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TRANSIENT STARTING OF A MAGNETRON AS DESCRIBED BY  
THE INHOMOGENEOUS VAN DER POL EQUATION

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RESEARCH LABORATORY OF ELECTRONICS  
MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
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Abstract

The solution to the inhomogeneous Van der Pol equation specifies the operation of a nonlinear oscillator perturbed by an external signal of similar frequency. The steady-state behavior of such an oscillator was first quoted by Van der Pol himself and is in agreement with other literature in the field. In this study, approximate and differential analyzer solutions are used to investigate the frequency and phase transients during starting, and distortion of the build-up envelope by the exciting signal. Results are essentially in agreement with results indicated in Technical Report No. 100, although details are different because reactive electronic beam-loading is neglected here.



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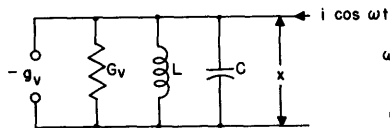
Introduction

A single-mode oscillator may be properly represented by a parallel RLC combination shunted by a negative conductance and susceptance, which are functions of the terminal voltage. The differential equation describing such a circuit is known as Van der Pol's equation and its solution shows a transient build-up, followed by steady-state sinusoidal oscillations. If the circuit is acted upon by a sinusoidal current source, a driving term is added to the equation. Its solution now will show the same steady-state synchronization behavior that has been discussed previously (1, 2). The conditions for synchronization can be found as a function of the ratio of the injected current to the oscillation amplitude, and the frequency difference between the oscillator and injected signal. This calculation has been made by B. Van der Pol (3) and is in agreement with other literature in the field. However, the transient conditions existing during the oscillator build-up have never been exhaustively studied. While the general form of the build-up envelope is well known, such things as the instantaneous phase and frequency, and the distortion of the envelope by the injected signal, have not been examined. The latter two aspects are of fundamental interest in a discussion of the transient starting of synchronized oscillators.

In a previous treatment by the author (4), a shape for the build-up envelope was assumed and the instantaneous phase calculated on a quasi-steady-state basis. The validity of this analysis may be more firmly established if the solution to the Van der Pol equation leads to similar results.

1. The Van der Pol Equation with a Driving Current

The form of Van der Pol's equation with which this section concerns itself is easily derived from the equivalent circuit shown in Fig. 1. It is to be noted that reactive loading effects of the electronic space charge have been neglected for the sake of simplicity.



$\omega_0 = \frac{1}{\sqrt{LC}}$  Fig. 1 The single mode equivalent circuit of an oscillator, neglecting reactive loading by the electronic space charge.  
 $g_V = g_V(x)$

The presence of this loading serves only to exaggerate the frequency pushing exhibited during the transient build-up. We are justified, therefore, in ignoring this factor so long as we are concerned only with the nature of the solution, and not with its quantitative details.

Returning to Fig. 1, the nodal equation for the voltage  $x$  may be written as

$$(G_V - g_V)x + \frac{1}{L} \int x dt + c \frac{dx}{dt} = i \cos \omega t \quad (1)$$

which, when differentiated with respect to time, becomes

$$\frac{d^2x}{dt^2} + \mu \left( G_V - g_V - x \frac{dg_V}{dx} \right) \frac{dx}{dt} + \omega_0^2 x = -E\omega^2 \sin \omega t \quad (2)$$

where  $\mu = 1/C$ ,  $\omega_0^2 = 1/\sqrt{LC}$ , and  $E = i/\omega C$ . Now the function relationship,  $g_V = g_V(x)$ , must be evaluated so that the equation may be reduced to one in  $x$  and  $t$  only.

## 2. Relation of Van der Pol's Equation to Actual Conditions in the Magnetron

From experimental observations, the form of the magnetron r-f voltage build-up is well known. The voltage initially shows an exponential growth, a behavior which reveals that the electronic conductance is constant during this period. That such is the case may be seen by considering Eq. A5a. This equation may be written as

$$t = \frac{dA/A *}{\frac{\omega_0}{2} \left( \frac{g}{\omega_0 C} - \frac{1}{Q_L} \right)}$$

and if  $g$  is constant, integrates directly, giving

$$A = e^{\frac{\omega_0}{2} \left( \frac{g}{\omega_0 C} - \frac{1}{Q_L} \right) t} \quad (3)$$

Hence, if  $g/\omega_0 C > 1/Q_L$ , the voltage increases exponentially.

Near the end of the build-up, the voltage "saturates" and approaches its final value asymptotically. It is shown in the Appendix that such behavior is characteristic when the electronic conductance is an inverse function of the voltage (Eq. A7).

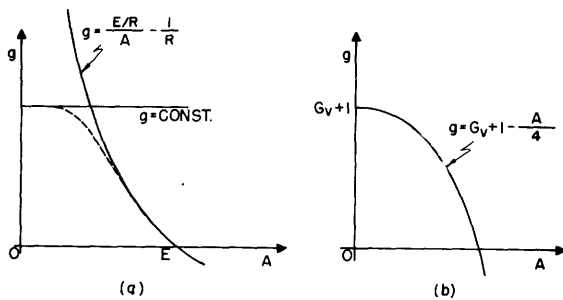


Fig. 2

- (a) Dotted curve shows a possible form of  $g(A)$  as interpolated from the limiting loci.
- (b) Square-law approximation to the locus shown in (a).

The region intermediate between these two extremes is one of transition, in which the voltage is described equally well by either boundary equation. Hence, we should like to find an expression for  $g_V$  which closely approximates a constant at small voltage, decreases at larger amplitude, and has a smooth transition in between. Figure 2a shows a possible form of this function as interpolated from the limiting conditions. One recognizes that this curve may be closely represented by the square-law function, as shown in

\*Note that  $A$  is the envelope function of  $x$ . In other words, we have assumed  $x \sim A \cos(\omega t - \phi)$  where  $A = A(t)$  and  $\phi = \phi(t)$ .

Fig. 2b. There it has been assumed that

$$g = G_V + 1 - \frac{A^2}{4} \quad (4)$$

As a function of the voltage  $x$ , Eq. 4 may be written as

$$g_V = G_V + 1 - \frac{x^2}{3} \quad (5)$$

The details of this transformation will be derived in Sec. 3. Now, substituting Eq. 5 into Eq. 2, we arrive at

$$\frac{d^2x}{dt^2} + \mu (x^2 - 1) \frac{dx}{dt} + \omega_0^2 x = -E\omega^2 \sin \omega t \quad (6)$$

This is the form of Van der Pol's equation with which we shall be concerned. Its form shows clearly that it represents a second-order system with nonlinear damping, driven by a sinusoidal forcing function.

### 3. An Approximate Analytical Solution to the Homogeneous Equation

If the driving current is reduced to zero, Eq. 6 becomes

$$\frac{d^2x}{dt^2} + \mu (x^2 - 1) \frac{dx}{dt} + \omega_0^2 x = 0 \quad (7)$$

One may obtain an approximate solution to this homogeneous equation by assuming  $x = A \cos(\omega_0 t - \phi)$ , where  $A$  and  $\phi$  are both functions of time. Van der Pol's original solutions were made under a similar assumption. His results, however, are not in the form most useful for this analysis. Hence, the purpose of this redundant presentation will be to formulate  $A(t)$  and  $\phi(t)$  in convenient terms.

When one substitutes the assumed solution into Eq. 7 he obtains

$$\begin{aligned} & \left\{ \frac{d^2A}{dt^2} - A(\omega_0 - \frac{d\phi}{dt}) + \mu \left( \frac{3A^2}{4} - 4 \right) \frac{dA}{dt} + A\omega_0^2 \right\} \cos(\omega_0 t - \phi) \\ & - \left\{ 2(\omega_0 - \frac{d\phi}{dt}) \frac{dA}{dt} - A \frac{d^2\phi}{dt^2} + \mu \left( \frac{A^3}{4} - 4A \right) (\omega_0 - \frac{d\phi}{dt}) \right\} \sin(\omega_0 t - \phi) \\ & + \mu \frac{A^2}{4} \frac{dA}{dt} \cos 3(\omega_0 t - \phi) - \mu \frac{A^3}{4} (\omega_0 - \frac{d\phi}{dt}) \sin 3(\omega_0 t - \phi) = 0 \quad (8) \end{aligned}$$

If the coefficients in this equation are slowly varying compared to the sines and cosines, each coefficient must be identically zero if the equation is to be true at all instants of time. This condition implies that during the transient build-up the amplitude and phase do not change appreciably during one r-f cycle. For instance, in S-band magnetrons, which have typical starting times of about  $10^{-7}$  second, containing 300 r-f cycles, the

assumption is quite well satisfied.

Let us, then, set the coefficient of  $\sin(\omega_0 t - \phi)$  equal to zero and neglect the second derivative,  $d^2\phi/dt^2$ , by the previous assumption.

$$\frac{dA}{dt} = -\mu \frac{A^3 - 4A}{8} \quad (9)$$

Equation 9 may be integrated directly after solving for  $dt$ . The solution is

$$t = \frac{1}{\mu} \ln \left[ \frac{A^2(4 - A_0^2)}{A_0^2(4 - A^2)} \right] \quad (10)$$

or, solving for  $A$

$$A = \frac{2}{\sqrt{1 + \frac{4 - A_0^2}{A_0^2} e^{-\mu t}}} \quad (11)$$

where  $A_0$  is the amplitude at time zero. This result is shown in Fig. 3 for various values of  $A_0$ , plotted against the dimensionless time  $\mu t$ . It is to be noted that the initial amplitude,  $A_0$ , is the critical factor fixing the time of starting, while  $\mu$  enters only as

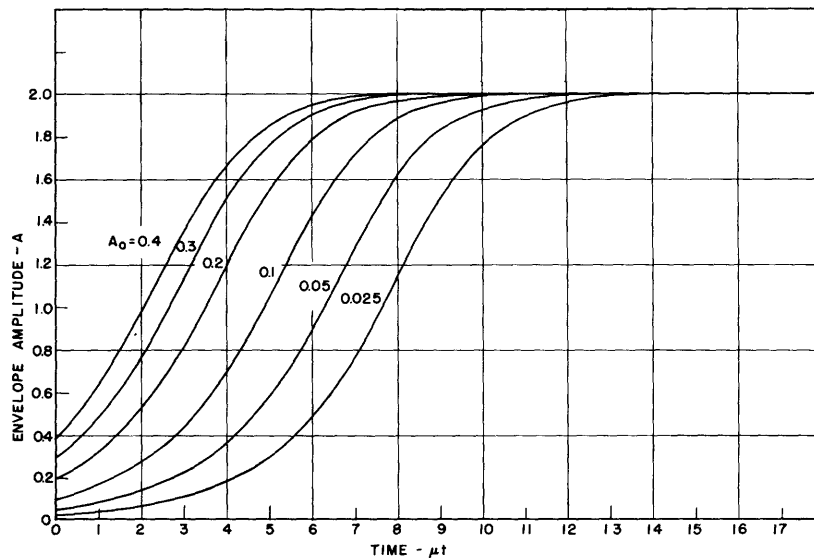


Fig. 3 Approximate envelopes of starting transient.

a scale factor. Also, so long as the assumption  $f_0 \gg \mu$  is valid, the value of  $\omega_0$  does not enter.

Consider Eq. 5, which expresses  $g_V$  as a function of  $x$ . Rewriting this equation

$$g_V x = G_V x + x - \frac{x^3}{3} \quad (12)$$



If the assumed solution for  $x$  is substituted, there results

$$g_V A \cos(\omega_0 t - \phi) = G_V A \cos(\omega_0 t - \phi) + A \cos(\omega_0 t - \phi) - \frac{A^3}{3} \left[ \frac{3}{4} \cos(\omega_0 t - \phi) - \frac{1}{4} \cos 3(\omega_0 t - \phi) \right] \quad (13)$$

The triple frequency term may be neglected, since the circuit is sharply tuned to the frequency  $\omega_0$ . Under this assumption  $g_V$  reverts to our former  $g$  and

$$g = G_V + 1 - \frac{A^3}{4} \quad (14)$$

Equation 11 shows that the steady-state amplitude,  $A$ , is equal to 2. Then the steady-state  $g$  is just  $G_V$ , and the total conductance shunting the circuit is  $(G_V - g)$  or zero. This result gives a quantitative check on our solution for  $A$  and shows the process by which Eq. 5 was reached from Eq. 4.

Returning to Eq. 8 and putting the coefficient of the cosine equal to zero, we have

$$\left(\frac{d\phi}{dt}\right)^2 - \mu \left(\frac{3A^2 - 4}{4A}\right) \frac{dA}{dt} - 2\omega_0 \frac{d\phi}{dt} = 0 \quad (15)$$

where the factor  $d^2A/dt^2$  has been neglected. This equation, when solved for the time derivative of  $\phi$ , becomes

$$\frac{d\phi}{dt} = \omega_0 - \sqrt{\omega_0^2 + \mu \left(\frac{3A^2 - 4}{4A}\right) \frac{dA}{dt}} \quad (16)$$

Substituting Eq. 11 for  $A$  and utilizing the fact that the instantaneous frequency  $\omega$  is  $\omega_0 - d\phi/dt$ , Eq. 16 may be written

$$\omega = \left[ \omega_0^2 + \frac{\mu 2}{2} \left(\frac{4 - A_0^2}{A_0^2}\right) e^{-\mu t} \frac{\left\{ 2 - \left(\frac{4 - A_0^2}{A_0^2}\right) e^{-\mu t} \right\}}{\left\{ 1 + \left(\frac{4 - A_0^2}{A_0^2}\right) e^{-\mu t} \right\}^2} \right]^{1/2} \quad (17)$$

Figure 4 shows the instantaneous frequency for various values of  $A_0$  where the approximation  $(1 + q)^{1/2} = 1 + q/2$  when  $q \ll 1$  has been made in Eq. 17. Like the build-up loci of Fig. 3, a change of  $A_0$  merely shifts the time scale and does not alter the form of the curves. Hence, the frequency and build-up for any  $A_0$  may be found from those shown by an appropriate shift on the time axis. It is of interest to note that the quantity  $\omega - \omega_0$  changes sign during the transient and that this change is coincident with the inflection point of the amplitude build-up.

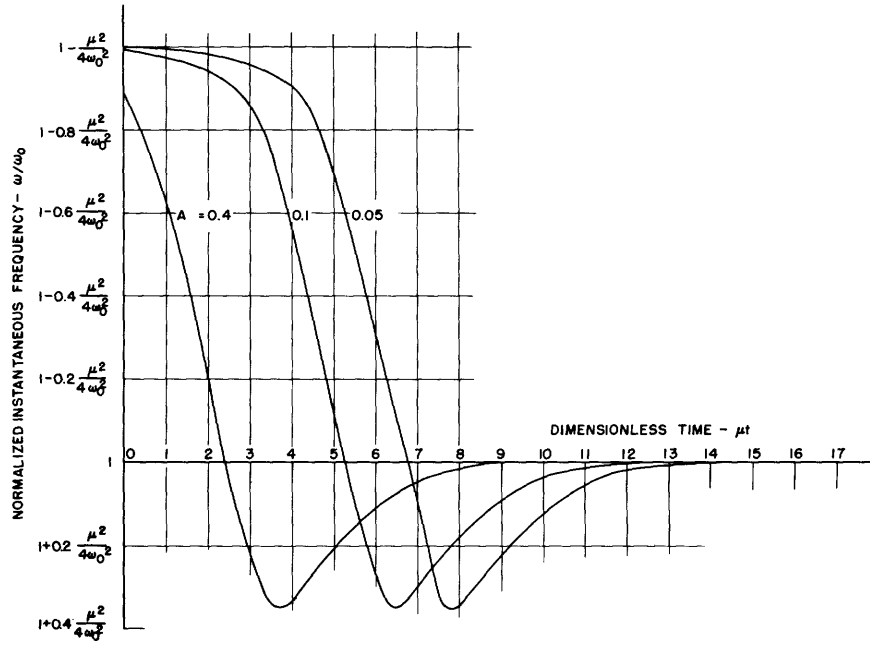


Fig. 4 Instantaneous frequency during starting.

#### 4. An Approximate Analytical Solution to the Inhomogeneous Equation

If one attempts to obtain an analytic solution to Eq. 6 he is eventually confronted with insoluble nonlinear differential equations in  $A$  and  $\phi$ . Some information may be obtained, however, by making the assumption that the driving current is small enough so that our former expression for  $A(t)$ , with a suitable value of  $A_0$  inserted, is still representative of the build-up envelope. Let us, then, make a solution for the inhomogeneous equation under this assumption.

Now, Eq. 6 may be written as

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1) \frac{dx}{dt} + \omega_0^2 x = -E\omega^2 \left[ \sin \phi \cos(\omega t - \phi) + \cos \phi \sin(\omega t - \phi) \right] \quad (18)$$

If we assume the solution  $x = A \cos(\omega t - \phi)$  and substitute in Eq. 18 we find it may be written in a form analogous to Eq. 8, in which the coefficient of  $\cos(\omega t - \phi)$  is

$$\frac{d^2A}{dt^2} - A \left( \omega - \frac{d\phi}{dt} \right)^2 + \mu \left( \frac{3A^2 - 4}{4} \right) \frac{dA}{dt} + A\omega_0^2 + E\omega^2 \sin \phi \quad .$$

As before, this coefficient may be set equal to zero if it is slowly varying. Neglecting the term  $d^2A/dt^2$ , we can obtain the following

$$\left( \frac{d\phi}{dt} \right)^2 - 2\omega \frac{d\phi}{dt} - \mu \left( \frac{3A^2 - 4}{4A} \right) \frac{dA}{dt} - (\omega_0^2 - \omega^2) - \frac{E\omega^2}{A} \sin \phi = 0$$

which, when solved for  $d\phi/dt$ , becomes

$$\frac{d\phi}{dt} = \omega \pm \sqrt{\omega^2 + \mu \left( \frac{3A^2 - 4}{4A} \right) \frac{dA}{dt} + (\omega_o^2 - \omega^2) + \frac{E\omega^2}{A} \sin \phi} .$$

From the known nature of the solution,  $d\phi/dt \ll \omega$ ; hence we may expand the one-half-power term in a power series and discard all but the first two terms.  $d\phi/dt$  becomes

$$\frac{d\phi}{dt} = -\frac{\mu}{2\omega} \left( \frac{3A^2 - 4}{4A} \right) \frac{dA}{dt} - \frac{\omega_o^2 - \omega^2}{2\omega} - \frac{E\omega}{2A} \sin \phi . \quad (19)$$

We may substitute Eq. 11 for A and thus reduce our equation to one in  $\phi$  and t alone.

$$\begin{aligned} \frac{d\phi}{dt} + \frac{E\omega}{4} \left[ 1 + \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \right]^{1/2} \sin \phi = \omega - \omega_o \\ - \frac{\mu}{4\omega} \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \frac{\left[ 2 - \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \right]}{\left[ 1 + \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \right]^2} \end{aligned} \quad (20)$$

where we have used the approximation  $(\omega^2 - \omega_o^2)/2\omega \approx \omega - \omega_o$ .

We should note here that when  $e^{-\mu t}$  becomes quite small, Eq. 20 reduces to

$$\frac{d\phi}{dt} + \frac{E\omega}{4} \sin \phi = \omega - \omega_o . \quad (21)$$

Hence, the solution to Eq. 20, after the effects of the starting transient die out, is exactly similar to that found for the steady state by other investigators (1, 2).

Examination shows that Eq. 20 is very much like Eq. 70 in Technical Report No. 100 (4), which was derived for quasi-steady-state build-up. Both contain a term expressing the effect of frequency pushing during starting. These terms are the functions of time located on the right-hand side of the equalities, and both show time variations exactly similar to the frequency variations during unsynchronized starting as expressed by Eq. 17, and Eq. 66 of Technical Report No. 100. These variations are different in detail, for we have neglected reactive beam-loading in the present analysis. The important conclusion to be drawn is that both the transient and quasi-steady-state analyses show this frequency disturbance to be present irrespective of the locking signal.

In addition, both equations show that the effect of the locking signal is enhanced during starting by a factor which is, in magnitude, merely the inverse of the r-f voltage envelope. Finally, the fact that these two analyses, each beginning from a different point of view, indicate similar behavior lends credence to the previous result.

An analytic solution for Eq. 20 may be obtained if one considers the special case,  $\omega - \omega_o = 0$ . Equation 20 then becomes

$$\frac{d\phi}{dt} = -\frac{E\omega}{4} \left[ 1 + \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \right]^{1/2} \sin \phi$$

$$- \frac{\mu^2}{4\omega_o} \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \frac{\left[ 2 - \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \right]}{\left[ 1 + \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \right]^2} . \quad (22)$$

Values of the constants in this equation corresponding to those used in Technical Report No. 100 are approximately as follows

$$S \rightarrow \frac{E\omega}{4} = 5 \times 10^6$$

$$k \rightarrow \mu = 10^8$$

$$\omega' \rightarrow \omega_o = 2\pi \times 3 \times 10^9$$

$$1 - \eta \rightarrow A_o/2 = 0.2 .$$

Now, the second term on the right in Eq. 22 has its minimum value at  $t = 0$  and increases rapidly thereafter, being nearly zero at the end of the starting transient. The minimum value is approximately  $-\mu^2/4\omega_o \sim -1.33 \times 10^5$ , assuming that  $(4 - A_o^2)/A_o^2 \gg 1$ . Even if this term remained at its minimum value during the entire starting period, it would correspond to a phase change of only  $-1.33 \times 10^5 \times 10^{-7} \approx -1^\circ$ . Hence, we may safely rewrite Eq. 22 as

$$\frac{d\phi}{dt} = -\frac{E\omega_o}{4} \left[ 1 + \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \right]^{1/2} \sin \phi . \quad (23)$$

By separating the variables and integrating, we obtain

$$\ln \tan \frac{\phi}{2} = -\frac{E\omega_o}{4} \int \left[ 1 + \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} \right]^{1/2} dt + C . \quad (24)$$

The integral may be evaluated by making the substitution

$$y^2 = 1 + \left( \frac{4 - A_o^2}{A_o^2} \right) e^{-\mu t} .$$

After utilizing the initial condition,  $\phi = \phi_o$  at  $t = 0$ , the final result is

$$\ln \tan \frac{\phi}{2} = \ln \tan \frac{\phi_0}{2} + \frac{E\omega}{2\mu} \left\{ (1 + \chi e^{-\mu t})^{1/2} - (1 + \chi)^{1/2} + \frac{1}{2} \ln \left[ \frac{(1 + \chi e^{-\mu t})^{1/2} - 1}{(1 + \chi e^{-\mu t})^{1/2} + 1} \frac{(1 + \chi)^{1/2} + 1}{(1 + \chi)^{1/2} - 1} \right] \right\} \quad (25)$$

where  $\chi = (4 - A_0^2)/A_0^2$ . The phase,  $\phi$ , is plotted as a function of time in Fig. 5 for two values of  $\chi$ . It is seen that after the envelope build-up goes to completion, the phase decreases exponentially for small  $\phi$ . The time constant of this exponential may be found from Eq. 25 by using the condition  $e^{-\mu t} \ll 1$ . Equation 25 then becomes

$$\tan \frac{\phi}{2} = \left[ \frac{\chi (1+\chi)^{1/2} + 1}{4(1+\chi)^{1/2} - 1} \right] \frac{E\omega}{4} e^{-\frac{E\omega}{2\mu} t} \left[ (1+\chi)^{1/2} - 1 \right] - \frac{E\omega}{4} t \tan \frac{\phi_0}{2} \quad (26)$$

As we have found previously, then, the time constant of the long-time phase transient is the inverse of the coefficient of the sine term in the differential equation.

For large initial angles, it is noted that the duration of the transient is greatly

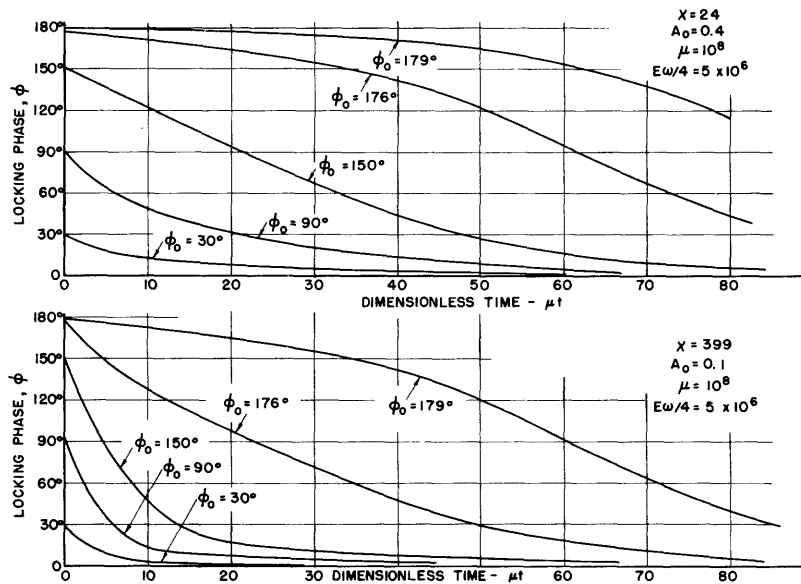


Fig. 5 Phase transient during starting.

prolonged. Hence, as in Technical Report No. 100, we may conclude that the conditions at the initial instant of starting are of prime importance in determining pulse-to-pulse coherence.

Before discussing the "goodness of phasing" we must choose a criterion. Let us define the parameter U, where

$$U = \frac{1}{N_P} \sum_{n=1}^{N_P} \int_0^{\tau_n} \left[ \phi_n(t) - \phi_{ss} \right]^2 dt \quad (27)$$

This definition is arbitrary, but reasonable. We have taken the square of the integrated phase deviation from the steady-state value and averaged it over a large number,  $N_P$ , of pulses, each of duration  $\tau_n$ . Now,  $U$  is a function of (1) static parameters of the oscillator, (2) ratio of the steady-state amplitude to the locking signal, (3) initial amplitude  $A_0$ , and (4) ratio of preoscillation noise to synchronizing power (which determines the initial phase as described in Technical Report No. 100). For a given oscillator, the static parameters are important only in relation to other oscillators. Hence, we may exclude this factor in the sequel. The ratio of steady-state to locking signal amplitude determines the time constant of the phase transient resulting from the phase deviations which exist when the r-f envelope first reaches its final value. This time constant decreases with increasing locking power. Equation 26 and Fig. 5 show that phase deviations existing when the envelope has reached its final value increase with  $A_0$  for a given initial phase deviation. On the other hand, the initial phase deviations decrease with increasing locking power. Consequently, the behavior of  $U$  as a function of locking power is somewhat complicated and will depend rather strongly on the individual case. We can, however, draw some generalized conclusions.

For a given oscillator, the value of  $U$  will, for the most part, decrease with increasing locking power. This decrease will not be uniform, however. It will be most rapid for small locking signals (small  $A_0$ ) and large locking signals (small initial phase deviations and time constant). More important is the possibility that  $U$  will increase or decrease rather slowly in the region of intermediate locking power. In other words, there can exist a situation in which increase of locking power does not improve, and may even impair, phasing.

##### 5. Differential Analyzer Solution to the Driven Van der Pol Equation and Distortion of Build-Up Envelope by Driving Signal

The driven Van der Pol equation (Eq. 6) may be solved exactly by an electronic computer of analog type. For particular values of the coefficients, the solutions are presented as traces on a cathode ray oscilloscope. It is from solutions of this sort that we may determine the distortion of the build-up envelope by the driving signal.

In examining such solutions we are faced with the problem of determining if the quantity  $|\omega_0 - \omega|$  is small enough to allow synchronization. If the simulated oscillator is operating in a locked condition, changes in its initial phase will not change its steady-state phase with respect to the locking signal. By utilizing this criterion we may ascertain the condition of the oscillator. Figure 6 shows Lissajous patterns between the locking signal and the oscillator output voltage. There it can be clearly seen that a  $90^\circ$  change in the initial phase does not alter the final phase, which is determined by the

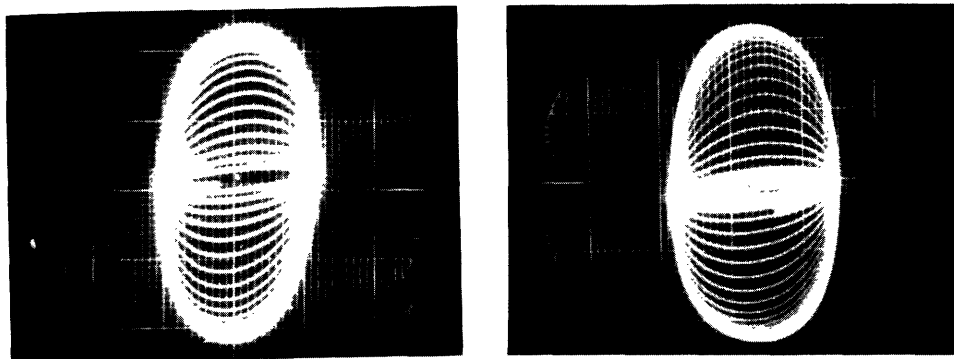


Fig. 6 Lissajous patterns between synchronized oscillator voltage and constant amplitude sinusoid showing insensitivity of steady-state phase to initial phase.

nearly identical outer traces. This fact may be further illustrated by using noise as the initial condition for the computer, which repeats its solution at a repetition rate of 60 times per second. In this case, successive solutions will have different initial conditions, and with no locking signal there will be no coherence from solution to solution. On the other hand, this incoherency is present only at the beginning of each solution when a locking signal is present, indicating a transient pull-in of the phase. These situations are illustrated in Fig. 7.

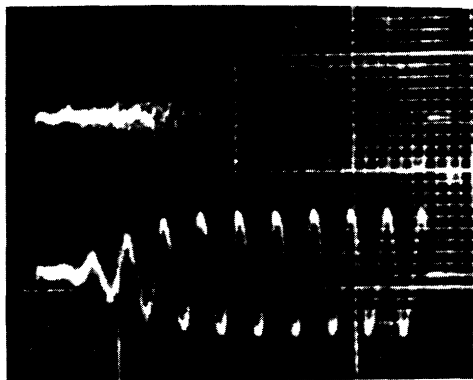


Fig. 7

Starting of an oscillator in the synchronized and unsynchronized cases with noise as initial condition.

With the assurance that our simulated oscillator is synchronized, we may proceed to examine solutions for given values of the parameters  $\mu$ ,  $\omega_o$ ,  $\omega$ , and  $E\omega/2A_f$ . It has been seen previously that when  $E\omega/2A_f = 0$  and  $f_o \gg \mu$ , the form of the build-up envelope is independent of  $f_o$ , and  $\mu$  enters merely as a scale factor. When  $E\omega/2A_f$  becomes finite, but not large, this condition will still be closely valid. For the solutions to be presented, the ratio  $f_o/\mu \sim 3.2$ ; hence, our conclusions as to distortion of the envelope will hold, in general, so long as  $f_o \gg \mu$ . For these solutions we have chosen the following values

$$\begin{aligned}\omega_o &= 2 \times 10^9 \text{ radians/sec} \\ \mu &= 10^8 \text{ sec}^{-1} \\ \omega_o - \omega &= 0\end{aligned}$$

and the synchronizing parameter  $E\omega/2A_f$  is variable over the range 0 to  $1.9 \times 10^7$ , which provides a maximum locking band of 3 megacycles. Figure 8 shows build-up envelopes for several values of the parameter. These curves were taken from computer solutions such as those shown in Fig. 9. Comparison of Figs. 9 and 3 shows that the build-up

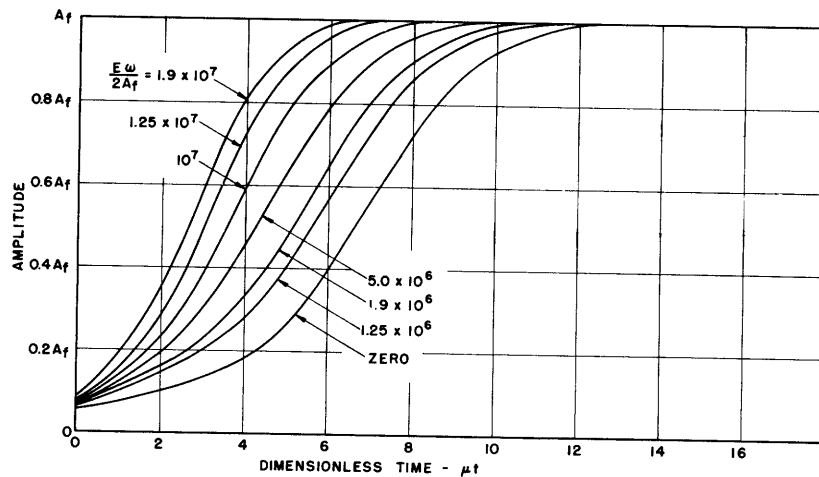


Fig. 8 Distortion of build-up enveloped by driving current as a function of the locking parameter.

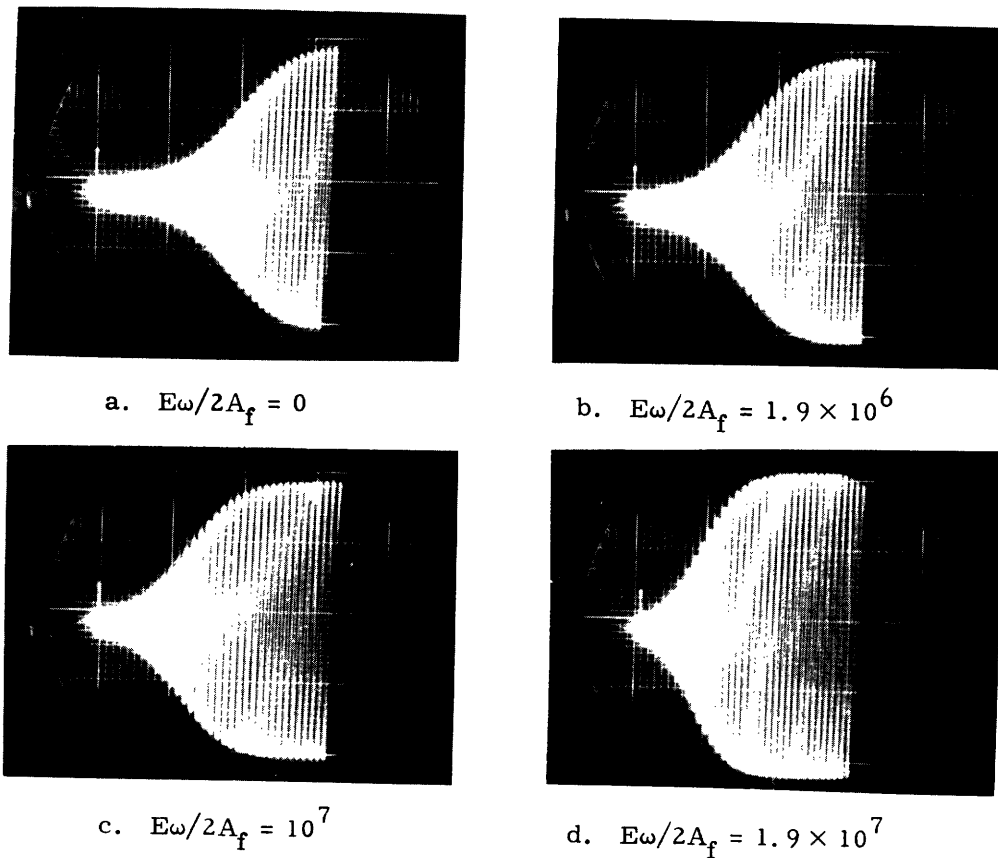


Fig. 9 Starting of an oscillator showing effect of synchronizing signal on build-up envelope.



curves with and without the driving signal are quite similar. Hence, our assumption that distortion of the envelope by the driving signal may be accounted for by utilizing a suitable value of  $A_0$  in Eq. 11 seems well justified. Therefore, the material in Sec. 4, which was derived under this assumption, may be considered reliable. Further, the results of Sec. 4 agree well with those found from the quasi-steady-state analysis in Technical Report No. 100. Since the solutions are mutually compatible, we may with confidence apply the conclusions drawn from them to the actual situation.

#### Appendix

The equivalent circuit usually used to characterize a magnetron is shown in Fig. 10. Here the normalized load admittance,  $G + jB$ , is coupled to the operating mode, represented by  $R$ ,  $C$ , and  $L$ . The factor  $K_C$  accounts for the transformer action of the

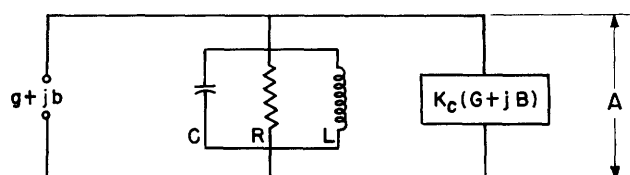


Fig. 10  
Single-mode equivalent circuit  
of a magnetron.

coupling loop or iris, and  $g + jb$  is the nonlinear admittance which characterizes the electronic discharge. If the magnetron is operating in the steady state, conservation of energy requires that the total admittance shunting any pair of terminals be zero. From this condition, we find the operating equations

$$\frac{g}{\omega_0 C} = \frac{1}{Q_0} + \frac{G}{Q_{\text{ext}}} \quad (\text{A1})$$

$$\frac{b}{\omega_0 C} = \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) + \frac{B}{Q_{\text{ext}}}$$

where  $\omega_0 = 1/\sqrt{LC}$  = resonant mode frequency,  $Q_0 = R/\omega_0 L = R\omega_0 C$  = quality factor of the mode, and  $Q_{\text{ext}} = \omega_0 C/K_C = 1/K_C \omega_0 L$  = loading effect of matched load. If the magnetron is in a transient condition, another term which accounts for energy storage in the mode must be added to the equations above. Specifically, the energy stored is

$$W = CA^2 \quad (\text{A2})$$

and the rate of energy storage is

$$\frac{dW}{dt} = 2CA \frac{dA}{dt} \quad (\text{A3})$$

At any instant this rate may be represented as a conductance shunting the circuit of Fig. 10. The magnitude of this shunting effect is

$$G_S A^2 = 2CA \frac{dA}{dt} \quad (\text{A4})$$

or

$$G_S = \frac{2C}{A} \frac{dA}{dt}$$

so that Eq. A1 becomes

$$\frac{g}{\omega_o C} = \frac{1}{Q_o} + \frac{G}{Q_{ext}} + \frac{2}{\omega_o} \frac{1}{A} \frac{dA}{dt} \quad (A5a)$$

$$\frac{b}{\omega_o C} = \left( \frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} \right) + \frac{B}{Q_{ext}} \quad (A5b)$$

Now if  $g$  is known as a function of  $A$ , then Eq. A5a may be solved to find  $A$  as a function of time. Assuming the empirical relation

$$g = \frac{E/R}{A} - \frac{1}{R} \quad (A6)$$

we find

$$A = \frac{E}{RC\omega_o} \frac{1}{\left( \frac{1}{RC\omega_o} + \frac{1}{Q_L} \right)} \left\{ 1 - e^{-\frac{\omega_o}{2} \left( \frac{1}{RC\omega_o} + \frac{1}{Q_L} \right) t} \right\} \quad (A7)$$

$$A = A_o (1 - e^{-kt})$$

where

$$A_o = \frac{E}{RC\omega_o} \frac{1}{\left( \frac{1}{RC\omega_o} + \frac{1}{Q_L} \right)}, \quad k = \frac{\omega_o}{2} \left( \frac{1}{RC\omega_o} + \frac{1}{Q_L} \right), \quad \text{and} \quad \frac{1}{Q_L} = \frac{1}{Q_o} + \frac{G}{Q_{ext}}.$$

Now we have found (5) that the coefficient  $N$  was

$$N = \frac{E\eta}{2CRA_o} \tan \alpha \quad (A8)$$

If we substitute our expression for  $A_o$ , we find

$$N = \frac{\omega_o}{2} \left( \frac{1}{RC\omega_o} + \frac{1}{Q_L} \right) \eta \tan \alpha = k\eta \tan \alpha \quad (A9)$$

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