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# A NOTE ON THE BEHAVIOR OF MUTUALLY COUPLED OSCILLATORS

E. E. DAVID, JR.

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RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS

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# A NOTE ON THE BEHAVIOR OF MUTUALLY COUPLED OSCILLATORS

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This report is based on work contained in a Doctoral thesis in the Department of Electrical Engineering.

Abstract

The operation of an arbitrary number of oscillators mutually coupled into an arbitrary number of loads is discussed. By utilizing the oscillator Rieke diagrams, the scattering equations of the interconnecting network, and the constraints furnished by the passive loads, the behavior of such a system may be predicted. Two simple, but practically important cases are discussed: (1) two oscillators operating into a matched parallel junction, and (2) the dual of this arrangement. The operation of these systems is critically dependent upon the oscillator spacing relative to the interconnecting network, although the tolerances on these spacings are not severe. Dissimilarities between the oscillators are not an important factor.

# A NOTE ON THE BEHAVIOR OF MUTUALLY COUPLED OSCILLATORS

## Introduction

There are practical and theoretical limitations on the power output obtainable from a single oscillator. These limitations are particularly severe when one considers the microwave region, for there the physical geometry of the device becomes increasingly important. The cathode area, for instance, is determined by factors other than the total desired emission current, which may be quite large, so large, in fact, as to require current densities greater than those obtainable from practical emitting surfaces. Nonetheless, great progress has been made in the production of large power at short wavelengths. The maximum output available from a single tube, however, is quite insufficient for many purposes. In applications such as the linear electron accelerator and very high power radar transmitters, it is desirable to operate two or more oscillators into a single network so that their powers combine in an additive fashion. One suspects immediately that this problem is closely akin to the oscillator synchronization problems considered in Technical Report No. 63 (1). It is, however, much more involved, for here each oscillator supplies a locking signal to each of the others. In addition, the network used for the interconnection affects the problem in an important way. A general solution, then, would be highly complex and is unnecessary for most purposes. We can show that the general problem is soluble and then carry out the solution for two simple but practically important cases. The results of this analysis may be used to predict the more involved cases.

## 1. The General Case of N Oscillators Operating into an M Terminal-Pair Network

Before considering some practically important cases of mutually-coupled oscillators, it would be satisfying to examine the most general situation. If such an examination shows that a unique solution is possible, one may use with confidence the results of simplified analyses to draw certain conclusions about the more complex cases. Suppose that N oscillators are operating into an M terminal-pair network; that is, there are N oscillators and M-N passive loads (these may or may not be frequency-sensitive). The network itself, assuming linearity, may be completely characterized by a set of M independent equations containing 2M independent variables. One possible set of these equations is

$$
\begin{aligned}
b_1 &= S_{11}a_1 + S_{12}a_2 + S_{13}a_3 \cdots + S_{1N}a_N + S_{1(N+1)}a_{N+1} \cdots + S_{1M}a_M \\
b_2 &= S_{21}a_1 + S_{22}a_2 + S_{23}a_3 \cdots + S_{2N}a_N + S_{2(N+1)}a_{N+1} \cdots + S_{2M}a_M \\
&\vdots \\
b_M &= S_{M1}a_1 + S_{M2}a_2 + S_{M3}a_3 \cdots + S_{MN}a_N + S_{M(N+1)}a_{N+1} \cdots + S_{MM}a_M
\end{aligned}
\tag{1}
$$

where

$$S_{ij} = \left[\frac{b_i}{a_j}\right] \quad \text{all other a's} = 0$$

is the scattering coefficient for the $i^{th}$ terminal-pair with respect to the $j^{th}$ terminal-pair, and $a_i$ and $b_i$ are the incident and reflected waves respectively for the $i^{th}$ terminal-pair. In order to obtain a solution, M additional relations are needed. These are provided by the constraints relating $a_i$ and $b_i$. For the M-N passive loads, the constraints take the form

$$\begin{aligned} a_{N+1} &= b_{N+1}\,\Gamma_{N+1} \\ a_{N+2} &= b_{N+2}\,\Gamma_{N+2} \\ &\vdots \\ a_M &= b_M\,\Gamma_M \end{aligned} \tag{2}$$

where $\Gamma_i = Z_i - 1/Z_i + 1$ is the reflection coefficient of the load at the $i^{th}$ terminal-pair and $Z_i$ is its impedance relative to the impedance of the line. The constraints imposed by the N oscillators are contained in the Rieke diagrams and may be represented functionally by

$$\begin{aligned} b_1 &= a_1 R_1 \\ b_2 &= a_2 R_2 \\ &\vdots \\ b_N &= a_N R_N \quad . \end{aligned} \tag{3}$$

Suppose that in such a system the only information available is the unperturbed frequency $\omega_{11}$ of one oscillator (hereafter referred to as No. 1) and the synchronized frequency of the system, $\omega$. By substituting Eq. 2 into Eq. 1 and transposing, there results

$$\begin{aligned} b_1 - S_{11}a_1 &= S_{12}a_2 + S_{13}a_3 \cdots S_{1N}a_N + S_{1(N+1)}a_{N+1} \cdots + S_{1M}a_M \\ -S_{21}a_1 &= -b_2 + S_{22}a_2 + S_{23}a_3 \cdots S_{2N}a_N + S_{2(N+1)}a_{N+1} \cdots + S_{2M}a_M \\ &\vdots \\ -S_{N1}a_1 &= -b_N + S_{N2}a_2 + S_{N3}a_3 \cdots S_{NN}a_N + S_{N(N+1)}a_{N+1} \cdots + S_{NM}a_M \\ -S_{(N+1)1}a_1 &= S_{(N+1)2}a_2 + S_{(N+1)3}a_3 \cdots S_{(N+1)N}a_N \\ &\quad + \left[S_{(N+1)(N+1)} - \frac{1}{\Gamma_{N+1}}\right] a_{N+1} \cdots S_{(N+1)M}a_M \\ &\vdots \\ -S_{M1}a_1 &= S_{M2}a_2 + S_{M3}a_3 \cdots S_{MN}a_N \\ &\quad + S_{M(N+1)}a_{N+1} \cdots \left[S_{MM} - \frac{1}{\Gamma_M}\right] a_M \quad . \end{aligned} \tag{4}$$

Now since oscillator No. 1 is operating at frequency $\omega$, $a_1$ and $b_1$ have a restricted range of values, which are specified by the appropriate frequency line on the Rieke diagram. There is, on each of the other Rieke diagrams, a corresponding locus, which is related to the first by Eqs. 4. In addition, these equations specify the ratios $|b_i/b_1|$

and $|a_i/a_1|$. This condition likewise determines a locus on each Rieke diagram. The intersection of the two loci on a Rieke diagram specifies the operating point, providing the ratio $|b_i/b_1|$ (or $|a_i/a_1|$) and $\Gamma_i = b_i/a_i$ at that point correspond to the same $a_1$ and $b_1$. Once the operating point for each oscillator has been located, the relative tuning and power output of each is determined. It is possible, of course, that there is no operating point which satisfies the conditions. This means that the initial assumptions of $\omega_{11}$ and $\omega$ are not possible ones. However, this situation arises only if the difference $|\omega_{11} - \omega|$ is quite large. Results of this type may be used to establish the limits of the locking band.

Further, one may use similar procedures to establish operating conditions when sets of information other than the one discussed above are available. For instance, the tuning of each oscillator and the operating frequency of the system desired might be known. In any case, it has been shown that there are 2M independent variables involved, and these are related by 2M equations. Thus a solution is possible.

2. Normal Parallel Operation of Two Similar Oscillators

The operation of two oscillators, identical except for their relative tuning, into a single load is usually accomplished by use of a circuit which places the tubes in parallel. Such a circuit may be a transmission line tee or an H-plane waveguide tee junction. Usually the load arm is made to have a characteristic impedance half of the main line connecting the oscillators. Further, in order to make the oscillators appear truly in parallel, they should be located an integral number of half-wavelengths from the tee reference planes. Under such conditions the oscillators are said to operate in a "normal" fashion.

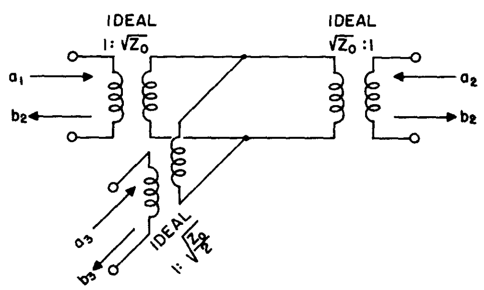


Fig. 1

Figure 1 shows an equivalent circuit which may be used to analyze the normal situation. From this circuit, the scattering equations are

$$\begin{aligned} b_1 &= -\tfrac{1}{2} a_1 + \tfrac{1}{2} a_2 + \tfrac{\sqrt{2}}{2} a_3 \\ b_2 &= \tfrac{1}{2} a_1 - \tfrac{1}{2} a_2 + \tfrac{\sqrt{2}}{2} a_3 \\ b_3 &= \tfrac{\sqrt{2}}{2} a_1 + \tfrac{\sqrt{2}}{2} a_2 \quad . \end{aligned} \tag{5}$$

In practice, the passive load is usually matched to its line, making $a_3 \equiv 0$. If this is the case, Eqs. 5 become

$$\begin{aligned} b_1 &= -\tfrac{1}{2} a_1 + \tfrac{1}{2} a_2 \\ b_2 &= \tfrac{1}{2} a_1 - \tfrac{1}{2} a_2 \\ b_3 &= \tfrac{\sqrt{2}}{2} a_1 + \tfrac{\sqrt{2}}{2} a_2 \quad . \end{aligned} \tag{6}$$

It is seen immediately that the condition $b_1 = -b_2$ must be satisfied regardless of the values of $a_1$ and $a_2$. Stated in another way, this condition means that the reflected powers of the oscillators must be equal if they are to operate at the same frequency. Let this condition be incorporated in either of the first two Eqs. 6; there results

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} = -2 \quad . \tag{7}$$

Note $\Gamma_1$ and $\Gamma_2$ are complex variables and may be written $\Gamma_1 = R_1 + jI_1$ and $\Gamma_2 = + jI_2$. Then Eq. 7 becomes, after separation of the real and imaginary parts

$$\frac{R_1}{|\Gamma_1|^2} + \frac{R_2}{|\Gamma_2|^2} + 2 = 0$$

$$\frac{I_1}{|\Gamma_1|^2} + \frac{I_2}{|\Gamma_2|^2} = 0 \quad . \tag{8}$$

Now Eqs. 8 must be satisfied at the operating point; however, these conditions carry the stipulation, imposed by the passive network, that the reflected waves be equal. This proviso, as well as Eqs. 8, must be applied to the Rieke diagrams.

As an aid to the computation, contours of constant $R/|\Gamma|^2$ and $I/|\Gamma|^2$ will be plotted in the reflection coefficient plane. These, of course, will be the same in both the $\Gamma_1$ and $\Gamma_2$ spaces and are a set of orthogonal circles. Figure 2 shows the unit circle of the reflection space with the orthogonal set plotted thereon.

Consider then the contours of constant frequency and reflecter power shown in Fig. 3. Such a plot is representative of an idealized reflex klystron (3) and will be used as the Rieke diagram of both oscillators in the computation. Note the diagram is symmetrical about the center frequency line; this, in general, is not the actual configuration. The analysis, however, will be affected in detail only by asymmetries. Now upon superposing Figs. 2 and 3 it is seen at once that the $R/|\Gamma|^2 = -1.0$ circle is the locus of points satisfying both Eqs. 8 and stipulation of equal reflected powers. Any two points, $\Gamma_1$ and $\Gamma_2$, lying simultaneously on this locus and the same reflected power contour represent possible operating points for the two oscillators. The frequency contour passing through each of these points determines the relative tuning of the oscillators. Just how this is accomplished may be seen by considering Fig. 4, which shows the $\Gamma_1$ and $\Gamma_2$ spaces with the operating locus and frequency contours. Typical operating points for synchronized operation are $0_1$ and $0_2$, lying respectively on the frequency contours $\omega_1 + \omega_{22}$ and $\omega_2 - \omega_{22}$ where $\omega_1$ and $\omega_2$ are the oscillator center frequencies (frequency into a matched load) and are determined by the relative tuning. Now since the tubes are synchronized, $\omega = \omega_1 + \omega_{22} = \omega_2 - \omega_{22}$, or $\omega_2 - \omega_1 = 2\omega_{22}$ and, in general, for any two operating points

$$\omega_2 - \omega_1 = \omega_{ii} + \omega_{jj} \tag{9}$$

where $\omega_{ii}$ and $\omega_{jj}$ are the frequency deviations of the contours passing through the points.

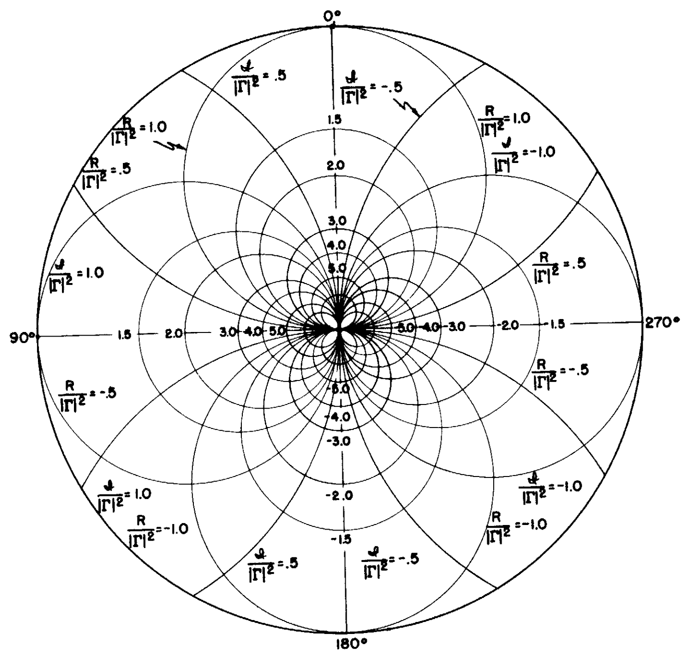


Fig. 2 Contours of constant $R/|\Gamma|^2$ and $\mathcal{X}/|\Gamma|^2$ inside unit circle of the reflection coefficient plane.

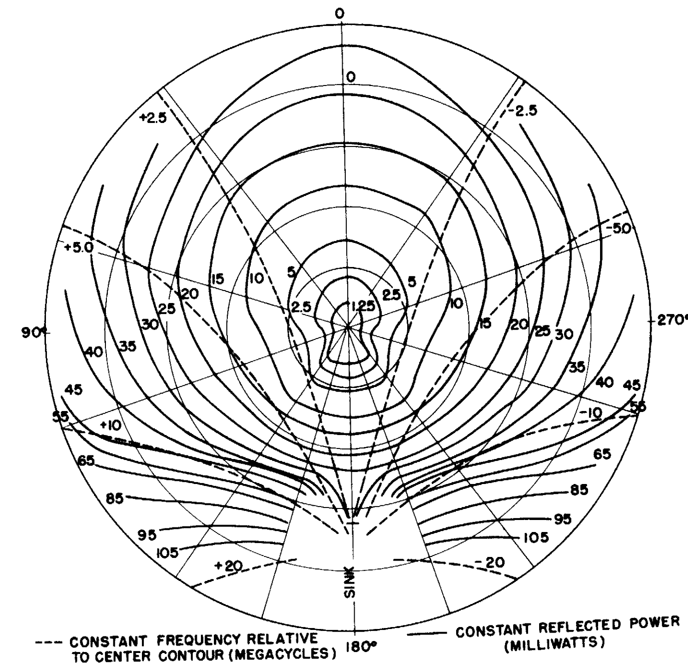


Fig. 3 Constant reflected power and frequency contours.

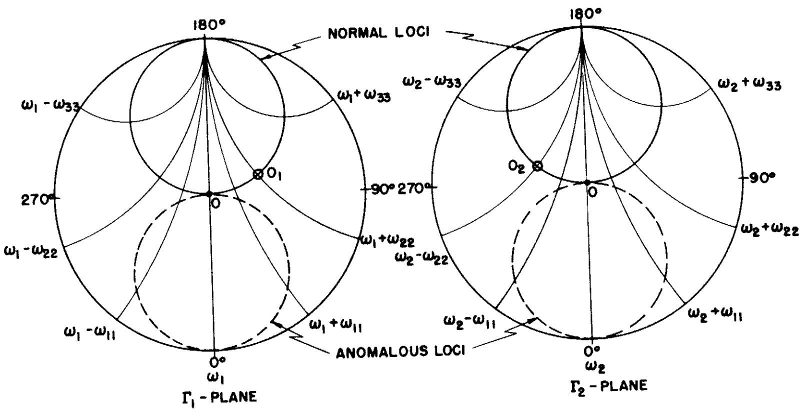


Fig. 4

The synchronized frequency is

$$\omega = \omega_1 + \omega_{ii} = \omega_2 - \omega_{jj} \quad . \tag{10}$$

This discussion makes the assumption that when an oscillator is tuned over the range indicated on its Rieke diagram, this diagram is not changed except for the absolute frequency of the contours. All available evidence indicates that this assumption is quite good for physical and electron-beam tunings, but may not be good for electronic tuning, such as large reflector voltage changes in a reflex klystron (3).

Now a complete picture of normal parallel operation may be constructed. If both oscillators are tuned to the same frequency, they operate at the point $\Gamma = 0$ (matched load). As either or both oscillator tunings are varied so that $|\omega_2 - \omega_1|$ increases, the operating points move outward on the circular loci, the value of $\omega_2 - \omega_1$ and $\omega$ at any time being given by Eqs. 9 and 10. Returning to Figs. 2 and 3, the maximum value of this difference, still maintaining synchronism, for this case is greater than 40 Mc. In general, this value is determined by the factors which become important in the "sink" region such as the series oscillator coupling impedance. Note that if the operating locus actually follows a circular path at large reflection coefficients, $|\omega_2 - \omega_1|$ could assume an infinitely large value. That is, at the point $\Gamma = -1$, the load admittance consists of an infinite susceptance and, hence, may "pull" the oscillator frequency an infinite amount. Actually, of course, even though the load at the oscillator output terminals may approximate a short circuit closely, the series coupling impedance places a lower limit on the effective load impedance, hence on the maximum-frequency pulling. Thus, in this region the shape of the locus is altered by factors not included in this analysis, and the frequency deviation introduced by an infinite susceptance at the output terminals determines the synchronization bandwidth.

There remains one additional matter to be discussed, namely that of the power available from the system. Figure 5 shows constant power output contours for each oscillator. Since the microwave junction is essentially lossless, it may be seen that the load power is the sum of the individual oscillator powers. Once the operating points have been determined, these powers, and hence the load power, are immediately calculable. Also, the latter decreases sharply when $|\omega_2 - \omega_1|$ becomes large; this characteristic is to be expected quite generally.

3. Anomalous Operation of Two Similar Oscillators

When the oscillators are not located at the tee reference planes, or an integral number of half-wavelengths from them, the operation is changed quite drastically. An extreme example of this sort occurs when the spacing is an odd number of quarter-wavelengths. This condition results in operation of the "anomalous" variety.

The analysis may be carried out exactly as for the normal case. However, the Rieke diagram of Fig. 3 must be referred to the tee reference planes. In order to accomplish this, the diagram should be rotated 180°, following Smith Chart computations.

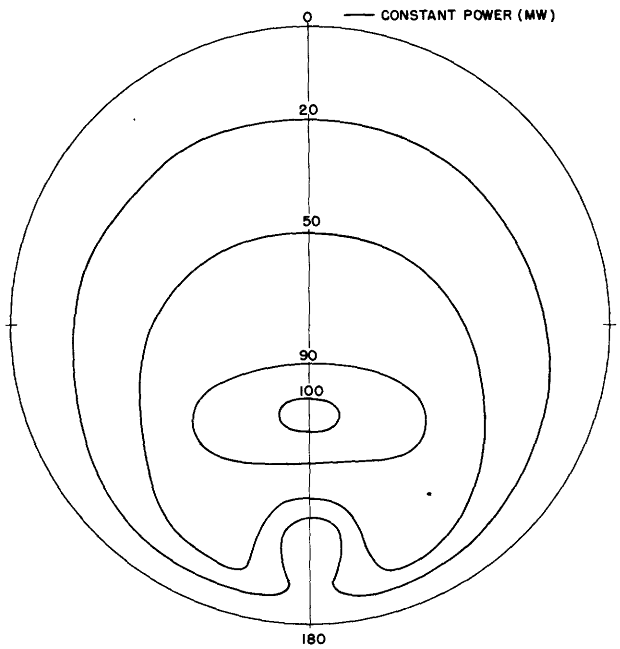

Fig. 5 Power output contours.

Again considering Figs. 2 and 3, it is seen that the operating locus is a circle, but displaced, on the Rieke diagram, 180° from the one previously discussed (see dotted locus in Fig. 4). Thus the operation is entirely similar except in two important respects: (1) the synchronization bandwidth is greatly reduced (7 as compared to > 40 Mc); and (2) each intersecting frequency contour now crosses the locus twice, giving two sets of possible operating points for each tuning of the oscillators. In order to resolve this ambiguity, it is necessary to examine each point with respect to stability.

The angle of the reflection coefficients for the oscillators may be written

$$\theta_1 = \phi_{r_1} - \phi_{i_1} \quad \text{and} \quad \theta_2 = \phi_{r_2} - \phi_{i_2} \tag{11}$$

where $\phi_r$ is the angle of the reflected wave at the tee terminals and $\phi_i$ is that of the incident wave. From the condition $b_1 = -b_2$, imposed by the junction, it is seen that $\phi_{r_1} = \phi_{r_2} \pm \pi$; so by subtracting Eqs. 11 and substituting

$$\theta_1 - \theta_2 \equiv \gamma = \phi_{i_2} - \phi_{i_1} \pm \pi \tag{12}$$

or differentiating with respect to time

$$\frac{d\gamma}{dt} = \frac{d\phi_{i_2}}{dt} - \frac{d\phi_{i_1}}{dt} \sim \Delta\omega_2 - \Delta\omega_1 \quad . \tag{13}^*$$

This equation states a fundamental law as illustrated in Fig. 6, which shows a vector representation of the oscillator outputs. Relative motion of the two vectors determines the quantity, $\Delta\omega_2 - \Delta\omega_1$. If the angle between the vectors is to increase ($d\gamma/dt > 0$),

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* The approximation involved here is a rather subtle one. Specifically, a change of frequency corresponds to the derivative of the phase angle of the total voltage (vector sum of incident and reflected voltages) rather than the derivative of the phase of the incident voltage alone. For most conditions, however, the magnitudes of these derivatives are not greatly different, and for all points inside the unit circle (incident wave > reflected wave) they will have the same sign, which will be our major concern.

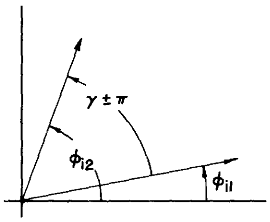


Fig. 6

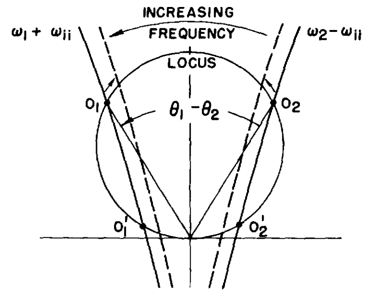


Fig. 7

$\Delta\omega_2$ must be greater than $\Delta\omega_1$, making $\Delta\omega_2 - \Delta\omega_1 > 0$ and inversely. Consider Fig. 7 which shows the operating locus and two sets of operating points illustrating the ambiguity. Suppose now that the oscillators are subject to a perturbation, such as might result from noise modulation of an electrode. Under these conditions, the frequency contours on which $0_1$ and $0_2$ lie shift to some new position, as indicated by the dotted loci in the figure. If the oscillators are to take up the new steady-state operating points, the angle $\theta_1 - \theta_2$ must decrease. Hence $d\gamma/dt = \Delta\omega_2 - \Delta\omega_1 < 0$ by the previous argument. Now when the frequency contours shift, $0_1$ finds itself in a region on the Rieke diagram where the frequency is greater than before the shift; thus $\Delta\omega_1 < 0$. For point $0_2$ the opposite is the case, so $\Delta\omega_2 < 0$. Then the condition $\Delta\omega_2 - \Delta\omega_1 < 0$ must certainly be satisfied, and the oscillators assume the new steady-state condition. A similar examination of points $0_1'$ and $0_2'$ show that $\Delta\omega_2 - \Delta\omega_1$ should be greater than zero, while actually this is not the case, since the signs of $\Delta\omega_2$ and $\Delta\omega_1$ remain the same as for $0_1$ and $0_2$. Hence points $0_1'$ and $0_2'$ are unstable, stable operation occurring at $0_1$ and $0_2$. For stability in general, then, the sign of $\Delta\omega_2 - \Delta\omega_1$, as determined by both the angle $\theta_1 - \theta_2$ and the position of the locus on the Rieke diagram, must be the same.

Quarter-wavelength spacing, therefore, results in a rather unusual condition. Specifically, when the two oscillators are tuned to the same frequency, they work into unity reflection coefficient (open circuit). As $|\omega_2 - \omega_1|$ is increased, the operating points move on a circular path toward smaller reflection coefficients. The locking bandwidth is determined by the frequency contours tangent to the locus. Also, the power available under these conditions is quite small, as may be seen by considering Fig. 5. In fact, when identically tuned, the oscillators give zero output. This anomaly is caused, qualitatively, by phase shifts introduced through the lengths of transmission line. With normal conditions and identical tuning, $\theta_1 - \theta_2 = \pi$ so that $\phi_{i_2} - \phi_{i_1}$, the relative phase of the two generated waves, is zero. Hence reinforcement results. With anomalous conditions and identical tuning, $\theta_1 - \theta_2 = 0$, so $\phi_{i_2} - \phi_{i_1} = \pi$ and destructive interference is produced. Therefore, no voltage appears across the load and no power is delivered.

## 4. General Parallel Operation of Two Similar Oscillators

Thus far the extreme examples of parallel oscillator operation have been considered in detail. One asks immediately about operation in the transition region between these extremes. More specifically, one would like to know the details of the operating loci when (1) the spacing of each oscillator from the tee reference planes is not an integral number of quarter-wavelengths, and (2) when the oscillators are asymmetrically spaced with respect to the junction.

Consider, then, the case of both oscillators located the same distance, d, from the tee and having the physical and electrical distances identical. That is, frequency sensitivity of the transmission line lengths is neglected. In order to determine the loci it is first necessary to refer the Rieke diagrams to the tee reference planes. This may be accomplished by rotation of the diagrams an appropriate amount with respect to the reflection coefficient plane. The loci may then be found through a point-by-point examination of Figs. 2 and 3. Again, each pair of points on the locus must satisfy the conditions imposed by the junction, namely Eqs. 8 and equal reflected powers. This procedure has been carried out for seven different spacings ranging from zero to quarter-wavelength. These loci are shown in Fig. 8. Those for spacings of quarter- to half-wavelength are mirror images of the ones shown, since the Rieke diagram is symmetrical with respect to the center frequency contour. Several facts are to be noted. Firstly, the bandwidth of synchronization is strongly dependent on the spacing. Figure 9 shows the manner of this dependence. Secondly, portions of the loci lie outside the unit circle, indicating there are points at which one oscillator absorbs more power than it delivers. Operation of this sort occurs on the dashed portions of the loci in Fig. 8. These parts are extrapolations of the data; an exact evaluation would require the "load characteristic" of the oscillator, that is, information about its output and frequency at points where the output wave is smaller than the reflected wave. Only points inside the unit circle were used in calculating the bandwidth values shown in Fig. 9. Lastly, in several cases, two sets of points correspond to the same frequency intersections. These ambiguities may be removed by utilizing the stability criterion presented in Sec. 3*. Regions of unstable operation are indicated by double lines in Fig. 8. Thus it is seen that no drastically new aspects of behavior are revealed. The transition from normal to anomalous operation is a systematic one. Finally, the line-length tolerances for normal operation are not severe in practical applications.

One must note that the single loci indicated in Fig. 8 are a special case in which the locus of both oscillators coincide. This situation exists only when the spacings relative to the tee are equal or differ by an integral number of half-wavelengths. In all other cases, two loci will be necessary to describe the system. The simplest condition of this type occurs when the spacings are respectively $n\lambda/2 + x$ and $m\lambda/2 - x$ where m and n are integers, not necessarily equal, and x is the deviation from integral half-wavelength

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*This criterion, as presented, does not apply to operating points outside the unit circle. It may be made pertinent to these cases if we determine the appropriate sign of $\Delta\omega_2 - \Delta\omega_1$ (in Eq. 13). This may be found by examining the vector diagram formed by the incident, reflected, and total voltage at the oscillator terminals for each specific case.

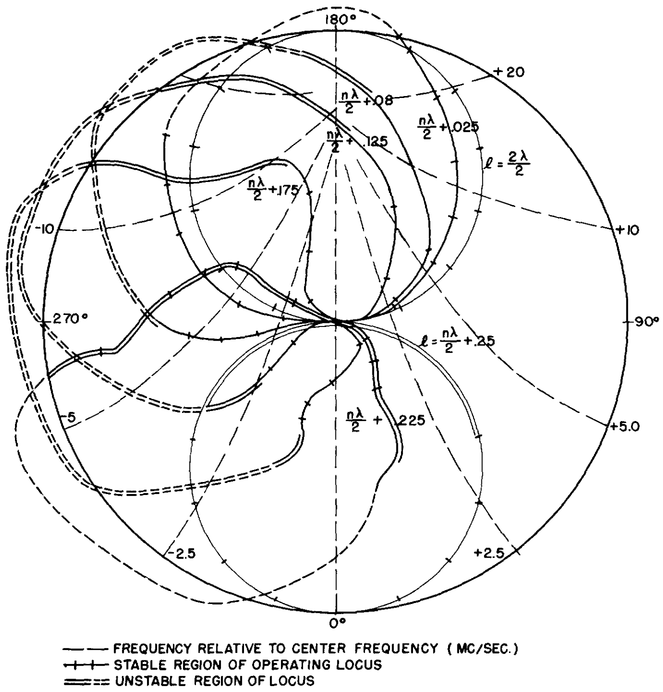


Fig. 8 Operating loci for line spacings nλ/2 to nλ/2 + λ/4.

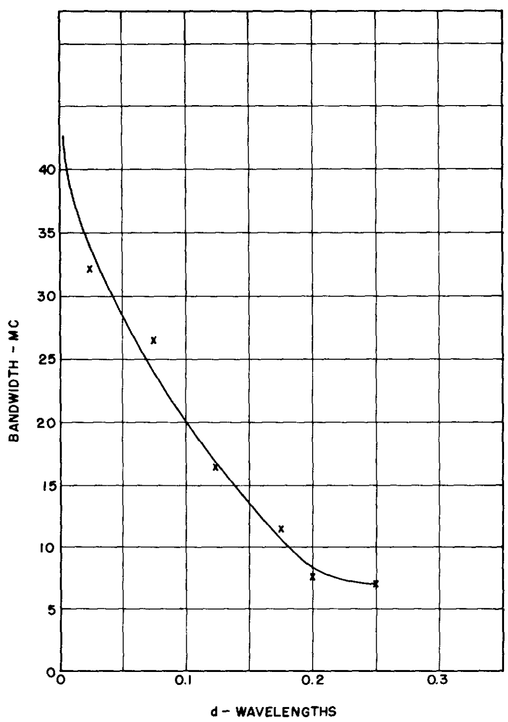


Fig. 9 Locking bandwidth as a function of oscillator spacing.

spacing. Again appropriately rotating the Rieke diagrams and satisfying the junction conditions, the operating loci are found to be circles, passing through the point $\Gamma = 0$ and tangent to the unit circle respectively at points $\lambda/4 + x$ and $\lambda/4 - x$ (see Fig. 10). Stability of operating points on dual loci is determined exactly as formerly. In other cases of unsymmetrical spacing, the loci shapes are distorted and shifted about on the Rieke diagram much as loci 2 through 6 in Fig. 8. These considerations, however, bring no drastic changes in the operation. Similarly, an unsymmetrical Rieke diagram will alter the discussion in detail only. In particular, the circular loci presented here will be somewhat distorted.

5. Series Operation of Two Similar Oscillators

Having considered the parallel operation of two oscillators, it is pertinent to examine the dual case, that of series operation. Oscillators may be placed in series by use of a branched transmission line or an E-plane waveguide tee junction. Usually the branch or load line has twice the characteristic impedance of the line connecting the two oscillators. An equivalent circuit for this type junction is shown in Fig. 11. The scattering equations as calculated from this circuit are

$$
\begin{aligned}
b_1 &= \frac{1}{2} a_1 + \frac{1}{2} a_2 + \frac{\sqrt{2}}{2} a_3 \\
b_2 &= \frac{1}{2} a_1 + \frac{1}{2} a_2 + \frac{\sqrt{2}}{2} a_3 \\
b_3 &= \frac{\sqrt{2}}{2} a_1 + \frac{\sqrt{2}}{2} a_2 \quad .
\end{aligned}
\tag{14}
$$

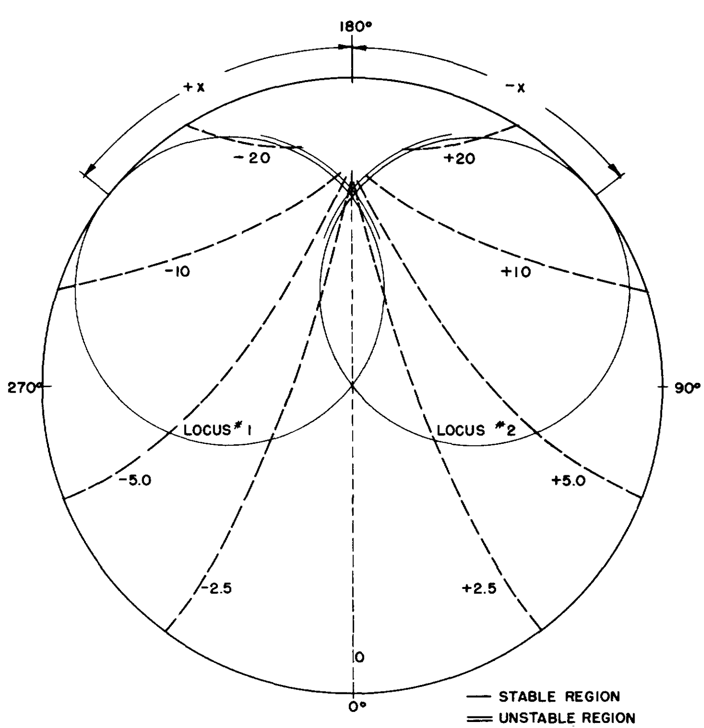


Fig. 10 Operating loci for asymmetrical spacing.

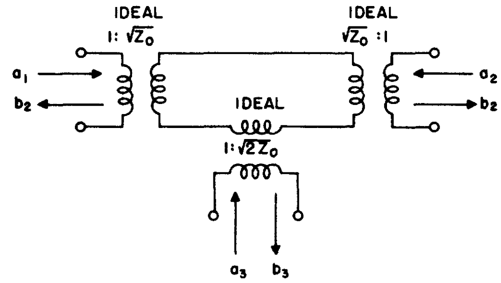


Fig. 11

Equivalent circuit of an idealized series junction.

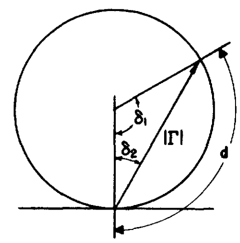

Fig. 12

Phase relationships during normal parallel operation.

Again assuming that the passive load is matched to its line or that $a_3 = 0$, it is seen that $b_1 = b_2$. Continuing the development as before

$$\frac{R_1}{|\Gamma_1|^2} + \frac{R_2}{|\Gamma_2|^2} - 2 = 0$$

$$\frac{I_1}{|\Gamma_1|^2} + \frac{I_2}{|\Gamma_2|^2} = 0 \quad . \tag{15}$$

These are identical to Eqs. 8 for the parallel case except for the sign of the sum of the real parts. Hence, it is seen immediately that the locus for series operation with oscillator spacing d is the same as for parallel operation with spacing $d \pm \lambda/4$. Hence, for normal operation with series connection the oscillators should be placed an odd number of quarter-wavelengths from the tee reference planes. In a similar manner, the entire discussion in Secs. 2, 3, and 4 may be applied to this case.

6. Phase Relationships during Normal Operation

It is of interest to compute the relative phase of oscillators during normal operation. This information, of course, is available for any specific case from the circle diagrams previously presented. However, one would like to know explicitly the dependence of the phase upon oscillator parameters. Consider, then, Fig. 12 showing the normal operating locus. Now the distance d is $d = \delta_1/2$, and $|\Gamma| = 2 \times 1/2 \sin \delta_1/2 = \sin d$. Also, note that the locus is a contour of constant load conductance; hence by the oscillator operating equations, the electronic conductance and susceptance are likewise constants. Then in the equation

$$\frac{b}{\omega_o C} = 2\left(\frac{\omega - \omega_o}{\omega_o}\right) + \frac{B}{Q_{ext}}$$

only B and $\omega$ may change. Along the operating locus, therefore, the load susceptance is related to the frequency in the following manner

$$B = -\frac{2\,Q_{ext}(\omega-\omega_C)}{\omega_o} \tag{16}$$

where $\omega_C$ is the center frequency. Also along the locus, $d = \tan^{-1} B/2$, or

$$d = \tan^{-1}\left[-\frac{Q_{ext}(\omega-\omega_C)}{\omega_o}\right] \quad .$$

Now, note from Fig. 12 and Eq. 12

$$\gamma = 2\delta_2 = 180° - \delta_1 = \phi_{i_2} - \phi_{i_1} \pm \pi$$

or

$$\phi_{i_2} - \phi_{i_1} = -\delta_1$$

and

$$\sin\left(\frac{\phi_{i_2} - \phi_{i_1}}{2}\right) = -\sin\frac{\delta_1}{2} = -\sin d = \sin\left\{\tan^{-1}\left[\frac{Q_{ext}(\omega-\omega_C)}{\omega_o}\right]\right\}$$

or

$$\phi_{i_2} - \phi_{i_1} = 2\tan^{-1}\left[\frac{Q_{ext}(\omega_1-\omega_2)}{2\omega_o}\right] \tag{17}$$

where the relation, $\omega - \omega_C = 1/2(\omega_1 - \omega_2)$, has been used. Equation 17 allows the evaluation of phase sensitivity and tuning tolerance as a function of oscillator bandwidth.

7. Practical Considerations

In most practical applications of mutually coupled oscillators, the center frequencies are made as closely equal as possible. Under such conditions, it is desired that the oscillators each operate into a matched load. For the systems just presented, this is the case so long as normal or near-normal operation is maintained. This fact may be verified by noting that all such operating loci have a stable point at $\Gamma = 0$, corresponding to $\omega_1 - \omega_2 = 0$. Such results were derived assuming the oscillators identical. In practice, the oscillators in question will deviate from this assumption. What will be the effect of this deviation?

It will be shown that when asymmetries between the oscillators exist, they will not work into matched loads even when tuned to the same frequency. Further, it will be seen that this effect is not large and may be compensated by altering the characteristic impedances of the tee arms. In order to show these things, we will assume that the matched condition exists and find the conditions imposed by the junction.

The scattering equations for a junction may be found from equivalent circuits such as shown in Figs. 1 and 11. Considering only the series junction (the development for parallel junctions is entirely similar), the first two equations are

$$b_1 = \left(\frac{-Z_1 + Z_2 + Z_3}{Z_1 + Z_2 + Z_3}\right)a_1 + \left(\frac{2\sqrt{Z_1 Z_2}}{Z_1 + Z_2 + Z_3}\right)a_2$$

$$b_2 = \left(\frac{2\sqrt{Z_1 Z_2}}{Z_1 + Z_2 + Z_3}\right) a_1 + \left(\frac{Z_1 - Z_2 + Z_3}{Z_1 + Z_2 + Z_3}\right) a_2 \tag{18}$$

where $Z_1$, $Z_2$, and $Z_3$ are the characteristic impedances respectively of terminals 1, 2, and 3; and it has been further assumed that the load impedance connected to arm 3 is matched, making $a_3 = 0$. For the matched condition, $b_1 = b_2 = 0$, or

$$(-Z_1 + Z_2 + Z_3)\, a_1 + (2\sqrt{Z_1 Z_2})\, a_2 = 0$$

$$(2\sqrt{Z_1 Z_2})\, a_1 + (Z_1 - Z_2 + Z_3)\, a_2 = 0 \quad . \tag{19}$$

Now $Z_1$, $Z_2$, and $Z_3$ are all positive-real quantities since they are characteristic impedances. Thus the ratio $a_1/a_2$ must be a real quantity if Eqs. 19 are to be satisifed. Therefore, let $a_2 = Ka_1$; then Eqs. 19 become

$$-Z_1 + Z_2 + Z_3 - 2K\sqrt{Z_1 Z_2} = 0$$

$$K(Z_1 - Z_2 + Z_3) + 2\sqrt{Z_1 Z_2} = 0 \quad . \tag{20}$$

By successive addition and subtraction, two expressions for $Z_3$ may be found

$$Z_3 = (Z_2 - Z_1)\,\frac{K+1}{K-1} - 2\sqrt{Z_1 Z_2}$$

and

$$Z_3 = (Z_2 - Z_1)\,\frac{K-1}{K+1} + 2\sqrt{Z_1 Z_2} \quad .$$

Now in the physical circuit these $Z_3$'s must be identical. This condition results in

$$Z_2 - Z_1 = \left(\frac{K^2 - 1}{K}\right)\sqrt{Z_1 Z_2} \quad . \tag{21}$$

This relation must be satisfied if the oscillators are to work into a matched load when identically tuned. Replacing this condition in either expression for $Z_3$

$$Z_3 = \left(\frac{K^2 + 1}{K}\right)\sqrt{Z_1 Z_2} \quad . \tag{22}$$

Thus, the load arm impedance is likewise uniquely specified. Note that if the oscillators are identical $K = 1$ and $Z_1 = Z_2$ and $Z_3 = 2Z_1 = 2Z_2$, which is the case previously considered.

That the effect of an asymmetry is not large may be seen from a numerical example. The value of K may be found by taking $K^2$ as the ratio of oscillator powers when both are operating at $\Gamma = 0$. For this example, take $K^2 = 0.81$, $K = 0.90$, and $Z_1 = 50$ ohms. By Eqs. 21 and 22, $Z_2 = 40.61$ ohms and $Z_3 = 90.5$ ohms. There is another, and perhaps more meaningful, computation which can be made to show the relative magnitude of this effect. Consider the case $Z_1 = Z_2$, $Z_3 = 2Z_2$, and $a_2 = -Ka_1$, where K is real. (This latter assumption is good only when $K \approx 1$.) From the scattering equations

$$b_1 = \frac{1}{2} a_1 + \frac{1}{2} a_2 = \frac{1}{2} a_1 (1 - K) = \frac{1}{2} a_2 (1 - \frac{1}{K})$$

or

$$\Gamma_1 = \frac{b_1}{a_1} = \frac{1}{2} (1 - K)$$

$$\Gamma_2 = \frac{b_2}{a_2} = \frac{1}{2} (1 - \frac{1}{K}) \quad . \tag{23}$$

For K = 0.9, then, $\Gamma_1 = 0.050$ and $\Gamma_2 = -0.055$. Thus, the resulting mismatch is quite small, even when there is a 20 percent difference in the oscillator power outputs. In conclusion, oscillators may be operated successfully in series or parallel for most practical uses even if only approximately alike.

## 8. Conclusions

The following conclusions concerning mutually coupled oscillators may be drawn from the previous discussion.

1. The general problem of N oscillators working into an M terminal-pair network may be solved by using the scattering equations for the network and Rieke diagrams of the oscillators. In order to obtain a solution in any particular case, however, an arduous trial-and-error examination may be necessary.

2. The operating conditions of mutually coupled oscillators are determined principally by:

   a. The interconnecting network

   b. Oscillator spacing relative to the network

   c. Relative oscillator tuning.

3. Dissimilarities between the oscillators, so long as they are not large, do not affect the operation appreciably if the tuning of each is nearly the same.

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