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Probabilistic Analysis of the 1-Tree Relaxation for the Euclidean Traveling Salesman Problem

Michel X. Goemans *   Dimitris J. Bertsimas †

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Abstract

We analyze probabilistically the classical Held-Karp lower bound derived from the 1-tree relaxation for the Euclidean traveling salesman problem (ETSP). We prove that, if n points are uniformly and independently distributed over the d-dimensional unit cube, the Held-Karp lower bound on these n points is almost surely asymptotic to $\beta_{HK}(d)\, n^{(d-1)/d}$, where $\beta_{HK}(d)$ is a constant independent of n. The result suggests a probabilistic explanation of the observation that the lower bound is very close to the length of the optimal tour in practice since the ETSP is almost surely asymptotic to $\beta_{TSP}(d)\, n^{(d-1)/d}$. The techniques we use exploit the polyhedral description of the Held-Karp lower bound and the theory of subadditive Euclidean functionals.

Key words. Probabilistic analysis, traveling salesman problem, linear relaxation, 1-trees, subadditive Euclidean functionals.

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1 Introduction

During the last two decades combinatorial optimization has been one of the fastest growing areas in the field of mathematical programming. Some of the major contributions were Lagrangian relaxation, polyhedral theory and probabilistic analysis.

The landmark in the development of Lagrangian relaxation (see Geoffrion [6] or Fisher [5]) for combinatorial optimization problems were the two papers for the traveling salesman problem (TSP) by Held and Karp [9], [10]. In the first paper, Held and Karp [9] proposed a Lagrangian relaxation based on the notion of 1-tree for the TSP. Using a complete characterization of the 1-tree polytope, which follows from a result of Edmonds [4] for matroids, they showed that this Lagrangian relaxation gives the same bound as the linear relaxation of a classical formulation of the TSP. In the second paper, Held and Karp [10] introduced a method, which is now known under the name of subgradient optimization (Held, Wolfe and Crowder [11]), to solve the Lagrangian dual. The 1-tree relaxation has been extensively and successfully used to devise branch and bound procedures to solve the TSP (see Held and Karp [10], Helbig Hansen and Krarup [8], Smith and Thompson [17], Volgenant and Jonker [20] or, for a survey, Balas and Toth [1]). These computational studies have shown that, on the average, the Held-Karp lower bound is extremely close to the length of the optimal tour. According to most of the above authors (see also Christofides [3] and Johnson [12]) the relative gap is often less or much less than 1%. On a theoretical ground, a result due to Wolsey [21] states that the Held-Karp lower bound is never less than 2/3 of the length of the optimal tour when the triangle inequality is satisfied. However, this worst-case analysis does not capture the efficiency of the bound in practice. The probabilistic analysis developed in this paper is aimed at shedding new light on the behavior of the Held-Karp lower bound.

The area of probabilistic analysis has its origin in the pioneering paper by Beardwood, Halton and Hammersley [2]. The authors characterize very sharply the asymptotic behavior of the TSP if the points are uniformly and independently distributed in the Euclidean plane or, more generally, in R^d. The potential importance of this early work was demonstrated in Karp [13]. Steele [18] analyzed probabilistically a general class of combinatorial optimization problems by developing the notion of subadditive Euclidean functionals. In Karp and Steele [14] the original proof of Beardwood et al. [2] is simplified using the Efron-Stein inequality. In Steele [19] an even simpler proof is offered using martingale inequalities. Martingale inequalities were first applied to the probabilistic analysis of combinatorial optimization problems by Rhee and Talagrand [16].

In this paper, we combine the combinatorial interpretation of the Held-Karp lower bound with the probabilistic techniques of Steele [18]. We prove that, if n points are uniformly and independently distributed over the d-dimensional unit cube, the Held-Karp lower bound on these n points divided by $n^{(d-1)/d}$ is almost surely asymptotic to a constant $\beta_{HK}(d)$. When $d = 2$, we prove the complete convergence of the Held-Karp lower bound divided by $\sqrt{n}$. We exploit extensively the fact that the bound can be viewed as the cost of the best convex combination of 1-trees such that each vertex has degree 2 on the average. Relying on computational studies for the TSP and the matching problem in the Euclidean plane, we estimate that the asymptotic gap $(\beta_{TSP} - \beta_{HK})/\beta_{TSP}$ is less than 3%. To our best knowledge this is the first time that a linear relaxation of a combinatorial optimization problem is analyzed probabilistically using subadditivity techniques.

The remaining of this paper is structured as follows. Section 2 reviews briefly the main results of the Held and Karp [9] paper. In section 3 we first prove that the Held-Karp lower bound is monotone and subadditive and then prove the main theorem. In section 4 we use a martingale inequality to derive some sharp bound for the Held-Karp lower bound and we establish its complete convergence.

2 Held-Karp lower bound

In this section we summarize the main results of Held and Karp [9]. They presented a lower bound on the length of the optimal tour to the symmetric traveling salesman problem on the complete undirected graph with vertex set V. This bound can be described in several equivalent ways.

First, it can be expressed as the optimal objective function value HK of the linear relaxation of the following standard formulation of the TSP:

$$Min \sum_{i \in V} \sum_{\substack{j \in V \\ j > i}} c_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{\substack{j \in V \\ j > i}} x_{ij} + \sum_{\substack{j \in V \\ j < i}} x_{ji} = 2 \qquad \forall i \in V \tag{2}$$

$$\sum_{i \in S} \sum_{\substack{j \in S \\ j > i}} x_{ij} \leq |S| - 1 \qquad \forall\, \emptyset \neq S \subset V \tag{3}$$

$$0 \leq x_{ij} \leq 1 \qquad \forall i, j \in V, j > i \tag{4}$$

$$x_{ij} \ integer \qquad \forall i, j \in V, j > i \tag{5}$$

In this program, x_{ij} indicates whether cities i and j are adjacent in the optimal tour; c_{ij} represents the cost of traveling from city i to city j or, by symmetry, from city j to city i.

We now give two alternative definitions of a 1-tree which constitutes the core of the other formulations.

Definition 1 *$T = (V, E)$ is a 1-tree (rooted at vertex 1) if T consists of a spanning tree on $V \setminus \{1\}$, together with two edges incident to vertex* 1.

From now on we shall always assume, unless otherwise stated, that the root node is identical for any 1-tree, say vertex 1.

Definition 2 *$T = (V, E)$ is a 1-tree if*

1. *T is connected*
2. *$|V| = |E|$*
3. *T has a cycle containing vertex 1*
4. *the degree in T of vertex 1 is 2.*

Held and Karp [9] highlighted the relation between the linear program (1)-(4) and the class of 1-trees. More precisely, they showed that the feasible solutions to (2)-(4) can be equivalently characterized as convex combinations of 1-trees such that each vertex has degree 2 on the average. Hence, we may rewrite (1)-(4) in the following way:

$$HK = Min \sum_{r=1}^{k} \lambda_r c(T_r) \tag{6}$$

subject to

$$\sum_{r=1}^{k} \lambda_r = 1 \tag{7}$$

$$\sum_{r=1}^{k} \lambda_r d_j(T_r) = 2 \qquad \forall j \in V \setminus \{1\} \tag{8}$$

$$\lambda_r \geq 0 \qquad r = 1, \ldots, k, \tag{9}$$

where

- $\{T_r\}_{r=1,\ldots,k}$ constitutes the class of 1-trees defined on the vertex set V,
- $c(G) = \sum_{e=(i,j)\in E} c_{ij}$ is the total cost of the subgraph $G = (V, E)$ and
- $d_j(T)$ denotes the degree in T of vertex j.

Finally, the most common approach to find the Held-Karp lower bound is to take the Lagrangian dual of (6)-(9) with respect to (8). We then obtain:

$$HK = \max_{\mu} L(\mu) \tag{10}$$

subject to

$$L(\mu) = \min_{r=1,\ldots,k} c_\mu(T_r) - 2\sum_{j\in V} \mu_j \tag{11}$$

where $c_\mu(T_r)$ is the cost of the 1-tree T_r with respect to the costs $c_{ij} + \mu_i + \mu_j$.

3 The main theorem

Let the n points $X^{(n)} = (X_1, X_2, \ldots, X_n)$ be uniformly and independently distributed in the d-cube $[0,1]^d$. Let $HK(X^{(n)})$ denote the Held-Karp lower bound on $X^{(n)}$ as defined by any of the formulations of section 2. We are interested in the behavior, as n tends to infinity, of $HK(X^{(n)})$. Steele [18] proved that the asymptotic behavior of a particular class of Euclidean functionals L defined on finite subsets of R^d to R can be characterized very sharply as follows:

Theorem 1 (Steele [18]) *Let L be a* monotone $[L(A\cup\{x\}) \geq L(A)\ \forall x \in R^d, \forall A \subset R^d]$, Euclidean $[L(ax_1, ax_2, \ldots, ax_n) = aL(x_1, x_2, \ldots, x_n), L(x_1+x, x_2+x, \ldots, x_n + x) = L(x_1, x_2, \ldots, x_n)]$ *functional of* finite variance $[Var[L(X^{(n)}] < \infty]$ *which satisfies the* subadditivity hypothesis*:*

If $\{Q_i : 1 \leq i \leq m^d\}$ is a partition of the d-cube $[0,1]^d$ into m^d identical subcubes with edges parallel to the axes then there exists a constant $C > 0$ such that $\forall m \in N \setminus \{0\}, \forall t > 0$, we have that

$$L(\{x_1, \ldots, x_n\} \cap [0,t]^d) \leq \sum_{i=1}^{m^d} L(\{x_1, \ldots, x_n\} \cap tQ_i) + Ctm^{d-1}.$$

Then there exists a constant $\beta_L(d)$ such that

$$\lim_{n\to\infty} \frac{L(X^{(n)})}{n^{(d-1)/d}} = \beta_L(d)$$

almost surely.

We emphasize that the critical property in theorem 1 is the subadditivity hypothesis. It can easily be seen that HK is a Euclidean functional with finite variance. Proposition 2 proves that the subadditivity hypothesis holds for the functional HK. The monotonicity of HK is proved in proposition 3. For these propositions the most useful formulation of the Held-Karp lower bound is (6)-(9). For convenience and clarity we denote by P(A) the program (6)-(9) corresponding to the set A of cities.

Proposition 2 *HK is subadditive, i.e.* $\exists C > 0, \forall m \in N \setminus \{0\}, \forall t > 0$:

$$HK(\{x_1, \ldots, x_n\} \cap [0, t]^d) \leq \sum_{i=1}^{m^d} HK(\{x_1, \ldots, x_n\} \cap tQ_i) + Ctm^{d-1}$$

for any finite subset of R^d.

Proof:

Using the fact that HK is a Euclidean functional, we may restrict ourselves to the case $t = 1$. Let $V = \{x_1, \ldots, x_n\} \cap [0, 1]^d$ and $V_i = \{x_1, \ldots, x_n\} \cap Q_i$ for $i = 1, \ldots, m^d$. Let $p = m^d$. We arbitrarily choose a root vertex 1_i in every V_i. Let $\{T_{i1}, T_{i2}, \ldots, T_{ik_i}\}$ be the class of 1-trees defined on V_i (with respect to the root 1_i). We consider the optimal solution $\{\lambda_{ir}\}_{r=1,\ldots,k_i}$ to $P(V_i)$, i.e. $\{\lambda_{ir}\}_{r=1,\ldots,k_i}$ satisfies:

$$\sum_{r=1}^{k_i} \lambda_{ir} = 1 \qquad (12)$$

$$\sum_{r=1}^{k_i} \lambda_{ir} d_j(T_{ir}) = 2 \qquad \forall j \in V_i \setminus \{1_i\} \qquad (13)$$

$$\lambda_{ir} \geq 0 \qquad r = 1, \ldots, k_i \qquad (14)$$

$$HK(V_i) = \sum_{r=1}^{k_i} \lambda_{ir} c(T_{ir}) \qquad (15)$$

From these optimal solutions we shall construct a feasible solution to $P(V)$ whose cost is less than

$$\sum_{i=1}^{p} HK(V_i) + Cm^{d-1} \tag{16}$$

where $C = 2\sqrt{d+3}$. For this purpose, we consider every possible combination of selecting one 1-tree in each subcube Q_i. There are $(\prod_{i=1}^{p} k_i)$ such combinations. Let us focus on one of them, say $\{T_{ir_i}\}_{i=1,\ldots,p}$. Let Λ be the indices $(r_1, r_2, \ldots, r_p)$ of the corresponding 1-trees. From these p 1-trees we shall construct a 1-tree T_Λ rooted at 1_1, spanning V and satisfying the following conditions:

$$d_j(T_\Lambda) = d_j(T_{ir_i}) \qquad if\ j \in Q_i \tag{17}$$

$$c(T_\Lambda) \leq \sum_{i=1}^{p} c(T_{ir_i}) + Cm^{d-1} \tag{18}$$

We claim that, by assigning a weight of $\lambda_\Lambda = \prod_{i=1}^{p} \lambda_{ir_i}$ to each 1-tree T_Λ we get a feasible solution to P(V) whose cost is less than (16). Indeed,

1. Using (12) recursively, we have:

$$\begin{aligned}
\sum_{\Lambda} \lambda_\Lambda &= \sum_{\Lambda=(r_1,\ldots,r_p)} \prod_{i=1}^{p} \lambda_{ir_i} \\
&= \sum_{r_1=1}^{k_1} \lambda_{1r_1} \sum_{r_2=2}^{k_2} \lambda_{2r_2} \cdots \sum_{r_p=p}^{k_p} \lambda_{pr_p} \\
&= 1.
\end{aligned} \tag{19}$$

2. Consider any vertex $j \in V$. Assume that $j \in Q_i$. We have that:

$$\begin{aligned}
\sum_{\Lambda} \lambda_\Lambda d_j(T_\Lambda) &= \sum_{\Lambda} \lambda_\Lambda d_j(T_{ir_i}) \\
&= \sum_{r_i=1}^{k_i} \lambda_{ir_i} d_j(T_{ir_i}) \prod_{j \in \{1,\ldots,p\} \setminus \{i\}} \sum_{r_j=1}^{k_j} \lambda_{jr_j} \\
&= \sum_{r_i=1}^{k_i} \lambda_{ir_i} d_j(T_{ir_i}) \\
&= 2
\end{aligned} \tag{20}$$

using (17), (12) and (13) respectively.

3. $\lambda_\Lambda \geq 0$ follows from (14).

1, 2 and 3 imply that the solution is feasible in $P(V)$. The cost of this solution is given by:

$$\begin{aligned}
\sum_\Lambda \lambda_\Lambda c(T_\Lambda) &\leq \sum_{\Lambda=(r_1,\ldots,r_p)} \lambda_\Lambda (\sum_{i=1}^{p} c(T_{ir_i})) + \sum_\Lambda C m^{d-1} \lambda_\Lambda \\
&= \sum_{i=1}^{p} \sum_{r_i=1}^{k_i} \lambda_{ir_i} c(T_{ir_i}) + C m^{d-1} \\
&= \sum_{i=1}^{p} HK(V_i) + C m^{d-1} \qquad (21)
\end{aligned}$$

using (18), (12), (19) and (15) respectively. The last point left in this proof is the construction of the 1-tree T_Λ satisfying (17) and (18). We proceed in 2 steps:

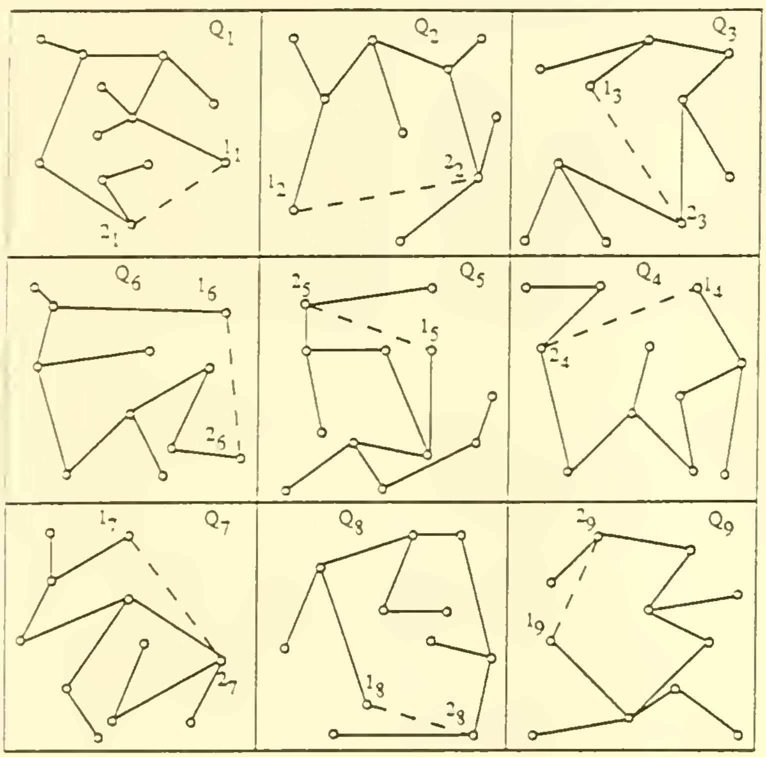

Figure 1: Step 1 in the construction of T_Λ.

1. (Figure 1) In each 1-tree T_{ir_i} ($i = 1, \dots, p$) we delete one of the 2 edges incident to the root 1_i, say $(1_i, 2_i)$. Note that typically 2_i depends on r_i.

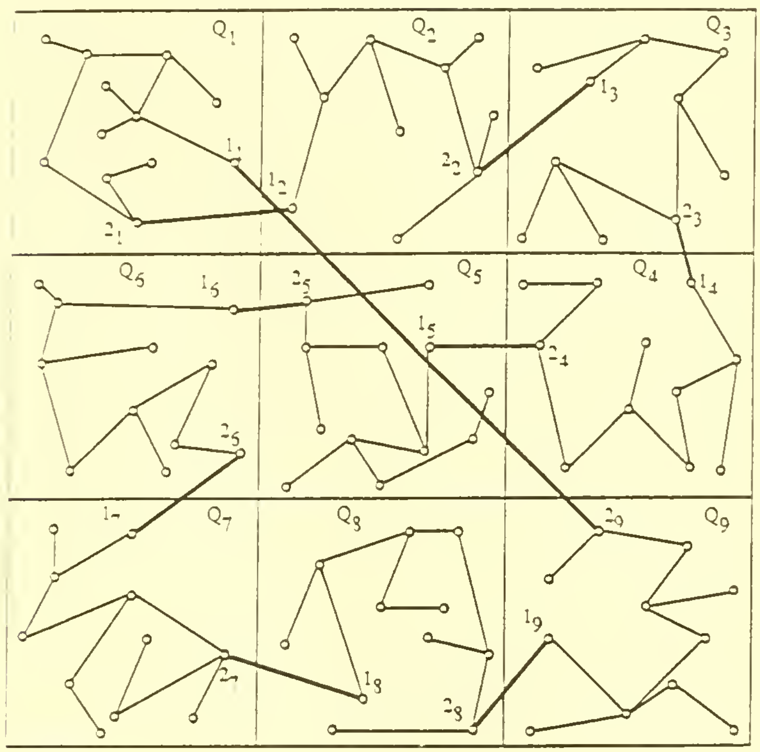

Figure 2: Step 2 in the construction of T_Λ.

2. (Figure 2) Assume that the numbering of the subcubes is such that the subcubes Q_i and Q_{i+1} ($i = 1, \dots, p-1$) are adjacent. Such a numbering clearly exists for every d. A possible numbering for the case $d = 2$ is represented in Figure 3. We now add the edges $(2_i, 1_{i+1})$ ($i = 1, \dots, p-1$) and the edge $(2_p, 1_1)$.

We first claim that the resulting subgraph $T_\Lambda = (V, E_\Lambda)$ is a 1-tree rooted at vertex 1_1. This follows from definition 2. Indeed T_Λ is clearly connected, the number of

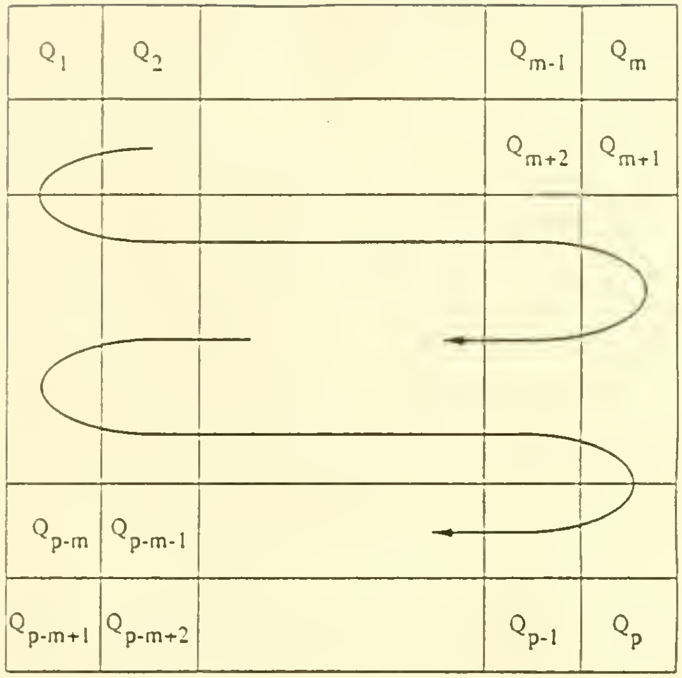

Figure 3: Numbering of the subsquares when $d = 2$.

edges of T_Λ is $|E_\Lambda| = \sum_{i=1}^{p}(|E_{ir_i}| - 1) + p = \sum_{i=1}^{p}|E_{ir_i}| = \sum_{i=1}^{p}|V_i| = |V|$, T_Λ has a cycle containing vertex 1_1 and the degree in T_Λ of vertex 1_1 is 2. Secondly, from the construction, it is evident that we have not changed the degree of any vertex. Therefore (17) holds. Finally, we have added $(p-1)$ edges of length at most $\sqrt{d+3}/m$ and one edge of length at most $\sqrt{d}$. Hence,

$$\begin{aligned} c(T_\Lambda) &\le \sum_{i=1}^{p} c(T_{ir_i}) + (m^d - 1)\frac{\sqrt{d+3}}{m} + \sqrt{d} \\ &= \sum_{i=1}^{p} c(T_{ir_i}) + \sqrt{d+3}\, m^{d-1} - \frac{\sqrt{d+3}}{m} + \sqrt{d} \\ &\le \sum_{i=1}^{p} c(T_{ir_i}) + Cm^{d-1} \end{aligned}$$

and therefore (18) is also satisfied. This completes the proof of proposition 2. □

We now prove the monotonicity of the functional HK.

Proposition 3 *If $n \geq 3$ then HK is monotone, i.e.*

$$\forall x_1, \ldots, x_{n+1} \in R^d : HK(x_1, \ldots, x_{n+1}) \geq HK(x_1, \ldots, x_n).$$

Proof:

Let $\{T_1, \ldots, T_k\}$ be the class of 1-trees defined on a set V of n+1 points. Without loss of generality we may assume that the optimal solution $\{\lambda_i\}_{i=1,\ldots,k}$ to $P(V)$ is a basic feasible solution and thus rational. Let $\lambda = \gcd(\lambda_1, \ldots, \lambda_k)$, i.e. λ is the greatest rational that divides λ_i for $i = 1, \ldots, k$. By duplicating T_i (λ_i/λ) times we get a multiset $\mathcal{S} = \{T_i\}_{i=1,\ldots,l}$ of 1-trees such that each 1-tree has weight λ in the optimal solution. For clarity we assume that two identical 1-trees can be differentiated and therefore every multiset can be seen as a set. Now assume that we want to remove vertex $(n+1)$. Let $V' = V \setminus \{n+1\}$. We shall construct a feasible solution to $P(V')$ whose cost is less than $HK(V)$. For this purpose, we first need to show that the optimal solution to $P(V)$ can be decomposed in such a way that $\mathcal{S}$ does not contain some particular 1-trees. Let $\mathcal{S}_\triangle = \{T : T \in \mathcal{S}, d_{n+1}(T) = 2, \exists j \in V : (1,j), (1, n+1), (j, n+1) \in T\}$. A possible candidate for $\mathcal{S}_\triangle$ is represented in Figure 4.

Claim 1 *Without loss of generality, $\mathcal{S}_\triangle$ can be assumed to be empty.*

Indeed, let $T \in \mathcal{S}_\triangle$ such that $(1,j), (1, n+1), (j, n+1) \in T$ and $d_{n+1}(T) = 2$. As $n + 1 \geq 4$, the degree of vertex j in T is at least 3. Therefore, since the degree of each vertex is 2 on the average, there exists a 1-tree $T' \in \mathcal{S}$ such that $d_j(T') = 1$. Let i_1 and i_2 be the two vertices adjacent to vertex 1 in T'. Without loss of generality, we may assume that $i_1 \neq n+1$. Moreover, since T' is a 1-tree with $d_j(T') = 1$, we have that $i_1 \neq j$ and $i_2 \neq j$. If we replace $(1,j)$ in T by $(1, i_1)$ and $(1, i_1)$ in T' by $(1,j)$, we get two 1-trees $\overline{T}$ and $\overline{T'}$ which are not in $\mathcal{S}_\triangle$. This

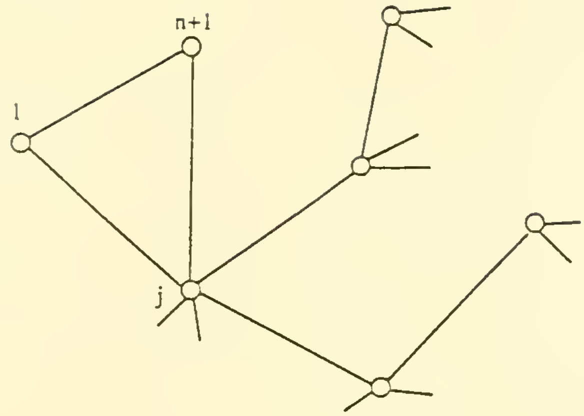

Figure 4: 1-tree in $\mathcal{S}_\triangle$.

basically follows from the fact that $(1, j) \not\in \overline{T}$ while $(1, n+1), (j, n+1) \in \overline{T}$ and that if $(1, n+1)$ and $(j, n+1)$ were both in $\overline{T'}$ and $d_{n+1}(\overline{T'}) = 2$ then T' would not be a 1-tree since $T' \setminus \{(1, i_1), (1, n+1)\}$ would be disconnected. But $\mathcal{S} \setminus \{T, T'\} \cup \{\overline{T}, \overline{T'}\}$ represents the same optimal solution as previously since T and T' have the same weight λ. Hence, by applying this procedure repeatedly, we see that we may assume, without loss of generality, that $\mathcal{S}_\triangle = \emptyset$.

Let $\mathcal{S}_i = \{T \in \mathcal{S} : d_{n+1}(T) = i\}$, $i = 1, 2$. We duplicate every 1-tree T in $\mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$ $(d_{n+1}(T) - 2)$ times and we associate to each copy a weight of $\lambda/(d_{n+1}(T) - 2)$ in order to keep the solution unchanged. Call $\mathcal{S}_3$ the resulting set. Note that the weight associated to the 1-trees in $\mathcal{S}_1$ or $\mathcal{S}_2$ is still λ while the weight associated to a 1-tree T in $\mathcal{S}_3$ is $\lambda/(d_{n+1}(T) - 2)$.

Claim 2 $|\mathcal{S}_1| = |\mathcal{S}_3|$.

Since vertex n+1 has degree 2 on the average, we have

$$\sum_{T\in\mathcal{S}_1}\lambda+\sum_{T\in\mathcal{S}_2}2\lambda+\sum_{T\in\mathcal{S}_3}d_{n+1}(T)\frac{\lambda}{d_{n+1}(T)-2}=2. \tag{22}$$

Now the claim follows by substracting the equality $\sum_{T\in\mathcal{S}_1}\lambda+\sum_{T\in\mathcal{S}_2}\lambda+\sum_{T\in\mathcal{S}_3}\frac{\lambda}{d_{n+1}(T)-2}=1$ twice from (22).

This means that we can regroup $\mathcal{S}_1$ and $\mathcal{S}_3$ into a set $\mathcal{S}_{13}$ of pairs (T_1,T_3) of 1-trees of $\mathcal{S}_1$ and $\mathcal{S}_3$ ($|\mathcal{S}_{13}|=|\mathcal{S}_1|=|\mathcal{S}_3|$). From $\mathcal{S}_2$ and $\mathcal{S}_{13}$ we shall construct a feasible solution to $P(V')$ whose total cost is less than $HK(V)$. More precisely, we associate to each 1-tree $T\in\mathcal{S}_2$ (to each pair $(T_1,T_3)\in\mathcal{S}_{13}$, respectively) a 1-tree T' (a pair (T_1',T_3') of 1-trees, respectively) defined on V' such that:

$$\lambda c(T')\le\lambda c(T) \tag{23}$$

$$\left(\lambda c(T_1')+\frac{\lambda}{d_{n+1}(T_3)-2}c(T_3')\le\lambda c(T_1)+\frac{\lambda}{d_{n+1}(T_3)-2}c(T_3),\ \text{resp.}\right)$$

$$\lambda d_j(T')=\lambda d_j(T)\qquad\forall j\in V' \tag{24}$$

$$\left(\lambda d_j(T_1')+\frac{\lambda}{d_{n+1}(T_3)-2}d_j(T_3')=\lambda d_j(T_1)+\frac{\lambda}{d_{n+1}(T_3)-2}d_j(T_3),\ \text{resp.}\right)$$

hold. Combining (23) and (24) we clearly see that, by keeping the old weights, we get a feasible solution to $P(V')$ whose cost is less than the cost of the optimal solution to $P(V)$ which is $HK(V)$.

The construction of T' and (T_1',T_3') is as follows:

1. $\mathbf{T}\in\mathcal{S}_2$

   Let $(i,n+1)$ and $(j,n+1)$ be the two edges incident to vertex $n+1$ in T. Let $T'=T\setminus\{(i,n+1),(j,n+1)\}\cup\{i,j\}$. The fact that T' is a 1-tree on V' follows from definition 2 and the fact that we can assume without loss of generality that $\mathcal{S}_\Delta=\emptyset$ (claim 2). Clearly (24) is satisfied and the triangle inequality implies that (23) holds.

2. $(\mathbf{T_1},\mathbf{T_3})\in\mathcal{S}_{13}$

   Let $i\ne 1$ be the unique vertex adjacent to $(n+1)$ in T_1. Let $\nu=d_{n+1}(T_3)\ge 3$.

Let $j_1, \ldots, j_\nu$ be the vertices adjacent to $n+1$ in T_3. We may assume without loss of generality that i is in the same connected component as j_1 when we remove the vertices 1 and $n+1$ in T_3. Moreover, if vertex 1 is adjacent to vertex $n+1$, we may assume that $j_2 = 1$ if and only if $(1, j_1) \notin T_3$. The

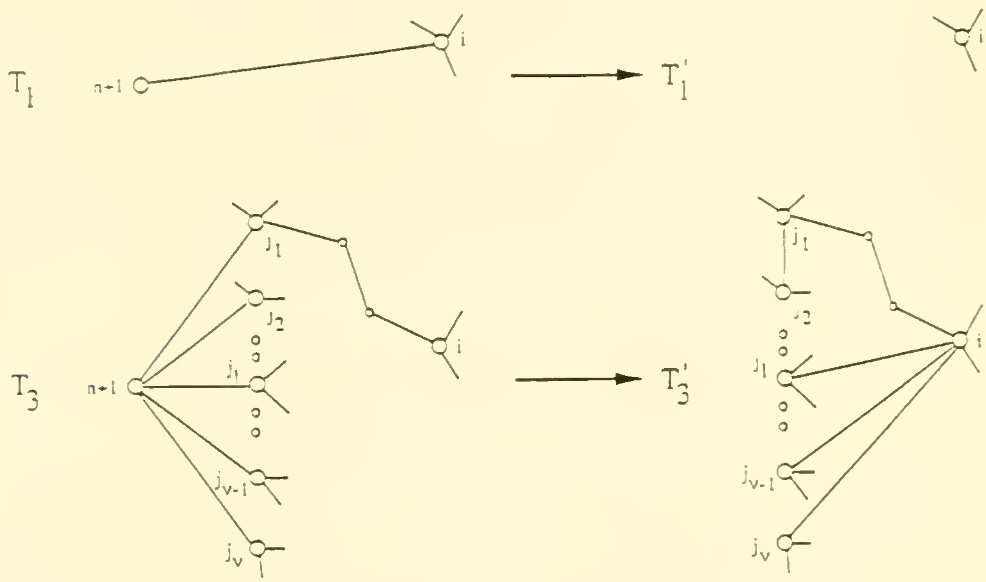

Figure 5: Construction of T_1' and T_3'.

transformation is the following (see Figure 5):

$$T_1' = T_1 \setminus \{(i, n+1)\}$$

$$T_3' = T_3 \setminus \{(j_1, n+1), \ldots, (j_\nu, n+1)\} \cup \{(j_1, j_2), (j_3, i), \ldots, (j_\nu, i)\}$$

The fact that T_1' is a 1-tree is obvious. We notice that none of the edges added to T_3 were already present in T_3. We then check that T_3' is connected, $|T_3'| = |T_3| - 1 = |V| - 1 = |V'|$, T_3' has a cycle containing vertex 1 and $d_1(T_3') = 2$. Hence, by definition 2, T_3' is a 1-tree. Using the triangle inequality, we have

$$\lambda c(T_1) + \frac{\lambda}{\nu - 2} c(T_3) - \lambda c(T_1') - \frac{\lambda}{\nu - 2} c(T_3')$$

$$
\begin{aligned}
&= \lambda c_{i,n+1} + \sum_{k=1}^{\nu} \frac{\lambda}{\nu-2} c_{j_k,n+1} - \frac{\lambda}{\nu-2} c_{j_1 j_2} - \sum_{k=3}^{\nu} \frac{\lambda}{\nu-2} c_{j_k i} \\
&= \frac{\lambda}{\nu-2}(c_{j_1,n+1} + c_{j_2,n+1} - c_{j_1 j_2}) + \\
&\quad \frac{\lambda}{\nu-2} \sum_{k=3}^{\nu} (c_{i,n+1} + c_{j_k,n+1} - c_{j_k i})
\end{aligned}
$$

and therefore (23) holds. Moreover, since

$$
d_j(T_1') - d_j(T_1) = \begin{cases} -1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}
$$

and

$$
d_j(T_3') - d_j(T_3) = \begin{cases} \nu - 2 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}
$$

we see that

$$
\lambda d_j(T_1') + \frac{\lambda}{\nu-2} d_j(T_3') - \lambda d_j(T_1) - \frac{\lambda}{\nu-2} d_j(T_3) = 0.
$$

Hence, (24) is satisfied.

This completes the proof of proposition 3. □

We may now deduce the asymptotic behavior of HK as a corollary to theorem 1 and propositions 2 and 3.

Theorem 4 *Let the n points $X^{(n)} = (X_1, \ldots, X_n)$ be uniformly and independently distributed in the d-dimensional unit cube. Then there exists a constant $\beta_{HK}(d)$ such that*

$$
\lim_{n \to \infty} \frac{HK(X^{(n)})}{n^{(d-1)/d}} = \beta_{HK}(d)
$$

almost surely.

A number of combinatorial optimization problems, like the Euclidean traveling salesman problem, the Euclidean minimum spanning tree problem and the Euclidean minimum weight matching problem, have a similar asymptotic behavior although with a different constant β (see Beardwood, Halton and Hammersley [2] and Papadimitriou [15]). It is therefore interesting to compare $\beta_{HK}(d)$ to the value of β for closely related combinatorial optimization problems. In particular, it is clear that $\beta_{HK}(d) \leq \beta_{TSP}(d)$. Moreover, since the value of the Held-Karp lower bound on n points is never less than the cost of the minimum spanning tree on a subset of n-1 points, $\beta_{HK}(d) \geq \beta_T(d)$ where $\beta_T(d)$ is the corresponding constant for the Euclidean minimum spanning tree problem. The relationship between $\beta_{HK}(d)$ and $\beta_M(d)$, where $\beta_M(d)$ is the constant for the Euclidean minimum weight matching problem, is a little less obvious. Using a complete characterization of the perfect matching polytope, we may express the cost M of the minimum weight matching as:

$$M = Min \sum_{i \in V} \sum_{\substack{j \in V \\ j > i}} c_{ij} y_{ij} \tag{25}$$

subject to

$$\sum_{\substack{j \in V \\ j > i}} y_{ij} + \sum_{\substack{j \in V \\ j < i}} y_{ji} = 1 \qquad \forall i \in V \tag{26}$$

$$\sum_{i \in S} \sum_{\substack{j \in S \\ j > i}} y_{ij} \leq \frac{|S| - 1}{2} \qquad \forall S \subset V,\ |S| \text{ odd} \tag{27}$$

$$0 \leq y_{ij} \qquad \forall i, j \in V, j > i \tag{28}$$

Substituting y_{ij} by $x_{ij}/2$ we get a relaxation of the linear program (1)-(4). Hence $M \leq \frac{HK}{2}$ which implies that $\beta_{HK}(d) \geq 2\beta_M(d)$. We thus obtain the following proposition:

Proposition 5 $\max(2\beta_M(d), \beta_T(d)) \leq \beta_{HK}(d) \leq \beta_{TSP}(d)$.

When $d = 2$, $\beta_M(d)$, $\beta_T(d)$ and $\beta_{TSP}(d)$ were estimated to be 0.35, 0.68 and 0.72 by Papadimitriou [15], Gilbert [7] and Johnson [12], respectively. Using proposition

5, we may therefore deduce that the asymptotic gap $(\beta_{TSP} - \beta_{HK})/\beta_{TSP}$ is approximately less than $(0.72 - 0.70)/0.70 \approx 3\%$. This suggests a probabilistic explanation of the observation that the Held-Karp lower bound is very close to the length of the optimal tour in practice.

4 Martingale inequality and the Held-Karp lower bound

In this section we use a martingale inequality to deduce a sharp bound on

$$Pr\{|HK(X^{(n)}) - E[HK(X^{(n)})]| > t\}$$

for the case $d = 2$, i.e. in the Euclidean plane. As a consequence, we shall be able to establish the finiteness of

$$\sum_{n=1}^{\infty} Pr\left\{\left|\frac{HK(X^{(n)})}{\sqrt{n}} - \beta_{HK}\right| > \epsilon\right\}$$

for all $\epsilon > 0$, i.e. the complete convergence of the Held-Karp lower bound. This result is stronger than the almost sure convergence of theorem 4. This section basically rests upon the martingale arguments developed by Rhee and Talagrand [16] for the TSP.

For each $1 \leq i \leq n$, we let A_i be the sigma field generated by X_j, $1 \leq j \leq i$. Let $HK_i = HK(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$. Clearly $E[HK_i|A_i] = E[HK_i|A_{i-1}]$. If $\Delta_i = HK(X^{(n)}) - HK_i$, then $d_i \triangleq E[HK(X^{(n)})|A_i] - E[HK(X^{(n)})|A_{i-1}] = E[\Delta_i|A_i] - E[\Delta_i|A_{i-1}]$. In this way, $HK(X^{(n)}) - E[HK(X^{(n)})] = \sum_{i=1}^{n} d_i$ and the sequence $(d_i)_{i \leq n}$ is a martingale difference sequence. We prove the following theorem.

Theorem 6 *There exists a constant γ such that for every n*

$$Pr\left\{\left|HK(X^{(n)}) - E\left[HK(X^{(n)})\right]\right| > t\right\} \leq 2e^{-\gamma t}.$$

Proof:

We apply the martingale inequality (see Rhee and Talagrand [16])

$$Pr\left\{\left|\sum_{i=1}^{n} d_i\right| > t\right\} \leq 2\exp(-\frac{t}{C_1 B})$$

where $B = \max_k \left\| E\left[\sum_{i=k}^{n} d_i^2 | A_k\right]\right\|_{\infty}^{1/2}$ and C_1 is a numerical constant. The goal is to prove that, for the Held-Karp lower bound, $B \leq C_2$ for some constant C_2. We first need the following lemma.

Lemma 7 *1.* $0 \leq \Delta_i \leq 2\sqrt{2}$ *for all* i

2. There exist constants C_3 *and* C_4 *such that, for* $k \leq i \leq n$ *and* $k < n-1$,

$$E[\Delta_i | A_k] \leq \frac{C_3}{\sqrt{n-k-1}}$$

and

$$E[\Delta_i^2 | A_k] \leq \frac{C_4}{n-k-1}.$$

Proof:

The Held-Karp lower bound on $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ can be viewed as a convex combination of 1-trees $(T_1, \ldots, T_q)$, rooted at vertex X_j, with corresponding multipliers $(\lambda_1, \ldots, \lambda_q)$. Let $\sigma(T)$ denote one of the two vertices adjacent to X_j in the 1-tree T. For every T_r $(r = 1, \ldots, q)$ we add the two edges (X_j, X_i) and $(X_i, \sigma(T_r))$ and delete $(X_j, \sigma(T_r))$. This produces a new 1-tree T_r' spanning $(X_1, \ldots, X_n)$ with the degree of each vertex unchanged and the degree of X_i being 2. The multiplier associated to T_r' is still λ_r. Because the degrees remain unchanged, we have constructed a feasible solution to $P(\{X_1, \ldots, X_n\})$. As a result,

$$\begin{aligned} HK(X^{(n)}) &\leq \sum_{r=1}^{q} \lambda_r c(T_r') \leq \sum_{r=1}^{q} \lambda_r \left(c(T_r) + 2|X_i - X_j|\right) \\ &= HK_i + 2|X_i - X_j| \end{aligned}$$

Thus, $\Delta_i \leq 2|X_i - X_j|$ for every $j \neq i$. Hence $\Delta_i \leq 2\sqrt{2}$ for any i, which, together with the monotonicity of the bound, proves the first part of the lemma. Moreover, when $k \leq i \leq n$ and $k < n-1$, we have that $\Delta_i \leq u$ where $u = 2\min\{|X_i - X_j| : k < j \leq n, j \neq i\}$. Since u is independent of A_k, we get that $E[\Delta_i|A_k] \leq E[u]$ and $E[\Delta_i^2|A_k] \leq E[u^2]$. As $Pr\{u > t\} \leq (1-\alpha t^2)^{n-k-1} \leq \exp(-\alpha(n-k-1)t^2)$ for some constant α (see Karp and Steele [14]), we easily get that

$$E[u] \leq \frac{C_3}{\sqrt{n-k-1}} \quad \text{and} \quad E[u^2] \leq \frac{C_4}{n-k-1}$$

for some constants C_3 and C_4, which proves the second part of the lemma. □

As a corollary of the above lemma and following exactly the same techniques as in Rhee and Talagrand [16], we can easily prove that

1. $E[d_i^2|A_k] \leq C_5$ for any $k \leq i \leq n$ and
2. $E[d_i^2|A_k] \leq \frac{C_6}{n-k-1}$ for $k < i < n-1$,

where C_5 and C_6 are constants. As a result, we can bound B as follows:

$$\begin{aligned} E\left[\sum_{i=k}^{n} d_i^2|A_k\right] &= E[d_k^2|A_k] + \sum_{k<i<n-1} E[d_i^2|A_k] + E[d_{n-1}^2|A_k] + E[d_n^2|A_k] \\ &\leq 3C_5 + \frac{C_6}{n-k-1}(n-k-2) \leq 3C_5 + C_6 \triangleq C_2 \end{aligned}$$

Hence, $B = \max_k \left\| E\left[\sum_{i=k}^{n} d_i^2|A_k\right]\right\|_\infty^{1/2} \leq C_2$. Letting $\gamma = \frac{1}{C_1 C_2}$, the theorem follows. □

Applying theorem 6 with $t = \epsilon\sqrt{n}$, we find that

$$Pr\left\{\left|HK(X^{(n)}) - E\left[HK(X^{(n)})\right]\right| > \epsilon\sqrt{n}\right\} \leq 2e^{-\gamma\epsilon\sqrt{n}}. \quad (29)$$

The complete convergence of the Held-Karp lower bound now follows from (29) and the fact that $E[HK(X^{(n)})]/\sqrt{n}$ tends to β_{HK} as n tends to infinity.

5 Concluding remarks

We analyzed probabilistically the Held-Karp lower bound for the TSP. Our result corroborates the observation that the lower bound is very close to the length of the optimal tour in practice. We would like to emphasize that we exploited the combinatorial interpretation of the Held-Karp lower bound and the theory of subadditive Euclidean functionals. We believe that the idea of combining polyhedral characterizations with probabilistic analysis has the potential to lead to very interesting results.

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