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Abstract

This paper analyzes the accumulated hedging errors generated by discretely rebalanced option hedges. We show that simple-minded generalizations of the prior research can underestimate the variance of the accumulated hedging errors significantly and that even with daily rebalancing, these accumulated hedging errors can introduce substantial risk in arbitrage strategies suggested by the Black-Scholes model. We show that the correlation between accumulated hedging errors for different options can be quite high so that the risk of arbitrage due to hedging errors can be substantially reduced by optimally combining options into portfolios. The results also suggest that tests of market pricing of traded options which are based on employing a portfolio approach are likely to be much better specified than the standard tests that focus on individual options.

The premise of the Black-Scholes option pricing model is that an option's payoff can be perfectly replicated by combining trading in the option's underlying security with borrowing or lending, and continually rebalancing this position. However, non-continuous trading in the underlying securities and transactions costs make the continuously rebalanced hedges that are assumed in the Black-Scholes model not possible. In reality, option hedges can, therefore, only be rebalanced discretely. This makes them no longer riskless and Figlewski (1989), through simulations, documents that discrete rebalancing exposes standard option arbitrage to large risks.

This paper deals with the risks due to hedging errors generated by discretely rebalanced option hedges over multiple rebalancing intervals. Boyle and Emanuel (1980) and Leland (1985) derive expressions for the variance of the hedging error over one instantaneous rebalancing interval for an option's delta hedge. However, most delta hedges are run for more than one rebalancing period. Hedging errors accumulated over the entire holding period are thus of importance. For example, an investment bank which has issued equity warrants, may have to delta hedge them over their entire life. Or, an arbitrageur, contemplating taking a position in an option and putting on a hedge, should be concerned with the future accumulated hedging errors up to maturity since the mispricing may not disappear till then. Moreover, to predict the variance of hedging error over an interval, one needs to assume that at the end of the rebalancing period, the option is priced in accordance with the Black-Scholes model. This is often not the case for any instant before the maturity date. To be able to deal with these situations, we derive expressions for the moments of the distribution of the accumulated hedging errors and show that in many cases, especially for out-of-the money options, simple generalizations of the Leland and Boyle and Emanuel single-period results can significantly underestimate the variance of the errors for the hedge portfolio.

The major contribution of this paper is to show how this hedging error induced risk over multiple rebalancing intervals can substantially be reduced by suitably combining options into portfolios.[1] When options with several different exercise prices are traded on the same underlying asset, intuition suggests that the hedging errors across these individual options would be correlated. We derive expressions governing the moments of the joint distribution of the accumulated hedging errors across different options and indeed find evidence of substantial correlation in these accumulated hedging errors. This suggests that option arbitrage can be made less risky if mispriced options are appropriately combined with other options. Consider, for example, an arbitrage trade employing an at-the-money index option with one month to maturity. For this hedge with daily rebalancing, the standard deviation of the accumulated hedging errors up to maturity is approximately 13% of the option's price. For a trader, this is a considerable amount. This standard deviation can be reduced by about 70% if the option is combined with another near-the-money option. If more options are used in the portfolio, the risk can be further reduced.

Our analysis also suggests that portfolio based tests of the efficiency of the options markets are more powerful than the traditional tests that examine whether individual options are mispriced. The portfolio methodology tests for the joint mispricing of options by taking advantage of the variance-covariance matrix of the accumulated hedging errors of individual options. Tests based on single options ignore the joint distribution of the hedging errors over the holding period and therefore may leave out from the test sample options that can reduce the risk of a strategy without reducing the abnormal returns. These tests also ignore the information about the correlation between the accumulated hedging errors, while the portfolio approach uses this information.

The paper is organized as follows. In section I we derive analytical expressions for the variance of the accumulated hedging errors for a discretely rebalanced delta hedged option. There we also discuss why simple generalizations of the single-period hedging errors in Boyle and Emanuel (1980) and Leland (1985) can often underestimate the dispersion of hedging errors over longer holding periods. In section II we derive the joint distribution for the accumulated hedging errors for a portfolio of delta hedged options. In section III we show that the effectiveness of hedging strategies used to exploit the mispricing of single options can be considerably improved by employing a portfolio approach. In Section IV we generalize the results of section II by deriving the joint distribution of the accumulated hedging errors between options written on different underlying assets. The implications of our results for option market efficiency tests are discussed in Section V. Finally, in section VI we present the conclusions.

I. The Variance of The Accumulated Hedging Errors for a Delta Hedged Option

To compute the variance of the accumulated hedging errors, we focus on the hedge portfolio consisting of positions in a European option and its underlying asset. Assume that trading takes place only at discrete intervals and that over a small interval, Δt, the underlying security value follows a process of the type:

$$\frac{\Delta S}{S} = \sigma U \Delta t^{1/2} + (\mu + \frac{1}{2}\sigma^2 U^2)\Delta t + (\mu\sigma U + \frac{1}{6}\sigma^3 U^3)\Delta t^{3/2} + O(\Delta t^2) \tag{1}$$

where ΔS is the change in the non-dividend paying asset's stock price over the rebalancing interval Δt and U is a normally distributed random variable with zero mean and variance equal to one. There exists a riskless asset paying a continuous rate of return r. Consider a zero-investment portfolio which combines with borrowing or lending a long position in an European call option hedged with m units of the underlying asset ($m = -C_S$). The value of this portfolio when the option has τ time to maturity is

$$H = mS + C(S,\tau) - B$$

where B equals C + mS and is the amount invested in the riskless asset. Over a discrete rebalancing interval, the change in this portfolio is

$$\Delta H = m\Delta S + \Delta C - Br\Delta t + O(\Delta t^2).$$

Following Boyle and Emanuel (1980), the hedging error over the interval Δt can be rewritten as

$$\Delta H = -C_\tau \Delta t + (1/2)C_{SS}\Delta S^2 + (C - C_S S)r\Delta t + C_{St}\Delta S\Delta t + (1/6)C_{SSS}\Delta S^3 + O(\Delta t^2). \quad (2)$$

Knowing that the option value satisfies the differential equation

$$0 = -C_\tau + (1/2)C_{SS}S^2\sigma^2 + (C-C_S S)r,$$

expression (2) can be rewritten as

$$\Delta H = -(1/2)C_{SS}S^2\sigma^2\Delta t + (1/2)C_{SS}\Delta S^2 + C_{St}\Delta S\Delta t + (1/6)C_{SSS}\Delta S^3 + O(\Delta t^2). \quad (3)$$

Substituting in (3) for the values of ΔS, ΔS^2, and ΔS^3 gives, after leaving out the terms of order higher than $\Delta t^{3/2}$,

$$\Delta H = (1/2)C_{SS}S^2\sigma^2(U^2-1)\Delta t + C_{SS}S^2\sigma U[\mu+(1/2)\sigma^2U^2]\Delta t^{3/2} + C_{St}S\sigma U\Delta t^{3/2} + (1/6)C_{SSS}S^3\sigma^3U^3\Delta t^{3/2} + O(\Delta t^2). \quad (4)$$

where U^2 is chi-squared distributed with one degree of freedom. Equation (4) shows that discrete rebalancing of the Black-Scholes hedge portfolio is risky even when $m = -C_s$ at each rebalancing since the stochastic terms in (4) cannot be hedged by taking a position in the underlying asset. At each rebalancing instant the hedging error requires additional borrowing or lending and the position is no longer zero-investment. The net wealth of a discretely rebalanced Black-Scholes hedge portfolio with zero-investment at its inception is therefore equal to the present value of the accumulated future hedging errors.

We now define

$$\lambda_t = (1/2)C_{SS}S_{t-1}^2\sigma^2$$

and

$$\theta_t = C_{SS}S_{t-1}^2\sigma U[\mu+(1/2)\sigma^2U^2] + C_{St}S_{t-1}\sigma U + (1/2)C_{SSS}S_{t-1}^3\sigma^3U^3$$

where C_{SS} , C_{St}, and C_{SSS} are all evaluated at time t-1 and S_{t-1} is the security price at the beginning of the t-th rebalancing interval. Define W_t

$= U_t^2 - 1$, for a long position in the option and $W_t = 1 - U_t^2$, for a short position in the option. Gilster (1990) shows that an option's beta over a discrete rebalancing interval is proportional to the length of the interval, Δt. Therefore, for sufficiently small rebalancing intervals, it is appropriate to compute the future value of the hedging errors at the riskless rate of return. Thus at maturity the future value of the hedging error for the first rebalancing interval is

$$(\lambda_1 W_1 \Delta t + \theta_1 \Delta t^{3/2}) R^{n-1} + O(\Delta t^2)$$

where $R = \exp(r\Delta t)$. Similarly, the future value of the hedging error over the second rebalancing interval is $(\lambda_2 W_2 \Delta t + \theta_2 \Delta t^{3/2}) R^{n-2} + O(\Delta t^2)$. The accumulated hedging errors at the maturity of the hedge portfolio is thus simply the sum of these future values of errors for the n rebalancing intervals.

$$\begin{aligned} \Sigma\Delta H = {} & \lambda_1 W_1 R^{n-1}\Delta t + \lambda_2 W_2 R^{n-2}\Delta t + \ldots + \lambda_i W_i R^{n-i}\Delta t + \ldots + \lambda_n W_n \Delta t \\ & + \theta_1 R^{n-1}\Delta t^{3/2} + \theta_2 R^{n-2}\Delta t^{3/2} + \ldots + \theta_n \Delta t^{3/2} \end{aligned} \tag{5}$$

where terms of order Δt^2 and higher have been ignored. At t_0, λ_1 and θ_1 are known since S_0 is known; all other λ coefficients are stochastic. Boyle and Emanuel (1980) and Leland (1985) who model the hedging errors over only one rebalancing interval need to assume that the option's market price at the end of the rebalancing interval equals the Black-Scholes value. However, one cannot be sure that the market prices the option according to Black-Scholes. In our analysis, we do not need to make these assumptions since the hedge portfolio is not unwound until the options' maturity.

We are now in a position to derive the expression for the variance of the accumulated hedging errors.

Proposition I: The variance of the accumulated hedging errors, valued at maturity, of the Black-Scholes hedge portfolio is:

$$V[\Sigma\Delta H] = 2\lambda_o^2\Delta t^2 R_1 + 2G\Delta t^3 R_2 + K_0\Delta t^3 R_1 + K\Delta t^4 R_2$$

where

$$\lambda_0 = \tfrac{1}{2}C_{SS}S_0^2\sigma^2,$$

$$G = \lambda_S^2 S_0^2\sigma^2 + 2\lambda_0\lambda_S S_0\mu + \lambda_0\lambda_{SS}S_0^2\sigma^2 - 2\lambda_0\lambda_\tau,$$

$$K_0 = X_0{}^2 + 15Y_0{}^2 + 6X_0Y_0,$$

$$\begin{aligned} K = {} & (2X_0X_S + 30Y_0Y_S + 6X_0Y_S + 6Y_0X_S)S_0\mu + (X_S{}^2 + X_0X_{SS} + 15Y_S{}^2 \\ & + 15Y_0Y_{SS} + 3X_0Y_{SS} + 6X_SY_S + 3Y_0X_{SS})S_0{}^2\sigma^2 - (X_0X_\tau + 15Y_0Y_\tau \\ & + 6X_0Y_\tau + 6Y_0X_\tau) \end{aligned}$$

and

$$X_0 = C_{SS}S_0^2\sigma\mu + C_{St}S_0\sigma,$$

$$Y_0 = (1/2)C_{SS}S_0^2\sigma^3 + (1/6)C_{SSS}S_0^3\sigma^3,$$

$$R_1 = (R^{2n}-1)/(R^2-1),$$

$$R_2 = [R^{2n}-R^2-(n-1)(R^2-1)]/(R^2-1)^2.$$

In the above expression, λ_S and λ_{SS} are the first and second partial derivatives of λ_0 with respect to the asset price and λ_τ is the first partial derivative of λ_0 with respect to time to maturity; X_S, X_{SS}, X_τ, Y_S, Y_{SS}, and Y_τ are interpreted similarly.

Proof: See Appendix.

Proposition I implies that the variance of the accumulated hedging errors depends on the option's degree of moneyness. The hedge ratio changes most with changes in the underlying asset for at-the-money options so that with discrete rebalancing, the variance of the hedging errors is highest for this case.

Proposition I also implies that the variance of the accumulated hedging errors is approximately proportional to the length of the rebalancing interval. This can be easily seen by assuming that the interest rate is zero. Then $V[\Sigma\Delta H] = 2\lambda_0^2\Delta t^2 n + G\Delta t^3 n(n-1) + K_0\Delta t^3 n + (1/2)K\Delta t^4 n(n-1)$, where $n = (\tau/\Delta t)$ and is the number of rebalancing intervals. For a sufficiently large n, this expression for $V[\Sigma\Delta H]$ is approximately equal to $(2\lambda_0^2 + G\tau)\tau\Delta t + O(\Delta t^2)$.

Table I provides a sense for the magnitude of the dispersion of the accumulated hedging errors at the maturity of the options. Assume that the current stock price is 100, the annual risk-free rate is 10%, the expected return on the stock is 15%, and the annual volatility is 15%. The calculations assume an European option and the underlying asset to be non-dividend paying.

Panels A through C respectively present the standard deviations of the accumulated hedging errors for a one-month call, a three-month call, and a

12-month call. Two rebalancing intervals, daily and weekly, are considered.[2,3] For each rebalancing interval, the standard deviations are provided based on Proposition I as well as using a simple generalization of the Leland and Boyle and Emanuel results. This generalization assumes that the hedging errors are independently and identically distributed through time.

The hedging errors computed using the simple model differ considerably from those based on the other model for deep in- and out-of-the-money options. This difference increases with maturity. For example, for the one-year maturity with daily rebalancing, the out-of-the money (X=120) call has a standard deviation of accumulated hedging errors of about \$0.32 according to our model while the simple model gives a value of about \$0.23, about 30% lower. For a long maturity option, when the option is away from the money, the hedging errors are likely to be quite small at the initial stages and the simple model significantly underestimates the true standard error by generalizing those hedging errors. The discussion that follows focuses only on the more accurate results from our model.

The variance of the accumulated hedging errors increases almost five-fold as one goes from daily to weekly rebalancing and this result obtains irrespective of the moneyness or maturity of the option. This is consistent with the implication of proposition I that the variance is of order Δt.

Even with daily rebalancing, hedging errors can be quite important. Consider an at-the-money (X=100) call with one month to maturity. The Black-Scholes option price is \$2.14. The standard deviation of accumulated hedging error for this call is about \$0.28 which is 13% of the price of the option. If such an option is mispriced by 5% then a trader following a 5% rule would take a hedged position in the option. If he plans to rebalance the hedge daily and if he has to carry this position to maturity, the probability of a loss is 35%.[5] Weekly rebalancing increases the hedging error induced risk. For example, for a one-year out-of-the-money call (X=120) with a Black-Scholes price of $2.72, a weekly rebalancing strategy results in a standard deviation of accumulated hedging errors of $0.69, approximately 25% of the option value.

The standard deviation of accumulated hedging errors is highest for options that are slightly out-of-the-money forward and lowest for deep in- and out-of-the-money options. This is reasonable since for deep in- and out-of-the-money options the hedge ratio does not change very much over small intervals of time and hence hedging errors accumulate at a small rate. This is not the case for near-the-money options for which the hedge ratio can change significantly over small periods of time and hence hedging errors grow at a fast rate.

Hedging errors are more serious for short-maturity options than for those with long maturities. For example, with daily rebalancing, for a one-year maturity, the (forward) at-the-money call (X=110) has a standard deviation of the accumulated hedging errors of $0.30 or only 5% of the option value, whereas for the one-month option the standard deviation for a near-the-money (X=105) call is $0.29 or about 24% of the Black-Scholes value. This results from the smaller gamma for long-maturity options.

II. The Covariance of Accumulated Hedging errors Across Different Options

In this section we derive the expression for the covariance of the accumulated hedging errors between two delta hedged options, when each is held until maturity. Proposition II provides the expression when both options are held in either short or long positions.

Proposition II: If two European options C and C^* with strike prices X and $X^*(\neq X)$ are both held until a common maturity in either short or long positions, the covariance of the accumulated hedging errors, $\Sigma\Delta H$ and $\Sigma\Delta H^*$, is given by the expression:

$$\mathrm{COV}[\Sigma\Delta H, \Sigma\Delta H^*] = 2\lambda_0\lambda_0^*\Delta t^2 R_1 + 2\hat{G}\Delta t^3 R_2 + \hat{K}_0\Delta t^3 R_1 + \hat{K}\Delta t^4 R_2$$

where $\hat{G} = (\lambda_0\lambda_S^* + \lambda_0^*\lambda_S)S_0\mu + 1/2(\lambda_0\lambda_{SS}^* + \lambda_0^*\lambda_{SS})S_0^2\sigma^2 + \lambda_S\lambda_S^*S_o^2\sigma^2 -$

$$- (\lambda_0\lambda_\tau^* + \lambda_0^*\lambda_\tau),$$

$$\hat{K}_0 = X_0X_0^* + 15Y_0Y_0^* + 3(X_0Y_0^* + X_0^*Y_0),$$

$$\begin{aligned}\hat{K} = {} & [(X_0X_S^* + X_0^*X_S) + 15(Y_0Y_S^* + Y_0^*Y_S) + 3(X_0Y_S^* + Y_0^*X_S + Y_0X_S^* + \\ & Y_SX_0^*)]S_0\mu + [(1/2)(X_0X_{SS}^* + X_0^*X_{SS} + 2X_SX_S^*) + (15/2)(Y_0Y_{SS}^* + Y_0^*Y_{SS} + \\ & 2Y_SY_S^*) + (3/2)(X_0Y_{SS}^* + 2X_SY_S^* + Y_0^*X_{SS} + Y_0X_{SS}^* + 2Y_SX_S^* + X_0^*Y_{SS})]S_0^2\sigma^2 \\ & - [(X_0X_\tau^* + X_0^*X_\tau + 15(Y_0Y_\tau^* + Y_0^*Y_\tau) + 3(X_0Y_\tau^* + Y_0^*X_\tau + Y_0X_\tau^* + X_0^*Y_\tau)]\end{aligned}$$

and λ_0, λ_0^*, X_0, X_0^*, Y_0, Y_0^*, as well as the partial derivatives are as defined in Proposition I with the recognition that the starred quantities refer to the option C^*.

Proof: See the Appendix.

The next proposition considers the case when one option is held short and the other is held long.

Proposition III: If two European options C and C^* with strike prices X and $X^*(\neq X)$ are held until a common maturity, one in a short and the other in a long position, the covariance of the accumulated hedging errors, $\Sigma\Delta H$ and $\Sigma\Delta H^*$, is given by the expression:

$$COV[\Sigma\Delta H, \Sigma\Delta H^*] = -2\lambda_o\lambda_o^*\Delta t^2 R_1 - 2\hat{G}\Delta t^3 R_2 - \hat{K}_0\Delta t^3 R_1 - \hat{K}\Delta t^4 R_2$$

with $\hat{G}$, $\hat{K}_0$, and $\hat{K}$ as given in Proposition II. λ_0, λ_0^*, X_0, X_0^*, Y_0, Y_0^*, as well as the partial derivatives are as defined in Proposition II.

Proof: See the Appendix.

It can be shown as an implication of these propositions that the covariance of the accumulated hedging errors is proportional to the length of the rebalancing interval. Propositions II and III together imply that when one of the options is held short and the other long, the resulting covariance is the same in magnitude but opposite in sign to that when both the options are held either short or long. Henceforth, the discussion assumes that both options are held either short or long so that the covariance is always positive.

Table II presents the correlations between accumulated hedging errors for options with different strike prices, using the same set of parameter values as in table I. Panels A and B present the data for one-month and three-month calls with daily rebalancing and panels C and D for a one-year call with daily and weekly rebalancing respectively.

Table II shows that the correlations between the accumulated hedging errors for the different options depend significantly on the relative moneyness of the options and the time to maturity, and do not change much with the rebalancing frequency. The correlations between hedging errors for options whose moneyness are close turn out to be substantial.

It is important to note that these high correlations are not a consequence of the systematic part of the hedging errors being correlated. This is because Gilster (1990) has shown that hedging errors are zero-beta as $\Delta t \rightarrow 0$ and at the same time, using Propositions I and II, it can be shown that as $\Delta t \rightarrow 0$, the correlation between hedging errors approaches one. This can be illustrated by the high correlations in table II for daily rebalancing and noting that in Gilster (1990) these hedging errors have betas close to zero.

The correlations decline as the options' moneyness become further apart. For example, for the three-month options, the correlation between the hedging errors for an in-the-money call (X=95) and a deep out-of-the money call (X=110) is 0.348 for daily rebalancing. Suppose that one option is in-the-money and the other is out-of-the-money. If the stock price goes up, the hedge ratio for the first option does not change by much but it changes considerably for the second option. This causes the turnover in the two corresponding hedge portfolios to deviate significantly and results in a low correlation in the accumulated hedging errors.

In summary, one finds that the accumulated hedging errors for European options sharing a common maturity and written on the same non-dividend paying asset display significant cross-correlations. These correlations decline as maturity increases but still remain quite large, especially for options whose moneyness is close, even for maturity as long as one year.

III. The Portfolio Approach to Option Arbitrage

The correlations between the accumulated hedging errors across options can be used in option arbitrage strategies to reduce the hedging error induced risk that results from infrequent rebalancing. Let n be the number of available options with maturity τ, all written on the same asset with current price S, let r be the continuously compounded risk-free rate to maturity τ, and ω an arbitrarily chosen wealth level. Let **S** and **D** respectively represent the variance-covariance matrix of the accumulated hedging errors and the vector containing the current mispricing of options. Let **a** represent the vector of positions taken in the options. The objective function then minimizes the standard deviation of the accumulated hedging errors for the option portfolio, subject to the constraint that the future value, at the options' maturity date, of the expected profits of the aggregate portfolio be ω. Formally, the quadratic optimization problem is:

$$\min_{\mathbf{a}} \sqrt{(\mathbf{a}'\mathbf{S}\mathbf{a})}$$

$$\text{St:} \quad \mathbf{a}'\mathbf{D}\exp(r\tau) = \omega$$

The solution to this problem is straight-forward and is given by

$$\mathbf{a}^{*} = [\omega/\exp(r\tau)](\mathbf{S}^{-1}\mathbf{D})/(\mathbf{D}'\mathbf{S}^{-1}\mathbf{D})$$

If the minimized standard deviation, $\sqrt{(\mathbf{a}^{*\prime}\mathbf{S}\mathbf{a}^{*})}$, is acceptable compared to the expected profits ω (i.e., it meets the trader's risk-return trade-off), then the arbitrageur sets the trade, otherwise not. For example, the arbitrageur may set up a trade only if the standard deviation is less than half of ω.

Consider, for example, a one-month call (option 1) with strike price of 100, which is underpriced by \$0.05. Assume that the trader has only one more option available to him for arbitrage, with X=102 and that this call (option 2) is fairly priced. Assume that ω=\$1 and the trader wants to rebalance daily. Then the optimal solution to the problem is: $a_1^* = 19.84$ and $a_2^* = -18.17$, so that the trader would thus buy 19.84 calls with strike price of 100 and write 18.17 calls with strike price of 102 and hedge this portfolio with the underlying stock. This trade would result in expected future profits of \$1 and the standard deviation of those profits turns out to be \$1.79. As opposed to the portfolio approach, if the trader only bought 19.84 calls with X=100, his expected future profits would still be \$1 but the standard deviation would be \$5.51 so that the optimal policy results in reducing the standard deviation by more than two-thirds.[5]

Consider the availability of one more option, say with X=98, which is overpriced by \$.05. Then, denoting this option as number three, the optimal policy with ω = \$1 is: $a_1^*=12.21$, $a_2^*=-6.19$ and $a_3^*=-7.63$ and the minimized σ is \$0.24. Thus a long position in the underpriced option and short positions in the fairly priced and overpriced options results in very little accumulation of hedging errors to maturity.

As another example, consider the one-year call with X=110 that is overpriced by \$0.10. For expected future profits of \$1, i.e., for ω = \$1, with weekly rebalancing, under standard arbitrage one has to write 9.09 calls. This results in a standard deviation of accumulated hedging errors of \$6.86, which is much higher than the level of expected profits. If a fairly priced option with X=100 is available then the portfolio approach entails combining the 9.09 calls written in the overpriced option with buying 11.24 calls with X=100 and the resulting standard deviation turns out to be \$2.69, a 60% reduction. If the X=100 call is underpriced, the improvement is even greater. For example, an underpricing of X=100 call by \$0.10 implies that the optimal portfolio consists of buying 5.27 X=100 calls and writing 3.82 X=110 calls and the standard deviation that results is \$1.16. Similar results obtain when other options, maturities, or rebalancing frequencies are considered.

The portfolio approach requires more options to be traded than the simple approach of only taking a position in the mispriced option. Employing the portfolio approach therefore involves a trade-off between higher set-up costs and lower rebalancing costs that result from the smaller net hedge ratio of the aggregate position. Nevertheless, the major conclusion is that instead of focusing on mispriced options individually, combining options into portfolios has significant potential in reducing the risk due to infrequent rebalancing.

IV. A Generalization to Options on Different Underlying Instruments

Consider two European options that have the same maturity date but different underlying instruments. Let ρ be the correlation between the returns of the underlying instruments. Then the joint distribution of the accumulated hedging errors for these options can be derived and proposition IV summarizes the results.

Proposition IV: If two European calls C and C^* with exercise prices of X and X^* that have the same maturity date but different underlying instruments, are both either held long or short, the covariance of the accumulated hedging errors, $\Sigma\Delta H$ and $\Sigma\Delta H^*$, is given by the expression:

$$COV[\Sigma\Delta H, \Sigma\Delta H^*] = 2\rho^2\lambda_0\lambda_0^*\Delta t^2 R_1 + 2\rho^2\hat{G}\Delta t^3 R_2 + \rho\hat{K}_0\Delta t^3 R_1 + \rho\hat{K}\Delta t^4 R_2$$

where $\hat{G} = \lambda_0\lambda_S^* S_0^*\mu^* + \lambda_0^*\lambda_S S_0\mu + (1/2)[\lambda_0\lambda_{SS}^*(S_0^*\sigma^*)^2 + \lambda_0^*\lambda_{SS}S_0^2\sigma^2] +$

$\lambda_S\lambda_S^* S_0 S_0^*\sigma\sigma^*\rho - (\lambda_0\lambda_\tau^* + \lambda_0^*\lambda_\tau)$,

$$\hat{K}_0 = X_0X_0^* + 3(2\rho^2+3)Y_0Y_0^* + 3(X_0Y_0^* + X_0^*Y_0),$$

$$\begin{aligned}\hat{K} = {} & [X_0X_S^* + 3(2\rho^2+3)Y_0Y_S^* + 3(X_0Y_S^* + Y_0X_S^*)]S_0^*\mu^* \\ & + [X_0^*X_S + 3(2\rho^2+3)Y_0^*Y_S + 3(X_0^*Y_S + Y_0^*X_S)]S_0\mu \\ & + (1/2)[X_0X_{SS}^* + 3(X_0Y_{SS}^* + Y_0X_{SS}^*) + 3(2\rho^2+3)Y_0Y_{SS}^*](S_0^*\sigma^*)^2 \\ & + (1/2)[X_0^*X_{SS} + 3(X_0^*Y_{SS} + Y_0^*X_{SS}) + 3(2\rho^2+3)Y_0^*Y_{SS}](S_0\sigma)^2 \\ & + [X_SX_S^* + 3(X_SY_S^* + Y_SX_S^*) + 3(2\rho^2+3)Y_SY_S^*]S_0S_0^*\sigma\sigma^*\rho \\ & - [X_0X_\tau^* + X_0^*X_\tau + 3(2\rho^2+3)(Y_0Y_\tau^* + Y_0^*Y_\tau) + 3(X_0Y_\tau^* + Y_0^*X_\tau + Y_0X_\tau^* + X_0^*Y_\tau)]\end{aligned}$$

and λ_0, λ_0^*, X_0, X_0^*, Y_0, Y_0^*, as well as the partial derivatives are as defined in proposition I with the recognition that the starred quantities refer to the option C^*.

Proof: See appendix.

Proposition IV reduces to proposition II for $\rho = 1$. Furthermore, it implies that the higher the correlation between the returns of the two underlying instruments, the higher is the covariance between the accumulated hedging errors of the two options. This has implications for reducing the hedging error induced risk in option arbitrage. Consider taking a position in an option on an instrument that has no other options available. Proposition IV then implies that the hedging risk can be effectively reduced by combing this option with others written on a security whose returns are highly correlated with the returns of the option's underlying security.

The possibility of employing options on other instruments to reduce the hedging risk becomes even more useful when the transactions costs on options on different securities vary significantly. Thus, if transactions costs are high on the options on a given security, in forming the portfolio one may substitute these options with those on an asset with high return correlation with the underlying instrument but whose options have lower transactions costs.

V. Implications of the Portfolio Approach for Tests of the Efficiency of Options Markets

Traditionally, tests of efficiency of options markets are based on examining whether options are individually efficiently priced. Our analysis in the previous sections suggests two reasons why such tests are likely to have low power.[6] First, the screening rules for including options in the test samples typically are simple percentage rules that do not take into account the notion that the standard deviation of the accumulated hedging errors depends on the option's degree of moneyness. For example, a commonly used screening rule to determine sufficient mispricing of an option is to choose the cutoff to be a particular percentage by which the option's price differs from its model value. This is the case in Trippi (1977), Chiras and Manaster (1978), and Blomeyer and Klemkosky (1983). Table I implies that in percentage terms, deep in-the-money options have the lowest standard deviation of accumulated hedging errors so that under a percentage rule such options should have the highest probability of earning abnormal returns. The rule, however, does not treat these options in a different manner from others, and may therefore result in option hedge portfolios that have a high probability of earning abnormal returns not being included in the test sample.

The second reason for the low power of traditional tests is that the test design does not consider all available information, e.g., the cross-sectional distribution of accumulated hedging errors across options. These tests only examine whether options are individually mispriced and therefore cannot capture market inefficiencies manifested in the joint pricing of options. Chiras and Manaster (1978), Whaley (1982, 1987), and others have used portfolios of options which consist of options that are individually mispriced. A more powerful portfolio approach would, however, explicitly consider joint pricing in the test design; it would determine whether all or most of the options traded on a stock are jointly efficiently priced. Given that the hedging errors of different options are correlated, the accumulated hedging errors as well as the amount of rebalancings needed on the portfolio of options are likely to be much smaller than when only an individual option is considered. The portfolio approach has more power because it can identify inefficiencies which tests based on mispricing of individual options ignore.

VI. Conclusions

In this paper, we have examined the risks due to hedging errors that result from discrete rebalancing of option hedges. We have derived expressions governing the joint distribution of the accumulated hedging errors across different options traded on the same stock. We find that for option maturities of one month or more, even with daily rebalancing the accumulated hedging errors on individual option hedges can be substantial and quite different from those derived over a discrete time interval generalized to many periods.

The hedging errors across the different options are found to be highly correlated, especially for options whose moneyness is close. This implies that the risk of option arbitrage can be significantly reduced by combining options into portfolios. Indeed the portfolio approach implies that option markets where there are many options available on different assets and with many different strike prices can have prices that are close to those implied by the Black-Scholes model. This is because arbitrageurs by employing a portfolio approach can neutralize the hedging error induced risk on individual options that arises from infrequent rebalancing.

Finally, we have also argued that the traditional tests of efficiency of option markets have low power since they are based on testing whether options are individually mispriced. For tests of market efficiency we recommend a more powerful portfolio approach that examines whether all the options traded on a stock are jointly efficiently priced. A possible extension of our work would be to examine how transactions costs affect the design of the hedge as well as the accumulated hedging errors both on individual options as well as portfolios of options.

Appendix

PROOF OF PROPOSITION I:

Applying the variance operator to equation (5) gives:

$$V[\Sigma\Delta H] = \sum_{j=1}^{n} V[\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2}]R^{2(n-j)}$$

$$+ \sum_{i<j}^{n} 2COV[\lambda_i W_i \Delta t + \theta_i \Delta t^{3/2}, \lambda_j W_j \Delta t + \theta_j \Delta t^{3/2}]R^{2n-i-j} \qquad (A.1)$$

We first show that all covariance terms in this expression equal zero. Write the covariance as

$$COV[\lambda_i W_i \Delta t + \theta_i \Delta t^{3/2}, \lambda_j W_j \Delta t + \theta_j \Delta t^{3/2}] =$$

$$E[(\lambda_i W_i \Delta t + \theta_i \Delta t^{3/2})(\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2})] - E[(\lambda_i W_i \Delta t + \theta_i \Delta t^{3/2})E(\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2})]$$

The first terms equal zero because, by definition, the U_i and U_j terms in W and θ are independent of one another. Also, $E[W_i W_j] = 0$. To see this, consider:

$$E[(U_i^2-1)(U_j^2-1)] = E[U_i^2]E[U_j^2] - E[U_i^2] - E[U_j^2] + 1 = 0.$$

Also, for any j,

$$E[\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2}] = E[W_j]\lambda_1 \Delta t + E[\theta_j]\Delta t^{3/2} = 0 \qquad (A.2)$$

since both $E[W_j] = 0$ and $E[\theta_j] = 0$. Thus, all the covariance terms in (A.1) are zero. Hence $V[\Sigma\Delta H]$ can be re-written as:

$$V[\Sigma\Delta H] = \sum_{j=1}^{n} V[\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2}]R^{2(n-j)} \qquad (A.3)$$

For each and every j,

$$V[\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2}] = E[(\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2})^2] - [E(\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2})]^2. \quad (A.4)$$

From (A.2), one can see that the second term in the right hand side of (A.4) equals zero. Furthermore, $E[W_j \theta_j] = 0$, since all terms comprise an odd power of U_j. Therefore,

$$\begin{aligned} V[\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2}] &= E[(\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2})^2] \\ &= E[\lambda_j^2 W_j^2 \Delta t^2 + 2\lambda_j W_j \theta_j \Delta t^{5/2} + \theta_j^2 \Delta t^3] \\ &= E[\lambda_j^2 W_j^2] \Delta t^2 + E[\theta_j^2] \Delta t^3 \\ &= 2E[\lambda_j^2] \Delta t^2 + E[\theta_j^2] \Delta t^3 \qquad (A.5) \end{aligned}$$

where the last expression results from the independence of λ_j and W_j and the fact that $E[W_j^2] = E[U_j^4 + 2U_j^2 + 1] = 2$. We now proceed in two steps. First, we examine the λ_j terms and then the θ_j terms. At the beginning of the first balancing interval $\lambda_1 = (1/2)C_{SS}S_0^2\sigma^2$, where C_{SS} is evaluated at the inception of the hedge, when the stock price is S_0. Defining $\lambda_0 = (1/2)C_{SS}S_0^2\sigma^2$, then $E[\lambda_1^2] = \lambda_0^2$. At the beginning of the second rebalancing period $\lambda_2 = (1/2)C_{SS}S_1^2\sigma^2$. λ_2 can be approximated by expanding λ around the initial stock price, S_0, and calendar time, t:

$$\lambda_2 = \lambda_0 + \lambda_S \Delta S + (1/2)\lambda_{SS}\Delta S^2 - \lambda_\tau \Delta t$$

where terms of order $\Delta t^{3/2}$ and higher are disregarded. λ_2^2 is then:

$$\lambda_2^2 = \lambda_0^2 + \lambda_S^2 \Delta S^2 + 2\lambda_0\lambda_S\Delta S + \lambda_0\lambda_{SS}\Delta S^2 - 2\lambda_0\lambda_\tau \Delta t.$$

Note that both ΔS and ΔS^2 refer to the stock price changes over the first rebalancing interval only. Applying the expectations operator gives:

$$E[\lambda_2^2] = \lambda_0^2 + \lambda_S^2 S_0^2 \sigma^2 \Delta t + 2\lambda_0 \lambda_S S_0 \mu \Delta t + \lambda_0 \lambda_{SS} S_0^2 \sigma^2 \Delta t - 2\lambda_0 \lambda_\tau \Delta t \qquad (A.6)$$

For the θ_j coefficients, let us first define

$$\theta_j^2 = (X_j U_j + Y_j U_j^3)^2 \Delta t^3$$

where

$$X_j = C_{SS} S_{j-1}^2 \sigma\mu + C_{St} S_{j-1} \sigma$$

and

$$Y_j = (1/2) C_{SS} S_{j-1}^2 \sigma^2 + (1/6) C_{SSS} S_{j-1}^3 \sigma^3 .$$

One can approximate θ_j using an expansion around the stock price, S_0, and calendar time. Hence, θ_j can be rewritten as

$$\theta_j = [X_0 + X_S \Delta S + (1/2) X_{SS} \Delta S^2 - X_\tau (j-1)\Delta t] U_j \Delta t^{3/2} +$$
$$[Y_0 + Y_S \Delta S + (1/2) Y_{SS} \Delta S^2 - Y_\tau (j-1)\Delta t] U_j \Delta t^{3/2},$$

where terms on the expansion with order $\Delta t^{3/2}$ and higher are disregarded. Squaring θ_j and taking its expectation gives

$$E(\theta_j^2) = K_0 \Delta t^3 + K(j-1)\Delta t^4$$

where

$$K_0 = X_0{}^2 + 15 Y_0{}^2 + 6 X_0 Y_0 ,$$

and

$$K = (2X_0X_S + 30Y_0Y_S + 6X_0Y_S + 6Y_0X_S)S_0\mu$$
$$+ (X_S{}^2 + X_0X_{SS} + 15Y_S{}^2 + 15Y_0Y_{SS} + 3X_0Y_{SS} + 6X_SY_S + 3Y_0X_{SS})S_0{}^2\sigma^2$$
$$- (X_0X_\tau + 15Y_0Y_\tau + 6X_0Y_\tau + 6Y_0X_\tau).$$

We can then rewrite (A.5) as

$$V[\lambda_jW_j\Delta t + \theta_j\Delta t^{3/2}] = 2\lambda_0^2\Delta t^2 + 2G(j-1)\Delta t^3 + K_0\Delta t^3 + K\Delta t^4$$

for all j=1, ..., n. Expression (A.3) is then

$$V[\Sigma\Delta H] = 2(\sum_{j=1}^{n} [\lambda_0{}^2 + G(j-1)\Delta t]R^{2(n-j)})\Delta t^2 + (\sum_{j=1}^{n} [K_0 + K(j-1)\Delta t]R^{2(n-j)})\Delta t^3$$

where from (A.6)

$$G = \lambda_S^2S_0^2\sigma^2 + 2\lambda_0\lambda_SS_0\mu + \lambda_0\lambda_{SS}S_0^2\sigma^2 - 2\lambda_0\lambda_\tau.$$

It can be easily shown that

$$\sum_{j=1}^{n} R^{2(n-j)} = (R^{2n}-1)/(R^2-1)$$

and

$$\sum_{j=1}^{n} (j-1)R^{2(n-j)} = [R^{2n}-R^2-(n-1)(R^2-1)]/(R^2-1)^2.$$

Consequently, the variance $V[\Sigma\Delta H]$ in (A.3) becomes:

$$V[\Sigma\Delta H] = 2\lambda_0^2\Delta t^2[(R^{2n}-1)/(R^2-1)] + 2G\Delta t^3[R^{2n}-R^2-(n-1)(R^2-1)]/(R^2-1)^2$$
$$+ K_0\Delta t^3[(R^{2n}-1)/(R^2-1)] + K\Delta t^4[R^{2n}-R^2-(n-1)(R^2-1)]/(R^2-1)^2.$$

QED.

We now list the partial derivatives of λ_0, X_0 and Y_0. Recall that

$$\lambda_0 = \frac{1}{2} C_{SS}S_0^2\sigma^2,$$

$$X_0 = C_{SS}S_0^2\sigma\mu + C_{St}S_0\sigma,$$

and

$$Y_0 = (1/2)C_{SS}S_0^2\sigma^3 + (1/6)C_{SSS}S_0^3\sigma^3.$$

The option's gamma is equal to $C_{SS} = N(d)/(S\sigma\tau^{\frac{1}{2}})$, where N(.) is the standard normal density function and d equals $[\ln(S/(Xe^{-r\tau}))/[\sigma\tau^{\frac{1}{2}}] + (1/2)\sigma\tau^{\frac{1}{2}}$. As $\lambda_0 = 1/2C_{SS}S_0^2\sigma^2$, it follows that λ_0 can also be written

$$\lambda_0 = (1/2)N(d)S\sigma\tau^{-\frac{1}{2}}.$$

To obtain the partial derivatives λ_S, λ_{SS} and λ_τ, we first note that

$$\partial N(d)/\partial d = -dN(d)$$

$$\partial d/\partial S = 1/(S\sigma\tau^{\frac{1}{2}})$$

$$\partial d/\partial\tau = -(1/2)\tau^{-1}(\bar{d})$$

where $(\bar{d})=d-\sigma\tau^{\frac{1}{2}}$. The partial λ_S is thus

$$\partial\lambda_0/\partial S = (1/2)N(d)\sigma\tau^{-\frac{1}{2}} + (1/2)\tau^{-\frac{1}{2}}S\sigma[\partial N(d)/\partial d](\partial d/\partial S)$$

and substituting the partials $\partial N(d)/\partial d$ and $\partial d/\partial S$ in the expression above gives

$$\lambda_S = (1/2)N(d)[\sigma\tau^{\frac{1}{2}}-d]\tau^{-1}$$

The partial λ_{SS} is

$$\partial\lambda_S/\partial S = (1/2)[\sigma\tau^{\frac{1}{2}}-d]\tau^{-1}(\partial N(d)/\partial d)(S\sigma\tau^{\frac{1}{2}})^{-1} - (1/2)N(d)\tau^{-1}(\partial d/\partial S)$$

$$= -(1/2)[\sigma\tau^{½}-d]\tau^{-1}dN(d)(S\sigma\tau^{½})^{-1} - 1/2N(d)\tau^{-1}(S\sigma\tau^{½})^{-1}$$

$$= (1/2)N(d)(S\sigma\tau^{3/2})^{-1}[d^2-d\sigma\tau^{½}-1]$$

The partial λ_τ is

$$\partial\lambda_0/\partial\tau = (1/2)S\sigma[\tau^{-½}(\partial N(d)/\partial\tau)-(1/2)\tau^{-3/2}N(d)]$$

using $\partial N(d)/\partial d = -dN(d)$ and $\partial d/\partial\tau = -(1/2)\tau^{-1}(\bar{d})$ in the expression, we have:

$$\partial\lambda_0/\partial\tau = (1/4)S\sigma\tau^{-3/2}N(d)[\bar{d}d-1].$$

Proceeding similarly, one finds that

$$X_S = \lambda_S[2\mu/\sigma - \bar{d}/\tau^{½}] -\lambda_0/(S\sigma\tau),$$

$$Y_S = \lambda_S[\sigma - (1/3)(d+\sigma\tau^{½})/\tau^{½}]- \lambda_0/(3S\sigma\tau),$$

$$X_{SS} = \lambda_{SS}[2\mu/\sigma - \bar{d}/\tau^{½}] - 2\lambda_S/(S\sigma\tau) + \lambda_0/(S^2\sigma\tau),$$

$$Y_{SS} = \lambda_{SS}[\sigma - (1/3)(d+\sigma\tau^{½})/\tau^{½}] - 5\lambda_S/(6S\sigma\tau) + \lambda_0/(3S^2\sigma\tau),$$

$$X_\tau = \lambda_\tau[2\mu/\sigma - \bar{d}/\tau^{½}] + (1/2)\lambda_0\tau^{-3/2}(\bar{d} + d),$$

and

$$Y_\tau = \lambda_\tau[\sigma - (1/3)(d+\sigma\tau^{½})/\tau^{½}] + (1/6)\lambda_0\tau^{-3/2}(\bar{d} + d).$$

PROOF OF PROPOSITION II:

By definition of the covariance operator

$$COV[\Sigma\Delta H,\Sigma\Delta H^*] = E[(\Sigma\Delta H)(\Sigma\Delta H^*)]-E[\Sigma\Delta H]E[\Sigma\Delta H^*]$$

From the proof of Proposition I, the second term in the right hand side of the above expression is zero. The covariance is then

$$COV[\Sigma\Delta H,\Sigma\Delta H^{*}] = E[(\Sigma\Delta H)(\Sigma\Delta H^{*})]$$

which can be rewritten as

$$COV[\Sigma\Delta H,\Sigma\Delta H^{*}] = \sum_{j=1}^{n} E[(\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2})(\lambda_j^{*} W_j^{*} \Delta t + \theta_j^{*} \Delta t^{3/2})]\ R^{2(n-j)}$$

which can be restated as

$$COV[\Sigma\Delta H,\Sigma\Delta H^{*}] = \sum_{j=1}^{n} E[\lambda_j \lambda_j^{*}]E[W_j W_j^{*}]R_j^{2(n-j)}\Delta t^2 + \sum_{j=1}^{n} E[\theta_j \theta_j^{*}]R_j^{2(n-j)}\Delta t^3$$

Recall from Proposition I how λ_j, λ_j^{*}, θ_j, and θ_j^{*} are expanded. Also, recall that $E[W_j W_j^{*}] = 2$ if both options are held short or long. Taking expectations and collecting terms gives:

$$COV[\Sigma\Delta H,\Sigma\Delta H^{*}] = 2[\ \sum_{j=1}^{n} E[\lambda_j \lambda_j^{*}]R_j^{2(n-j)}\Delta t^2 + \sum_{j=1}^{n} E[\theta_j \theta_j^{*}]R_j^{2(n-j)}\Delta t^3$$

and from the proof of Proposition I, it is easily shown that the $COV[\Sigma\Delta H,\Sigma\Delta H^{*}]$ satisfies the expression in Proposition II. *QED*.

PROOF OF PROPOSITION III:

Same as for Proposition II, except that when one option is held short and the other option is held long $E[W_j W_j^{*}]=E[(U_j^2-1)(1-U_j^2)] = -2$. *QED*.

PROOF OF PROPOSITION IV:

By definition of the covariance operator,

$$COV[\Sigma\Delta H, \Sigma\Delta H^*] = E[(\Sigma\Delta H)(\Sigma\Delta H^*)] - E[\Sigma\Delta H]E[\Sigma\Delta H^*]$$

From the proof of Proposition I, the second term in the right hand side of the above expression is zero. The covariance is then

$$COV[\Sigma\Delta H, \Sigma\Delta H^*] = E[(\Sigma\Delta H)(\Sigma\Delta H^*)]$$

which can be rewritten as

$$COV[\Sigma\Delta H, \Sigma\Delta H^*] = \sum_{j=1}^{n} E[(\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2})(\lambda_j^* W_j^* \Delta t + \theta_j^* \Delta t^{3/2})] \; R^{2(n-j)}$$

where

$$E[(\lambda_j W_j \Delta t + \theta_j \Delta t^{3/2})(\lambda_j^* W_j^* \Delta t + \theta_j^* \Delta t^{3/2})] = E[\lambda_j \lambda_j^* W_j W_j^* \Delta t^2 + \theta_j \theta_j^* \Delta t^3]$$

so that

$$COV[\Sigma\Delta H, \Sigma\Delta H^*] = \sum_{j=1}^{n} E[\lambda_j \lambda_j^*]E[W_j W_j^*]R_j^{2(n-j)}\Delta t^2 + \sum_{j=1}^{n} E[\theta_j \theta_j^*]R_j^{2(n-j)}\Delta t^3 \qquad (A.7)$$

We now proceed in two steps. First, we examine the first term on the R.H.S. of the above expression and then the θ_j terms. Recall that if the calls are held long, $W_j = U_j^2 - 1$ and $W_j^* = U_j^{*2} - 1$ (if the calls are held short, $W_j = 1 - U_j^2$ and $W_j^* = 1 - U_j^{*2}$. The correlation between U_j and U_j^* is denoted ρ. Following the proof of Proposition I, multiplying $\Sigma\Delta H$ by $\Sigma\Delta H^*$ and taking expectations yields

$$\sum_{j=1}^{n} E[\lambda_j \lambda_j^*]E[W_j W_j^*]R_j^{2(n-j)}\Delta t^2 = 2\rho^2 \sum_{j=1}^{n} E[\lambda_j \lambda_j^*]R_j^{2(n-j)}\Delta t^2$$

since $Cov(U_j U_j^*) = \rho$ implies $E[W_j W_j^*] = 2\rho^2$ (with a negative sign when one call is held short and the other long). By approximating λ_j and λ_j^* with expansions around S_0 and S_0^*, and taking expectations of the product gives

$$\sum_{j=1}^{n} E[\lambda_j\lambda_j^*]E[W_jW_j^*]R_j^{2(n-j)}\Delta t^2 = 2\rho^2[\sum_{j=1}^{n}\lambda_0\lambda_0R_1\Delta t^2 + \sum_{j=1}^{n}\hat{G}R_2\Delta t^3] \qquad (A.8)$$

where $\hat{G}$ is as provided in the statement of proposition IV.

To evaluate the second term in the R.H.S. of (A.7), recall, from Proposition I, how θ_j and θ_j^* can be written in terms of X and Y and X and Y^* and how X_j and Y_j can be approximated by higher order expansions around S_0 and S_0^*. Multiplying θ_j and θ_j^*, taking expectations, and collecting terms, terms, one obtains expressions involving the terms $E(U_jU_j^*)$, $E(U_j^3U_j^*)$, $E(U_jU_j^{*3})$, and $E(U_j^3U_j^{*3})$.

It can be shown that $E(U_jU_j^*) = \rho$, $E(U_j^3U_j^*) = E(U_jU_j^{*3}) = 3\rho$, and $E(U_j^3U_j^{*3}) = 3\rho(2\rho^2+3)$ so that $E[\theta_j\theta_j^*]$ can be rewritten as

$$E[\theta_j\theta_j^*] = \rho\hat{K}_0\Delta t^3R^{2(n-j)} + \rho\hat{K}\Delta t^4R^{2(n-j)}$$

where $\hat{K}_0$ and $\hat{K}$ are as given in the statement of proposition IV. Summing up over all j,

$$\sum_{j=1}^{n} E[\theta_j\theta_j] = \rho\hat{K}_0\Delta t^3R_1 + \rho\hat{K}\Delta t^4R_2 \qquad (A.9)$$

Combining (A.8) and (A.9) gives the covariance of the accumulated hedging errors as stated in Proposition IV. *QED.*

References

Black, F. and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 163-175.

Blomeyer, E.C. and R.C. Klemkosky, 1983, Tests of Market Efficiency for American Call Options, *Option Pricing*, M. Brenner, ed., (D.C. Heath and Company, Lexington, MA), 101-121.

Boyle, P.P. and D. Emanuel, 1980, Discretely Adjusted Option Hedges", *Journal of Financial Economics* 8, 259-282.

Chiras, D., and S. Manaster, 1978, The Information Content of Option Prices and a Test of Market Efficiency, *Journal of Financial Economics* 6, 213-234.

Figlewski, S., 1989, Option Arbitrage in Imperfect Markets, Journal of Finance 44, 1289-1311.

Gilster, J., 1990, The Systematic Risk of Discretely Rebalanced Option Hedges, *Journal of Financial and Quantative Analysis* 25, 507-516.

Leland, H.E., 1985, Option Pricing and Replication with Transactions Costs, *Journal of Finance* 40, 1283-1301.

Merton, R.C., 1973, Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science* 3, 141-183.

Trippi, R.R., 1977, A Test of Option Market Efficiency Using a Random-Walk Valuation Model, *Journal of Economics and Business* 29, 93-98.

Whaley, R.E., 1982, Valuation of American Call Options on Dividend-paying Stocks, *Journal of Financial Economics* 10, 28-58.

______. 1986, Valuation of American Future Options: Theory and Empirical Tests, *Journal of Finance* 41, 127-150.

Wiggins, J.B., 1987, Option Values Under Stochastic Volatility: Theory and Empirical Estimates, *Journal of Financial Economics* 19, 351-372.

Footnotes

1. Combining options into portfolios to reduce the risk of hedging errors caused by discrete rebalancing is distinct from using multiple options to reduce the risks resulting from having more than one state variables. The latter possibility has been discussed in the previous literature [See, for example, Wiggins (1987), who considers employing not one but two options plus the stock to form a hedge portfolio when the volatility is stochastic].

2. We assume daily rebalancing is equivalent to 20 rebalancings every month. Similarly, weekly rebalancing requires 4 rebalancings in a month.

3. For the one-month option, we do not report the numbers for weekly rebalancing since the underlying assumption that hedging errors are zero-beta is unlikely to hold in that situation.

4. This assumes that the hedging errors are normally distributed. With a standard deviation of 13%, $P[\text{Profits} < 0 \mid E(\text{Profits}) = 5\%] = 0.35$.

5. Using panel B, Table I, the standard deviation of the accumulated hedging errors for 19.84 calls with strike price of 100 is \$0.2778x19.84 = \$5.51.

6. Gilster (1990) notes a related problem with some of the empirical tests in that the hedging errors of the delta-neutral hedge may not be zero-beta.

Table I
Dispersion of Accumulated Hedging Errors

The table presents the standard deviation of the accumulated hedging errors to maturity. The current stock price is set at 100, the risk-free rate at 10%, the expected return on the stock at 15% and the standard deviation of the stock return at 15%. Daily and weekly rebalancing respectively correpsond to 240 and 48 rebalancings per year. The simple model is a simple generalization of the Boyle and Emanuel and the Leland models.

Panel A: One-month Call

		Daily Rebalancing	
Strike Price	B-S Option Price	Our Model	Simple Model
98	3.44	0.2360	0.2175
100	2.14	0.2778	0.2670
102	1.21	0.2869	0.2640
104	0.60	0.2629	0.2133

Panel B: Three-month Call

		Daily Rebalancing		Weekly Rebalancing	
Strike Price	B-S Option Price	Our Model	Simple Model	Our Model	Simple Model
95	7.84	0.1836	0.1581	0.4116	0.3540
100	4.28	0.2648	0.2550	0.6255	0.5703
105	2.72	0.2909	0.2600	0.7032	0.5815
110	1.61	0.2470	0.1786	0.6288	0.3995

Panel C: One-year Call

		Daily Rebalancing		Weekly Rebalancing	
Strike Price	B-S Option Price	Our Model	Simple Model	Our Model	Simple Model
90	18.75	0.1237	0.0985	0.2814	0.2200
100	11.36	0.2278	0.2074	0.5101	0.4637
110	5.98	0.3045	0.2661	0.6753	0.5951
120	2.72	0.3180	0.2347	0.6935	0.5254

Table II
Correlations Between Accumulated Hedging Errors of Different Options

The table presents the correlations between the accumulated hedging errors for different options on the same asset with the same maturity but different exercise prices. The current stock price is set at 100, the risk-free rate at 10%, the expected return on the stock at 15% and the standard deviation of the stock return at 15%. Daily and weekly rebalancing respectively correpsond to 240 and 48 rebalancings per year.

Panel A: One-month Call, Daily Rebalanced

Strike Price	98	100	102	104
98	1.000	0.944	0.795	0.587
100		1.000	0.946	0.801
102			1.000	0.949
104				1.000

Panel B: Three-month Call, Daily Rebalanced

Strike Price	95	100	105	110
95	1.000	0.901	0.635	0.348
100		1.000	0.901	0.694
105			1.000	0.934
110				1.000

Panel C: One-Year Call, Daily Rebalanced

Strike Price	90	100	110	120
90	1.000	0.916	0.688	0.399
100		1.000	0.911	0.726
110			1.000	0.945
120				1.000

Panel D: One-Year Call, Weekly Rebalanced

Strike Price	90	100	110	120
90	1.000	0.909	0.666	0.394
100		1.000	0.910	0.714
110			1.000	0.937
120				1.000

Date Due

Lib-26-67