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A RATIONAL ANTICIPATIONS GENERAL EQUILIBRIUM
ASSET PRICING MODEL:
THE CASE OF DIFFUSION INFORMATION[†]

by
Chi-fu Huang*

May 1983

Sloan School of Management Working Paper No. 1553-84

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A RATIONAL ANTICIPATIONS GENERAL EQUILIBRIUM
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Abstract

We prove existence of an equilibrium in a continuous trading economy with diffusion information. Equilibrium asset price processes are Ito integrals. Under some regularity conditions, equilibrium asset price processes and the vector of “state variable processes” generating information form a vector diffusion process. A martingale representation technique is used to characterize agents’ optimal portfolio rules in an equilibrium context.

1. Introduction and summary

In the Intertemporal Capital Asset Pricing Model, developed by Merton [34] and extended by Breeden [4], the story goes roughly as follows: Let there exist a (perhaps endogenously determined) vector diffusion process $\{Y(t)\}$ that describes “states of the world”. Assume that prices for traded assets can be represented by stochastic differential equations of the Ito type with coefficients that are functions of $Y(t)$ and t at each time t , and assume that the price processes and Y together form a vector diffusion process. Taking asset prices as given, each agent maximizes his expected utility of life-time consumption. Markovian stochastic dynamic programming is then used to characterize agents’ consumption-investment choices over time. Assuming that an equilibrium exists, restrictions on equilibrium asset prices are derived by inverting the “aggregate demand” for assets, calculated by adding the first order optimality conditions for agents’ dynamic programs.

The two main assumptions made above are the existence of an equilibrium in a continuous trading economy and the finite dimensionality of Y . The latter is an assumption involving endogenous properties of an economic equilibrium. Cox, Ingersoll, and Ross [10], in a production economy, identify the finite dimensionality of Y . They assume, however, that there is a single representative agent in the economy and that an equilibrium exists in which prices are “smooth” functions of Y . They also rely on Markovian stochastic dynamic programming methods.

The purpose of this essay is three-fold. First, the existence of an equilibrium is established in a Merton/Breeden-like economy. Equilibrium asset prices are Ito integrals

whose coefficients are “nonanticipative functionals”. Furthermore, the equilibrium allocation will be shown to be Pareto efficient. Second, conditions are provided under which the value of these “nonanticipative functionals” at each time t can be written as functions of $Z(t)$ and t , where $\{Z(t)\}$ is a finite-dimensional vector of exogenously specified diffusion “state variable processes”. In particular, the vector of equilibrium price processes for traded assets and the vector of state variable processes Z together form a vector diffusion process. Finally, a martingale-representation technique, similar to one proposed by Cox [9], is used to characterize agents’ optimal portfolio behavior. It is argued that this martingale-representation technique is a more powerful tool for equilibrium analysis than Markovian stochastic dynamic programming.

In Section 2 of this paper, a continuous-time frictionless pure exchange economy under uncertainty with time span $[0, T]$ is formulated. It is assumed that there is one perishable consumption commodity in the economy, consumed only at times 0 and T . Agents, finite in number, are characterized by their endowments at times zero and T and by their consumption preferences. Agents are endowed with a common information structure \mathbf{F} generated by an exogenously specified vector diffusion process Z . That is, Z describes the evolution of the exogenous uncertain environment.

It is assumed that there are at most a finite number of traded *long-lived* securities in zero net supply. Each agent’s problem is to manage a portfolio of long-lived securities so as to maximize consumption preferences. The equilibrium concept used is Radner’s [37] *equilibrium of plans, prices, and price expectations*.

Given the nice properties of \mathbf{F} , a diffusion filtration, Section 3 shows that if we select long-lived securities appropriately, an equilibrium exists and the equilibrium allocation is Pareto efficient. Furthermore, markets are complete in equilibrium in the sense that all contingent claims not traded can be replicated by trading on long-lived securities. (Equivalently, all contingent claims are priced by *arbitrage*.) The equilibrium price processes for all contingent claims are Ito integrals whose integrands are nonanticipative functionals (Theorem 3.3.1 and its corollary). The existence proof exploits the machinery developed in Duffie and Huang [13].

If time T aggregate endowment is *path-independent*,¹ if agents have time-additive utility functions which are C^3 with bounded derivatives, and if agents are at interior maxima (Assumptions 4.2.1, 4.2.2), then the societal shadow prices for time T consumption are path-independent (Proposition 4.2.1). Here we have exploited the fact that a representative agent with a time-additive utility function which is C^3 with bounded derivatives on an open subset of the real line can be constructed to support the equilibrium at the aggregate endowment point (Propositions 4.1.1 and 4.1.2). Since the equilibrium price of a contingent claim is, very roughly, the *product* of its payoff and the societal shadow prices, it then follows that the equilibrium price process of any contingent claim with a *nice* payoff structure has a *nice* representation (Proposition 4.2.2), which can be described by a partial differential equation with a boundary condition. Conversely, any contingent claim whose price process has a *nice* representation must have a *nice* payoff structure (Proposition 4.2.3). The meaning of *nice* in each context will be made precise. In addition, when a claim's payoff structure is nice, and when some regularity conditions are satisfied, the equilibrium price process of this claim forms, with Z , a diffusion process.

Markovian stochastic dynamic programming is a useful tool in characterizing an agent's dynamic choice either in a purely microeconomic context under uncertainty (cf. Merton [32,33]) or in an equilibrium setting with a representative agent (cf. Cox, Ingersoll, and Ross [10]). Sufficient conditions for the existence of an optimal control are quite severe; for example, the space of admissible controls is compact (cf. Bismut [3], Chapter IV). Cox [8] recently proposed an alternative using a martingale-representation argument. (This method is vaguely foreshadowed in the earlier literature. See, for example, Harrison and Kreps [20, Section 3] and Kreps [26].) In Cox's work, however, the space of admissible controls is a linear space. (This is implicit in Cox's setup.)

When Markovian stochastic dynamic programming is used in an equilibrium setting, as in Merton [34] and Breeden [4], the purpose is both to depict agents' optimal dynamic choices and to characterize equilibrium relations among asset prices. A finite dimensional vector diffusion process Y is assumed to exist, one whose value at each time t along with an

¹We say that time T aggregate endowment is *path-independent* if it is a function of $Z(T)$.

agent's wealth are sufficient statistics for this agent's dynamic program. In this approach, it is shown that, in equilibrium, agents hold the market portfolio, the riskless asset, and those portfolios most highly correlated with each component of Y , respectively. Equilibrium relations among asset prices are then shown to be determined by Y and the market portfolio. The above procedure is flawed, however. Before an equilibrium is established, and equilibrium relations among asset prices are examined, it is not known whether such a process Y exists. Thus, stochastic control of the Markovian type should be used to depict agents' optimal dynamic choices *after* an equilibrium is established and characterized.

In Section 5, we fix a *version* of the equilibrium established in Section 3 and argue that after an equilibrium is established and its stochastic nature characterized, the martingale-representation technique proposed by Cox [9] seems to be a very useful way to describe agents' optimal portfolio behavior. (The meaning of *version* is that we choose a particular set of long-lived securities.) It is shown that agents hold pure *hedging securities* and a *numeraire security* (Proposition 5.1). This characterization is valid when Markovian dynamic programming is *not* applicable.

Recall that, in our setting, Z is the vector diffusion process that generates agents' information. Denoting the price system for long-lived securities by S , we show that the state-price vector process (Z, S) is a diffusion process provided certain regularity conditions are satisfied (Assumptions 4.2.1, 4.2.2, 4.2.3). If an agent's time T endowment is path-independent (with respect to Z), the number of shares of a hedging security held by this agent at each time t depends only upon $Z(t)$. The number of shares of the numeraire security held by this agent at each time t depends only upon $Z(t)$ and $S(t)$. Thus this agent's optimal portfolio rule is path-independent with respect to Z and S . The functional relation between this agent's optimal portfolio rule for pure hedging securities and Z is described by a system of partial differential equations with boundary conditions (Proposition 5.3). Agents in the economy can *all*, however, have *path-dependent* optimal portfolio rules, with respect to (Z, S) , without destroying the strong Markov property of (Z, S) . Thus, summing up agents' first order conditions from their dynamic programs to characterize equilibrium asset price relations may be ineffective. Concluding remarks are given in Section 6.

2. The economy

In this section we present a model of a continuous-time frictionless pure exchange economy under uncertainty with time span $[0, T]$. Let (Ω, \mathcal{F}, P) be a complete probability space. Each $\omega \in \Omega$ denotes a complete description of the exogenous environment. The set of trading dates is $[0, T]$, where T is a strictly positive real number. Agents are endowed with a common probability measure on the measurable space (Ω, \mathcal{F}) , denoted by P .

2.1. The information structure

We assume that there is defined on the basic probability space, (Ω, \mathcal{F}, P) , an N -dimensional Standard Brownian Motion $W = \{W(t); 0 \leq t \leq T\}$. The component processes $W_1(t), \dots, W_N(t)$ are independent one-dimensional Standard Brownian Motions. Let \mathcal{F}_t^w be the ²tribe² generated by $\{W(s); 0 \leq s \leq t\}$. We assume that $\mathcal{F}_T^w = \mathcal{F}$ and that \mathcal{F}_t^w is augmented by all P -negligible sets $\forall t \in [0, T]$. It is clear that \mathcal{F}_0 is almost trivial and that the filtration $\mathbf{F}^w = \{\mathcal{F}_t^w; 0 \leq t \leq T\}$ is increasing, that is, $\mathcal{F}_t^w \subseteq \mathcal{F}_s^w$, if $t \leq s$. For the fact that \mathbf{F}^w is a *continuous information structure*, see the ~~previous~~ Essay.I of Huang [24].

If $\sigma = (\sigma_{m,n})$ is a matrix, we write $|\sigma|^2 = \text{tr}(\sigma\sigma^\top)$, where \top denotes the inner product, and “tr” denotes trace. Let

$$\sigma(y, t) : R^N \times [0, T] \rightarrow R^{N \times N} \text{ and } \mu(y, t) : R^N \times [0, T] \rightarrow R^N$$

be given functions, continuous in y and t ,³ such that

$$|\mu(y, t) - \mu(\bar{y}, t)| \leq K^* |y - \bar{y}|, \quad |\sigma(y, t) - \sigma(\bar{y}, t)| \leq K^* |y - \bar{y}| \quad (2.1.1a)$$

(a Lipschitz condition), and

$$|\mu(y, t)|^2 \leq K^2(1 + |y|^2), \quad |\sigma(y, t)|^2 \leq K^2(1 + |y|^2) \quad (2.1.1b)$$

²“Tribe” may be read as “sigma-field” or “sigma-algebra”, but the former term seems simpler and is more modern.

³This implies that σ and μ are measurable with respect to the product Borel tribe on $R^N \times [0, T]$.

(a “growth condition”), for some constants K^* and K . We assume that the $N \times N$ matrix $\sigma(\mathbf{y}, t)$ is nonsingular for each \mathbf{y} and t . Let Z be a (measurable) process adapted to \mathbf{F}^w satisfying the Ito integral equation

$$Z(t) = Z(0) + \int_0^t \mu(Z(s), s) ds + \int_0^t \sigma(Z(s), s) dW(s) \quad (2.1.2)$$

for $0 \leq t \leq T$, where $Z(0)$ is a constant N -vector.⁴ Theorem 9.3.1 in Arnold [1] ensures that Z is the unique solution of (2.1.2) and is a diffusion process with drift vector $\mu(\mathbf{y}, t)$ and diffusion matrix $\sigma(\mathbf{y}, t)$. Here we should note that the above statement implies the conditions:

$$\int_0^T |\mu(Z(t), t)| dt < \infty \text{ a.s.},$$

and

$$\int_0^T |\sigma(Z(t), t)|^2 dt < \infty \text{ a.s.}$$

Let \mathcal{F}_t^z be the tribe generated by $\{Z(s); 0 \leq s \leq t\}$ augmented by all the P -negligible sets of \mathcal{F} . The filtration \mathbf{F}^w is at least as fine as \mathbf{F}^z since the process Z is adapted to \mathbf{F}^w . In fact, \mathbf{F}^z is equivalent to \mathbf{F}^w . For this point see Harrison and Kreps [20]. We shall therefore use \mathbf{F} to denote both \mathbf{F}^z and \mathbf{F}^w from now on. Furthermore, unless it is clearly otherwise from the context, all processes will be adapted to \mathbf{F} .

Agents in the economy are endowed with the common information structure \mathbf{F} . By construction, \mathbf{F} is increasing and is a continuous information structure. The interpretation is that the exogenous uncertain environment can be described by an N -dimensional Brownian motion W , which agents in the economy may not actually observe directly. Agents can, however, observe a vector of “state variable processes” Z , whose evolution over time depends upon W in an unpredictable fashion. That is, the information content of the vector of state variable processes can only be less than that of W . The structure of the state variable process, however, provides agents with the same information as if they could in fact observe

⁴ A vector random process $Y = \{Y(t); t \in [0, T]\}$, is called measurable if, when viewed as a mapping defined on $\Omega \times [0, T]$, for all Borel sets B in its range, $\{(\omega, t) : Y(\omega, t) \in B\} \in \mathcal{F} \otimes \mathcal{B}([0, T])$, where $\mathcal{B}([0, T])$ is the Borel tribe of $[0, T]$. A vector (measurable) random process Y is said to be adapted to \mathbf{F}^w if for every $t \in [0, T]$ the random variable $Y(t)$ is \mathcal{F}_t^w -measurable. For brevity, such a random process will be denoted $Y = \{Y(t), \mathcal{F}_t^w, t \in [0, T]\}$ and called \mathbf{F}^w -adapted or *nonanticipative*.

the evolution of W directly, since Z and W generate the same information. The reader is cautioned to note that the vector of state variable processes is exogenously specified.

2.2. The consumption space

It is assumed that there is a single perishable consumption commodity in the economy, consumed only at times 0 and T . The consumption space for agents is $V = R \times L^2(P)$, where $L^2(P)$ denotes the space of square-integrable random variables defined on (Ω, \mathcal{F}, P) . Thus $(r, x) \in V$ represents r units of consumption at time zero and $x(\omega)$ units of consumption at time T in state ω . We endow V with the product topology τ generated by the Euclidean topology on R and the $L^2(P)$ -norm topology on $L^2(P)$.

The set $L_+^2(P)$ is defined as $\{x \in L^2(P) : P\{\omega \in \Omega : x(\omega) \geq 0\} = 1\}$, while V_+ denotes the set $\{(r, x) \in V : r \in [0, \infty), x \in L_+^2(P)\}$. For any $x \in L^2(P)$, we will write $x \geq 0$ if $x \in L_+^2(P)$; $x > 0$ if $x \geq 0$ and $x \neq 0$; and $x \gg 0$ if $x \in L_+^2(P)$ and $P\{\omega \in \Omega : x(\omega) > 0\} = 1$. Similarly, for $v = (r, x) \in V$ we will write $v \geq 0$ if $v \in V_+$; $v > 0$ if $v \geq 0$ and $v \neq 0$; and $v \gg 0$ if $r > 0$ and $x \gg 0$. For $v, z \in V$, the relation $v \geq z$ is taken to mean that $v - z \geq 0$, and likewise for the other relations just defined on V .

2.3. Agents

There are a finite number of agents in the economy indexed by $i = 1, 2, \dots, I$. Each agent i is characterized by a consumption set V_i , endowments $\hat{v}_i = (\hat{r}_i, \hat{x}_i) \in V_i \subseteq V$, and consumption preferences represented by a von Neumann-Morgenstern utility function $U_i : V_i \rightarrow R$ of the form:

$$U_i(r, x) = \int_{\Omega} u_i(r, x(\omega))P(d\omega).$$

We assume that for every $i = 1, 2, \dots, I$:

(1) $V_i = V_+$;

(2) $u_i(r, y) : R_+ \times R_+ \rightarrow R$ is continuous in r , concave and strictly increasing in r and y , with finite right hand partial derivatives with respect to r and y denoted $\frac{\partial}{\partial r}^+ u_i(r, y)$ and $\frac{\partial}{\partial y}^+ u_i(r, y)$, respectively, and $\frac{\partial}{\partial r}^+ u_i(0, x(\omega)) \in L^1(P) \forall x \in L_+^2(P)$.

(3) there exist $(\hat{r}_i, \hat{x}_i) \in V_+$ and $\epsilon_i > 0$ such that, for every $(s, y) \in V$ and $k \in R_+$, if $U_i(r - k\hat{r}_i + s, x - k\hat{x}_i + y) \geq U_i(r, x)$ then $(s^2 + \int_{\Omega} y^2(\omega)P(d\omega))^{\frac{1}{2}} \geq k\epsilon_i$.

Note that (2) above implies that U_i is τ -continuous, and *strictly increasing* in the sense that $U_i(\tau + s, x + y) > U_i(\tau, x)$ whenever $(s, y) > 0$; and (3) implies that U_i is *proper*. (For a general definition of *properness* see Mas-Colell [30].) Note also that a sufficient condition for (3) is that for all i , $\frac{\partial}{\partial y}^+ u_i(\tau, y)$ is bounded away from zero for all $(\tau, y) \in R_+^2$, $\frac{\partial}{\partial r}^+ u_i(0, x(\omega)) \in L^2(P)$, for all $x \in L_+^2(P)$, and (2). We shall further assume that

(4) $\hat{v}_i > 0$ for every $i = 1, 2, \dots, I$, $\sum_i \hat{v}_i \gg 0$, and $\sum_{i=1}^I \hat{x}_i(\omega) \geq \underline{c}$ for almost every $\omega \in \Omega$, where \underline{c} is a strictly positive real number; and

(5) there is at least one agent, say agent 1, such that $\lim_{y \rightarrow \infty} \frac{\partial}{\partial y}^+ u_1(\tau, y) > 0$, for all $\tau \in R_+$.

We interpret (5) to mean agent 1's marginal utility for time T consumption is bounded away from zero.

2.4. Traded long-lived securities

There are a finite number of *long-lived securities* (cf. Kreps [25]) traded in the economy, indexed by $j = 1, 2, \dots, J$. Each long-lived security is represented by an element $d_j \in L^2(P)$. The holder of one share of security j is entitled to $d_j(\omega)$ units of consumption at time T in state ω . The traded securities are in zero net supply. Agents have zero initial endowments of these long-lived securities.

2.5. The admissible price systems and trading strategies

Before any discussion of the admissible price systems and trading strategies, some technical definitions are in order.

A continuous random process γ is called an *Ito process* (relative to the vector Brownian Motion W) if there exists a nonanticipative process ζ , and an N -vector (row) nonanticipative process φ , such that

$$P \left\{ \int_0^T |\zeta(t)| dt < \infty \right\} = 1, \quad (2.5.1)$$

$$P \left\{ \int_0^T |\varphi(t)|^2 dt < \infty \right\} = 1, \quad (2.5.2)$$

and, with probability one for all $t \in [0, T]$,

$$\gamma(t) = \gamma(0) + \int_0^t \zeta(s) ds + \int_0^t \varphi(s) dW(s). \quad (2.5.3)$$

Returning back to economics, an admissible price system S , is a J -vector Ito process S of the form:

$$S_j(t) = S_j(0) + \int_0^t \zeta_j(s)ds + \int_0^t \beta_j(s)dW(s) \quad \forall j = 1, 2, \dots, J,$$

with probability one for all $t \in [0, T]$, with $S(t)$ representing the J -vector of relative prices of the J traded long-lived securities at time t .⁵

Given an admissible price system S , an admissible trading strategy θ is a J -vector (column) nonanticipative process such that

$$E\left(\int_0^T \theta(t)^\top \beta(t) \beta(t)^\top \theta(t) dt\right) < \infty, \quad (2.5.4)$$

where the j -th row of β is β_j , and the following conditions hold:

1. for every $t \in [0, T]$, the stochastic integral $\int_0^t \theta(s)^\top dS(s)$ is well-defined in the Ito sense, that is,

$$\int_0^T |\theta(t)^\top \zeta(t)| dt < \infty, \quad a.s.$$

and

$$\int_0^T |\theta(t)^\top \beta(t) \beta(t)^\top \theta(t)|^2 dt < \infty \quad a.s.;$$

2. for every $t \in [0, T]$,

$$\theta(t)^\top S(t) = \theta(0)^\top S(0) + \int_0^t \theta(s)^\top dS(s) \quad a.s. \quad (2.5.5)$$

Let $\Theta[S]$ denote the set of admissible trading strategies with respect to the price system S . By virtue of (2.5.5), elements of $\Theta[S]$ are all *self-financing* trading strategies. Equation (2.5.5) says that the initial value of the trading strategy (portfolio) θ plus the amount of

⁵Since \mathbf{F} is a Brownian filtration, from ^{Essay I of Huang [14]} ~~the previous essay~~ we know any equilibrium price system can be represented as an Ito integral. Thus taking admissible price systems to be the set of Ito processes is not restrictive at all, once we know that an equilibrium exists.

capital gain or loss by any time t is equal to the value of the strategy (portfolio) at that time. That is, after the initial investment, there is neither withdrawal of funds out of nor investment into the portfolio. (This is the obvious budget constraint.) By the linearity of stochastic integrals and a simple application of the Cauchy-Schwarz inequality, $\Theta[S]$ can easily be shown to be a linear space.

2.6. Equilibrium

Each agent's problem in the economy is to manage a portfolio of long-lived securities so as to maximize preferences on consumption at times zero and T . An *equilibrium of plans, prices, and price expectations* (of Radner [37]) is an admissible price system S , admissible trading strategies $\left\{(\theta_i^*)_{i=1}^I\right\}$, one for each agent, and a price a for the consumption good at time zero (relative to the prices of the securities, given by $S(0)$), such that, for all $i = 1, 2, \dots, I$,

$$\sum_{i=1}^I \theta_i^*(t) = 0, \quad \forall t \in [0, T], \quad a.s.,$$

and $(\hat{r}_i - \theta_i^*(0)^\top S(0)/a, \hat{x}_i + \theta_i^*(T)^\top d)$ is U_i -maximal in the set

$$\{(\hat{r}_i - \theta(0)^\top S(0)/a, \hat{x}_i + \theta(T)^\top d) \in V_i : \theta \in \Theta[S]\}.$$

3. The existence of an equilibrium

The continuous-trading model of financial markets formulated in the previous section has been popular among financial theorists for more than a decade. It has always been *assumed* in the literature that an equilibrium exists and that the equilibrium price system is a vector Ito process (cf. Merton [34], Breeden [4], Cox, Ingersoll, and Ross [10]). Huang [24] (Essay I) studied a similar model and showed that, if indeed an equilibrium exists in the continuous-trading economy, then equilibrium asset prices are Ito integrals if information is generated by (multidimensional) Brownian motion. The existence of an equilibrium for an economy of one representative agent was demonstrated. In this section we use the machinery developed in Duffie and Huang [12] to show that if the J traded long-lived

securities are chosen appropriately, an equilibrium for our economy exists. Furthermore, the equilibrium allocation will be Pareto efficient. The proof will be constructive in the sense that a finite number of long-lived securities are selected, their prices are announced, a trading strategy for each agent is assigned, markets are shown to be clear, and the allocation is shown to be Pareto efficient. Before we carry out this proof, we analyze an analogous Arrow-Debreu economy.

3.1. An Arrow-Debreu economy

Suppose that at time zero not only the spot market for the consumption good but also markets for all “state contingent claims”⁶ are open in an economy whose consumption space and agents are as described in Sections 2.3 and 2.4. We will then show the existence of an Arrow-Debreu equilibrium, defined as a strictly positive τ -continuous linear functional $\psi : V \rightarrow R$ and an allocation $\{v_i^* = (r_i^*, x_i^*) \in V_i, i = 1, 2, \dots, I\}$ such that

$$\sum_{i=1}^I v_i^* = \sum_{i=1}^I \hat{v}_i \text{ a.s.}$$

and v_i^* is U_i -maximal in the set $\{v \in V_i : \psi(v) \leq \psi(\hat{v}_i)\}$, for each $i = 1, 2, \dots, I$. The linear functional ψ gives Arrow-Debreu equilibrium prices. Here we should note that if $L^2(P)$ is infinite dimensional, then an Arrow-Debreu style economy consists of an infinite number of markets at time zero. There is no incentive, however, for markets to reopen after time zero. (For this point in a finite dimensional commodity space, see Arrow [2].)

Proposition 3.1.1: Let $\mathcal{E} = (V_i, U_i, \hat{v}_i, i = 1, 2, \dots, I)$ be an economy satisfying the conditions of Sections 2.2 and 2.3. Then there exists an Arrow-Debreu equilibrium $((v_i^*)_{i=1}^I, \psi)$.

Proof: Given the assumptions of Sections 2.2 and 2.3, the existence of a *quasi-equilibrium* follows from Mas-Colell [30].⁷ A quasi-equilibrium is a τ -continuous positive linear functional $\psi : V \rightarrow R$ and an allocation $\{v_i^* \in V_i, i = 1, 2, \dots, I\}$ such that

⁶A state contingent claim is an element of $L^2(P)$.

⁷Since V is a reflexive Banach space, the *Closeness Hypothesis* in Mas-Colell [30] is automatically satisfied.

$\sum_{i=1}^I v_i^\circ = \sum_{i=1}^I \hat{v}_i$ almost surely and

$$U_i(v) \geq U_i(v^\circ) \text{ implies } \psi(v) \geq \psi(v^\circ).$$

To show that ψ and $\{v_i^\circ, i = 1, 2, \dots, I\}$ form an equilibrium, we must show that ψ is strictly positive and that

$$U_i(v) > U_i(v_i^\circ) \text{ implies } \psi(v) > \psi(v_i^\circ) \quad \forall i.$$

By the assumption that $\sum_{i=1}^I \hat{v}_i \gg 0$ ((4) in Section 2.3), we know $\psi(\sum_i \hat{v}_i) > 0$, and therefore $\psi(v_{i'}) > 0$ for some $i' \in \{1, 2, \dots, I\}$. Suppose $v \in V_+$, $U_{i'}(v) > U_{i'}(v_{i'}^\circ)$, and $\psi(v) = \psi(v_{i'}^\circ)$. By continuity of preferences, there exists $0 < \eta < 1$ such that

$$U_{i'}(\eta v) > U_{i'}(v_{i'}^\circ).$$

Hence,

$$\psi(\eta v) \geq \psi(v_{i'}^\circ) = \psi(v) > 0.$$

This implies that $\eta \geq 1$, an obvious contradiction. Therefore $U_{i'}(v) > U_{i'}(v_{i'}^\circ)$ implies $\psi(v) > \psi(v_{i'}^\circ)$. Since $U_{i'}$ is strictly increasing, ψ is strictly positive. Then, by the assumption that $\hat{v}_i > 0$, we know $\psi(v_i^\circ) > 0$ for all i . We can repeat the above arguments to show that v_i° is U_i -maximal for all i . Therefore, ψ and $\{v_i^\circ \in V_i, i = 1, 2, \dots, I\}$ comprise an Arrow-Debreu equilibrium. ■

In an economic equilibrium only relative prices are determined. We can therefore freely choose any contingent claim having a positive price as the numeraire. In particular, the following characterization is possible.

Proposition 3.1.2: Suppose the Arrow-Debreu equilibrium of Proposition 3.1.1 has as numeraire the contingent claim paying one unit of the consumption good at time T in every state. That is, $\psi(0, 1_\Omega) = 1$. Then there exists a probability measure Q on (Ω, \mathcal{F})

uniformly absolutely continuous with respect to P ,⁸ such that

$$\psi(0, x) = E^*(x), \quad \forall x \in L^2(P), \quad (3.1.1)$$

where $E^*(\cdot)$ denotes expectation under Q .

Proof: Since ψ is a r -continuous strictly positive linear functional on V , there exists $(a, \xi) \in R \times L^2(P)$ with $a > 0$ and $\xi > 0$ such that for all $(r, x) \in V$

$$\begin{aligned} \psi(r, x) &= ar + \int_{\Omega} x(\omega)\xi(\omega)P(d\omega), \\ &\equiv ar + \phi(x). \end{aligned}$$

That is, $\phi(x)$ gives the equilibrium price at time zero for a claim which pays $x(\omega)$ at time T in state ω , and the price of one unit of consumption good is a . By assumption we know that

$$\psi(0, 1_{\Omega}) = \int_{\Omega} \xi(\omega)P(d\omega) = 1.$$

Defining

$$Q(A) = \int_A \xi(\omega)P(d\omega), \quad \forall A \in \mathcal{F},$$

Q is a probability measure on (Ω, \mathcal{F}) equivalent to P . It is a probability measure since it is countably additive and $Q(\Omega) = E(\xi) = 1$; Q is equivalent to P since ξ is strictly positive.

Therefore,

$$\begin{aligned} \psi(0, x) &= \phi(x) = \int_{\Omega} x(\omega)\xi(\omega)P(d\omega) \\ &= \int_{\Omega} x(\omega)Q(d\omega) \\ &\equiv E^*(x). \end{aligned}$$

Now consider agent 1. We know $v_1^* = (r_1^*, x_1^*)$ must solve the following concave programming problem:

$$\max_{v \in V_+} U_1(v)$$

⁸ Q is said to be uniformly absolutely continuous with respect to P if it is equivalent to P and the Radon-Nikodym derivative dQ/dP is bounded above and below away from zero.

$$\text{s.t. } \psi(v) - \psi(\hat{v}_1) \leq 0.$$

Since $V_i = V_+$ and $\psi(\hat{v}_i) > 0$ for all i , we know that the Slater condition holds. (A good reference for the Slater condition is Section 14E of Holmes [23].) It then follows from the *saddle-point* theorem that v_1^* is a solution for the above concave programming problem only if there exists a nonnegative number λ_1 such that for any $v \in V_+$

$$U_1(v) - U_1(v_1^*) \leq \lambda_1 \left[\psi(v) - \psi(v_1^*) \right]. \quad (3.1.1)$$

(For this point, see Section 14F of Holmes [23].) We claim that λ_1 is strictly positive. Suppose this is not the case. Take any $v \in V_+$ with $v > 0$. Since V_+ is a positive cone, we have $v_1^* + v \in V_+$. Since U_1 is strictly increasing, we have $U_1(v_1^* + v) - U_1(v_1^*) > 0$. This contradicts (3.1.1).

Now we proceed to prove that ξ is bounded away from zero. Since V_+ is a positive cone, we know that $(r_1^*, x_1^* + k1_A) \in V_+$ for all $A \in \mathcal{F}$ and $k \in R_+$. So we have,

$$\begin{aligned} U_1(r_1^*, x_1^* + k1_A) - U_1(r_1^*, x_1^*) &\leq \lambda_1 \left[\psi(r_1^*, x_1^* + k1_A) - \psi(r_1^*, x_1^*) \right] \\ &= \lambda_1 k \int_A \xi(\omega) P(d\omega). \end{aligned}$$

Dividing both side of the above equation by k and letting k approach zero, we get, by (2) in Section (2.3) and the Lebesgue convergence theorem:

$$\int_A \frac{\partial^+}{\partial y} u_1(r_1^*, x_1^*(\omega)) P(d\omega) \leq \lambda_1 \int_A \xi(\omega) P(d\omega). \quad (3.1.2)$$

Equivalently,

$$\int_A \left(\lambda_1 \xi(\omega) - \frac{\partial^+}{\partial y} u_1(r_1^*, x_1^*(\omega)) \right) P(d\omega) \geq 0. \quad (3.1.3)$$

Since (3.1.3) holds for every $A \in \mathcal{F}$, we can take $A = \{\omega \in \Omega : \lambda_1 \xi(\omega) - \frac{\partial^+}{\partial y} u_1(r_1^*, x_1^*(\omega)) < 0\}$. For (3.1.3) to hold it must be that $P(A) = 0$. That is,

$$\lambda_1 \xi(\omega) \geq \frac{\partial^+}{\partial y} u_1(r_1^*, x_1^*(\omega)) \text{ a.s.} \quad (3.1.4)$$

By (5) in Section 2.3, we know $\frac{\partial^+}{\partial y} u_1(r_1^*, x_1^*(\omega))$ is bounded away from zero. Dividing the above expression by λ_1 , we have the desired result that ξ is bounded away from zero.

Next we want to show that ξ is bounded above. Recall the assumption ((4) in Section 2.3) that $\sum_i \hat{x}_i \geq \underline{c}$, where \underline{c} is a strictly positive constant. Let $A_i = \{\omega \in \Omega : x_i^*(\omega) \geq \underline{c}/I\}$. Then $A_i \in \mathcal{F}$ for all i and $\Omega = \bigcup_i A_i$. For any real numbers $k \in (0, \underline{c}/I)$ and $B \in \mathcal{F}$, $(r_i^*, x_i^* - k1_{A_i \cap B}) \in V_+$. So (3.1.1) implies that

$$U_1(r_i^*, x_i^* - k1_{A_i \cap B}) - U_1(r_i^*, x_i^*) \leq -\lambda_1 k \int_{A_i \cap B} \xi(\omega) P(d\omega).$$

Dividing both sides of the above expression by $-k$ and letting it approach zero gives

$$\int_{B \cap A_i} \frac{\partial^-}{\partial y} u_i(r_i^*, x_i^*(\omega)) P(d\omega) \geq \lambda_i \int_{B \cap A_i} \xi(\omega) P(d\omega) \quad \forall B \in \mathcal{F},$$

where $\frac{\partial^-}{\partial y} u_i(r, y)$ denotes the left hand partial derivative of $u_i(r, y)$ with respect to y . This implies that

$$\begin{aligned} \lambda_i \xi(\omega) &\leq \frac{\partial^-}{\partial y} u_i(r_i^*, x_i^*(\omega)) \\ &\leq \frac{\partial^+}{\partial y} u_i(r_i^*, 0) \text{ for almost every } \omega \in A_i, \end{aligned} \tag{3.1.5}$$

the second inequality of which follows from concavity. Now let $K = \sup_i \frac{\partial^+}{\partial y} u_i(r_i^*, 0)/\lambda_i$. By (2) in Section 2.3, K is finite. Since $\xi \leq K$ a.s., the proof is complete. ■

Proposition 3.1.2 states that Arrow-Debreu prices can be normalized so that the price for any claim $x \in L^2(P)$ at time zero is given by $E^*(x)$, its expected value under a probability Q uniformly absolutely continuous with respect to P .

The uniform absolute continuity of Q with respect to P can be illustrated as follows. Interpret ξ to be societal shadow prices for time T consumption. In equilibrium, we know ξ must be greater than or equal to each agent's shadow prices for time T consumption, or otherwise, agents can increase their utilities *cheaply*. In particular, ξ must be greater than or equal to agent 1's shadow prices, which are constant multiples of his marginal utilities for time T consumption. Since agent 1's marginal utilities for time T consumption

are assumed to be bounded away from zero, ξ is bounded away from zero. On the other hand, by the assumption that time T aggregate endowment is bounded away from zero, in (almost) every state there is at least one agent who is consuming a nontrivial amount. For an agent to consume a nontrivial amount in a particular state, it is necessary, roughly, that his shadow price for consumption in that state be no less than the societal shadow price. By the assumption that all the agents' marginal utilities are bounded above, we then know ξ is bounded above.

3.2. Some Theorems on the representation of square-integrable martingales

It is well known that any square-integrable martingale adapted to a Brownian motion filtration can be represented as an Ito integral (Kunita and Watanabe [28]). We will make use of the following result, originally developed by Fujisaki, Kallianpur, and Kunita [15], in the context of non-linear filtering.

Lemma 3.2.1: Let there be defined on a complete probability space $(\Omega', \mathcal{F}', P')$ an N -dimensional process Y . Let the filtration generated by Y be denoted by $\mathbf{F}^y = \{\mathcal{F}_t^y, t \in [0, T]\}$. Assumed that $\mathcal{F}_T^y = \mathcal{F}'$, that for every $t \in [0, T]$ \mathcal{F}_t^y is augmented by all the P' -negligible sets, and that \mathcal{F}_0^y is almost trivial. Suppose that Y can be represented by an Ito integral as

$$Y(t) = Y(0) + \int_0^t h(s) ds + W^y(t),$$

where W^y is an N -dimensional standard Brownian motion adapted to \mathbf{F}^y , and where $\{h(t), \mathcal{F}_t^y, t \in [0, T]\}$ is an N -vector \mathbf{F}^y -adapted process defined on $(\Omega', \mathcal{F}', P')$ satisfying

$$E_{P'} \int_0^T |h(t)|^2 dt < \infty, \quad (3.2.1)$$

where $E_{P'}$ denotes the expectation under P' . Then any square-integrable martingale m defined on $(\Omega', \mathcal{F}', P')$ adapted to \mathbf{F}^y can be represented as

$$m(t) = m(0) + \int_0^t h(s)^\top dW^y(s) \quad \forall t \in [0, T],$$

for some h satisfying (3.2.1).

Proof: See Theorem 3.1 of Fujisaki, Kallianpur, and Kunita [14]. ■

The process Y of Lemma 3.2.1 is a generalized diffusion on $(\Omega', \mathcal{F}', P')$. (For the definition of a generalized diffusion see Chapter 5 of Liptser and Shiriyayev [29].) Lemma 3.2.1 states that any square-integrable martingale on $(\Omega', \mathcal{F}', P')$ adapted to the filtration generated by the generalized diffusion Y can be represented as a stochastic integral with respect to the Brownian motion W^y .

The information structure \mathbf{F} for our economy is generated by an N -dimensional Brownian motion W defined on (Ω, \mathcal{F}, P) . Therefore, any square-integrable martingale on (Ω, \mathcal{F}, P) adapted to \mathbf{F} can be represented as an Ito integral with respect to W (cf. Kunita and Watanabe [28]). The next proposition shows that if we substitute for P the probability measure Q constructed in Proposition 3.1.2, then any square-integrable martingale defined on (Ω, \mathcal{F}, Q) adapted to \mathbf{F} can still be represented as an Ito integral with respect to N independent Brownian motions on (Ω, \mathcal{F}, Q) adapted to \mathbf{F} . That is, the *multiplicity* of the filtration \mathbf{F} is invariant under a substitution of a uniform absolute continuous probability measure. (For a discussion of *multiplicity*, see Duffie and Huang [13].)

Proposition 3.2.1: There exists an N -dimensional Brownian motion W^* defined on (Ω, \mathcal{F}, Q) adapted to \mathbf{F} such that any square-integrable martingale m on (Ω, \mathcal{F}, Q) adapted to \mathbf{F} can be represented as

$$m(t) = m(0) + \int_0^t h(s)^\top dW^*(s), \quad a.s.$$

where $\{h(t), \mathcal{F}_t, t \in [0, T]\}$ is an N -vector nonanticipative functional such that

$$E^* \int_0^T |h(t)|^2 dt < \infty. \quad (3.2.2)$$

Remark: We will henceforth denote the set of \mathbf{F} -adapted N -vector processes defined on (Ω, \mathcal{F}, Q) satisfying (3.2.2) by $H(Q)$. The analogous set defined under probability measure P is denoted by $H(P)$. Since Q is uniformly absolutely continuous with respect to P , $H(P) = H(Q)$.

Remark: Since P and Q are equivalent, we use *a.s.* to denote both $P - a.s.$ and $Q - a.s.$, unless a distinction is needed.

Proof: For $\xi = dQ/dP$, let us define

$$\xi(t) = E(\xi | \mathcal{F}_t) \text{ a.s.},$$

where $\{\xi(t)\}$ is taken to be a continuous version of $\{E(\xi | \mathcal{F}_t)\}$.⁹ Then there exists $\rho \in H(P)$ such that

$$\xi(t) = 1 + \int_0^t \rho(s)^\top dW(s) \quad (3.2.3)$$

(Kunita and Watanabe [28]). From Proposition 3.1.2 we know that ξ is strictly positive and bounded away from zero. It follows from Ito's Lemma that $\xi(t)$ can be represented as

$$\xi(t) = \exp \left\{ \int_0^t \eta(s)^\top dW(s) - \frac{1}{2} \int_0^t |\eta(s)|^2 ds \right\},$$

where $\eta(t) \equiv \rho(t)/\xi(t)$. Let

$$W^*(t) = W(t) - \int_0^t \eta(s) ds, \quad t \in [0, T]. \quad (3.2.4)$$

Girsanov's Fundamental Theorem 1 ([17]) states that W^* is an N -dimensional Brownian motion on (Ω, \mathcal{F}, Q) . It is clear that W^* is adapted to \mathbf{F} . By rearranging the last expression,

$$W(t) = \int_0^t \eta(s) ds + W^*(t), \quad t \in [0, T] \quad (3.2.5)$$

is a generalized diffusion on (Ω, \mathcal{F}, Q) and generates \mathbf{F} . Furthermore,

$$\begin{aligned} E^* \left(\int_0^T |\eta(t)|^2 dt \right) &= E \left(\xi \int_0^T \frac{|\rho(t)|^2}{\xi^2(t)} dt \right) \\ &\leq \frac{\text{ess sup } \xi}{(\text{ess inf } \xi)^2} E \left(\int_0^T |\rho(t)|^2 dt \right) < \infty. \end{aligned} \quad (3.2.6)$$

It then follows from Lemma 3.2.1 that any square-integrable martingale $\{m(t), \mathcal{F}_t, t \in [0, T]\}$ on (Ω, \mathcal{F}, Q) can be represented as

$$m(t) = m(0) + \int_0^t h(s)^\top dW^*(s), \text{ a.s.}$$

⁹This is possible since \mathbf{F} is a continuous information structure. For details, see ~~the previous~~ ~~essay~~. Essay I of Huang [24].

for some $h \in H(Q)$. This was to be shown. ■

Remark: Relation (3.2.4) defines W^\bullet as an Ito process (under P) since

$$\begin{aligned} E \int_0^T |\eta(t)|^2 dt &= E \int_0^T \frac{|\rho(t)|^2}{\xi^2(t)} dt \\ &\leq \frac{1}{(\text{ess inf } \xi)^2} E \int_0^T |\rho(t)|^2 dt < \infty \end{aligned} \quad (3.2.8)$$

implies that

$$\int_0^T |\eta(t)| dt \leq T^{\frac{1}{2}} \left(\int_0^T |\eta(t)|^2 dt \right)^{\frac{1}{2}} < \infty \quad P - a.s.$$

by the Cauchy-Scharwz inequality.

Remark: Let $h \in H(P)$. Then

$$m(t) \equiv m(0) + \int_0^t h(s)^\top dW(s) \quad a.s.,$$

where $m(0)$ is a constant, is a square-integrable martingale under P . Similarly, for $h \in H(Q)$,

$$m(t) \equiv m(0) + \int_0^t h(s)^\top dW^\bullet(s) \quad a.s.$$

is a square-integrable martingale under Q . See Chapter 4 of Liptser and Shiriyayev [29].

3.3. A constructive proof of the existence of an equilibrium

In this sub-section we prove the existence of an equilibrium for our continuous-trading economy. The proof is essentially that given as Proposition 5.1 in Duffie and Huang [13], recast in an economy with diffusion process information. In that paper it was shown that if the tribe \mathcal{F} is separable, if an equilibrium in the Arrow-Debreu economy exists, and if the *equilibrium price measure* Q is uniformly absolutely continuous with respect to P , then the Arrow-Debreu equilibrium can be implemented by continuous trading of at most a countable number of long-lived securities. We show here, however, that in our economy an equilibrium with a *finite* number of long-lived securities exists.

Theorem 3.3.1: An equilibrium exists in the continuous-trading economy formulated in Section 2 with $N + 1$ long-lived securities having payoff structures:

$$(d_j)_{j=1}^N = \int_0^T \hat{\beta}(t) dW^*(t)$$

$$d_{N+1} = 1_\Omega,$$

where $\hat{\beta} : \Omega \times [0, T] \rightarrow R^{N \times N}$ is any function that is nonanticipating, nonsingular for all $t \in [0, T]$ a.s. with $E\left(\int_0^T |\hat{\beta}(t)|^2 dt\right) < \infty$. A set of equilibrium price processes for these $N + 1$ long-lived securities is

$$\begin{aligned} S_j(t) &= E^*(d_j | \mathcal{F}_t) \\ &= \int_0^t \hat{\beta}_j(s) dW^*(s) \\ &= \int_0^t \hat{\beta}_j(s) dW(s) - \int_0^t \hat{\beta}_j(s) \eta(s) ds, \quad a.s. \quad j = 1, 2, \dots, N, \end{aligned} \tag{3.3.1}$$

$$S_{N+1}(t) = 1,$$

for all $t \in [0, T]$, where W^* and η are defined Proposition 3.2.1 and equation (3.2.4), respectively, and where $\hat{\beta}_j$ denotes the j -th row of $\hat{\beta}$. Furthermore, the equilibrium allocation is Pareto efficient.

Proof: Let $\{(r_i^*, x_i^*) \in V_+; i = 1, 2, \dots, I\}$ be an equilibrium allocation in the Arrow-Debreu economy and ψ be the strictly positive and τ -continuous linear functional on V that gives equilibrium prices as in Proposition 3.1.2:

$$\psi(r, x) = ar + E^*(x),$$

where a is the price for time zero consumption, and $E^*(x)$ gives the equilibrium price at time zero for claims $x \in L^2(P)$. By the definition of an Arrow-Debreu equilibrium we know that (r_i^*, x_i^*) is U_i -maximal in the set

$$\{(r, x) \in V_+ : \psi(r, x) \leq \psi(\hat{r}_i, \hat{x}_i)\}$$

for every $i = 1, 2, \dots, I$. By the assumption that agents' preferences are strictly increasing, it is obvious that

$$\psi(r_i^*, x_i^*) = \psi(\hat{r}_i, \hat{x}_i).$$

Equivalently,

$$\begin{aligned} & \psi(r_i^* - \hat{r}_i, x_i^* - \hat{x}_i) \\ &= a(r_i^* - \hat{r}_i) + E^*(x_i^* - \hat{x}_i) = 0. \end{aligned} \tag{3.3.3}$$

The price for $x_i^* - \hat{x}_i$ at time zero is $a(\hat{r}_i - r_i^*)$. That is, at time zero agent i pays $(\hat{r}_i - r_i^*)$ units of consumption good to buy the claim $x_i^* - \hat{x}_i$. In what follows we show that an equilibrium exists in our continuous trading economy with the Arrow-Debreu equilibrium allocation $\{(r_i^*, x_i^*) \in V_+; i = 1, 2, \dots, I\}$. The proof is completed by the following program:

1. Verify that d is well-defined and lies in $L^2(P)$.
2. Show that S is an admissible price system.
3. Allocate an admissible trading strategy to each agent and show that markets clear and the Arrow-Debreu allocation is attained.
4. Show there is no incentive for any agent to deviate from his or her allocated trading strategy.

Now we proceed exactly as outlined above.

Step 1. We want to show that d is well-defined. Note that $E(\int_0^T |\hat{\beta}(t)|^2 dt) < \infty$ implies $E^*(\int_0^T |\hat{\beta}(t)|^2 dt) < \infty$, which in turn implies $\int_0^T |\hat{\beta}(t)|^2 dt < \infty$ Q -almost surely, by the uniform absolute continuity of P with respect to Q . Thus, d is well-defined under P and Q . It follows from the second remark after Proposition 3.2.1 that d is square-integrable under Q , and therefore square-integrable under P , again by the absolute continuity of P with respect to Q .

Step 2. First we show that S is well-defined. The second line of (3.3.1) follows from the second remark after Proposition 3.2.1 and S is well-defined under Q . The stochastic integral in the third line of (3.3.1) is well-defined under P , since $P\{\int_0^T |\hat{\beta}(t)|^2 dt < \infty\} = 1$. Finally, it follows from (3.2.8) and the Cauchy-Scharwz inequality that the Lebesgue integral in the third line of (3.3.1) is well-defined. Thus the second and the third lines of

(3.3.1) are equal from the definition of W^* . Therefore the second line is also well-defined under P , and S is an admissible price system.

Step 3. We allocate an admissible trading strategy $\theta^i \in \Theta[S]$ to each agent i and show that with the announced price system agents obtain their Arrow-Debreu allocations almost surely and markets clear.

Denoting $x_i^* - \hat{x}_i$ by e_i , we know from Proposition 3.2.1 that there exists an N -vector nonanticipative process $\{h^i(t), t \in [0, T]\}$ with $h^i \in H(Q)$ such that

$$e_i = E^*(e_i) + \int_0^T h^i(t)^\top dW^*(t) \quad a.s.,$$

since, by the fact that ξ is bounded above, $e_i \in L^2(Q)$. For every $t \in [0, T]$, and $j = 1, 2, \dots, N$, let

$$\theta_j^i(t) = h^i(t) \beta_j^{-1}(t),$$

where $\hat{\beta}_j^{-1}$ denotes the j -th column of the inverse of β , and let

$$\theta_{N+1}^i(t) = E^*(e_i) + \sum_{j=1}^N \int_0^t \theta_j^i(s) dS_j(s) - \sum_{j=1}^N \theta_j^i(t) S_j(t).$$

We claim that the $N + 1$ -vector process θ^i is an admissible trading strategy. Let $\beta : \Omega \times [0, T] \rightarrow R^{(N+1) \times N}$ be such that $\beta_j(t) = \hat{\beta}_j(t) \forall t \in [0, T]$ a.s. for all $j = 1, 2, \dots, N$, and $\beta_{N+1}(t) = \mathbf{0} \forall t \in [0, T]$, where $\beta_j(t)$ denotes the j -th row of $\beta(t)$, and where $\mathbf{0}$ denotes an N row vector of zeros. First we want to show that

$$E \left(\int_0^T (\theta^i(t)^\top \beta(t) \beta(t)^\top \theta^i(t)) dt \right) < \infty. \quad (3.3.4)$$

Note that

$$\begin{aligned} \int_0^T \theta^i(t)^\top \beta(t) \beta(t)^\top \theta^i(t) dt &= \int_0^T h^i(t)^\top \hat{\beta}^{-1}(t) \hat{\beta}(t) \hat{\beta}(t)^\top \hat{\beta}^{-1}(t)^\top h^i(t) dt \\ &= \int_0^T h^i(t)^\top h^i(t) dt = \int_0^T |h^i(t)|^2 dt \end{aligned}$$

by construction. Because $h \in H(Q)$ we have

$$E^* \left(\int_0^T \theta^i(t)^\top \beta(t) \beta(t)^\top \theta^i(t) dt \right) < \infty. \quad (3.3.5)$$

Thus, (3.3.4) follows since ξ is bounded away from zero. Next, we want to show that the stochastic integral $\int_0^T \theta^i(t)^\top dS(t)$ is well-defined under probability measure P . Once we show that the above stochastic integral is well-defined under probability measure Q , we are done, since it follows from Memin [31] that the definition of a stochastic integral is invariant under an equivalent change of measure. Under probability measure Q ,

$$\begin{aligned} \int_0^T \theta^i(t)^\top dS(t) &= \int_0^T h^i(t)^\top \hat{\beta}^{-1}(t) \hat{\beta}(t) dW^*(t) \\ &= \int_0^T h^i(t)^\top dW^*(t), \end{aligned} \quad (3.3.6)$$

which is well-defined, since $h \in H(Q)$. Therefore $\theta^i \in \Theta[S]$ since by construction it is also self-financing.

Finally, by construction and relation (3.3.6),

$$\begin{aligned} &\theta^i(0)^\top S(0) + \int_0^T \theta^i(t)^\top dS(t) \\ &= \theta_{N+1}^i(0) + \int_0^T h^i(t)^\top dW^*(t) \\ &= E^*(e_i) + \int_0^T h^i(t)^\top dW^*(t) = e_i \text{ a.s.} \end{aligned}$$

Thus, for an initial investment of $E^*(e_i)/a$ units of consumption good, the admissible trading strategy θ^i attains e_i units of consumption good at time T almost surely. Let θ^i thus chosen be agent i 's trading strategy for $i = 1, 2, \dots, I-1$. For agent I , set

$$\theta^I = - \sum_{i=1}^{I-1} \theta^i.$$

Since $\Theta[S]$ is a linear space, θ^I is admissible. Markets for long-lived securities then clear by construction. Furthermore,

$$\int_0^T \theta^I(t)^\top dS(t) + \theta_{N+1}^I(0) = - \sum_{i=1}^{I-1} e_i = e_I \quad a.s.$$

by the fact that, in an Arrow-Debreu equilibrium, $\sum_{i=1}^I e_i = 0$, *a.s.*

If all agents follow these assigned trading strategies, each agent i at time zero consumes

$$\begin{aligned} \hat{r}_i - E^*(e_i)/a &= \hat{r}_i + (r_i^* - \hat{r}_i) \\ &= r_i^*, \end{aligned}$$

where the first line follows from (3.3.3). At time T agent i consumes

$$\begin{aligned} \hat{x}_i + e_i &= \hat{x}_i + (x_i^* - \hat{x}_i) \\ &= x_i^* \quad a.s. \end{aligned}$$

Thus every agent gets his or her Arrow-Debreu allocation.

Step 4. Now we want to show that there is no incentive for agents to diverge from their assigned trading strategies. First, let us note that for any $h \in H(Q)$,

$$\int_0^t h(s)^\top dW^*(s) \quad t \in [0, T]$$

is a square-integrable martingale under Q (cf. the second remark after the proof of Proposition 3.2.1). It then follows from (3.3.5) that, for any $\theta \in \Theta[S]$,

$$\int_0^t \theta(s)^\top dS(s) = \int_0^t \theta(s)^\top \beta(s) dW^*(s)$$

is also a square-integrable martingale under Q . Next suppose there is one agent, say agent i , and $(r, x) \in V_+$, with $\theta \in \Theta[S]$ such that $U_i(r, x) > U_i(r_i^*, x_i^*)$ and

$$\begin{aligned} x - x_i^* &= \theta_{N+1}(0) + \int_0^T \theta(t)^\top dS(t) \\ &= a(r_i^* - r) + \int_0^T \theta(t)^\top dS(t). \end{aligned}$$

Taking expectations with respect to Q ,

$$\begin{aligned} E^*(x - x_i^*) &= a(r_i^* - r) + E^*\left(\int_0^T \theta(t)^\top dS(t)\right) \\ &= a(r_i^* - r), \end{aligned}$$

the second line follows from the fact that $\int_0^t \theta(s)^\top dS(s)$ is a square-integrable martingale under Q . But, in an Arrow-Debreu equilibrium, $U_i(r, x) > U_i(r_i^*, x_i^*)$ implies

$$a(r_i^* - r) < E^*(x - x_i^*),$$

a contradiction. Therefore there is no incentive for agents to diverge from assigned trading strategies.

The equilibrium allocation is Pareto efficient since agents get their Arrow-Debreu equilibrium allocations. ■

Given an Arrow-Debreu equilibrium, we can find $N + 1$ long-lived securities which “support” it in a continuous trading economy. It should be clear from the above proof that in selecting securities, we can restrict our attention to selecting the nonanticipative matrix process β . As long as $\beta(\omega, t)$ is nonsingular for all $t \in [0, T]$ and for almost every $\omega \in \Omega$, and satisfies the given regularity conditions, a continuous trading equilibrium exists with a Pareto efficient allocation.

Corollary 3.3.1: Under the conditions of Theorem 3.3.1, the given $N + 1$ long-lived securities complete markets. That is, any contingent claim not traded can be replicated by continuously trading on these securities. Let $x \in L^2(P)$. Under the chosen price system, the price of a claim to x at any time t is $E^*(x \mid \mathcal{F}_t)$, which can be represented as an Ito integral.

Proof: See Step 3 in the proof of the Theorem. ■

4. Properties of equilibrium price system when agents have von Neumann-Morgenstern utility functions

In the previous section we demonstrated an equilibrium in our continuous-trading economy and showed that the equilibrium allocation is Pareto efficient. The equilibrium price system in equation (3.3.1) is an Ito process whose coefficients are nonanticipative processes. That is, at any time $t \in [0, T]$, the values of η and $\hat{\beta}$ generally depend upon the past history of the economy. The Intertemporal Capital Asset Pricing Model of Merton [34], extended by Breeden [4], assumes, however, that the equilibrium price system analogous to (3.3.1) is not only an Ito process but is of the form:

$$S(t) = S(0) + \int_0^t \beta(\hat{Z}(s), s) dW(s) + \int_0^t \zeta(\hat{Z}(s), s) ds,$$

where \hat{Z} is some vector of unspecified diffusion processes. It is further assumed that (\hat{Z}, S) forms a vector diffusion process, or that (\hat{Z}, S) has the strong Markov property. We now will show conditions guaranteeing that \hat{Z} can be taken to be the “state variable process” Z , implying that the values of nonanticipative processes β and ζ at any time $t \in [0, T]$ can be written as a (Borel measurable) functions of $Z(t)$ and t . Furthermore, (Z, S) is a vector diffusion process. More generally, we will exhibit conditions under which the price process for any contingent claim having a certain payoff structure, adjoined to the state variable process, forms a vector diffusion process.

From Essay I of Huang [24] the fact that equilibrium prices are Ito integrals is an inherent property of diffusion/Brownian motion information structures. Conditions under which (Z, S) is a vector diffusion process are not as naturally posed.

This section flows as follows. In a market equilibrium with a Pareto efficient allocation in which agents have von Neumann-Morgenstern time-additive utility functions, we can construct a representative agent with a von Neumann-Morgenstern time-additive utility function who “supports” the equilibrium by consuming aggregate endowments. In that case, the properties of the equilibrium price for a particular contingent claim will be determined entirely by its payoff structure, the representative agent’s preferences, and aggregate endowments. When these “aggregates” have simple structure, so will equilibrium

prices.

4.1. Construction of a representative agent

In a market equilibrium of either the Arrow-Debreu or Radner [37] type, one can always design a representative agent who supports the equilibrium price system at the aggregate endowment point (Kreps [26, 27]). This is almost vacuous, since the equilibrium prices themselves give a “representative agent”. The statement that there is a representative agent takes on content (and has value) if one can show that this agent has nicely structured preferences. When individual agents have preferences given by von Neumann-Morgenstern time-additive utility functions and share given subjective probability assessments over states of the world, and when the equilibrium allocation is Pareto efficient, the representative agent can be chosen to be an expected utility maximizer having a time-additive utility function. Prescott and Mehra [36] assert that this is so. In a model with a finite dimensional commodity space, Constantinides [8] provides a detailed construction. The extension of Constantinides’ construction to economies with infinite dimensional commodity spaces (such as ours) is conceptually straightforward but needs to be formalized.

We assume that u_i is separable. That is, there exist functions $f_i : R_+ \rightarrow R$ and $g_i : R_+ \rightarrow R$ such that $u_i(r, y) = f_i(r) + g_i(y)$. Then U_i can be represented as

$$U_i(r, x) = f_i(r) + \int_{\Omega} g_i(x(\omega))P(d\omega).$$

Assumption (2) in Section 2.3 implies that for every $i = 1, 2, \dots, I$,

- (a) f_i and g_i are strictly increasing, concave, continuous, and with finite hand derivatives everywhere.

We also assume that for every $i = 1, 2, \dots, I$,

- (b) f_i is strictly concave and differentiable and g_i is strictly concave, three times continuously differentiable with bounded derivatives, and the second derivative of g_i is bounded away from zero.¹⁰

¹⁰The assumption of bounded derivatives can be relaxed by assuming Lipschitz-like conditions on the derivatives. See Theorem 3 on page 293 of Gihman and Skorohod [16].

For every $i = 1, 2, \dots, I$, we know (r_i^*, x_i^*) solves the following concave program:

$$\begin{aligned} & \max_{(r, x) \in V_+} U_i(r, x) \\ & \text{s.t. } a(r - \hat{r}_i) + \phi(x - \hat{x}_i) \leq 0. \end{aligned} \quad (4.1.1)$$

By (4) in Section 2.3, $a\hat{r}_i + \phi(\hat{x}_i) > 0$. That is, the Slater condition holds. By the saddle-point theorem there exists a nonnegative real number λ_i such that, for every $(r, x) \in V_+$ and every $b \in R_+$,

$$\begin{aligned} & f_i(r_i^*) + \int_{\Omega} g_i(x_i^*(\omega))P(d\omega) + b\left(a(\hat{r}_i - r_i^*) + \int_{\Omega} \xi(\omega)(\hat{x}_i(\omega) - x_i^*(\omega))P(d\omega)\right) \\ & \geq f_i(r_i^*) + \int_{\Omega} g_i(x_i^*(\omega))P(d\omega) + \lambda_i\left(a(\hat{r}_i - r_i^*) + \int_{\Omega} \xi(\omega)(\hat{x}_i(\omega) - x_i^*(\omega))P(d\omega)\right) \\ & \geq f_i(r) + \int_{\Omega} g_i(x(\omega))P(d\omega) + \lambda_i\left(a(\hat{r}_i - r) + \int_{\Omega} \xi(\omega)(\hat{x}_i(\omega) - x(\omega))P(d\omega)\right). \end{aligned} \quad (4.1.2)$$

Using the arguments following (3.1.1) in the proof for Proposition 3.1.2, we know $\lambda_i > 0$. Then (4.1.2) implies that

$$a(\hat{r}_i - r_i^*) + \int_{\Omega} \xi(\omega)(\hat{x}_i(\omega) - x_i^*(\omega))P(d\omega) = 0. \quad (4.1.3)$$

From the arguments used to derive (3.1.4), we have

$$g'_i(x_i^*(\omega)) \leq \lambda_i \xi(\omega) \text{ a.s.}, \quad (4.1.4)$$

where g'_i denotes the derivative of g_i . Let B_n be the set $\{\omega \in \Omega : x_i^*(\omega) \geq \frac{1}{n}\}$, let $A \in \mathcal{F}$, and finally let $(r, x) = (r_i^*, x_i^* - k1_{A \cap B_n}) \in V_+$ for some strictly positive scalar k , with $k < 1/n$. From the argument used to obtain (3.1.5) we get

$$\int_{A \cap B_n} g'_i(x_i^*(\omega))P(d\omega) \geq \lambda_i \int_{A \cap B_n} \xi(\omega)P(d\omega), \quad n = 1, 2, \dots$$

Since $B \equiv \{\omega \in \Omega : x_i^*(\omega) > 0\} = \bigcup_{n=1}^{\infty} B_n$, we have

$$\int_{A \cap B} g'_i(x_i^*(\omega))P(d\omega) \geq \lambda_i \int_{A \cap B} \xi(\omega)P(d\omega) \quad \forall A \in \mathcal{F}. \quad (4.1.5)$$

Let $B^+ = B \cap \{\omega \in \Omega : g'_i(x_i^*(\omega)) > \lambda_i \xi(\omega)\}$ and $B^- = B \cap \{\omega \in \Omega : g'_i(x_i^*(\omega)) < \lambda_i \xi(\omega)\}$. Applying the argument used to obtain (4.1.4) to the sets B^+ and B^- we get $P\{B^+ \cup B^-\} = 0$. That is, for those $\omega \in \Omega$ such that $x_i^*(\omega) > 0$,

$$g'_i(x_i^*(\omega)) = \lambda_i \xi(\omega) \text{ a.s.} \quad (4.1.6)$$

It is clear from the calculus that

$$\begin{aligned} f'_i(r_i^*) &= \lambda_i a \quad \text{if } r_i^* > 0, \text{ and} \\ f'_i(r_i^*) &\leq \lambda_i a \quad \text{if } r_i^* = 0. \end{aligned} \quad (4.1.7)$$

Therefore (4.1.3), (4.1.4), (4.1.6), and (4.1.7) are necessary conditions for $(r_i^*, x_i^*) \in V_+$ to be a solution to (4.1.1).

Now define $F : R_+ \rightarrow R$ and $G : R_+ \rightarrow R$ by

$$\begin{aligned} F(r) &= \max_{(y_i)_{i=1}^I \in R_+^I} \sum_{i=1}^I \frac{1}{\lambda_i} f_i(y_i) \\ \text{s.t. } &\sum_{i=1}^I y_i \leq r, \end{aligned} \quad (4.1.8)$$

and

$$\begin{aligned} G(r) &= \max_{(y_i)_{i=1}^I \in R_+^I} \sum_{i=1}^I \frac{1}{\lambda_i} g_i(y_i) \\ \text{s.t. } &\sum_{i=1}^I y_i \leq r. \end{aligned} \quad (4.1.9)$$

One can easily verify that $F(\cdot)$ and $G(\cdot)$ are strictly increasing and strictly concave. Furthermore, the following lemma shows that $F(\cdot)$ and $G(\cdot)$ are differentiable.

Lemma 4.1.1: $F(\cdot)$ and $G(\cdot)$ are differentiable. In addition,

$$\begin{aligned} F'(\hat{R}) &= a \\ G'(\hat{X}(\omega)) &= \xi(\omega) \text{ a.s.,} \end{aligned} \quad (4.1.10)$$

where $\hat{R} \equiv \sum_{i=1}^I \hat{r}_i$ and where $\hat{X} \equiv \sum_{i=1}^I \hat{x}_i$.

Proof: Consider $F(\cdot)$ first. Let r of (4.1.8) be any strictly positive real number and let $(y_i)_{i=1}^I$ be the solution of the concave program of (4.1.8). Then there must exist $j \in \{1, 2, \dots, I\}$ such that $y_j > 0$. Let k be any positive real number. Then

$$F(r+k) - F(r) \geq \frac{1}{\lambda_j}(f_j(y_j+k) - f_j(y_j)),$$

since giving the “extra amount” k to agent j is certainly feasible. Dividing both sides of the above expression by k and letting k approach zero, we get

$$D^+F(r) \geq \frac{1}{\lambda_j}f'_j(y_j),$$

where $D^+F(r)$ denotes the right hand derivative of F at r , which exists since F is concave and $r > 0$. When k lies in $(0, y_j)$, we have

$$F(r-k) - F(r) \geq \frac{1}{\lambda_j}(f_j(y_j-k) - f_j(y_j)).$$

Dividing both sides of the above expression by $-k$ and letting k approach zero, we get

$$D^-F(r) \leq \frac{1}{\lambda_j}f'_j(y_j),$$

where $D^-F(r)$ denotes the left hand derivative of F at r . Thus,

$$\frac{1}{\lambda_j}f'_j(y_j) \geq D^-F(r) \geq D^+F(r) \geq \frac{1}{\lambda_j}f'_j(y_j).$$

Therefore $D^+F(r) = D^-F(r) = \frac{1}{\lambda_j}f'_j(y_j)$, and so $F'(r)$ exists. Thus, $F(\cdot)$ is differentiable on $(0, \infty)$. Suppose the right hand derivative of $F(\cdot)$ at zero does not exist. Then, by strict concavity of $F(\cdot)$, there exists a sequence of strictly positive real numbers $\{k_n\}$, with $k_n \downarrow 0$, such that

$$\frac{F(k_n) - F(0)}{k_n} > n, \quad \forall n = 1, 2, \dots$$

Now it follows from monotonicity and concavity of f_i that

$$n < \frac{F(k_n) - F(0)}{k_n} < \sum_{i=1}^I \frac{1}{\lambda_i}f'_i(0), \quad \forall n = 1, 2, \dots,$$

which contradicts the fact that $f'_i(0)$ is finite for all i . Therefore $F(\cdot)$ is differentiable at zero. Identical arguments show that $G(\cdot)$ is also differentiable.

Next we show that $F'(\hat{R}) = a$ and that $G'(\hat{X}(\omega)) = \xi(\omega)$ a.s. By strict concavity of f_i , we know $(r_i^*)_{i=1}^I$ is the unique set of positive real numbers such that

$$F(\hat{R}) = \sum_{i=1}^I \frac{1}{\lambda_i} f_i(r_i^*).$$

Since $\hat{R} > 0$,

$$F'(\hat{R}) = \frac{1}{\lambda_j} f'_j(r_j^*)$$

for some $j \in \{1, 2, \dots, I\}$ with $r_j^* > 0$. Substituting from (4.1.7) gives

$$F'(\hat{R}) = a.$$

Similar arguments show $G'(\hat{X}(\omega)) = \xi(\omega)$ a.s. ■

The concavity and differentiability of $F(\cdot)$ and $G(\cdot)$ also imply that $F'(\cdot)$ and $G'(\cdot)$ are bounded. Now we are ready to prove the main results of this subsection:

Proposition 4.1.1: (\hat{R}, \hat{X}) is the unique solution of the following program:

$$\begin{aligned} \max_{(r,x) \in \mathbb{R}_+ \times L_+^2(P)} \quad & F(r) + \int_{\Omega} G(x(\omega))P(d\omega) \\ \text{s.t.} \quad & ar + \phi(x) \leq a\hat{R} + \phi(\hat{X}). \end{aligned}$$

Proof: This assertion follows directly from (4.1.10). ■

Proposition 4.1.1 implies that the equilibrium established in the previous section can be supported by a representative agent whose preferences are represented by the functional $U(r, x) = F(r) + \int_{\Omega} G(x(\omega))P(d\omega)$, and whose consumption is constrained by aggregate endowments.

If each agent's time T equilibrium consumption is in the quasi-interior¹¹ of $L_+^2(P)$ and if time aggregate endowments satisfy a regularity condition, we can say a bit more.

¹¹ $L_+^2(P)$ has an empty interior. The quasi-interior of $L_+^2(P)$ is the set $\{x \in L_+^2(P) : x \gg 0\}$.

Proposition 4.1.2: Suppose $x_i^* \gg 0$ for every $i = 1, 2, \dots, I$. Let $\underline{c} = \text{ess inf } \hat{X}$. If $P\{\hat{X} > \underline{c}\} = 1$, then $G(\cdot)$ is three times continuously differentiable on (\underline{c}, ∞) with bounded derivatives. Furthermore, there exist C^2 functions with bounded derivatives $c_i : (\underline{c}, \infty) \rightarrow R_+$, $i = 1, 2, \dots, I$, such that $x_i^*(\omega) = c_i(\hat{X}(\omega))$ for almost every $\omega \in \Omega$.

Proof: By the assumptions $x_i^* \gg 0 \forall i \in \{1, 2, \dots, I\}$ and $P\{\hat{X} > \underline{c}\} = 1$, we know the solution of

$$\begin{aligned} \max_{(y_i)_{i=1}^I \in R_+^I} \quad & \sum_{i=1}^I \frac{1}{\lambda_i} g_i(y_i) \\ \text{s.t.} \quad & \sum_{i=1}^I y_i \leq r \end{aligned}$$

lies in the interior of R_+^I if $r > \underline{c}$, for otherwise there exists a $i' \in \{1, 2, \dots, I\}$ such that $x_{i'}^* = 0$ on a set of strictly positive probability measure. Suppose $r > \underline{c}$. Necessary and sufficient conditions for $(y_i)_{i=1}^I$ to solve the above program are:

$$\frac{1}{\lambda_i} g'_i(y_i) = \frac{1}{\lambda_I} g'_I(r - \sum_{i=1}^{I-1} y_i)$$

for $i = 1, 2, \dots, I-1$. Now let

$$y_i(y_1, \dots, y_{I-1}, r) \equiv \frac{1}{\lambda_i} g'_i(y_i) - \frac{1}{\lambda_I} g'_I(r - \sum_{i=1}^{I-1} y_i)$$

for $i = 1, 2, \dots, I-1$. Since each g_i is three times continuously differentiable on R_+ , each y_i is twice continuously differentiable on R_+^{I-1} . Let J be the Jacobian of $(y_i)_{i=1}^{I-1}$ with respect to the first $I-1$ arguments. It is easy but tedious to check that the determinant of J is non-trivial from the fact that each g_i is strictly concave. It then follows from the implicit function theorem (see, for example, Hestenes [22], p.172) that there exist $I-1$ twice continuously differentiable functions $c_i : (\underline{c}, \infty) \rightarrow R_+$, $i = 1, 2, \dots, I-1$ such that $y_i = c_i(r)$. That is,

$$\frac{1}{\lambda_i} g'_i(c_i(r)) = \frac{1}{\lambda_I} g'_I(r - \sum_{i=1}^{I-1} c_i(r)). \quad (4.1.11)$$

Define $c_I : (\underline{c}, \infty) \rightarrow R_+$ by $c_I(r) = r - \sum_{i=1}^{I-1} c_i(r)$, clearly twice continuously differentiable.

For $r \in (\underline{c}, \infty)$, we have

$$G(r) = \sum_{i=1}^I \frac{1}{\lambda_i} g_i(c_i(r)).$$

We want to show that G is three times continuously differentiable with bounded derivatives on (\underline{c}, ∞) . From Proposition 4.1.1, $G(\cdot)$ is differentiable on (\underline{c}, ∞) and

$$\begin{aligned} G'(r) &= \sum_{i=1}^I \frac{1}{\lambda_i} g'_i(c_i(r)) c'_i(r) \\ &= \frac{1}{\lambda_i} g'_i(c_i(r)) \sum_{i=1}^I c'_i(r) \\ &= \frac{1}{\lambda_i} g'_i(c_i(r)). \end{aligned} \tag{4.1.12}$$

The second and the third lines of the above expression follow from (4.1.6) and the fact that $\sum_{i=1}^I c'_i(r) = 1$, respectively. Equation (4.1.12) reconfirms (4.1.10) and is the so called “envelope condition”. Now the thrice continuous differentiability of G on (\underline{c}, ∞) follows from the thrice continuous differentiability of g_i on R_+ and twice continuous differentiability of c_i on (\underline{c}, ∞) .

Next, we want to show that G has bounded derivatives on (\underline{c}, ∞) . Given Proposition 4.1.1, we only have to show the boundedness for G'' and G''' . If we can show that each c_i has bounded first and second derivatives, then the boundedness of G'' and G''' on (\underline{c}, ∞) follows from the chain rule for differentiation. From (4.1.11),

$$c'_i(r) = \frac{g'_i(c_i(r))/g''_i(c_i(r))}{\sum_{j=1}^I g'_j(c_j(r))/g''_j(c_j(r))}. \tag{4.1.13}$$

Therefore c'_i is bounded since it is clear from (4.1.13) that $1 > c'_i > 0$ for all $i \in \{1, 2, \dots, I\}$. (Recall that $g''_i < 0$ for all i .) Differentiating (4.1.11) with respect to r again, the boundedness of c''_i follows from the boundedness of g'_i , g''_i , g'''_i , and the fact that g''_i is bounded above away from zero. ■

The fact that the price system (3.1.1) completes markets makes the above characterization possible. As should be clear from Section 3, every complete markets equilibrium

in our continuous trading economy corresponds to an Arrow-Debreu equilibrium. In an Arrow-Debreu economy, equilibrium prices are determined by agents' preferences and (the distribution of) endowments. Thus, equilibrium prices in the continuous trading economy should depend only upon the distribution of endowments and not on the *distribution of wealth* over time. In the construction of the representative agent, (4.1.8), the weights on agents, $\frac{1}{\lambda_i}$, $i = 1, 2, \dots, I$, are constants reflecting the distribution of endowments. If the distribution of wealth over time were important, then these weights would be random variables, and the representative agent's utility function would be state dependent.

The usefulness of a representative agent is apparent from (4.1.10). Societal shadow prices can be linked directly to aggregate endowments. In our pure exchange economy, aggregate endowments are exogenously specified. We can therefore derive useful properties of ξ by making assumptions about \hat{X} . This in turn allows us to deduce conclusions about asset price behavior and optimal portfolio choice.

4.2. Equilibrium price processes and “state variable process” form a vector diffusion process

In this subsection we show a set of conditions ensuring that the price process for a contingent claim, adjoined to the vector of state variable processes, forms a vector diffusion process.

We adopt the notation:

$$D_y^\alpha = \frac{\partial^\alpha}{\partial y^\alpha} = \frac{\partial^{\alpha_1 + \alpha_2 + \dots + \alpha_n}}{\partial y_1^{\alpha_1} \dots \partial y_n^{\alpha_n}}; \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

for positive integers $\alpha_1, \alpha_2, \dots, \alpha_n$. If $g : R^N \times [0, T] \rightarrow R$ is C^1 , the vector $(\partial g / \partial y_1, \dots, \partial g / \partial y_n)$ is denoted by $D_y g$ or by g_y . The following assumptions are made throughout this subsection.

Assumption 4.2.1: In equilibrium, $x_i^* \gg 0$ for every $i = 1, 2, \dots, I$. Furthermore, $P\{\hat{X} > \underline{c}\} = 1$, where we recall \underline{c} from (4) of Section 2.3.

Assumption 4.2.2: There exists a C^2 function $L : R^N \rightarrow R_+$ with bounded partial derivatives, such that $\hat{X} = L(Z(T))$ a.s.

Assumption 4.2.3: $D_{\mathbf{y}}^{\alpha}\mu(\mathbf{y}, t)$ and $D_{\mathbf{y}}^{\alpha}\sigma(\mathbf{y}, t)$ exist and are continuous for $|\alpha| \leq 2$, and there exist constants K_0 and K_1 such that

$$|D_{\mathbf{y}}^{\alpha}\mu(\mathbf{y}, t)| + |D_{\mathbf{y}}^{\alpha}\sigma(\mathbf{y}, t)| \leq K_0(1 + |\mathbf{y}|^{K_1}), \quad |\alpha| \leq 2.$$

Assumption 4.2.3 is a regularity condition. Assumption 4.2.2 says that the time T aggregate endowment is *path-independent*: The value of time T aggregate endowment depends only upon the value of the state variable process at that time. This latter assumption can be relaxed to some extent by enlarging the state space (cf. Section 4.3), but some sort of path-independence is needed. The first half of Assumption 4.2.1 is the least satisfactory part of this essay. In Section 3, we used the fact that each agent's consumption set is the positive orthant of V to establish the existence of an equilibrium. In the first half of Assumption 4.2.1, however, we assume that in the equilibrium, the constraint that agents' consumption lies in the positive orthant of V is *not* binding. The second half of Assumption 4.2.1 is a technical assumption.

Assumptions 4.2.1 and 4.2.2 together imply that ξ , which we interpret to be societal shadow prices at time zero for time T consumption, is path-independent and a smooth function of $Z(T)$. A more useful result is actually possible:

Proposition 4.2.1: There exists a function $\delta : R^N \times [0, T] \rightarrow R$ such that $\delta_{\mathbf{y}}$ and $D_{\mathbf{y}}^2\delta$ are continuous and bounded in \mathbf{y} and t ; δ_t exists; and $\delta(\mathbf{y}, t)$ satisfies the following partial differential equation and boundary condition:

$$\begin{aligned} \delta_{\mathbf{y}}^{\top}\mu + \delta_t + \frac{1}{2}\text{tr}(\delta_{\mathbf{y}\mathbf{y}}\sigma\sigma^{\top}) &= 0, \\ \lim_{t \uparrow T} \delta(\mathbf{y}, t) &= G^t(L(\mathbf{y})). \end{aligned} \tag{4.2.1}$$

In addition,

$$\xi(t) = \delta(\mathbf{z}(t), t)$$

(where $\{\xi(t), t \in [0, T]\}$ is the martingale defined in the proof of Proposition 3.2.1.). Finally,

$\xi(t)$ can be represented as:

$$\xi(t) = \exp \left\{ \int_0^t \eta(Z(s), s)^\top dW(s) - \frac{1}{2} \int_0^t |\eta(Z(s), s)|^2 ds \right\}, \text{ a.s. } \forall t \in [0, T],$$

where $\eta(y, t)$ is continuous in y and t , satisfying the Lipschitz and growth conditions:

$$\begin{aligned} |\eta(y, t) - \eta(\bar{y}, t)| &\leq K |y - \bar{y}|, \\ |\eta(y, t)|^2 &\leq K^2(1 + |y|^2) \end{aligned} \tag{4.2.2}$$

for some positive constant K , and

$$\eta(Z(t), t) = \frac{\delta_y(Z(t), t)^\top \sigma(Z(t), t)}{\delta(Z(t), t)}. \tag{4.2.3}$$

Proof: From (4.1.10), Assumptions 4.2.1 and 4.2.2, and Proposition 4.1.2, there exists a twice continuously differentiable function $\mathcal{G} : R^N \rightarrow R$ with bounded partial derivatives such that, redefining ξ on a null set, we have

$$\xi(\omega) = \mathcal{G}(Z(\omega, T)) \quad \forall \omega \in \Omega. \tag{4.2.4}$$

It follows from Assumption 4.2.3 and Theorem 9.4 in Chapter 2 of Part II of Gihman and Skorohod [16] that there exists a function $\delta(y, t)$ such that $\delta_y, D_y^2 \delta$ are continuous in y and t ; δ_t exists; $\delta_y, D_y^2 \delta$ are bounded; and

$$\xi(t) = \delta(Z(t), t).$$

It also follows from Theorem 8.4 in Chapter 2 of Part II of Gihman and Skorohod [16] that $\xi(t)$ can be represented as

$$\xi(t) = \delta(Z(t), t) = 1 + \int_0^t \rho(Z(s), s)^\top dW(s) \quad \text{a.s.},$$

where

$$\rho(Z(t), t)^\top = \delta_y(Z(t), t)^\top \sigma(Z(t), t), \tag{4.2.5}$$

and where $\delta(y, t)$ satisfies (4.2.1). It is clear that

$$E \int_0^T |\rho(Z(t), t)|^2 dt < \infty$$

since ξ is square-integrable. Ito's Lemma then yields

$$\xi(t) = \exp \left\{ \int_0^t \eta(Z(s), s)^\top dW(s) - \frac{1}{2} \int_0^t |\eta(Z(s), s)|^2 ds \right\} \text{ a.s. } \forall t \in [0, T],$$

where

$$\begin{aligned} \eta(Z(t), t)^\top &\equiv \rho(Z(t), t)^\top / \delta(Z(t), t) \\ &= \frac{\delta_y(Z(t), t)^\top \sigma(Z(t), t)}{\delta(Z(t), t)}. \end{aligned}$$

The fact that $\eta(y, t)$ is continuous in y and t follows from the continuity of $\delta(y, t)$, $\delta_y(y, t)$, and $\sigma(y, t)$. Finally, (4.2.2) follows from the fact that $\xi(t)$ is bounded above and below away from zero, the fact that $\sigma(y, t)$ satisfies (2.1.1a) and (2.1.1b), and the boundedness of $\delta_y(Z(t), t)$. ■

If we interpret $\xi(t)$ as the societal shadow prices at time t for time T consumption, then Proposition 4.2.1 shows that the shadow prices for time T consumption over time is path-independent. It follows that the payoff structure of a claim determines whether or not the equilibrium price process forms, with the state variable processes, a vector diffusion process.

Proposition 4.2.2: For any contingent claim $x \in L^2(P)$ satisfying

$$x = E^\bullet(x) + \int_0^T \kappa(Z(t), t)^\top dW^\bullet(t) \text{ a.s.}, \quad (4.2.6)$$

where $\kappa(y, t) : R^N \times [0, T] \rightarrow R^N$ is such that $E(\int_0^T |\kappa(Z(t), t)|^2 dt) < \infty$, the equilibrium price process for a claim to x , denoted $\{x(t)\}$, can be represented as

$$\begin{aligned} x(t) &= E^\bullet(x) + \int_0^t \kappa(Z(s), s)^\top dW^\bullet(s) \\ &= E^\bullet(x) - \int_0^t \kappa(Z(s), s)^\top \eta(Z(s), s) ds + \int_0^t \kappa(Z(s), s)^\top dW(s) \text{ a.s.} \end{aligned} \quad (4.2.7)$$

Furthermore, suppose that $\kappa(\mathbf{y}, t)$ is continuous in \mathbf{y} and t , and that both $\kappa(\mathbf{y}, t)$ and $\kappa(\mathbf{y}, t)^\top \eta(\mathbf{y}, t)$ satisfy Lipschitz and growth conditions. Then $\{(Z(t), x(t)), t \in [0, T]\}$ is a diffusion process under P .

Proof: First we note that by uniform absolute continuity of P with respect to Q , $E^*(\int_0^T |\kappa(Z(t), t)|^2 dt) < \infty$. It then follows from the corollary after Theorem 3.3.1, the second remark after the proof of Proposition 3.2.1, and Proposition 4.2.1 that the equilibrium price for x at time t is

$$\begin{aligned} E^*(x \mid \mathcal{F}_t) &= E^*(x) + E^*\left(\int_0^T \kappa(Z(s), s)^\top dW^*(s) \mid \mathcal{F}_t\right) \\ &= E^*(x) + \int_0^t \kappa(Z(s), s)^\top dW^*(s) \\ &= E^*(x) - \int_0^t \kappa(Z(s), s)^\top \eta(Z(s), s) ds + \int_0^t \kappa(Z(s), s)^\top dW(s) \quad a.s. \end{aligned}$$

Now suppose that $\kappa(\mathbf{y}, t)$ and $\kappa(\mathbf{y}, t)^\top \eta(\mathbf{y}, t)$ satisfy Lipschitz and growth conditions. Theorem 6.2.2 in Arnold [1] ensures that $\{(Z(t), x(t)), t \in [0, T]\}$ is the unique solution for the system of stochastic differential (integral) equations (2.1.2) and (4.2.7). Finally, by the continuity of $\mu(\mathbf{y}, t)$, $\sigma(\mathbf{y}, t)$, $\eta(\mathbf{y}, t)$, and $\kappa(\mathbf{y}, t)$, Theorem 9.3.1 of Arnold [1] ensures that $\{(Z(t), x(t)), t \in [0, T]\}$ is a vector diffusion process under P . ■

As an immediate consequence, we have:

Corollary 4.2.2: If the random matrix $\hat{\beta}(\omega, t)$ (cf. Theorem 3.3.1) can be written as $\hat{\beta}(Z(\omega, t), t)$, if $\hat{\beta}(\mathbf{y}, t)$ is continuous in \mathbf{y} and t , and if $\hat{\beta}(\mathbf{y}, t)$ and $\hat{\beta}(\mathbf{y}, t)^\top \eta(\mathbf{y}, t)$ satisfy Lipschitz and growth conditions, then the equilibrium price system for long-lived securities, S , can be represented as

$$S(t) = S(0) + \int_0^t \beta(Z(s), s) dW(s) - \int_0^t \beta(Z(s), s) \eta(Z(s), s) ds \quad a.s.,$$

and forms, with the state variable process, a vector diffusion process. (Recall that the $N + 1$ -th row of $\hat{\beta}(t)$ is a vector of zeros.)

Proposition 4.2.2 states that if the payoff structure of a claim is *nice*, then its equilibrium price has a *nice* representation. (Here we use *nice* to mean of the forms (4.2.6) and

(4.2.7), respectively). On the other hand, the fact that the equilibrium price process of a claim has a nice representation does not necessarily imply that adjoining to that process the state variable processes forms a vector diffusion process, unless some Lipschitz and growth conditions are satisfied. One natural question to follow is whether a claim's payoff structure is nice whenever its price process has a nice representation. The following proposition is a little stronger than the converse of Proposition 4.2.2.

Proposition 4.2.3: Suppose the equilibrium price for a claim to $x \in L^2(P)$ can be represented as

$$x(t) = x(0) + \int_0^t \zeta(s) ds + \int_0^t \kappa(Z(s), s)^T dW(s) \quad a.s., \quad (4.2.8)$$

where ζ is a nonanticipative process, and where $\kappa(y, t) : R^N \times [0, T] \rightarrow R^N$ is Borel measurable with $E(\int_0^T |\kappa(Z(t), t)|^2 dt) < \infty$. Then $\zeta(\omega, t)$ can be written as $\zeta(Z(\omega, t), t)$ (Borel measurable) for almost every $t \in [0, T]$ and almost every $\omega \in \Omega$, and

$$x = E^*(x) + \int_0^T \kappa(Z(t), t)^T dW^*(t) \quad a.s.$$

Proof: First let us rewrite (4.2.8) as

$$x(t) = x(0) + \int_0^t (\zeta(s) + \kappa(Z(s), s)^T \eta(Z(s), s)) ds + \int_0^t \kappa(Z(s), s)^T dW^*(s) \quad a.s. \quad (4.2.9)$$

Arguments in Step 2 of the proof of Theorem 3.3.1 show that the above expression is well-defined under Q . By Corollary 3.3.1 we know that, under probability measure Q , $\{x(t)\}$ is a square-integrable martingale. Next note that $E(\int_0^T |\kappa(Z(t), t)|^2 dt) < \infty$ implies that $E^*(\int_0^T |\kappa(Z(t), t)|^2 dt) < \infty$ by the fact that ξ is bounded above. It then follows from the second remark after the proof of Proposition 3.2.1 that the stochastic integral of (4.2.9) is a square-integrable martingale under Q . Thus the Lebesgue integral in (4.2.9) is a square-integrable martingale under Q as well. Furthermore, as a function of t , it is absolutely continuous. Now note the following: Any continuous martingale is either of unbounded variation or is a constant throughout (cf. Dellacherie and Meyer [12]). Therefore, we must have $\zeta(\omega, t) = -\kappa(Z(\omega, t), t)^T \eta(Z(\omega, t), t)$ for almost all $t \in [0, T]$ and almost every $\omega \in \Omega$.

Substituting into (4.2.9) we have

$$x(t) = x(0) + \int_0^t \kappa(Z(s), s)^\top dW^*(s) \quad a.s.$$

Finally note that $x(T) = x$ *a.s.*, so $x(0) = E^*(x)$. ■

Propositions 4.2.2 and 4.2.3 together imply that for the equilibrium price of a claim to have a *nice* representation it is necessary and sufficient that its payoff structure is *nice*. Note also that, in the above characterization, the equilibrium price of a claim at any time t may depend upon the past realizations of the state variable process, even if it has a *nice* representation. The following proposition formalizes the intuitive notion that if the payoff structure of a claim is path-independent, then its equilibrium price process must also be path-independent.

Proposition 4.2.4: The equilibrium price, at each time $t \in [0, T]$, of a consumption claim whose payoff is a Borel measurable function of $Z(T)$, can be written as a Borel measurable function of $Z(t)$.

Proof: The arbitrage value for a claim $x \in L^2(P)$ at time t is $E^*(x | \mathcal{F}_t)$ almost surely. By the definition of conditional expectation we have

$$E^*(x | \mathcal{F}_t) = E(x\xi | \mathcal{F}_t)/\xi(t) \quad a.s.$$

(For the above fact see, for example, Harrison [19].) If x depends only on $Z(T)$, it then follows from the definition of the Markov property (see Chung [7], p.2-3.) and a monotone class argument (see Chapter 2 of Chung [6] or see Williams [38], p.122) that

$$E(x\xi | \mathcal{F}_t) = E(x\xi | Z(t)) \quad a.s.,$$

since by Assumptions 4.2.1 and 4.2.2, ξ also depends only upon $Z(T)$. Finally, it follows from the Lemma on page 299 of Chung [6] that there exist two Borel measurable functions ϕ and φ such that

$$E(x\xi | Z(t)) = \phi(Z(t)) \quad a.s.$$

and

$$\xi(t) = \varphi(Z(t)) \text{ a.s.}$$

Thus the assertion follows from the fact that the ratio of two Borel measurable functions is Borel measurable, and the fact that $\xi(t)$ is strictly positive. ■

Special cases of the above proposition are (multiple contingency) options written on the final values of the state variable process. The equilibrium prices of these call options at each time t are Borel measurable functions of the value of the state variable process at that time.

Remark: If the payoff structure of a claim is not only path-independent but is C^2 with bounded partial derivatives, then this is a special case of Proposition 4.2.2.

In this subsection we have shown that under some conditions the equilibrium price system S together with the “state variable process” Z forms a vector diffusion process. In continuous trading models of financial markets it has always been assumed that the equilibrium price system together with certain unspecified processes, which may be endogenous, forms a vector diffusion process. Here, however, we have provided a set of conditions under which those unspecified processes are identified as the vector exogenously specified “state variable processes” Z .

The results of this subsection depend crucially upon Assumptions 4.2.1, 4.2.2, and 4.2.3. As mentioned earlier, Assumption 4.2.2 can be relaxed to some extent while maintaining characterizations of equilibrium prices of the same flavor. When time T aggregate endowments are not path-independent neither are societal shadow prices for time T consumption. In that case, the equilibrium price of a consumption claim with a *nice* payoff structure will *not* have a *nice* representation. Here we still use *nice* to mean of the forms stated in Propositions 4.2.2 and 4.2.3. In the next subsection, we show that if time T aggregate endowment is not path-independent but can be represented in a certain form, then, by enlarging the state space, we can make everything *nice* again.

4.3. An augmented system

Let us first suppose that the aggregate endowment \hat{X} at time T is of the form:

$$\hat{X} = \hat{X}_0 + \int_0^T \hat{\mu}(Z(t), t) dt + \int_0^T \hat{\sigma}(Z(t), t)^T dW(t) \quad a.s., \quad (4.3.1)$$

where $\hat{\mu}(y, t) : R^N \times [0, T] \rightarrow R$ and $\hat{\sigma}(y, t) : R^N \times [0, T] \rightarrow R^N$ are continuous in y and t , satisfy Lipschitz and growth conditions, $D_y^\alpha \hat{\mu}(y, t)$, $D_y^\alpha \hat{\sigma}(y, t)$ exist and are continuous if $|\alpha| \leq 2$ with

$$|D_y^\alpha \hat{\mu}(y, t)| + |D_y^\alpha \hat{\sigma}(y, t)| \leq K_0(1 + |y|^{K_1}), \quad |\alpha| \leq 2,$$

where \hat{X}_0 , K_0 , and K_1 are positive constants. Now define the process $\{\hat{X}(t), t \in [0, T]\}$ by

$$\hat{X}(t) = \hat{X}(0) + \int_0^t \hat{\mu}(Z(s), s) ds + \int_0^t \hat{\sigma}(Z(s), s) dW(s). \quad (4.3.2)$$

Thus, $\hat{X}(T) = \hat{X}$ a.s. Then $\left\{ \left(\hat{X}(t), Z(t) \right), t \in [0, T] \right\}$ is the unique solution for the system of stochastic differential (integral) equations (2.1.2) and (4.3.2), and is a vector diffusion process (Theorem 6.2.2 and 9.3.1 of Arnold [1]). We have the following result, analogous to Proposition 4.2.2.

Proposition 4.3.1: For any contingent claim $x \in L^2(P)$ of the form:

$$x = E^\bullet(x) + \int_0^T \kappa(Z(t), \hat{X}(t), t)^T dW^\bullet(t) \quad a.s.,$$

where $\kappa(y, t) : R^{N+1} \times [0, T] \rightarrow R^N$ satisfies $E(\int_0^T |\kappa(Z(t), \hat{X}(t), t)|^2 dt) < \infty$, if in equilibrium the assumptions of Section 4.2 hold, then the equilibrium price process of a claim to x , $\{x(t)\}$, can be represented as

$$x(t) = E^\bullet(x) + \int_0^t \zeta(Z(s), \hat{X}(s), s) ds + \int_0^t \kappa(Z(s), \hat{X}(s), s)^T dW(s) \quad a.s.,$$

where $\zeta(y, t) : R^{N+1} \times [0, T] \rightarrow R$ is Borel measurable. If, in addition, $\zeta(y, t)$ and $\kappa(y, t)$ satisfy Lipschitz and growth conditions, then $\{Z(t), \hat{X}(t), x(t)\}$ is a vector diffusion process.

Proof: The proof is similar to those for Propositions 4.2.1 and 4.2.2, so we omit it. \blacksquare

The above proposition once again signifies that, with complete markets (or when the equilibrium allocation is Pareto efficient), it is the nature of aggregate endowments that determines the properties of societal shadow prices, which in turn, together with the payoff structure of consumption claims, determine the properties of equilibrium prices.

5. The characterization of optimal portfolio rules

In intertemporal portfolio theory, where price processes are taken as primitives, agents' optimal trading strategies are typically computed using stochastic dynamic programming (cf. Merton [32,33]). Merton [34] and Breeden [4,5] characterized agents' optimal portfolio behavior in an equilibrium context by summing up agents' first order conditions from their dynamic programs. Cox [9] recently proposed an alternative using a martingale representation argument. (This method is vaguely foreshadowed in the earlier literature. See, for example, Harrison and Kreps [20, Section 3] and Kreps [26].) We will illustrate, in our equilibrium setting, a technique similar to that proposed by Cox. Properties of agents' optimal trading strategies which are difficult to come by using dynamic programming will be established. We contend, and hope herein to demonstrate, that the technique using martingale representation is not just an alternative to the dynamic programming; it is a better technology.

In establishing the existence of an equilibrium in our continuous-trading economy, we picked $N + 1$ long-lived securities, N of which could be characterized by an $N \times N$ nonsingular matrix process, $\hat{\beta}$, satisfying certain regularity conditions (Section 3.3). We mentioned that any such process would do the job. For this section, to simplify things, we analyze a particular version of the equilibrium by choosing $\hat{\beta}$ to be the identity matrix. (The $(N + 1)$ -th long-lived security is still a claim to 1_{Ω} .) It then follows from Theorem 3.3.1 that the equilibrium price system is

$$S_j(t) = W_j(t) - \int_0^t \eta_j(s) ds \quad a.s. \quad j = 1, 2, \dots, N,$$

$$S_{N+1}(t) = 1$$

for all $t \in [0, T]$. Once again the above set of long-lived securities completes markets, and a claim to $x \in L^2(P)$ has a value at time t of $E^*(x | \mathcal{F}_t)$ almost surely.

Some preliminary definitions and remarks are in order. Let m and \hat{m} be two square-integrable \mathbf{F} -adapted martingales under Q . They are square-integrable semimartingales under P by the uniform absolute continuity of P with respect to Q . (See Meyer [35] and Duffie and Huang [13].) Let $\langle m, \hat{m} \rangle$ denote the unique \mathbf{F} -adapted process of bounded variation vanishing at time zero which satisfies

$$m(t)\hat{m}(t) - \langle m, \hat{m} \rangle_t \quad t \in [0, T],$$

is an \mathbf{F} -adapted square-integrable martingale under Q . (See Dellacherie and Meyer [12].) The joint variation process $[m, \hat{m}]$ is equivalent to $\langle m, \hat{m} \rangle$ when m and \hat{m} are continuous (Meyer [35]). Since \mathbf{F} is a continuous information structure, any \mathbf{F} -adapted martingale is continuous (cf. the previous essay). We will therefore always use $[m, \hat{m}]$ to denote $\langle m, \hat{m} \rangle$. It is known that the joint variation process is invariant under a substitution of an equivalent probability measure (Memin [31]). Therefore, under P , $[m, \hat{m}]$ is the joint variation process for the two square-integrable semimartingales m and \hat{m} .

Recall the notation from Theorem 3.3.1 $e_i \equiv x_i^* - \hat{x}_i$. That is, e_i is the difference between agent i 's time T equilibrium allocation and endowment. We know from the same proof that, given the price system S , agent i pays $E^*(e_i)/a$ units of consumption at time zero in exchange for the portfolio $\theta^i(0)$, where a is the unit price of time zero consumption. zero Arrow-Debreu allocation at that time. By following the trading strategy $\{\theta^i(t), t \in (0, T)\}$, which is budget feasible throughout, he gets $e_i(\omega)$ units of consumption at time T in state ω in exchange for the portfolio he is holding at that time and consume $x_i^*(\omega)$.

In the proof of Theorem 3.3.1, we demonstrated I optimal trading strategies, one for each agent. They were chosen from a set of "admissible trading strategies" which forms a linear space. The primary tool used is *martingale representation* (Proposition 3.2.1). Using the theory of optimal control, however, sufficient conditions for the existence of optimal trading strategies usually involve compactness of the admissible set of controls (cf. Bismut [3], Chapter IV).

Now define a continuous process $\{e_i^\bullet(t), \mathcal{F}_t, t \in [0, T]\}$ by

$$e_i^\bullet(t) = E^\bullet(e_i | \mathcal{F}_t) \text{ a.s.}$$

This is the value process for agent i 's time T optimal *net trade*. It is clear that the above defined \mathbf{F} -adapted process is a square-integrable martingale under Q and that $e_i^\bullet(T) = e_i$ almost surely. The following proposition shows that the optimal trading strategies have *hedging* properties.

Proposition 5.1: For any agent in the economy, say agent i , we have for all $t \in [0, T]$,

$$\begin{aligned} \theta_j^i(t) &= d[e_i^\bullet, W_j^\bullet]_t/dt \\ &= d[e_i^\bullet, W_j]_t/dt \quad j = 1, 2, \dots, N, \end{aligned} \tag{5.1}$$

and

$$\theta_{N+1}^i(t) = e_i^\bullet(t) - \sum_{j=1}^N \theta_j^i(t) S_j(t). \tag{5.2}$$

Proof: The first line of (5.1) follows from Theorems 5.4 and 5.5 and the note after them in Liptser and Shirayev [29]. In fact, it defines θ_j . The second line in (5.1) follows from the fact that the joint variation process of a continuous process and a continuous bounded variation process is zero throughout, the fact that the joint variation process is invariant under a substitution of an equivalent probability measure (Memin [31]), and the definition of W^\bullet .

Equation (5.2) follows from the construction in Section 3.3. ■

Remark: Since we have taken $\hat{\beta}$ to be the identity matrix, $(\theta_j^i)_{j=1}^N = h^i$ for all i in the proof of Theorem 3.3.1.

Roughly speaking, we could say that the price process for the j -th ($j < N$) long-lived security is *locally* perfectly corrected with the j -th underlying source of uncertainty, W_j , since $d[W_j^\bullet, W_j]_t/dt = 1$ for all $t \in [0, T]$. The number of shares of the j -th ($j < N$) security that agent i optimally chooses to hold at time t is equal to, again roughly speaking, the *local joint variability* of e_i^\bullet and W_j at that time. If at time t , e_i^\bullet is negatively correlated with W_j (that is, $d[e_i^\bullet, W_j]_t/dt < 0$, which means, roughly, that if W_j goes up in the next

instant, *ceteris paribus* agent i would like the value of his optimal net trade to go down) agent i holds a negative amount of security j , and *vice versa*.

Since the first N long-lived securities can hedge against the underlying uncertainty perfectly, we will henceforth call them *hedging securities*. The $(N+1)$ -th long-lived security will be termed the *numeraire security*. The above analysis then implies that agents hold hedging securities purely for hedging motives and use the numeraire security to transfer wealth across time. If we define the *market portfolio* to be the aggregate value of the long-lived securities, then the value of the market portfolio is zero throughout by the fact that long-lived securities are in zero net supply. Our results are thus largely consistent with previous characterizations of agents' optimal trading strategies made with Markovian stochastic dynamic programming: Agents, in equilibrium, hold the market portfolio, the riskless asset, and portfolios most highly correlated with the variables which together with agents' wealth, are sufficient statistics for agents' Markov dynamic program (cf. Merton [34], Breeden [4, 5]). In our present setting, Markovian stochastic dynamic programming may not be applicable. Nevertheless, Proposition 5.1 illustrates the hedging properties of agents' optimal trading strategies. Even if Markovian stochastic dynamic programming can be applied in our present setting, the number of hedging portfolios (securities) held by agents as characterized by dynamic programming may be substantially larger than N . This last point will soon be clarified.

In the above analysis we have characterized hedging properties of agents' optimal trading strategies *without* the assumptions made in Section 4.2. We will now re-adopt Assumptions 4.2.1, 4.2.2, and 4.2.3 and ask the following question: Can $\theta^i(t)$ be written as a function of $Z(t)$ and $S(t)$? (Equivalently, is $(Z(t), S(t))$ a sufficient statistic at time t for agents' dynamic choice problems?) If this is true, what is the functional relation? If not, what additional assumptions are required to develop such a relationship? Before proceeding, we take note of the following fact.

Proposition 5.2: Given Assumptions 4.2.1, 4.2.2, and 4.2.3, and $\hat{\beta}$ equals the identity matrix, (Z, S) is a vector diffusion process.

Proof: Since the identity matrix is certainly continuous and satisfies Lipschitz and

growth conditions, the assertion follows from the fact that $\eta(\mathbf{y}, t)$ is continuous and satisfies Lipschitz and growth conditions, and from Proposition 4.2.2. ■

Note that agent i 's optimal net trade at time T is itself a contingent claim, with a value at time t of $e_i^*(t)$. If e_i^* meets the conditions of Proposition 4.2.4, then (Z, e_i^*) is a vector diffusion process. It may then follow that $\theta^i(t)$ is a function of $Z(t)$ and $S(t)$ alone. The following proposition formalizes this.

Proposition 5.3: Suppose there exists a C^2 function $\pi_i : R^N \rightarrow R$ with bounded partial derivatives such that $\hat{x}_i(\omega) = \pi_i(Z(\omega, T))$ a.s.. Then

$$\begin{aligned} \left(\theta_j^i(t)\right)_{j=1}^N &= \left(\theta_j^i(Z(t), t)\right)_{j=1}^N \\ &= \frac{\nu_{\mathbf{y}}(Z(t), t)^\top \sigma(Z(t), t)}{\delta(Z(t), t)} - \frac{\nu(Z(t), t) \delta_{\mathbf{y}}(Z(t), t)^\top \sigma(Z(t), t)}{\delta^2(Z(t), t)}, \end{aligned} \quad (5.3)$$

where $\delta(\mathbf{y}, t)$ satisfies (4.2.1); $\nu(\mathbf{y}, t) : R^N \times [0, T] \rightarrow R$ is such that $D_{\mathbf{y}}^2 \nu$ and $D_{\mathbf{y}} \nu$ exist and are continuous in \mathbf{y} and t ; ν_t exists; and $\nu(\mathbf{y}, t)$ satisfies the following partial differential equation and boundary condition:

$$\begin{aligned} \nu_{\mathbf{y}}^\top \mu + \nu_t + \frac{1}{2} \text{tr}(\nu_{\mathbf{y}\mathbf{y}} \sigma \sigma^\top) &= 0, \\ \lim_{t \uparrow T} \nu(\mathbf{y}, t) &= (c_i(L(\mathbf{y})) - \pi_i(\mathbf{y})) \mathcal{G}(\mathbf{y}), \end{aligned} \quad (5.4)$$

where c_i is as in Proposition 4.1.2, L as in Assumption 4.2.2, and \mathcal{G} as in (4.2.4). As for the numeraire security,

$$\theta_{N+1}^i(t) = \frac{\nu(Z(t), t)}{\delta(Z(t), t)} - \sum_{j=1}^N \theta_j^i(Z(t), t) S_j(t).$$

Proof: From Proposition 4.1.2 we know that for almost every $\omega \in \Omega$, $x_i^*(\omega)$ is a C^2 function of $\hat{X}(\omega)$ with bounded derivatives. Assumption 4.2.2 together with the assumption that $\hat{x}_i(\omega)$ is almost surely a C^2 function of $Z(\omega, T)$ with bounded partial derivatives implies that $e_i(\omega) \equiv x_i^*(\omega) - \hat{x}_i(\omega)$ is almost surely a C^2 function of $Z(\omega, T)$ with bounded partial

derivatives. Ignoring a null set we can write

$$e_i = \hat{\pi}_i(Z(T)),$$

where $\hat{\pi}_i : R^N \rightarrow R$ is C^2 with bounded partial derivatives. By the definition of conditional expectation and arguments similar to those in Proposition 4.2.1, we have

$$\begin{aligned} e_i^*(t) &= E^*(e_i | \mathcal{F}_t) \\ &= \frac{E(e_i \xi | \mathcal{F}_t)}{\xi(t)} \\ &= \frac{\nu(Z(t), t)}{\delta(Z(t), T)}, \\ &= \frac{E(e_i \xi) + \int_0^t \nu_y(Z(s), s)^\top \sigma(Z(s), s) dW(s)}{1 + \int_0^t \delta_y(Z(s), s)^\top \sigma(Z(s), s) dW(s)} \end{aligned} \tag{5.5}$$

where $\delta(y, t)$ satisfies (4.2.2), and $\nu(y, t)$ is such that $D_y \nu, D_y^2 \nu$ exist and are continuous in y and t , ν_t exists, and $\nu(y, t)$ satisfies (5.4). The numerator and the denominator of (5.5) are diffusion processes. By Ito's Lemma we have:

$$d\left(\frac{\nu}{\delta}\right) = \frac{\nu_y^\top \sigma}{\delta} dW - \frac{\nu \delta^\top \sigma}{\delta^2} dW + \frac{\nu \operatorname{tr}(\delta_y^\top \sigma \sigma^\top \delta_y)}{\delta^3} dt - \frac{\operatorname{tr}(\delta_y^\top \sigma \sigma^\top \nu_y)}{\delta^2} dt. \tag{5.6}$$

Substituting (4.2.5) into (5.6) yields

$$d\left(\frac{\nu}{\delta}\right) = \left(\frac{\nu_y^\top \sigma}{\delta} - \frac{\nu \delta_y^\top \sigma}{\delta^2} \right) dW^*,$$

where we recall that W^* is an \mathbf{F} -adapted N -dimensional Brownian motion under \mathcal{Q} . Equivalently,

$$e_i = E^*(e_i) + \int_0^T \left(\frac{\nu_y(Z(t), t)^\top \sigma(Z(t), t)}{\delta(Z(t), t)} - \frac{\nu(Z(t), t) \delta_y(Z(t), t)^\top \sigma(Z(t), t)}{\delta^2(Z(t), t)} \right) dW^*(t).$$

It is now obvious that the row vector $(\theta_j^i(t))_{j=1}^N$ can be chosen to be

$$\left(\frac{\nu_y(Z(t), t)^\top \sigma(Z(t), t)}{\delta(Z(t), t)} - \frac{\nu(Z(t), t) \delta_y(Z(t), t)^\top \sigma(Z(t), t)}{\delta^2(Z(t), t)} \right),$$

or (5.3).¹² The characterization of θ_{N+1}^i follows from Step 3 in the proof of Theorem 3.3.1.

Proposition 5.3 states that if agent i 's endowment is path-independent and smooth, then under the assumptions of Section 4.2, knowledge of $(Z(t), S(t))$ is sufficient for his dynamic choice problem at time t . His "indirect utility function" at t will be a function of only $Z(t)$, $S(t)$, and t , with $Z(t)$ being a sufficient statistic for the value of his net trade. (This follows since $E^*(e_i | \mathcal{F}_t)$ is a function of $Z(t)$ and t (cf. (5.5)).)

When the i -th agent's time T endowment is not path-independent, $\theta^i(t)$ will, in general, depend on the historical realizations of Z or S even when (Z, S) forms a vector diffusion process. That is, the fact that (Z, S) is a vector diffusion process may not render past history irrelevant for agent i 's portfolio decisions. Conversely, the fact that agents have path-dependent optimal dynamic decision rules may not destroy the Markovian nature of (Z, S) . A simple example illustrates how this might come about. Imagine an economy with two agents identical except that they have different time T endowments. One agent's time T endowment, \hat{x}_1 , equals the minimum of value attained by some state variable, say Z_1 , over the interval $[0, T]$. The second agent has time T endowment equal to the difference between some smooth function of $Z(T)$ and the first agent's time T endowment. (Note that implicit in the above two statements are the conditions: (i) The minimum value of Z_1 is positive and nonzerod, and (ii) time T aggregate endowments are strictly greater than the minimum of Z_1 .) Thus the aggregate endowment at time T is path-independent, implying (Z, S) forms a diffusion process (provided certain regularity conditions are satisfied). But, in equilibrium, each agent must hedge against his or her time T endowment realization which varies in ways that depend upon the entire path of Z_1 . Each consumes an amount that depends on $Z(T)$ only, but their net trades are much more complex.

In our exchange economy, aggregate endowments at time T are the essential determinants of asset price dynamics, rather than properties of agents' optimal portfolio rules. Agents in the economy may *all* have path-dependent decision rules, but so long as Assumptions

¹²The I optimal trading strategies, $(\theta^i)_{i=1}^I$, one for each agent, are unique over the measure $dt \times dP$ (Chapter 5 of Liptser and Shiriyayev [29]). But elements of a particular equivalence class over this measure may not be indistinguishable processes.

4.2.1, 4.2.2, and 4.2.3 hold, (Z, S) obeys the strong Markov property.

Given Proposition 4.3.1, the following is easy to prove.

Proposition 5.4: Suppose that agent i 's time T endowment \hat{x}_i satisfies the Ito integral:

$$\hat{x}_i = \hat{x}_i(0) + \int_0^T \hat{\mu}_i(Z(t), t) dt + \int_0^T \hat{\sigma}_i(Z(t), t)^\top dW(t), \quad a.s., \quad (5.7)$$

where $\hat{\mu}_i(\mathbf{y}, t) : R^N \times [0, T] \rightarrow R$ and $\hat{\sigma}_i(\mathbf{y}, t) : R^N \times [0, T] \rightarrow R^N$ are continuous in \mathbf{y} and t ; satisfy Lipschitz and growth conditions; and $D_{\mathbf{y}}^\alpha \hat{\mu}_i(\mathbf{y}, t)$ and $D_{\mathbf{y}}^\alpha \hat{\sigma}_i(\mathbf{y}, t)$ exist and are continuous for $|\alpha| \leq 2$, with

$$|D_{\mathbf{y}}^\alpha \hat{\mu}_i(\mathbf{y}, t)| + |D_{\mathbf{y}}^\alpha \hat{\sigma}_i(\mathbf{y}, t)| \leq K_0(1 + |\mathbf{y}|^{K_1}), \quad |\alpha| \leq 2,$$

where $\hat{x}_i(0)$, K_0 , and K_1 are positive constants. Furthermore, suppose the process $\{\hat{x}_i(t), t \in [0, T]\}$ is defined by

$$\hat{x}_i(t) = \hat{x}_i(0) + \int_0^t \hat{\mu}_i(Z(s), s) ds + \int_0^t \hat{\sigma}_i(Z(s), s)^\top dW(s) \quad a.s. \quad (5.8)$$

Then $\theta^i(t)$ can be written as a function of $Z(t)$, $S(t)$, and $\hat{x}_i(t)$, and $\{(Z(t), S(t), \hat{x}_i(t)), t \in [0, T]\}$ is a diffusion process.

Proof: The arguments are similar to those of Propositions 4.3.1 and 5.3. ■

When agent i 's time T endowment is *well-behaved*, the path-independence of his decision rule can be revived by augmenting (Z, S) with the process defined in (5.8). In that case, agent i 's indirect utility for *net trade* is a function of $Z(t)$, $S(t)$, and $\hat{x}_i(t)$, which are sufficient statistics for agent i 's dynamic choice at that time. Using the traditional characterization of equilibrium asset prices and optimal portfolio rules, we would have concluded that the $\hat{x}_i(t)$ is important in the price formation process, and that agent i faces $N + 1$ sources of uncertainty to hedge against.

In this section, we have characterized properties of agents' optimal portfolio rules in an equilibrium setting, primarily using *martingale representation* techniques. Some results have been derived in an environment where Markovian dynamic programming cannot be

applied. Even when Markovian dynamic programming is applicable, summing up agents' first order optimality conditions may not give the right answer. We contend, along with Kreps [26], that the stochastic control machinery "is not really necessary in the portfolio management problem". The martingale connection of equilibrium asset prices developed by Harrison and Kreps [20] makes available a rich theory of martingales to financial theorists. The martingale representation technique demonstrated in this section is not only an alternative to stochastic control, as suggested by Cox [9], but also a more powerful one, especially in an equilibrium setting.

6. Concluding remarks

This paper addresses three issues: First, can we prove the existence of an equilibrium in a Merton/Breeden-like continuous-trading economy? Second, under what conditions do the vector price process and the vector state variable process together form a vector diffusion process? Third, is dynamic programming an appropriate tool in characterizing agents' optimal portfolio rules in an equilibrium setting? Answers are provided for the first two questions. With respect to the third one, it is argued that martingale representation techniques might be more powerful. Dynamic programming has not yet been shown to yield a consistent characterization of heterogeneous-agent equilibrium asset prices. It seems from our work that dynamic programming is indeed an inappropriate tool for this problem.

The economy considered in this paper is a continuous time Radner economy in which agents consume only at two time points. Questions related to the Consumption Capital Asset Pricing Model, such as whether agents would optimally choose their consumption rates to be Ito integrals when information is generated by diffusion process, cannot be addressed here. When there is no intermediate consumption, the behavior of equilibrium asset prices is totally determined by the arrival of information (cf. ^{Essay I of HOANG [24].} ~~our previous essay~~). When agents are allowed to consume continuously, it seems that prices would be determined jointly by how agents choose to consume over time and by the way information is revealed. This is ongoing research of the authors.

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