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RETAIL OUTLET LOCATION:
A Model of the Distribution Network Aggregate Performance

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1. INTRODUCTION

Location of sales outlets is of major concern to such organizations as oil companies, banks, etc. The problem is often approached in two steps. The total market is divided into regions, and the first stage amounts to deciding in which regions to expand or to contract. Thus for example in period t, the company (i) will add n_i^{1t} outlets in area 1, n_i^{2t} in area 2, etc. We will refer to the first step as the aggregate location problem. The second stage consists in choosing specific sites for these new outlets. In practice, these decisions are regarded as rather independent, especially because they are made at different levels in the organization. The number of new outlets is a corporate decision, whereas specific sites are selected at the regional level, subject however to approval by the corporate headquarters. Many companies feel that, at least for the time being, hierarchical linking of the aggregate and detailed problems is neither worth the effort nor the cost.[1]

In this paper our sole concern will be with aggregate location. Our procedure will be closely related to a model developed by Hartung and Fisher [7]. However, their work lacked robustness, and suffered from a variety of deficiencies in the estimation of the model parameters. In section 2 we will review the Hartung-Fisher (hereafter H-F) model and the various weaknesses associated with it. In section 3 the estimation problems will be examined. In section 4 we will propose various changes to the model which will

make it robust, and we will use data from a major oil company in a European country to estimate the parameters and validate our approach.

2. THE HARTUNG-FISHER MODEL[2]

The market is reduced to a quasi duopoly, that is, we consider our brand, i, versus competitive brands taken together, c. Buying behavior is described as a first order Markov chain. The transition probabilities are defined as follows :

λ_i = probability that a person who *usually* buys brand i in period t-1, will *usually* buy brand i in period t.

Adding the word "usually" broadens the definition used by H-F because it allows for incidental purchases of a competitive brand. This is important for a product such as gasoline. Take a person who usually buys brand i. On a given day he is running out of gas on a thruway and tanks at the next service area. If the brand is not i, we should not conclude yet that brand switching has occurred.

The other transition probabilities are similarly defined :

σ_i = probability of switching from c to i

λ_c = probability of remaining a buyer of a competitive brand

σ_c = probability of switching from i to c.

And, $\lambda_i + \sigma_c = 1$, $\lambda_c + \sigma_i = 1$.

Market share of brand i in period t is

$$(1) \quad m_{i,t} = \lambda_i \cdot m_{i,t-1} + \sigma_i \cdot m_{c,t-1}$$

In steady state, $m_{i,t} = m_{i,t-1} = m_{i,e}$, and thus

$$(2) \quad m_{i,e} = \sigma_i / (1 - \lambda_i + \sigma_i) = \sigma_i / (\sigma_i + \sigma_c)$$

The transition probabilities will be functions of the decision variables of company i, such as advertising expenditures a_i, the number of sales outlets d_i, price p_i, and the corresponding competitive decision variables a_c, d_c, p_c. That is, for example,

$$\lambda_i = l_i \, (a_i, d_i, p_i, a_c, d_c, p_c)$$

$$\sigma_i = s_i \, (a_i, d_i, p_i, a_c, d_c, p_c) \quad .$$

H-F consider only the number of sales outlets as determinants of the transition probabilities. The following functions were postulated, k_1 and k_2 are positive constants.

$$(3) \quad \lambda_i = k_1 \cdot d_i / (d_c + d_i)$$

$$(4) \quad \sigma_i = k_2 \cdot d_i / (d_c + d_i)$$

For a given value of d_c, both λ_i and σ_i are increasing with d_i, and show decreasing returns. The limits of λ_i and of σ_i for $d_i \to \infty$ are k_1 and k_2 respectively. However, k_1 and k_2, and hence λ_i and σ_i are not restricted to be between zero and one. For example, H-F obtained empirical estimates of $k_1 = 4.44$, and of $k_2 = 0.64$. This would imply that for values of $d_i / (d_c + d_i) > 1/4.44$, λ_i becomes larger than one. Since λ_i is a probability, its dependence on d_i and d_c should be constrained in such a way that its value will lie in the [0,1] interval. The H-F model therefore lacks robustness. H-F are well aware of that. They state that,

> "Unless $k_1 = k_2 = 1.0$, λ_i and σ_i are not probabilities for all values of $d_i/(d_c + d_i)$. However, equations (3) and (4) can be assumed to represent probabilities if $d_i/(d_c + d_i)$ is restricted,
>
> $$d_i/(d_c + d_i) < \min(1.0,\ 1/k_1,\ 1/k_2)$$
>
> More general functions can be substituted for equations (3) and (4) without invalidating later results. The authors found that for their problem equations (4) were sufficiently accurate".[3]

Section 4 will be devoted to examining ways through wich the model can be made robust.

Let us now relate market share to the parameters k_1 and k_2. Substituting (3) and (4) for λ_i and σ_i in (2) gives

$$(5) \qquad m_{i,e} = \frac{k_2\, d_i}{d_c + (1 + k_2 - k_1)\, d_i}$$

Let q_i = sales of i, and Q = industry sales. Market share $m_i = q_i/$ Note that H-F replace $m_{i,e}$ in (5) by q_i/Q. We should observe that this implies the assumption that observed market share is equal to steady state market share. We will return to this issue in section .

Substituting q_i/Q for $m_{i,e}$ in (5) and rearranging terms, H-F obtain,

$$(6) \qquad q_i/d_i = k_2 \cdot Q \,/\, (d_c + \mu \cdot d_i) \;,$$

where $\mu = 1 + k_2 - k_1$, and q_i/d_i = the average sales per outlet. From (6) it follows that for a given value of d_c, and with $\mu < 0$, average sales per outlet will increase with d_i, and goes to infinity when $(d_c + \mu \cdot d_i)$ goes to zero. With $k_1 = 4.44$ and $k_2 = 0.64$, $\mu = -2.80$, and therefore, the model would predict infinite sales per outlet when $d_i/d_c = 1/2.8$. For values $d_i/d_c > 1/2.8$, the model would predict negative sales. The function pattern is depicted in Figure 1.

Insert Figure 1 about here

On the other hand if $\mu > 0$, the model average sales per outlet will decrease when the number of outlets increases. Lambin applied the H-F model on a brand of gasoline in a European

country [10, chapter 7], and found values of $k_1 = .170$, $k_2 = .127$, and $\mu = 0.957$. However, even with $\mu > 0$, one may still run into difficulties. With d_i large (for a given d_c), q_i approaches $k_2 \cdot Q/\mu$ If $k_1 > 1$, this would imply that when d_i becomes large company i's sales exceed industry sales.

The next issue deals with the estimation of the parameters k_1 and k_2. Equation (6) is nonlinear in k_1 and k_2, and could not be estimated by linear regression. H-F applied a series of transformations to (6) and ended up with the following function

$$\bar{q}_i/\bar{Q} = \alpha + \beta(q_i/Q) \tag{7}$$

where $\bar{q}_i = q_i/d_i$ = average sales per outlet for firm i

$\bar{Q} = Q/(d_c + d_i)$ = industry average sales per outlet

$\alpha = k_2$

$\beta = k_1 - k_2$

H-F use linear regression to estimate α and β. The parameters k_1 and k_2 are then uniquely determined from α and β. There are various problems associated with the estimation procedure used by H-F. These will be examined in section 3.

The estimated coefficients k_1 and k_2 are then used in an aggregate retail outlet location model in which the objectiv is to maximize discounted return of firm (i).

Let J = the number of regions $(j=1,\ldots,J)$

T = the time horizon $(t=0,\ldots,T)$

τ = the discount factor

r_i^{jt} = return per unit sold in region j, period t

d_i^{jt}, $\bar{q}_i^{jt}$, ... as d_i, $\bar{q}_i$ but superscripted for regions and time

d_i^{j0} = existing number of outlets at time 0

l_i^{jt} = upper limit on the number of new outlets in region j, and in period t

c_i^{jt} = construction cost of a new outlet in region j

b_t = budget limit for building new outlets in period t

n_i^{jt} = number of new outlets constructed in region j, period t

The return maximization model is then

$$\text{Max} \sum_j \sum_t \left(\frac{1}{1+\tau}\right)^t r_i^{jt} \; \bar{q}_i^{jt} \; d_i^{jt}$$

subject to :

$$\sum_j c_i^{jt} \, n_i^{jt} \leq b_t \qquad \forall t$$

$$\bar{q}_i^{jt} = k_2 \cdot Q^{jt} \, / \, [d_c^{jt} + \mu \cdot d_i^{jt}] \qquad \forall j,t$$

$$d_i^{jt} = d_i^{j0} + \sum_{k=1}^{t} n_i^{jk} \qquad \forall j,t$$

$$n_i^{jt} \leq l_i^{jt} \qquad \forall j,t$$

The reader will notice that with $\mu < 0$, $\bar{n}_i^{jt}$ is an increasing function[4] of d_i^{jt} and therefore unless r_i^{jt} is carefully specified as a *net* return and unless the discounted construction costs, i.e. $\sum_j \sum_t \left(\frac{1}{1 + \tau}\right)^t c_i^{jt} n_i^{jt}$, are deduced, the objective function is monotone increasing and the solution is merely given by the constraints. It is also important to note that even when the discounted construction costs are deduced, the objective function may still be monotone increasing, for $d_i^{jt} < |d_c^{jt}/\mu|$.

3. ISSUES IN ESTIMATING THE H-F MODEL

The parameters k_1 and k_2 are obtained from the estimated values of α and β in equation (7).

First we observe that market share (q_i/Q) is on the right hand side of the equation while in fact the number of retail outlets is expected to have a causal effect on market share and not vice versa. Furthermore, with $\bar{q}_i = q_i/d_i$, and $\bar{Q} = Q/(d_c + d_i)$, the left hand side of equation (7) is $\bar{q}_i/\bar{Q} = (q_i/Q)(1 + d_c/d_i)$. Or, equation (7) can be rewritten as :

$$(8) \qquad m_i(1 + d_c/d_i) = \alpha + \beta \cdot m_i \ ,$$

with m_i on both sides of the equation.

From a causality point of view we would like to have d_i/d_c on the right and m_i on the left of the equation. Some simple manipulations of equation (8) result in the following linear equation,

(9) $$1/m_i = \gamma + \delta \cdot (d_c/d_i)$$

where $\gamma = (1 - \beta)/\alpha$, and $\delta = 1/\alpha$. Or in terms of k_1 and k_2, $k_2 = 1/$ and $k_1 = (1 + \delta - \gamma)/\delta$.

Yet we would rather think in terms of d_i and d_c determining m_i rather than $1/m_i$. So rather than trying to lineariz the market share function, we can estimate it directly as obtained in equation (5) (with m_i replacing $m_{i,e}$) by nonlinear estimation methods.

Below we will compare estimation of k_1 and k_2 obtained from equations (7), (9), and (5). The estimations were performe on the TROLL system[5]. The estimation procedure in TROLL is based on Marquardt's algorithm for least-squares estimation of nonlinear parameters [12].

We estimated equations (7), (9), and (5) for a major brand of gasoline in a European country. There are 35 quarterly observations, from the first quarter of 1962 to the third quarter of 1970. The statistical results for the nonlinear model are based on its linearized form around the optimum, i.e. the minimum of the residual sum of squares[6]. Let $\theta = (\theta_1 \ldots \theta_p \ldots \theta_P)'$ = vector of parameters ; $X_t = (X_{1t} \ldots X_{pt} \ldots X_{Pt})'$ = vector of observations. The general model is written as,

$$y_t = f(X_t, \theta) + \varepsilon_t$$

where ε_t is the disturbance term.

Let $\hat{\theta}$ be the final least-squares estimate of θ. With θ close to $\hat{\theta}$, $E(y_t)$ can be approximated by a first-order Taylor expansion about $\hat{\theta}$,

$$(10) \qquad E(y_t) \cong f(X_t, \hat{\theta}) + \sum_p \left(\frac{\partial f(X_t, \theta)}{\partial \theta_p} \right)_{\theta = \hat{\theta}} (\theta_p - \hat{\theta}_p)$$

with $$\hat{f}_{pt} = \left(\frac{\partial f(X_t, \theta)}{\partial \theta_p} \right)_{\theta = \hat{\theta}} \quad \text{, and}$$

$$\hat{f}_t = f(X_t, \hat{\theta}) - \sum_p \hat{f}_{pt} \; \hat{\theta}_p \; ,$$

equation (10) can be written as a linear function of the parameters

$$(11) \qquad z_t \cong \sum_p \theta_p \cdot \hat{f}_{pt} + \varepsilon_t$$

where, $z_t \cong y_t - \hat{f}_t$. The statistics for the nonlinear model are similar to those of a linear regression. Thus, if we define the following (T x P) - matrix :

$$F = \begin{pmatrix} \hat{f}_{11} & \hat{f}_{21} & \cdots & \hat{f}_{P1} \\ \hat{f}_{12} & \hat{f}_{22} & \cdots & \hat{f}_{P2} \\ \vdots & & & \\ \hat{f}_{1T} & \hat{f}_{2T} & \cdots & \hat{f}_{PT} \end{pmatrix}$$

where T is the number of observations available, then the estimated variance-covariance matrix of $\hat{\theta}$ is $V(\hat{\theta}) = s^2 \cdot (F'F)^{-1}$, in which s^2 is the residual mean square.[7]

Note that s^2 is no longer an unbiased estimate of σ^2, the disturbance variance and that even when the error term ε is normally distributed, $\hat{\theta}$ is no longer normally distributed. As a result, the usual t-, F-, and Durbin-Watson- statistics are not valid in general. However, these statistics will be reported here ; they should therefore be regarded as mere comparison values.

The results for brand i are presented in Table I. Both $\hat{k}_1$ and $\hat{k}_2$ are highly significant. Yet they are of little use. First, the high value of $\hat{k}_1$ would restrict the meaningful outlet-share of brand i to less than 5 per cent. Yet, over the whole period of observations, its outlet share was higher than 8 per cent. Worse even, $\hat{k}_2$ is negative. The values of the Durbin-Watson statistic indicates autocorrelation of the residuals, and hence problems with the model specification. There are several possible explanations. One is that important additional explanatory variables may have been left out. A previous study of these data shows that this is not the case[8]. More likely reasons are the misspecific tion of the transition probability functions, and the assumption that observed market shares are equilibrium values.

In other cases one may have better luck. For example, we applied the H-F model on another brand (a)[9], and found results similar to those in the H-F paper as illustrated in Table II. The Durbin-Watson statistic, however, again indicates significant autocorrelation[10].

Insert Tables I and II about here

In the next section we will explore two ways in which the model can be made robust. We will apply both these model formulations to the data for brand i.

4. ALTERNATIVE FORMULATIONS

In this section we will redefine the transition probability functions. First, as exponential functions of the relative number of outlets in section 4.1., and next as logistic functions in section 4.2. How to make use of these models will be further examined in section 4.3.

4.1. Exponential model

Let $D_i = d_i/d_c$, and $D_c = d_c/d_i$.
Define λ_i as,

(12) $\lambda_i = 1 - EXP(-a_i \cdot D_i)$

where $a_i > 0$. If there are no outlets for brand i, $\lambda_i = 0$. The larger d_i/d_c, the larger λ_i is. If d_c is zero, $\lambda_i = 1$. Also when d_i/d_c approaches infinity, λ_i approaches one. Equation (12) thus relates the relative number of outlets D_i, to the transition probability λ_i, in a robust way.

Similarly, λ_c is defined as

(13) $\lambda_c = 1 - EXP(-a_c \cdot D_c)$

where $a_c > 0$. And therefore, $\sigma_c = EXP(-a_i \cdot D_i)$, $\sigma_i = EXP(-a_c \cdot D_c$

Market share at time t is related to market share at time t-1,

$$(14) \quad m_{i,t} = [1 - EXP(-a_i \cdot D_{i,t}) - EXP(-a_c \cdot D_{c,t})] \cdot m_{i,t-1} + EXP(-a_c \cdot D_{c,t})$$

Assume now for a moment, as H-F did, that observed market share values are equilibrium values, for given values of $D_{i,t}$ and $D_{c,t}$. That is,

$$(15) \quad m_{i,t,e} = EXP(-a_c \cdot D_{c,t}) / [EXP(-a_i \cdot D_{i,t}) + EXP(-a_c \cdot D_{c,t})]$$

Applying a logit transformation to (15) results in a linear model[11],

$$(16) \quad \log[m_{i,t,e} / (1 - m_{i,t,e})] = a_i \cdot D_{i,t} - a_c \cdot D_{c,t} .$$

The results of the estimation of a_i and a_c for brand i are presented in Table III. The estimated $\hat{a}_i$ is negative which would seem to indicate that the assumption on equilibrium is not at all warranted

It should be clear that many assumptions - not satisfied in this case - have to be made in order to accept the equilibrium form (16). For example, provided that

a) - the consumption patterns are adequately stable,

b) - the unit-time period is sufficiently long so that the disruptive effects of a marketing campaign - launched in period

[t-1,t] - on the steady-state market shares can resorb in a new equilibrium achieved within the same period [t-1,t],

c) - the unit-time period is short enough so that no competitive reaction can interfere with this new equilibrium,

we may retain equation (15). The restrictive and somewhat contradictory character of this non-exhaustive set of assumptions explain why we should turn to dynamic forms. This does not imply that we have to disregard the steady-state aspects when we are about to make decisions on where to add new outlets. The results in Table III merely indicate the market dynamics should be taken into account in estimating the parameters.

Insert Table III about here

Equation (14) is intrinsically nonlinear, and was estimated in two different ways. First, we used TROLL. Secondly, we applied the Sequential Unconstrained Minimization Technique (SUMT)[12]. Nonlinear programming can be used for nonlinear estimation in the following way. Let the model be,

$$y_t = f(X_t, \theta) + \varepsilon_t ,$$

where ε_t is the disturbance term. Minimizing the sum of squares is achieved by solving the nonlinear programming problem below,

$$\text{(17)} \qquad \min_{\theta} \; \Sigma_{t=1}^{t=T} \; \varepsilon_t^2(\theta)$$

$$\text{s.t.} \qquad y_t - f(X_t, \theta) - \varepsilon_t = 0 \quad , \quad t=1, \ldots, T$$

Insert Table IV and Figure 2 about here

Table IV shows the results of the estimation of equation (14) using TROLL and SUMT. Figure 2 illustrates λ_i and λ_c as functions of D_i and D_c respectively.

With the current number of outlets, D_i = , and D_c = , λ_i =

λ_c =

These values are very high, which is to be expected for a well developed market. With these values of λ_i and λ_c, predicted equilibrium market share is . A one per cent increase in $d_i/(d_i + d_c)$, and assuming d_c remains constant, equilibrium market share would increase by per cent. Whether such an increase in share would be worthwile depends on industry sales volume, unit profit, and the number of new outlets needed to increase outlet share by one percent.

More interesting of course is to look at the problem on a regional basis. For this particular product, we have information on four different regions. The company's outlet share varies from a high of about 10 per cent in one region to a low of about 5 per cent in another region. Incremental yearly regional sales per outlet added are shown in Table V for each of the four regions. Adding outlets in region contributes the highest marginal return.

It is quite possible that the response parameters differ accross regions. For example, rural areas might be distinguished from metropolitan areas. In our particular application in-

formation by region was available only on an annual basis over a period of six years. This was insufficient for the purpose of estimating response coefficients by region.

Insert Table V about here

4.2. Logistic model

Market share as a function of relative number of outlets is often thought of as having an S shaped form. Various oil companies, for example, have been able to observe such S curves in plots of sales or market share as functions of the number or the relative number of outlets[13].

The theoretical arguments in favor of such a S shaped relationship at the market share level - already introduced in the form of equation (15) - may also hold at the transition probability level. Thus, at one extreme, if the oil company has too few filling-stations consumers will notice them too infrequently and will often be obliged to tank up at other companies stations; as a consequence, their loyalty will be very low. As the number of stations increases consumers will be able to tank up at the company petrol pumps located in various geographical areas and their loyalt will be enhanced accordingly. At the other extreme, if the company continues to extend its distribution network, each new station cons tructed will have to attract consumers from the remaining hard co of competitors customers.

Estimating the parameters of equation (20) will raise one issue. If the historical data come from a stable market, i.e. our outlet shares and market shares show relatively little variability, there will be severe multicollinearity problems. As mentioned in section 3, the independent variables in the linear equation derived from a first order Taylor expansion are first order derivatives evaluated at the current solution. Let the current solution be $a_i = a_i^o$, $b_i = b_i^o$, $a_c = a_c^o$, and $b_c = b_c^o$. To illustrate the multicollinearity problem, consider the derivatives with respect to a_i and b_i, evaluated at $a_i = a_i^o$ and $b_i = b_i^o$.

$$(22) \qquad \frac{\partial m_{i,t}(a_i = a_i^o \; ; \; b_i = b_i^o)}{\partial a_i} = - m_{i,t-1} \, D_{i,t}^{b_i^o} \Big/ \left(a_i^o + D_{i,t}^{b_i^o} \right)^2$$

$$(23) \qquad \frac{\partial m_{i,t}(a_i = a_i^o \; ; \; b_i = b_i^o)}{\partial b_i} = m_{i,t-1} \cdot a_i^o \cdot D_{i,t}^{b_i^o} \cdot \ln D_{i,t} \Big/ \left(a_i^o + D_{i,t}^{b_i^o} \right.$$

With limited variation in $D_{i,t}$, there will be high correlation between $\partial m_{i,t}/\partial a_i$ and $\partial m_{i,t}/\partial b_i$.

An important issue in nonlinear estimation is finding good initial values. We used the following procedure. In stead state, the elasticity of market share with respect to the relative number of outlets is,

$$(24) \qquad \eta_{m_{i,t,e}, D_{i,t}} = b_i \cdot \lambda_i \cdot m_{c,t,e} \quad .$$

For these reasons, a S shaped curve relating transition probabilities to the relative number of retail outlets seems justified. Hence we can define λ_i as

$$(18) \qquad \lambda_i = D_i^{b_i} / (a_i + D_i^{b_i})$$

where a_i and b_i are positive. With $d_i = 0$, $\lambda_i = 0$. With $d_c = 0$, $\lambda_i = 1$. λ_i increases with D_i and as D_i gets larger and larger, λ_i approaches one, according to a S shaped pattern.

Similarly λ_c is defined as

$$(19) \qquad \lambda_c = D_c^{b_c} / (a_c + D_c^{b_c})$$

with a_c and b_c positive constants. The switching probabilities are defined as $\sigma_c = a_i/(a_i + D_i^{b_i})$, $\sigma_i = a_c/(a_c + D_c^{b_c})$.

Market share at time t is then,

$$(20) \qquad m_{i,t} = [(D_{i,t}^{b_i} \cdot D_{c,t}^{b_c} - a_i\, a_c) / (a_i + D_{i,t}^{b_i}) \cdot (a_c + D_{c,t}^{b_c})]$$

$$\cdot\, m_{i,t-1} + [a_c / (a_c + D_{c,t}^{b_c})]$$

If D_i remains equal to $D_{i,t}$, and D_c to $D_{c,t}$, steady state market share would be,

$$(21) \qquad m_{i,t,e} = a_c(a_i + D_{i,t}^{b_i}) / [a_i(a_c + D_{c,t}^{b_c}) + a_c(a_i + D_{i,t}^{b_i})]$$

To estimate initial values for the parameters, we assumed a value of 0.5 for this elasticity. For a value of λ_i = .99, and a given value for $m_{c,t,e}$, we obtained b_i = 0.56. Substituting this value into equation (18), and solving for a_i, gave a_i = .0143. Applying the same reasoning to obtain initial values for a_c and b_c resulted in a_c = 0.625 and b_c = 2.46.

With these initial values TROLL failed to find an optimal solution. Divergence occurred for these and for all other initial values which we tried. The SUMT program performed better. The main reason is probably that the method for minimizing the unconstrained penalty function is the Newton-Raphson method, a second order procedure, whereas Marquardt's estimation method only uses first order derivatives[14]. Furthermore, all our SUMT computations were done in double precision. This may be particularly important in view of the fact that there is a real multicollinearity problem. The SUMT estimates are shown in Table VI. The t statistics are very poor, as expected. Nevertheless, the coefficients all have the correct sign, and the magnitudes are reasonable. After all, the estimated values are not too different from our initial subjective estimates.

Insert Table VI about here

With the current values of D_i and D_c, λ_i = , λ_c = . Predicted steady state market share is , compared to an actual market share of . Table VII shows the incremental sales for adding a new outlet in each of the regions.

Insert Table VII about here

4.3. Model application

Tables V and VII show respectively how the exponential and logistic models can be applied to compare incremental sales from investment in outlets in various regions.

The ultimate measure of performance is the comparison of regional profits rather than sales [15]. The cost structure may differ from one region to another. For example, transportation costs, and cost of purchasing land will vary accross regions.

The optimization model (8) proposed by H-F could be adjusted for the exponential transition probability functions of section 4.1. We replace $\bar{q}_i^{jt} \cdot d_i^{jt}$ in the objective function by $Q^{jt} \cdot m_{i,t}^j$. The constraint $\bar{q}_i^{jt} = k_2 \cdot Q^{jt} / (d_c^{jt} + \mu \cdot d_i^{jt})$ is replaced by [16] :

$$m_{i,t}^j = \left(1 - e^{-a_i^j \cdot D_{i,t}^j} - e^{-a_c^j \cdot D_{c,t}^j}\right) m_{i,t-1}^j + e^{-a_c^j \cdot D_{c,t}^j}$$

or if our interest is only in steady state results without concern for the transient behavior,

$$m_{i,t}^j = e^{-a_c^j \cdot D_{c,t}^j} \bigg/ \left(e^{-a_i^j \cdot D_{i,t}^j} + e^{-a_c^j \cdot D_{c,t}^j}\right) .$$

With the logistic model, complications would arise in the optimization, because the transition probability functions

are convex for some range of D_i (or D_c), and concave elsewhere.

Nevertheless the model will remain useful. Instead of trying to find the optimal allocation over time, the model could be applied to evaluate various outlet expansion plans.

5. CONCLUSION

In this paper we have presented a detailed study of the aggregate retail outlet location model developed by Hartung and Fisher. We found that their general procedure was sound, but that the specifics suffered from a variety of problems regarding the estimation of the model parameters, and the robustness of the response functions. Remedial action was proposed, and was applied to a brand of gasoline in a European country.

TABLE I

Estimation of k_1 and k_2 for brand i

	Coefficient Value	t Statistic	$\hat{k}_1$	$\hat{k}_2$	R^2	$\bar{R}^2$	F(2,33)	Durbin-Watson
Equation (7)			21.83	- 0.57	.877	.870	234.7	0.38
$\hat{\alpha}$	- 0.57	- 6.63						
$\hat{\beta}$	20.26	15.32						
Equation (9)			24.87	- 0.94	.617	.594	53.1	0.60
$\hat{\gamma}$	26.54	17.25						
$\hat{\delta}$	- 1.07	- 7.29						
Equation (5)			26.26	- 1.03	.588	.563	47.0	0.65
$\hat{k}_1$	26.26	11.88						
$\hat{k}_2$	- 1.03	- 6.66						

TABLE II

Estimation of k_1 and k_2 for brand a

	Coefficient Value	t Statistic	$\hat{k}_1$	$\hat{k}_2$	R^2	$\bar{R}^2$	F(2,36)	Durbin-Watson
Equation (7)			5.55	0.56	.42	.39	26.6	0.49
$\hat{\alpha}$	0.56	7.72						
$\hat{\beta}$	4.99	5.16						
Equation (9)			1.55	0.87	.59	.57	51.9	0.35
$\hat{\gamma}$	0.37	0.20						
$\hat{\delta}$	1.15	7.20						
Equation (5)			2.75	0.78	.69	.67	79.4	0.54
$\hat{k}_1$	2.75	2.32						
$\hat{k}_2$	0.78	8.07						

TABLE III

Steady State Estimation of Exponential Model
for brand i

	Coefficient Value	t statistic
$\hat{a}_i$	- 26.241	- 20.89
$-\hat{a}_c$	- .037	- 3.25

$R^2 = .700$ $\bar{R}^2 = .681$ $F(2,33) = 38.4$ $DW = 0.49$

TABLE IV

Exponential Model for Brand i

	Coefficient Value	t Statistic	R^2	$\bar{R}^2$	F(2,32)	DW
TROLL						
$\hat{a}_i$	59.08	2.75	.70	.68	37.3	2.68
$\hat{a}_c$	0.73	7.30				
SUMT						
$\hat{a}_i$	59.05	2.76	.70	.69	37.3	2.68
$\hat{a}_c$	0.73	7.63				

TABLE VI

SUMT Estimates for Logistic Model

	Coefficient Value	t statistic
$\hat{a}_i$	.00276	.019
$\hat{b}_i$	.8394	.044
$\hat{a}_c$	.8598	.012
$\hat{b}_c$	2.683	.068

R^2 = .7049 $\bar{R}^2$ = .675 F(4,30) = 17.92 DW = 2.68

FOOTNOTES

1 Based on discussions with various major oil companies. The concept of linked hierarchical models has been examined by Crowston and Scott-Morton [1], and has been applied in the area of operational planning and control by Green [6], Newson [15], and Shwimer [16]. A possible application in the public sector has been proposed by Hausman and Naert [8].

2 Our notation will differ from H-F.

3 See H-F [7, p. B 234, footnote 2]. The quote is given in our notation.

4 At least within the limits discussed above.

5 TROLL is an interactive estimation and simulation package developed by the Computer Research Center, National Bureau of Economic Research, Inc. For a description of the estimation method see [3].

6 For a further discussion on this method, we may refer for example to Draper and Smith [2, pp. 267 et sq.] and GOLDFELD and QUANDT [5, pp. 49-57].

7 Provided that the linearized form (11) of the model is valid around $\hat{\theta}$, the final estimate of θ.

8 A test of hypothesis conducted by LAMBIN [11].

9 For brand a, 38 observations were available.

10 At this stage an additional qualification ought to be made about H-F estimation procedure. The similarity of the results obtained from regression analysis applied on equations (9) and (5) as opposed to the relative dissimilarity observed between those derived from equations (7) and (5) should cause no surprise, since equation (7) contains m_i, a stochastic variable, on both sides (as evidenced by equation (8)). Hence we should expect biased estimates of α and β from small samples ; furthermore in this case $\hat{\alpha}$ and $\hat{\beta}$ are inconsistent since the residuals indicate a strong autocorrelation among the error terms.

All this adds up to pointing out the bias created in estimating k_1 and k_2 via equation (7).

Notice that no error term was introduced in equation (5), had one be added the linearization of (5) would have been impossible.

11 For additional discussion on linearizing such nonlinear models, see Naert and Bultez [14].

12 For a theoretical exposition of SUMT, see Fiacco and Mc Cormick [4]. The computer program is described in Mylander et al. [13].

13 Many factors making the S shaped curve plausible in this industry are reported by Kotler [9, pp. 96-97].

14 See Wilde and Beightler [17, pp. 22-24] for a discussion of the Newton-Raphson method. The SUMT routine has various options with regard to the technique for minimizing the penalty function.One of these is the Newton-Raphson method.

15 Assuming that profit maximization is the objective.

16 Different types of regions, e.g. metropolitan,suburban, rural, migh have different parameters, therefore a_i and a_c are superscripted by regions.

R E F E R E N C E S

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