ . We will call  $W(t)$  and  $z(t^-)$  the state variables at  $t$ , since they represent the status of the agent *before* making his consumption decision at  $t$ . The agent faces the problem of choosing a consumption pattern  $C^*$  – the optimal control – from the space  $X_+$  – the admissible controls – to maximize his utility,  $U(z(0^-), 0; (C, W(T)))$ , subject to the dynamics given in (1) and (4), and given that his initial wealth is  $W(0)$  and given  $z(0^-)$ .

Formally, let  $\mathcal{A}(W(0), 0) \subset X_+ \times \mathfrak{R}_+$  be the space of all pairs of consumption pattern  $C$  and final wealth  $W(T)$  that satisfy the budget constraint of (1) starting with  $W(0)$ . The agent's problem is to find:

$$\sup_{(C, W(T)) \in \mathcal{A}(W(0), 0)} U(z(0^-), 0; (C, W(T))). \quad (6)$$

A solution to (6) exists if the supremum is finite and is attained by some  $(C^*, W^*(T)) \in \mathcal{A}(W(0), 0)$ .

At times, we will also consider a sub-problem of (6) at some  $t \in [0, T]$ : given  $\{C(s); s \in [0, t^-]\}$  and  $W(t)$  determined by (1) (the budget constraint), solve

$$\sup_{(C^t, W(T)) \in \mathcal{A}(W(t), t)} U(z(t^-), t; C^t, W(T)) \equiv \int_t^T u(z(s), s) ds + V(W(T)), \quad (7)$$

where  $C^t$  is an increasing and right-continuous function on  $[t, T]$  representing accumulated consumption starting from  $t$  with the convention that  $C^t(s) = 0$  for all  $s < t$ , where  $\mathcal{A}(W(t), t)$  denotes the consumption pattern  $C^t$  and final wealth  $W(T)$  that satisfy the budget constraint (1) on  $[t, T]$  with an initial value  $W(t)$ , where

$$z(s) = z(t^-)e^{-\beta(s-t)} + \int_{t^-}^s e^{-\beta(s-\xi)} dC^t(\xi) \quad \text{for } s \geq t. \quad (8)$$

It is clear that once  $W(t)$  and  $z(t^-)$  are known,  $\{C(s); s \in [0, t^-]\}$  has no impact on the choices of  $C^t$  in (7) and this justifies the notation  $U(z(t^-), t; C^t, W(T))$ . Moreover, if  $(C^*, W^*(T))$  is a solution to (6), then  $\{C^t(s) = C^*(s) - C^*(t^-); s \in [t, T]\}$  is a solution to (7) with  $z(t^-)$  and  $W(t)$  corresponding to  $(C^*, W^*(T))$ .

### 3 Heuristic Derivation of Necessary Conditions

We now use heuristic arguments to derive a set of necessary conditions for a solution to (6). These conditions will imply that an optimal consumption pattern can have gulps only at  $t = 0$ .

Suppose  $(C^*, W^*(T))$  is an optimal solution to (6) and let the corresponding wealth and average past consumption, or *the state variables*, be  $(W^*(t), z^*(t^-))$  for  $t \in [0, T]$ . At any time  $t$  and starting with wealth  $W(t)$  and average past consumption  $z(t^-)$ , let the *value function*  $J(W(t), z(t^-), t)$  be the maximum possible attainable satisfaction over the remaining period  $[t, T]$ . In other words, let:

$$J(W(t), z(t^-), t) = \sup_{(C^t, W(T)) \in \mathcal{A}(W(t), t)} \left[ \int_t^T u(z(s), s) ds + V(W(T)) \right], \quad (9)$$

where  $\{z(s); s \in [t, T]\}$  is defined according to (8).

Assume that  $J$  is finite and differentiable in  $W$  and  $z$ . The marginal value of wealth at time  $t$ ,  $J_W(W, z, t) \equiv \frac{\partial J(W, z, t)}{\partial W}$ , measures the changes in the maximum attainable satisfaction that results from a very small change in wealth when everything else is kept constant. Similarly, the marginal value of the average past consumption  $z$  at time  $t$ ,  $J_z(W, z, t) \equiv \frac{\partial J(W, z, t)}{\partial z}$ , measures the change in the value function for infinitesimal changes in  $z$ , *ceteris paribus*.

Along the optimal solution,  $J$  should satisfy the following conditions:

### 1 – Bellman Optimality Principle

The Bellman optimality principle; see, for example, Fleming and Rishel (1975), implies that for all times  $t$  and  $\tau \in [0, T]$  such that  $t < \tau$ ,

$$J(W^*(t), z^*(t^-), t) = \int_t^\tau u(z^*(s), s) ds + J(W^*(\tau), z^*(\tau^-), \tau). \quad (10)$$

That is: an optimal policy on  $[t, T]$  remains optimal over any subperiod  $[\tau, T]$ . In particular, this principle implies that along the optimal solution, the value function  $J(W^*, z^*, t)$  is a continuous function of time even at times of jumps. Right-continuity follows from

$$J(W^*(\tau), z^*(\tau^-), \tau) = \lim_{\Delta \downarrow 0} J(W^*(\tau + \Delta), z^*(\tau^- + \Delta), \tau + \Delta) \equiv J(W^*(\tau^+), z^*(\tau), \tau) \quad \text{for all } \tau, \quad (11)$$

and left-continuity is easy to see.

For brevity of notation, we will use  $\bar{J}(t)$  to denote  $J(W^*(t), z^*(t^-), t)$  and similarly use  $\bar{J}_W(t)$  and  $\bar{J}_z(t)$  to denote the partial derivatives of  $\bar{J}(t)$  with respect to  $W$  and  $z$ , respectively.

### 2 – Continuous Marginal Value of $W$ and $z$

Along the optimal policy, given the interest rates, the marginal value of  $\epsilon$  units of wealth at time  $t \in [0, T)$  must be equal to the marginal value of  $\epsilon e^{\int_t^{t+\Delta t} r(s) ds}$  units of wealth at time  $t + \Delta t$  for small  $\Delta t > 0$  or else some changes in  $C^*$  will be called for to change  $W^*$  and  $z^*$ .

That is, we must have

$$\bar{J}_W(t)\epsilon = \bar{J}_W(t + \Delta t)\epsilon e^{\int_t^{t+\Delta t} r(s) ds}. \quad (12)$$

Letting  $\Delta t \downarrow 0$ , we get  $\bar{J}_W(t) = \bar{J}_W(t^+)$ . Thus  $\bar{J}_W(t)$  must be right-continuous on  $[0, T)$ . Along the same line of arguments, we show that  $\bar{J}_W(t)$  must be left-continuous on  $(0, T]$  and thus  $\bar{J}_W(t)$  is continuous on  $[0, T]$ .

Similarly, by the Bellman optimality principle, a marginal value of  $\epsilon$  increase in  $z(t^-)$  at  $t$  must be equal to its marginal contribution to the felicity function on  $(t, t + \Delta t]$  and to the value function at time  $t + \Delta t$ . That is,<sup>5</sup>

$$\bar{J}_z(t)\epsilon = \int_t^{t+\Delta t} u_z(z^*(s), s)\epsilon e^{-\beta(s-t)} ds + \bar{J}_z(t + \Delta t)\epsilon e^{-\beta\Delta t}. \quad (13)$$

Letting  $\Delta t \rightarrow 0$ , we conclude that  $\bar{J}_z(t)$  must be right-continuous on  $[0, T)$ . The left-continuity of  $\bar{J}_z(t)$  is established by noting that

$$\bar{J}_z(t - \Delta t)\epsilon = \int_{t-\Delta t}^t u_z(z^*(s), s)\epsilon e^{-\beta(s-t+\Delta t)} ds + \bar{J}_z(t)\epsilon e^{-\beta\Delta t} \quad (14)$$

and letting  $\Delta t \rightarrow 0$ .

### 3- Dynamics of Marginal Values

By the hypothesis that  $r(t)$  is continuous in  $t$ , (12) implies, as  $\Delta t \rightarrow 0$ , that

$$\frac{d\bar{J}_W(t)}{dt} = -r(t)\bar{J}_W(t).$$

Next note that if  $z^*(t)$  is continuous on some time interval  $(t - \Delta t, t + \Delta t)$ , then

$$\frac{\int_t^{t+\Delta t} u_z(z^*(s), s)e^{-\beta(s-t)} ds}{\Delta t} \rightarrow u_z(z^*(t), t)$$

and

$$\frac{\int_{t-\Delta t}^t u_z(z^*(s), s)e^{-\beta(s-t+\Delta t)} ds}{\Delta t} \rightarrow u_z(z^*(t), t).$$

Hence, in a time interval over which  $C^*$  has no ‘‘gulps’’, (13) implies, as  $\Delta t \rightarrow 0$ , that

$$\frac{d\bar{J}_z(t)}{dt} = -u_z(z^*(t), t) + \beta\bar{J}_z(t).$$

Also note that at  $t = T$ , the marginal value of wealth,  $\bar{J}_W(T)$  is exactly equal to the derivative of  $V$  evaluated at the terminal wealth  $W^*(T)$ . In addition, any unit of the good consumed at  $t = T$ , with no change in wealth, has no effect on the agent’s total satisfaction and hence  $\bar{J}_z(T) = 0$ .

### 4- Appropriate Times for ‘‘Gulps’’

Applying Bellman’s optimality principle in (10) on an interval  $(\tau_i, \tau_{i+1})$  when our solution

<sup>5</sup>We use  $u_z(z, t)$  to denote  $\frac{\partial u(z, t)}{\partial z}$ .



prescribes consumption at rates, we get the following Bellman equation:

$$\max_{c(t)} \left\{ u(z^*(t), t) + \bar{J}_W[r(t)W^*(t) - c(t)] + \bar{J}_z\beta[c(t) - z^*(t)] + \bar{J}_t \right\} = 0 \quad \text{for all } t \in (\tau_i, \tau_{i+1}). \quad (15)$$

Note that the Bellman equation is *linear* in the consumption rate  $c(t)$ . Hence, the consumption rate that satisfies (15) is given by:

$$\begin{aligned} \bar{c}(t) &= 0 & \text{when } \bar{J}_W(t) - \beta\bar{J}_z(t) > 0, \\ \bar{c}(t) &\in [0, \infty] & \text{when } \bar{J}_W(t) - \beta\bar{J}_z(t) = 0, \\ \bar{c}(t) &= \infty & \text{when } \bar{J}_W(t) - \beta\bar{J}_z(t) < 0. \end{aligned}$$

Bellman equation, therefore, suggests that over the periods in which consumption is at rates, it cannot be the case that ( $\beta$  times) the marginal value of  $z$  is greater than the marginal value of  $W$ . It also suggests that a “gulp” of consumption might be prescribed at any moment  $\tau$  when  $\bar{J}_W(\tau) - \beta\bar{J}_z(\tau) \leq 0$ . Suppose that we prescribe a “gulp” at  $\tau$  when  $\bar{J}_W(\tau) - \beta\bar{J}_z(\tau)$  is strictly negative. Since the marginal values of  $W$  and  $z$  are continuous, the function  $\bar{J}_W(t) - \beta\bar{J}_z(t)$  will be strictly negative on an interval  $(\tau - \epsilon, \tau + \epsilon)$  for some  $\epsilon > 0$ . Hence our candidate policy should prescribe a jump for all points  $t \in (\tau - \epsilon, \tau + \epsilon)$ . Since this cannot happen as there can only be a countable number of jumps, our policy might prescribe a “gulp” only at a moment  $\tau$ , on this particular interval, such that  $\bar{J}_W(\tau) - \beta\bar{J}_z(\tau) = 0$ .

### 5- No “Gulps” after $t = 0$

Now suppose that our policy calls for a “gulp” of size  $\Delta$  at time  $\tau > 0$ . Note that by right continuity of  $\bar{C}$ , we cannot have two successive jumps at any one moment. This, together with the continuity of the marginal values of  $W$  and  $z$ , implies that there are two strictly positive numbers  $\epsilon_1, \epsilon_2$  such that:

$$\begin{aligned} \bar{J}_W(t) - \beta\bar{J}_z(t) &\geq 0 & \text{on } (\tau - \epsilon_1, \tau), \\ \bar{J}_W(t) - \beta\bar{J}_z(t) &= 0 & \text{when } t = \tau, \\ \bar{J}_W(t) - \beta\bar{J}_z(t) &\geq 0 & \text{on } (\tau, \tau + \epsilon_2). \end{aligned}$$

Assuming that  $\bar{J}_W$  and  $\bar{J}_z$  are differentiable functions of time, this implies that  $\bar{J}_W(t) - \beta\bar{J}_z(t)$  is decreasing just to the left of  $\tau$  and increasing just to the right of  $\tau$ . This condition, however, cannot hold for any size of jump  $\Delta$ . From the dynamics of  $\bar{J}_W$  and  $\bar{J}_z$ , we get that:

$$\frac{d}{dt}[\bar{J}_W(\tau^+) - \beta\bar{J}_z(\tau^+)] - \frac{d}{dt}[\bar{J}_W(\tau) - \beta\bar{J}_z(\tau)] = \beta[u_z(z^*(\tau) + \beta\Delta, \tau) - u_z(z^*(\tau), \tau)]. \quad (16)$$

But by strict concavity of the felicity function in  $z$ , this last quantity is strictly negative. Hence, if  $\frac{d}{dt}[\bar{J}_W - \beta\bar{J}_z]$  were negative just before the jump, it would be strictly negative just after the jump, thus violating the condition that  $\bar{J}_W - \beta\bar{J}_z$  be increasing just after a “gulp”. Note that our analysis is not valid for  $t = 0$  and  $t = T$ , thus  $C^*$  does not have any gulps, except possibly at  $t = 0$  and  $t = T$ . A gulp at  $t = T$ , however, is clearly sub-optimal since the agent cannot derive any consumption satisfaction from it and the final wealth is also reduced.

All the above considerations lead us to conclude that  $C^*$  should take the form of a possible “gulp” at  $t = 0$ , followed by periods of consumption mixed with periods of no consumption. If we should prescribe a jump  $\Delta C^*(0)$  at  $t = 0$ , then the size of the jump should maximize the value function immediately after the jump. That is, the jump size  $\Delta C^*(0)$  should solve the following problem

$$\max_{\epsilon} J(W(0) - \epsilon, z(0^-) + \beta\epsilon, 0^+).$$

From the first order condition, we then know that  $\bar{J}_W(0^+) = \beta\bar{J}_z(0^+)$ . Moreover, after the initial possible jump at  $t = 0$ , the following condition should be satisfied for all  $t \in (0, T]$ ; at times of no consumption:

$$u(z^*(t), t) + \bar{J}_W r(t)W^*(t) + \bar{J}_z \beta z^*(t) + \bar{J}_t = 0, \quad (17)$$

together with

$$\beta\bar{J}_z(t) - \bar{J}_W(t) < 0, \quad (18)$$

and at the times when consumption occurs at rates, we must have

$$[\bar{J}_W(t) - \beta\bar{J}_z(t)]c^*(t) = 0. \quad (19)$$

More generally, consumption only occurs when  $\bar{J}_W(t) = \beta\bar{J}_z(t)$ . During these times the strategy of no consumption is sub-optimal and hence

$$u(z^*(t), t) + \bar{J}_W r(t)W^*(t) + \bar{J}_z \beta z^*(t) + \bar{J}_t \leq 0. \quad (20)$$

The necessary conditions can be expressed more compactly by stating that the value function should at all all times satisfy:

$$\max\{u(z, t) + \bar{J}_W W r(t) - \bar{J}_z \beta z + \bar{J}_t, \beta\bar{J}_z - \bar{J}_W\} = 0. \quad (21)$$

In the following section, we provide conditions sufficient for a candidate solution to be optimal.

## 4 Sufficiency

The necessary conditions in (18) and (19) simply say that the agent should adopt a consumption policy of a possible initial “gulp” followed by periods of consumption mixed with periods of no consumption. If the agent starts with a consumption “gulp”, then immediately after  $t = 0$ , the marginal value of wealth  $W$  should be equal to ( $\beta$  times) the marginal value of past consumption  $z$ . Otherwise, it would benefit the agent to change the value of the initial “gulp”.

During the periods in which the agent consumes at rates, he does this in such a manner that the marginal value of wealth is always equal to ( $\beta$  times) the marginal value of average past consumption. In other words, the agent chooses his consumption so that the marginal value of any additional unit of the good at any time is equalized between the two competing uses available to him, namely investing to increase his wealth and consuming to increase his average past consumption. If this condition is not satisfied in a particular policy, then the agent will find it attractive to deviate from that policy by reallocating the consumption good in the activity with the higher marginal value.

When the agent is not consuming, it must be the case that the marginal utility of wealth is at least as large as ( $\beta$  times) the marginal utility of the average past consumption. In other words, during these periods the agent gains more in overall satisfaction by investing *all* his wealth in the riskless asset and by temporarily refraining from consumption. The increased wealth will provide more consumption and bequeath value in the future than immediate consumption. Of course, when the marginal values of the good in both uses is again equalized, the agent may start consuming again. Finally, the necessary conditions instruct the agent to follow a policy such that at any moment after  $t = 0$ , the marginal value of the good if consumed,  $\beta\bar{J}_z$ , is never strictly greater than the marginal value of the good if invested,  $\bar{J}_W$ .

Note that, by the necessary conditions, when consumption occurs, there is no telling whether it is at “rates” or whether it includes singular components. However, the reader will find out later that the closed form solution we construct in the following section for a particular felicity function shows that the optimal consumption policy contains no singular component.

We now provide sufficient conditions for a consumption policy to be optimal. We argue in two steps. First, we show that there is a *value function*  $J^*(W, z, t)$  which is an upper bound on the satisfaction over  $[t, T]$  that an agent can obtain starting at  $t$  with wealth  $W$  and an average past consumption  $z$ . Second, we give conditions under which a consumption policy achieves this upper bound and hence it is indeed the optimal policy.

Let the agent at any time  $t$  have wealth  $W(t)$  and average past consumption  $z(t^-)$ . Assume that the agent has the opportunity to exchange his wealth and average past consumption for a level of satisfaction  $J^*(W(t), z(t^-), t)$ . In other words, there is a “utility equivalent” function  $J^*(W, z, t)$  defined for all levels of wealth and average past consumption and for all times in  $[0, T]$  that provides the agent at time  $t$  with lump-sum satisfaction for the period  $[t, T]$ . Therefore the agent has the option of choosing any budget feasible consumption plan till any time  $t$ , and then exchanging his wealth and average past consumption at that time for a lump-sum payoff in units of utility. We will show that if  $J^*(W, z, t)$  satisfies certain conditions, then exchanging  $W(0)$  and  $z(0^-)$  at time zero for  $J^*(W(0), z(0^-), 0)$  is better for the agent than adopting any feasible consumption policy till any time  $t$  and then receiving a lump-sum utility at that time.

Consider the situation in which the agent decides to terminate his consumption at time  $t \in [0, T]$ , and exchange his state variables at  $t$ ,  $W(t)$  and  $z(t^-)$ , for  $J^*(W(t), z(t^-), t)$ . In such a case, the agent’s total satisfaction will be given by:

$$\int_0^t u(z(s), s) ds + J^*(W(t), z(t^-), t). \quad (22)$$

To ensure that the value in (22) is the correct value of utility for all possible plans, we impose the following boundary conditions on  $J^*$ . First, note that the agent might decide not to terminate a given consumption plan at all. At time  $T$ , he would receive  $V(W(T))$  and his total satisfaction will be given by (5). We require that at  $T$ ,  $J^*(W(T), z(T^-), T) = V(W(T))$ , and thus the value in (22) is the utility for a consumption plan followed completely till time  $T$ . Second, note that the agent might follow a consumption plan that exhausts all his wealth strictly before time  $T$ , and then “live on” the satisfaction derived from past consumption. To ensure that the value given by (22) is the corresponding utility from such a plan, we impose the boundary condition that  $J^*(0, z, t) = \int_t^T u(ze^{-\beta(s-t)}, s) ds + V(0)$ .

The following theorem shows that the agent prefers receiving  $J^*(W(0), z(0^-), 0)$  to the utility gained from any consumption plan terminated at any time  $t \in (0, T)$ .

**Theorem 1** *Assume that there exists a differentiable function  $J^*(W, z, t): \mathfrak{R}_+ \times \mathfrak{R}_+ \times [0, T] \rightarrow \mathfrak{R} \cup \{-\infty\}$ , concave in  $W$  and  $z$ , whose partial derivatives are continuous functions of time, that satisfies the following differential equation:*

$$\max \left\{ u(z, t) + J_W^* W r(t) - J_z^* \beta z + J_t^*, \beta J_z^* - J_W^* \right\} = 0 \quad \forall t \in [0, T], \quad (23)$$

together with the boundary conditions:

$$J^*(W, z, T) = V(W) \quad \text{and}$$

$$J^*(0, z, t) = \int_t^T u(ze^{-\beta(s-t)}, s) ds + V(0),$$

then

$$\int_0^t u(z(s), s) ds + J^*(W(t), z(t^-), t) \leq J^*(W(0), z(0^-), 0) \quad \forall C \in \mathcal{A}(W(0), 0),$$

where  $\{z(s); 0 \leq s \leq t^-\}$  and  $W(t)$  are the average past consumption and time  $t$  wealth, respectively, associated with  $C$ .

PROOF. Assume that the agent adopts a consumption plan  $C \in \mathcal{A}(W(0), 0)$  that terminates at  $t$ , and let the points of discontinuity of  $C$  on  $(0, t)$  be  $\tau_1, \tau_2, \dots$ . The wealth process and the average past consumption associated with  $C$  is  $\{W(s); 0 \leq s \leq t\}$  and  $\{z(s); 0 \leq s \leq t\}$ , respectively. Recall that  $W$  is left-continuous and  $z$  is right-continuous. For convenience of notation, we at times denote  $J^*(W(t), z(t^-), t)$ ,  $J_W^*(W(t), z(t^-), t)$ ,  $J_z^*(W(t), z(t^-), t)$ , and  $J_s^*(W(t), z(t^-), t)$  by  $J^*(t)$ ,  $J_W^*(t)$ ,  $J_z^*(t)$ , and  $J_s^*(t)$ , respectively. We have

$$\begin{aligned} & \int_0^t u(z(s), s) ds + J^*(W(t), z(t^-), t) \\ = & \int_0^t u(z(s), s) ds + J^*(W(0), z(0^-), 0) \\ & + \sum_{i=0} \int_{\tau_i^+}^{\tau_{i+1}^-} (J_W^*(s)rW(s)ds - J_W^*(s)dC(s) + J_z^*(s)\beta(dC(s) - z(s)ds) + J_s^*(s)ds) \\ & + \sum_{i=0} (J^*(\tau_i^+) - J^*(\tau_i)), \end{aligned} \tag{24}$$

where the equality follows from fundamental theorem of calculus and we have set  $\tau_0 = 0$ . Note that the boundary conditions for  $J^*$  ensure that the left-hand expression is the correct expression of utility even for consumption plans that do not terminate before  $T$ , or those plans for which the wealth reaches zero strictly before  $T$ .

By the hypothesis that  $J^*$  is concave in its first two arguments, we know that

$$\begin{aligned} & J^*(\tau_i^+) - J^*(\tau_i) \\ \leq & J_W^*(\tau_i)(W(\tau_i^+) - W(\tau_i)) + J_z^*(\tau_i)(z(\tau_i) - z(\tau_i^-)) \\ = & J_W^*(\tau_i)(C(\tau_i^-) - C(\tau_i)) + J_z^*(\tau_i)\beta(C(\tau_i) - C(\tau_i^-)). \end{aligned}$$

Substituting this into (24) gives

$$\begin{aligned} & \int_0^t u(z(s), s) ds + J^*(W(t), z(t^-), t) \\ \leq & J^*(W(0), z(0^-), 0) + \int_0^t [u(z(s), s) + J_W^*(s)W(s)r(s) - J_z^*(s)\beta z(s) + J_s^*(s)] ds \end{aligned}$$

$$\begin{aligned}
& + \int_{0^-}^t [\beta J_z^*(s) - J_W^*(s)] dC(s) \\
\leq & J^*(W(0), z(0^-), 0),
\end{aligned}$$

where the second inequality follows from the hypothesis that the integrands are negative and  $C$  is increasing. ■

Now, if we can show that there exists an actual policy  $(C^*, W^*(T)) \in \mathcal{A}(W(0), 0)$  such that

$$U(z(0^-), 0; (C^*, W^*(T))) = J^*(W(0), z(0^-), 0),$$

then  $(C^*, W^*(T))$  is the optimal policy.

In fact, we will prove more than that. In the following theorem, we provide a characterization of a budget feasible policy  $C^*$ , starting from any time  $t$  and any state  $W(t), z(t^-)$ , which guarantees that the associated value function is equal to the function  $J^*(W, z, t)$  defined in theorem 1.

**Theorem 2** *Suppose that there exists a differentiable function  $J^*(W, z, t): \mathfrak{R}_+ \times \mathfrak{R}_+ \times [0, T] \rightarrow \mathfrak{R} \cup \{-\infty\}$ , concave in  $W$  and  $z$ , whose partial derivatives are continuous functions of time, which solves the differential equation:*

$$\max \left\{ u(z, t) + J_W^* W r(t) - J_z^* \beta z + J_t^*, \beta J_z^* - J_W^* \right\} = 0 \quad \forall t \in [0, T], \quad (25)$$

together with the boundary conditions:

$$\begin{aligned}
J^*(W, z, T) &= V(W) \quad \text{and} \\
J^*(0, z, t) &= \int_t^T u(z e^{-\beta(s-t)}, s) ds.
\end{aligned}$$

Starting from any time  $\tau \in [0, T]$ , with  $W(\tau)$  and  $z(\tau^-)$ , assume that there exists a policy  $C^* \in \mathcal{A}(W(\tau), \tau)$ , which is continuous on  $(\tau, T]$ , with a possible jump  $\Delta C^*(\tau)$ , and whose associated state variables are  $W^*(t), z^*(t^-)$ , such that for all  $t \in (\tau, T]$ :

$$u(z^*, t) + J_W^* W^* r(t) - J_z^* \beta z^* + J_t^* = 0 \quad \text{and} \quad (26)$$

$$\int_t^{t+\epsilon} [J_W^*(W^*, z^*, t) - \beta J_z^*(W^*, z^*, t)] dC^*(t) = 0 \quad \text{for all } \epsilon > 0, \quad (27)$$

and

$$J^*(W(\tau), z(\tau^-), \tau) = J^*(W(\tau) - \Delta C^*(\tau), z(\tau^-) + \beta \Delta C^*(\tau), \tau), \quad (28)$$

then  $U(z(t^-), t; (C^*, W^*(T))) = J^*(W(t), z(t^-), t)$  for all  $(W(t), z(t^-), t) \in \mathfrak{R}_+ \times \mathfrak{R}_+ \times [0, T]$ . In particular,  $U(z(0^-), 0; (C^*, W^*(T))) = J^*(W(0), z(0^-), 0)$ . It then follows from theorem 1 that  $C^*$  attains the maximum of  $U(z(0^-), 0; (C, W(T)))$  over the set  $\mathcal{A}(W(0), 0)$ .

PROOF. Recall from the definition of  $U(z(t^-), t; (C^*, W^*(T)))$  that for  $\tau \leq t < s \leq T$ , we have:

$$U(z^*(t^-), t; (C^*, W^*(T))) = \int_t^s u(z^*(\xi), \xi) d\xi + U(z^*(s^-), s; (C^*, W^*(T))). \quad (29)$$

Now consider the following:

$$\begin{aligned} & \int_t^s u(z^*(\xi), \xi) d\xi + J^*(W^*(s), z^*(s^-), s) \\ = & \int_t^s u(z^*(\xi), \xi) d\xi + J^*(W^*(t^+), z^*(t), t) + \int_t^s (J_W^* dW^* + J_z^* dz^* + J_\xi^* d\xi) \\ = & J^*(W^*(t^+), z^*(t), t) + \int_t^s (u(z^*(\xi), \xi) + J_W^* W^*(\xi) r(\xi) - J_z^* \beta z^*(\xi) + J_\xi^*) d\xi \\ & + \int_t^s (\beta J_z^*(W^*, z^*, \xi) - J_W^*(W^*, z^*, \xi)) dC^*(\xi) \\ = & J^*(W^*(t^+), z^*(t), t) \\ = & J^*(W^*(t), z^*(t^-), t), \end{aligned} \quad (30)$$

where the first equality follows from the fundamental theorem of calculus. The second equality follows from the dynamics of  $W^*$  and  $z^*$  and the assumption that  $C^*$  is continuous on  $(\tau, T]$ , and the third equality follows from equations (26) and (27). The last equality follows from the continuity of  $C^*$  for all  $t \in (\tau, T]$ , and from condition (28), when  $t = \tau$ .

From equations (29) and (30), we conclude that for all  $\tau \leq t < s \leq T$ ,

$$\begin{aligned} & U(z^*(t^-), t; (C^*, W^*(T))) - U(z^*(s^-), s; (C^*, W^*(T))) \\ = & J^*(W^*(t), z^*(t^-), t) - J^*(W^*(s), z^*(s^-), s) \quad \text{or} \end{aligned}$$

$$U(z^*(t^-), t; (C^*, W^*(T))) - J^*(W^*(t), z^*(t^-), t) = m \quad \text{for all } t \in [\tau, T],$$

where  $m$  is a constant independent of  $t$ . Now, let  $\hat{T} = \min\{t \in [0, T]: W^*(t) = 0\}$ , where we have used the convention that if the minimum does not exist, we set the minimum to be  $T$ . Note from the boundary condition on  $J^*$  that:  $J^*(W^*(\hat{T}), z^*(\hat{T}^-), \hat{T}) = U(z^*(\hat{T}^-), \hat{T}; (C^*, W^*(T)))$ . Thus we conclude that  $m = 0$ , and the proof is now complete. ■

## 5 Optimal Consumption Policy

We provide a closed form solution for the optimal consumption problem formulated in section 2 with infinite horizon in a world of constant interest rate  $r$ . The felicity function is  $u(z, t) =$

$\frac{1}{\alpha}e^{-\delta t}z^\alpha$ , where  $\alpha < 1$ , and where the discount factor  $\delta \geq 0$  captures the agent's impatience. In other words, the agent seeks to maximize

$$U(z(0^-), 0, C) \equiv \int_0^\infty \frac{1}{\alpha} e^{-\delta t} z(t)^\alpha dt, \quad (31)$$

given  $W(0)$  and  $z(0^-)$  and where  $z(t)$  is given by equation (3). The sufficiency theorem for this problem, which is a modified version of theorem 2, is given in the following corollary, whose proof is omitted.

**Corollary 1** *Suppose that there exists a concave, continuously differentiable function  $J^*(W, z): \mathfrak{R}_+ \times \mathfrak{R}_+ \rightarrow \mathfrak{R} \cup \{-\infty\}$ , which solves the differential equation:*

$$\max \left\{ \frac{z^\alpha}{\alpha} + J_W^* W r - J_z^* \beta z - \delta J^*, \beta J_z^* - J_W^* \right\} = 0, \quad (32)$$

together with the boundary conditions:

$$J^*(0, z) = \int_t^\infty e^{-\delta s} \frac{1}{\alpha} (ze^{-\beta(s-t)})^\alpha ds \quad \text{and} \quad (33)$$

$$\lim_{T \uparrow \infty} e^{-\delta T} J^*(W(T), z(T)) = 0, \quad (34)$$

for all feasible policies. Starting from any time  $\tau \in [0, \infty)$ , with  $W(\tau)$  and  $z(\tau^-)$ , assume that there exists a policy  $C^* \in \mathcal{A}(W(\tau), \tau)$ , which is continuous, with a possible jump  $\Delta C^*(\tau)$ , and whose associated state variables are  $W^*(t), z^*(t^-)$ , such that for all  $t \in (\tau, \infty)$ :

$$\frac{z^{*\alpha}}{\alpha} + J_W^* W^* r - J_z^* \beta z^* - \delta J^* = 0 \quad \text{and} \quad (35)$$

$$\int_t^{t+\epsilon} [J_W^*(W^*, z^*, t) - \beta J_z^*(W^*, z^*, t)] dC^*(t) = 0 \quad \text{for all } \epsilon > 0, \quad (36)$$

and

$$J^*(W(\tau), z(\tau^-)) = J^*(W(\tau) - \Delta C^*(\tau), z(\tau^-) + \beta \Delta C^*(\tau)), \quad (37)$$

then  $U(z(t^-), t; C^*) = J^*(W(t), z(t^-))$  for all  $(W(t), z(t^-)) \in \mathfrak{R}_+ \times \mathfrak{R}_+$ .

In particular,  $U(z(0^-), 0; C^*) = J^*(W(0), z(0^-))$ . It then follows from theorem 1 that  $C^*$  attains the maximum of  $U(z(0^-), 0; C)$  over the set  $\mathcal{A}(W(0), 0)$ .

We will show that the key feature of the solution is a critical ratio  $k^* > 0$  of wealth  $W$  to average past consumption  $z$ . If the agent starts at time zero with  $\frac{W(0)}{z(0^-)}$  strictly less than  $k^*$ , then the optimal behavior is to invest all the wealth in the riskless asset and wait while  $W$  increases and  $z$  declines till the ratio  $\frac{W}{z}$  reaches  $k^*$ . From then on, the agent consumes at the



rate which keeps the ratio  $\frac{W}{z}$  equal to  $k^*$  forever. If, on the other hand, the agent starts with  $\frac{W(0)}{z(0^-)}$  strictly greater than  $k^*$ , then the optimal behavior is to take a “gulp” of consumption, reducing  $W$  and increasing  $z$ , to bring  $\frac{W}{z}$  immediately to  $k^*$ . Following this gulp, the optimal consumption occurs at the rate that keeps the ratio  $\frac{W}{z}$  equal to  $k^*$  forever. The critical ratio  $k^*$  depends on the interest rate  $r$ , the impatience of the agent captured by  $\delta$ , the concavity of the felicity function captured by  $\alpha$ , and the rate of decay of past consumption captured by  $\beta$ .

Our construction of the critical ratio  $k^*$  and hence the optimal solution is in two steps. First, we examine candidate solutions of the  $k$ -ratio form. For any  $k > 0$ , the  $k$ -ratio policy is the policy of keeping  $\frac{W}{z}$  equal to  $k$  forever, after an initial “gulp” if  $\frac{W(0)}{z(0^-)} > k$ , or after a period of no consumption if  $\frac{W(0)}{z(0^-)} < k$ . We will show that the value function associated with any  $k$ -ratio policy, denoted  $J^k(W, z)$ , satisfies:

$$\begin{aligned} \frac{z^\alpha}{\alpha} + rW J_W^k - \beta z J_z^k - \delta J^k &= 0 \quad \text{if } \frac{W}{z} \leq k \quad \text{and} \\ J_W^k - \beta J_z^k &= 0 \quad \text{if } \frac{W}{z} \geq k. \end{aligned}$$

Second, we show that there is a unique value  $k^* > 0$  such that the associated value function,  $J^*(W, z)$ , satisfies the differential *inequality*

$$\max \left\{ \frac{z^\alpha}{\alpha} + J_W^* rW - J_z^* \beta z - \delta J^*, \beta J_z^* - J_W^* \right\} = 0, \quad (38)$$

together with the boundary condition that

$$J(0, z) = \frac{1}{\alpha} \int_0^\infty e^{-\delta t} (ze^{-\beta t})^\alpha dt = \frac{1}{\alpha(\alpha\beta + \delta)} z^\alpha. \quad (39)$$

It then follows from theorem 1, that the *ratio* policy associated with  $k^*$  is the optimal solution for the agent’s problem. We record our solution in the following statements.

**Assumption 1** *The parameters of the problem satisfy*

$$\delta > \alpha r \quad \text{and} \quad (1 - \alpha)\beta > \delta - r.$$

This assumption ensures the existence of a solution to the infinite horizon program. It rules out the case when the agent is so patient that he prefers to wait and accumulate wealth for the longest possible time. This behavior leads to infinite utility. Now fix a  $k$ -ratio policy.

**Lemma 1** Suppose that  $\frac{W(0)}{z(0^-)} = k$ . The consumption rate required to keep  $\frac{W(t)}{z(t)} = k$  for all  $t \geq 0$  is

$$c(t) = \frac{(\tau + \beta)}{(1 + \beta k)} W(t),$$

and the corresponding utility is

$$\frac{1}{\alpha \left[ \delta - \alpha \beta \left( \frac{\tau k - 1}{\beta k + 1} \right) \right]} z(0^-)^\alpha.$$

**PROOF.** Using the dynamics of  $W$  and  $z$ , and equating  $\frac{d}{dt} \left( \frac{W}{z} \right)$  to zero, we can compute  $c$ . Another simple computation produces the associated utility. Note that by assumption 1, the total utility is strictly positive. ■

**Lemma 2** Suppose that  $\frac{W(0)}{z(0^-)} < k$ . Consider the policy of no consumption till  $t^*$  when  $\frac{W(t^*)}{z(t^*)} = k$ , and then consuming to keep the ratio  $\frac{W(t)}{z(t)} = k$ , for  $t > t^*$ . The total utility from this policy is

$$\frac{z(0^-)^\alpha}{\alpha(\delta + \alpha\beta)} + \frac{z(0^-)^\alpha}{\alpha} \left[ \frac{W(0)}{kz(0^-)} \right]^{\frac{\delta + \alpha\beta}{\tau + \beta}} \left\{ \frac{1}{\delta - \alpha\beta \left( \frac{\tau k - 1}{\beta k + 1} \right)} - \frac{1}{\delta + \alpha\beta} \right\}. \quad (40)$$

**PROOF.** The reader can easily verify that  $t^* = \log \left[ \frac{kz(0^-)}{W(0)} \right]^{\frac{1}{\tau + \beta}}$ . A direct computation establishes the lemma. ■

**Proposition 1** The value function  $J^k(W, z)$  associated with any  $k$ -ratio policy is given by:

$$J^k(W, z) = \begin{cases} \frac{z^\alpha}{\alpha(\delta + \alpha\beta)} + \frac{z^\alpha}{\alpha} \left[ \frac{W}{kz} \right]^{\frac{\delta + \alpha\beta}{\tau + \beta}} \mathcal{A} & \text{if } \frac{W}{z} \leq k \\ \frac{1}{\alpha} \left( \frac{z + \beta W}{1 + \beta k} \right)^\alpha \mathcal{B} & \text{if } \frac{W}{z} \geq k \end{cases}$$

where

$$\mathcal{A} = \frac{1}{\delta - \alpha\beta \left( \frac{\tau k - 1}{\beta k + 1} \right)} - \frac{1}{\delta + \alpha\beta} \quad \text{and}$$

$$\mathcal{B} = \frac{1}{\delta - \alpha\beta \left( \frac{\tau k - 1}{\beta k + 1} \right)}.$$

Furthermore,  $J^k$  is continuous, concave, has continuous first derivatives and satisfies

$$\begin{aligned} \frac{z^\alpha}{\alpha} + \tau W J_W^k - \beta z J_z^k - \delta J^k &= 0 \quad \text{if } \frac{W}{z} \leq k, \\ J_W^k - \beta J_z^k &= 0 \quad \text{if } \frac{W}{z} \geq k, \end{aligned} \quad (41)$$

together with the boundary condition

$$J(0, z) = \frac{1}{\alpha(\alpha\beta + \delta)} z^\alpha.$$

PROOF. Suppose that  $\frac{W}{z} > k$ . The size of the initial “gulp” required to bring the ratio immediately to  $k$  is

$$\Delta = \frac{W - kz}{1 + \beta k},$$

and we define  $J^k(W, z) \equiv J^k(W - \Delta, z + \beta\Delta)$ . Concavity of  $J^k$  in both  $W$  and  $z$  follows from assumption 1, and the rest of the proposition can be easily verified by direct computations. ■

This proposition shows that *any* ratio policy has a value function which satisfies the differential equations (41) in the relevant parts of the domain. The value function of the optimal solution, however, satisfies the differential *inequality* (38) over the whole domain. There is exactly one value  $k^*$  that produces the optimal policy.

**Proposition 2** *Let*

$$k^* = \frac{\frac{r-\delta}{\beta} + (1 - \alpha)}{\delta - \alpha r},$$

and note that  $k^* > 0$  by assumption 1. The value function  $J^*$  associated with the  $k^*$ -ratio policy is continuous, concave, has continuous first derivatives and satisfies the differential inequality:

$$\max \left\{ \frac{z^\alpha}{\alpha} + J_W^* rW - J_z^* \beta z - \delta J^*, \beta J_z^* - J_W^* \right\} = 0,$$

together with the boundary condition that

$$J(0, z) = \frac{1}{\alpha(\alpha\beta + \delta)} z^\alpha.$$

It then follows from theorem 1, that the  $k^*$ -ratio policy is the optimal solution for the agent's problem.

**Remark 1**  $k^*$  is the unique ratio with the property that the associated value function  $J^*$  is twice continuously differentiable over the whole domain  $(W, z)$ .

PROOF. In light of proposition 1, we only need to prove that

$$\begin{aligned} \frac{z^\alpha}{\alpha} + J_W^* rW - J_z^* \beta z - \delta J^* &\leq 0 & \text{if } \frac{W}{z} \geq k^* & \text{ and} \\ \beta J_z^* - J_W^* &\leq 0 & \text{if } \frac{W}{z} \leq k^*. \end{aligned}$$

To prove the first inequality, consider any point  $\underline{x} \equiv (W, z)$  such that  $W \geq k^*z$ . Referring to Figure 1, let  $\underline{a}$  be the point on the intersection of the line  $W = k^*z$  and the straight line passing through the point  $\underline{x}$  with slope  $\frac{dW}{dz} = -\frac{1}{\beta}$ . In other words,  $\underline{a}$  is the point on the boundary  $W = k^*z$ , to which one would jump if one starts at  $\underline{x}$ . By construction,  $J^*(\underline{x}) = J^*(\underline{a})$ .

Let

$$f(W, z) = \frac{z^\alpha}{\alpha} + J_W^*rW - J_z^*\beta z,$$

and note that  $f(\underline{a}) - \delta J^*(\underline{a}) = 0$ . Applying the fundamental theorem of calculus along the straight line connecting  $\underline{a}$  and  $\underline{x}$ , we obtain

$$f(\underline{x}) - f(\underline{a}) = \int_{\underline{a}}^{\underline{x}} (f_W dW + f_z dz).$$

Noting that along the line connecting  $\underline{a}$  and  $\underline{x}$ , we have  $dz = -\beta dW$ , and that  $dW > 0$  in the direction from  $\underline{a}$  to  $\underline{x}$ , it then follows that  $f_W - \beta f_z \leq 0$  is sufficient to conclude that

$$\frac{z^\alpha}{\alpha} + J_W^*rW - J_z^*\beta z - \delta J^* \leq 0 \quad \text{if} \quad \frac{W}{z} \geq k^*.$$

Computing  $f_W - \beta f_z$  in the region  $W \geq k^*z$ , the reader can easily verify that  $f_W - \beta f_z \leq 0$  for  $k^* = \frac{\frac{r-\delta}{\beta} + (1-\alpha)}{\delta - \alpha r}$ .

To prove the second inequality, consider the function  $J_W^* - \beta J_z^*$  in the region where  $W \leq k^*z$ . We can write

$$\begin{aligned} J_W^* - \beta J_z^* &= z^{\alpha-1} g\left(\frac{k^*z}{W}\right) \quad \text{where} \\ g(y) &= \mathcal{A} \left[ \frac{y}{k^*} \left( \frac{\delta + \alpha\beta}{r + \beta} \right) + \frac{\beta(\delta - \alpha r)}{r + \beta} \right] y^{-\frac{\delta + \alpha\beta}{r + \beta}} - \left[ \frac{\beta}{\alpha\beta + \delta} \right], \end{aligned}$$

where  $\mathcal{A}$  is given in proposition 1. Note that  $g(1) = 0$ , and that  $1 - \frac{\delta + \alpha\beta}{r + \beta} > 0$ , by assumption 1, hence  $g(y) \uparrow \infty$  as  $y \uparrow \infty$ .

Computing the derivative of  $g$ , we get

$$\frac{dg}{dy} = \frac{\mathcal{A}(\delta + \alpha\beta)}{r + \beta} y^{-\frac{\delta + \alpha\beta}{r + \beta}} \left[ \left( \frac{1}{k^*} - \frac{\delta + \alpha\beta}{k^*(r + \beta)} \right) - \frac{\beta(\delta - \alpha r)}{y(\beta + r)} \right].$$

Substituting  $k^*$  in the expression for  $\frac{dg}{dy}$ , we conclude that  $\frac{dg}{dy} = 0$  for  $y = 1$ , and that  $\frac{dg}{dy} > 0$  for  $y > 1$ . It then follows that  $J_W^* - \beta J_z^* \geq 0$ , for all points  $(W, z)$  such that  $W \leq k^*z$ . ■

We have thus shown that the nature of the optimal consumption policy is exerting control over the ratio of wealth to average past consumption to keep it at or below a critical level. Any policy of keeping the ratio at or below a fixed constant satisfies many of the sufficient conditions

for optimality. However, the differential inequality (38), which is the crucial condition, is satisfied by only one critical ratio  $k^*$ . We summarize the solution in the following proposition.

**Proposition 3** *The optimal solution is to consume at rate  $c^*(t) = \frac{\delta - \alpha r}{1 - \alpha} W^*(t)$ , after an initial gulp of size  $\Delta = \frac{1}{(1 - \alpha)(r + \beta)} \left[ (\delta - \alpha r)W(0) + \left( \frac{r - \delta}{\beta} + (1 - \alpha) \right) z(0^-) \right]$ , if  $\frac{W(0)}{z(0^-)} < k^*$ , or after a period of no consumption of length  $t^* = \log \left[ \frac{k^* z(0^-)}{W(0)} \right]^{\frac{1}{r + \beta}}$ , if  $\frac{W(0)}{z(0^-)} > k^*$ .*

It is interesting to compare this consumption policy to that adopted by a consumer who starts with the same initial wealth  $W(0)$  and the same past consumption experience  $z(0^-)$  who seeks to maximize:

$$\int_0^\infty e^{-\delta t} c(t)^\alpha dt,$$

where we assume that  $\delta > \alpha r$  and  $1 - \alpha > 0$ . It can be easily verified that his optimal consumption rate is  $c_a^*(t) = \frac{\delta - \alpha r}{1 - \alpha} W_a^*(t)$ .

The propensity to consume is the same for agents with time-additive and non-time-additive preferences. However, the quantity of consumption and the trajectory of wealth are different. Consider the case when  $\delta = r$ . An agent with time-additive preferences will consume  $c_a^* = rW_a^*(t)$ , and hence his wealth at any time will be a constant equal to his initial wealth, see figure 2. A consumer with non-time additive preferences, however, will either start by consuming a “gulp” of size  $\frac{rW(0) + z(0^-)}{r + \beta}$ , if his wealth is too high relative to  $z(0^-)$ , and then keep the level of his wealth constant forever, or he will wait for  $t^* = \frac{1}{r + \beta} \left[ \log \frac{z(0^-)}{rW(0)} \right]$ , if his wealth is too low relative to  $z(0^-)$ , and then start consuming the interest on his wealth forever.

The parameter  $\beta$ , which controls the rate of “decay” of previous consumption, determines the size of the initial “gulp” or the length of the initial waiting period, and hence the level of the subsequent constant wealth. The higher the value of  $\beta$ , the smaller the size of the initial gulp, if the solution calls for a “gulp”, and the shorter the initial waiting period, in case the solution entails a waiting period. This is an intuitive result since a higher  $\beta$  implies that past consumption experience has “smaller” effect on current, and hence total, satisfaction. Therefore, when the optimal solution entails an initial waiting period to accumulate wealth, the effect of  $z(0^-)$  decays faster with higher  $\beta$  leading to shorter waiting period before the ratio  $\frac{W}{z}$  reaches its critical value. Similarly, when the optimal solution calls for an initial “gulp”, higher  $\beta$  means that the effect of the “gulp” will decay faster and hence the “gulp” contributes less to future satisfaction. Therefore, agents with higher  $\beta$  choose smaller initial “gulps”.

## 6 Concluding Remarks

In this paper, we have provided sufficient conditions for a consumption policy to be optimal for a class of time-nonseparable preferences that treat consumptions at nearly adjacent dates to be almost perfect substitutes. We demonstrated our general theory by explicitly solving in closed form the optimal consumption policy for a particular felicity function.

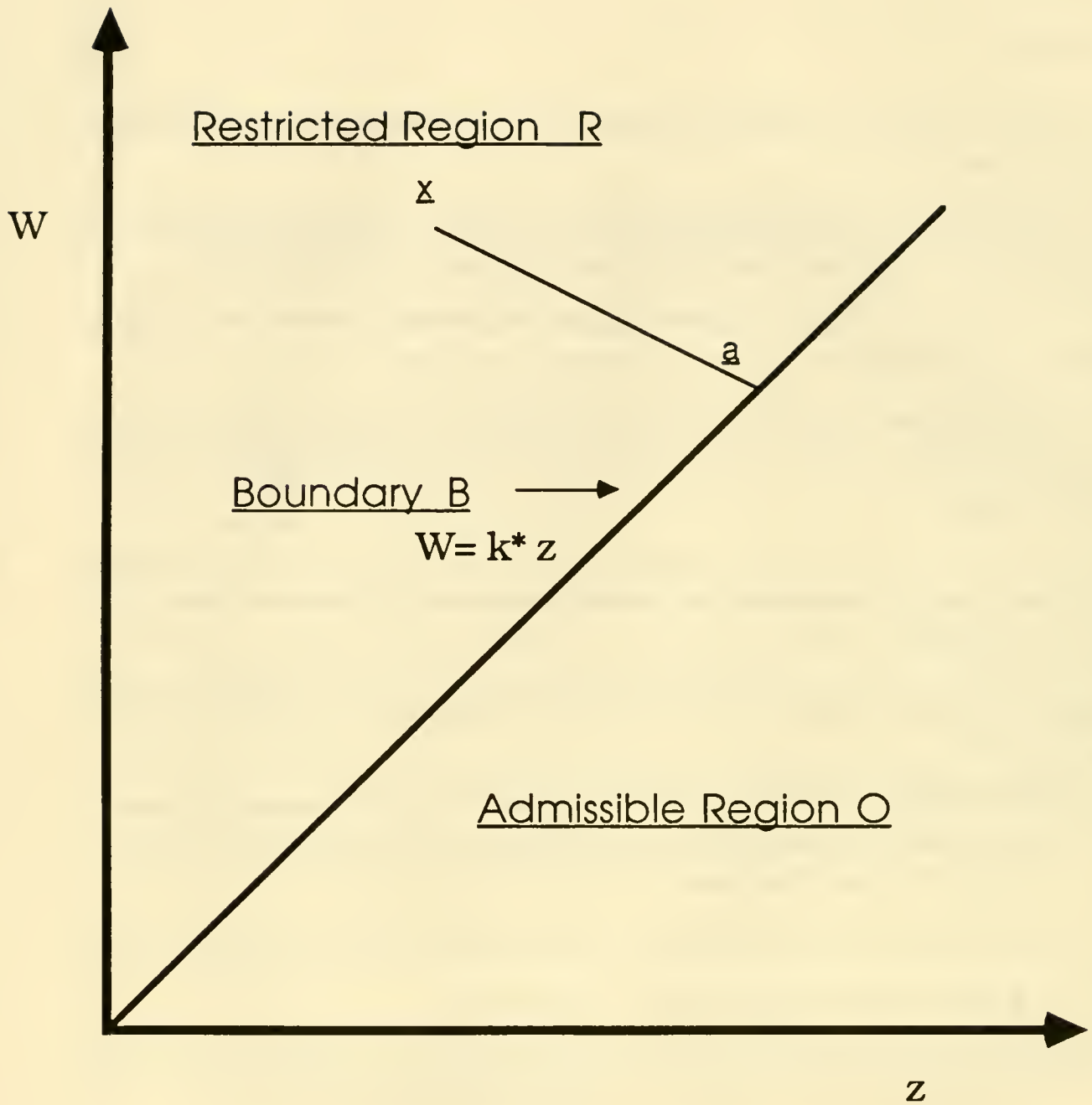
The heuristically derived necessary conditions for optimality in Section 3 can be justified rigorously. We refer the reader to Blaquière (1985) for details. We also remark that our conclusion that the optimal consumption policy does not involve any “gulps” after  $t = 0$  depends crucially on the assumption of continuous interest rates. If the interest rate is not a continuous function of time, a different consumption policy with intermediate gulps might be optimal.

Finally, we remark that the closed form solution we report in section 5 can be developed differently. Observing that the objective functional is homogeneous of degree  $\alpha$  in  $z$ , and that the dynamics of  $W$  and  $z$  are linear, we can write  $J^*(W, z) = z^\alpha h^*(\frac{W}{z})$ , for some function  $h^*: \mathfrak{R}_+ \rightarrow \mathfrak{R}$ . Substituting this form of  $J^*$  into inequality (32), we obtain an ordinary differential inequality that can be solved and the critical ratio  $k^*$  can be determined by imposing the condition that the solution of the differential inequality,  $h^*$ , is a twice continuously differentiable function. We elected to present our step-by-step construction of the candidate solutions and the optimal policy to illustrate more clearly the nature of the control problem.

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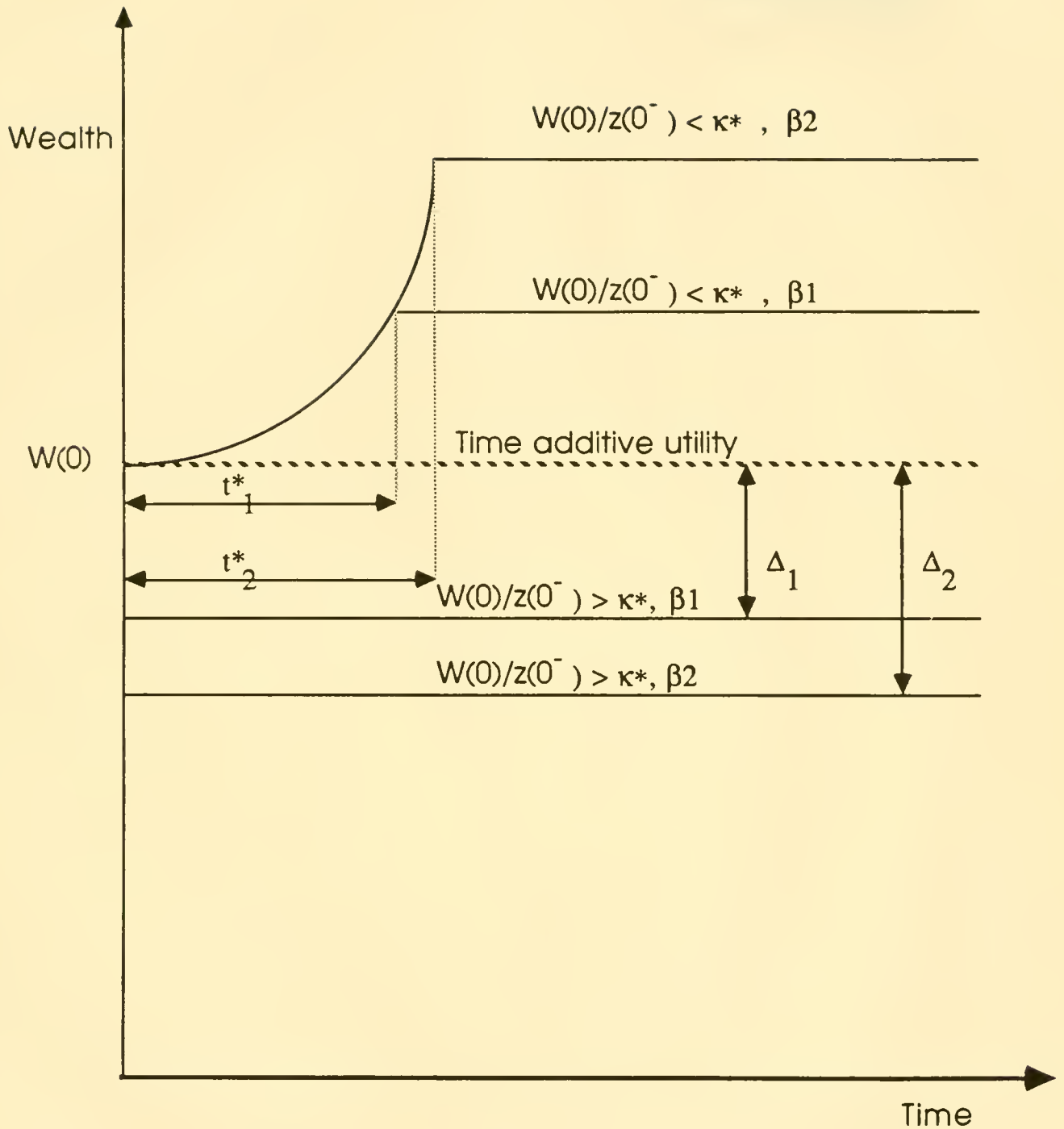
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The State Space showing the Boundary B.  
For all times after  $t=0$ , the optimal policy  
restricts the state to the admissible region O.

Figure 1





Trajectory of wealth along the optimal path for time-additive and non-time-additive preferences assuming

$$\delta = r , \beta_1 > \beta_2$$

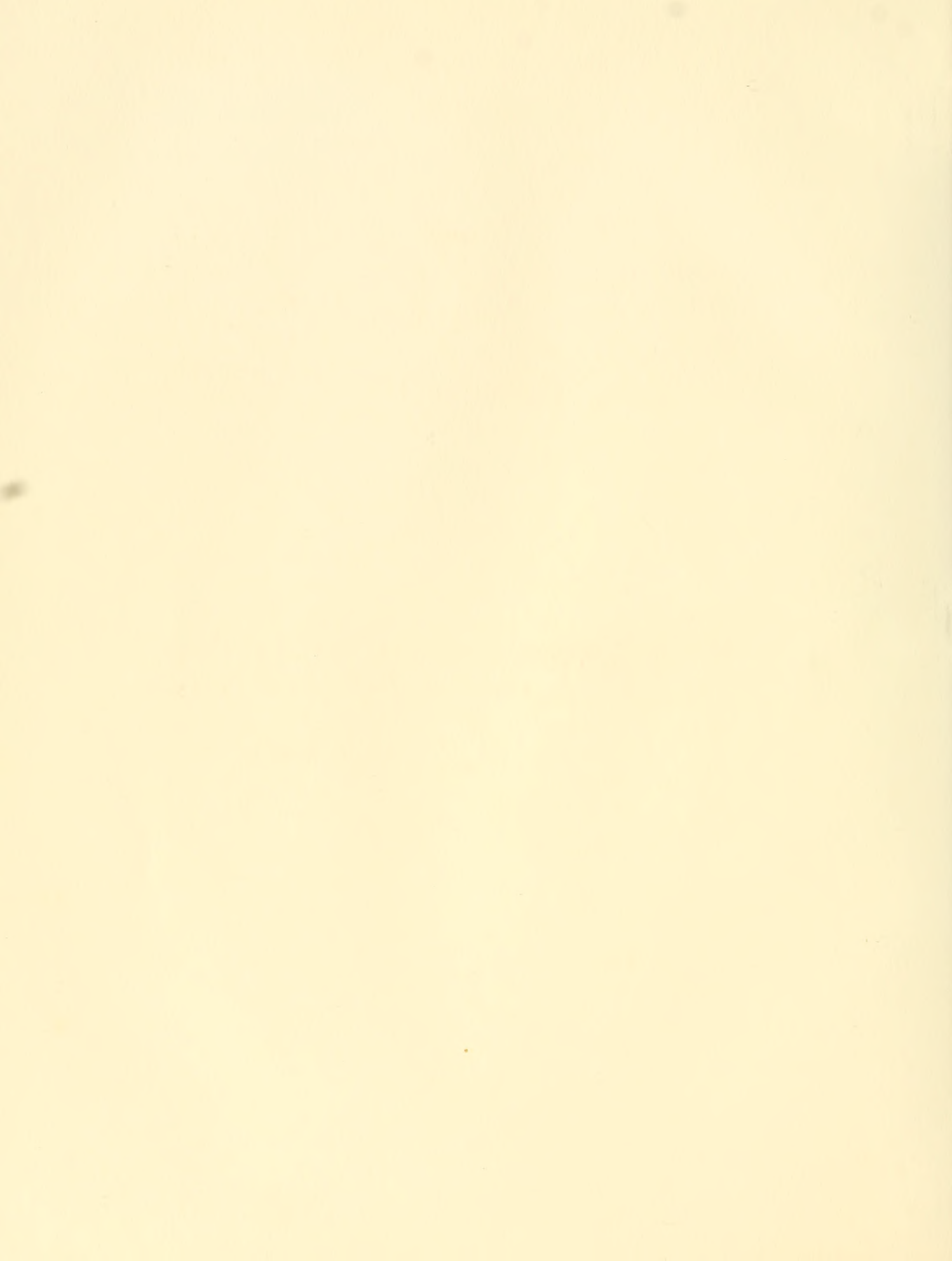
Figure 2

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