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THE SELECTION OF NETWORK FUNCTIONS TO APPROXIMATE PRESCRIBED FREQUENCY CHARACTERISTICS

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Abstract

A successive-approximations method is applied to the selection of network functions having desired magnitude and phase variation with frequency. The successive trial functions are specified in terms of the location of their poles and zeros in the complex frequency plane. The first approximation is an estimate of suitable pole and zero positions which only roughly fulfills the needs of the problem. It can be made on the basis of known solutions to other problems or through use of a set of curves which are presented.

The deviation of characteristics of the first estimate from the desired characteristics is determined, and changes in pole and zero positions are evaluated which tend to erase these deviations in the succeeding approximation. A set of normalized functions is presented, relating changes in the magnitude and phase characteristic to changes in the pole and zero positions. At the outset of the adjustment procedure, rough calculations based on the normalized functions suffice to obtain rapid improvement in the approximation. A far more precise but also more complicated algebraic process is applied to determine appropriate pole and zero shifts as the approximation becomes closer. This process adjusts the magnitude and phase characteristics simultaneously. The whole procedure is very flexible and convenient, permitting the accommodation of practical constraints not possible with other methods.

THE SELECTION OF NETWORK FUNCTIONS TO APPROXIMATE PRESCRIBED FREQUENCY CHARACTERISTICS

1.0 Introduction

The approximation problem is a component problem of almost every network design. The typical network design is the choice of a network which exhibits prescribed response characteristics within tolerable limits. The prescribed response characteristics are frequently graphical plots of the phase and magnitude of the desired system function given as functions of frequency. Since system functions of lumped-element linear networks are rational functions of frequency, the first problem of the designer is to select a realizable rational function which has approximately the prescribed magnitude and phase variation with frequency. The choice of the rational system function is the approximation problem, and a solution to this problem is the subject of this report.

The method of solution of the approximation problem presented is one of successive approximations and it has a number of unusual characteristics of practical value. In the first place, one chooses the complexity of the approximating function (the number of poles and zeros) at the outset and successive adjustments always leave the number of poles and zeros unchanged. One can approximate to magnitude and phase characteristics simultaneously. The approximating method is extremely flexible; one can easily impose constraints on the form of network to be obtained or on the nature of its elements.

The solution of the approximation problem is naturally divided into two parts. The first part of the solution is the restriction of attention to the class of functions which are realizable for a circuit of the type desired. For instance, if the system function is to be a driving-point reactance, one's choice is limited to reactance functions, functions having simple poles only on the imaginary axis of the complex frequency plane and having positive real residues in those poles. Once one's attention is restricted to the class of realizable functions, the second part of the approximation problem is to select from this class of functions one which fulfills the requirements of phase and magnitude variation within tolerable limits.

The difficulty of the approximation problem is dependent, among other things, upon the constraints which are imposed on the class of functions to be considered. Increasing the number or complexity of constraints ordinarily makes the approximation problem more difficult. Practically, it is very desirable to be able to impose constraints to get a network of a particular configuration, or containing elements of a prescribed nature, etc. Clearly there is a need for flexibility in the approximation procedure so that constraints can be most effectively accommodated.

The system function of a lumped-element network, being a rational function of frequency, can be expressed as follows:

$$F(\lambda) = \frac{p(\lambda)}{q(\lambda)} = \frac{a_o + a_1\lambda + \ldots a_m\lambda^m}{b_o + b_1\lambda + \ldots b_n\lambda^n} = \frac{h(\lambda - \lambda_{o1})(\lambda - \lambda_{o2})\ldots(\lambda - \lambda_{om})}{(\lambda - \lambda_{p1})(\lambda - \lambda_{p2})\ldots(\lambda - \lambda_{pn})} \quad . \qquad (1)$$

The quantities $\lambda_{o1}, \ldots \lambda_{om}$ are the zeros of the function and $\lambda_{p1}, \ldots \lambda_{pn}$ are the poles. It is apparent that the system function is specified within a constant multiplier by specification of the poles and zeros of the function. The poles and zeros represent the most convenient "common denominator" in terms of which requirements and constraints can be expressed. The constraint that a network be of a specified form may ordinarily be translated into a constraint on the pole and zero positions. For instance, lossless ladder networks terminated in resistance have transfer impedances with zeros on the imaginary axis of the λ-plane and poles inside the left half of the λ-plane. Furthermore, the behavior of the network expressed in terms of the magnitude and phase of the system function is readily related to the pole and zero positions of the system function. Accordingly, the approximation problem is, in these terms, to determine in the first part the possible range in the complex plane of positions of poles and zeros in which range the specified constraints apply. The second part is the selection, from the restricted range, of poles and zeros which lead to the most satisfactory network characteristics. The method of selection employed consists of a preliminary choice of pole and zero positions followed by successive adjustments of these positions. Ordinarily, the preliminary function chosen, or first estimate, only roughly approximates the desired characteristics. The successive adjustments, specified in terms of shifts in pole and zero positions, are made to reduce the deviation of the trial characteristics from the desired characteristics. The method of successive approximations is really a systematic fitting procedure with each successive trial producing a better fit with a function of the same complexity. The fitting procedure is unique in that both phase and magnitude characteristics can be fitted simultaneously*.

Once the restrictions imposed by physical realizability conditions on the poles and zeros of the approximating function have been determined, the remainder of the approximation procedure consists of two steps. The first step is the choice of the first approximation and the second is the process of successive adjustments leading to the final result. The two steps are distinct and can best be examined separately.

2.0 The First Approximation

The first approximation can be made on the basis of known solutions to similar problems which are roughly suitable to the given problem, or it can be made purely on the basis of a set of curves presented at a later point in this report. When a designer uses the solution of a similar problem as a starting point, wide experience and ingenuity have

*The implicit relation between magnitude and phase characteristics is well known. If the magnitude characteristic is completely specified over the whole frequency range from zero to infinity, the corresponding minimum-phase characteristic is implicitly and inflexibly determined. If one specifies the magnitude characteristic over a fraction of the spectrum only (as is ordinarily sufficient for practical purposes), or specifies it less exactly, more freedom is obtained in choice of the phase characteristic. The approximation method described here uses the available freedom to obtain suitable approximations to both phase and magnitude characteristics over a fraction of the complete spectrum.

a strong effect in decreasing the labor since a good first approximation reduces the amount of work to be done in the "fitting" step. A brief survey of previous contributions to the approximation problem is useful in providing a stock of first estimates on which to draw for particular problems.

2.1 Survey of Previous Contributions

The problem which has received the most attention and yielded the simplest and most satisfactory solution is that concerned with filters in which the characteristic desired is that sketched in Fig. 1. Early low-pass designs of such filters (due to Campbell and Zobel) did not make a systematic use of an approximation procedure with rational functions. Cauer (1, 2) and Bode (3, 2) introduced methods of obtaining rational functions identifiable as the reactance of the branches of symmetrical lattices for which lattices the propagation function exhibits filtering characteristics. Since the quantities on which Bode and Cauer primarily focused attention in the approximation procedure were not the rational functions indicated in Fig. 1, but rather some related functions (image impedance and index functions), their work is more suggestive than directly useful to the problem at hand. Darlington (4) applied an approximation procedure to rational functions, and the results are directly useful. He used Tschebyscheff polynomials in obtaining functions of ω (the radian frequency), which approximate the characteristic of Fig. 1. This approximation gives uniform tolerance in one range and monotonic behavior in the other (Fig. 2a). Darlington also used Jacobian elliptic functions to provide uniform-tolerance behavior (called Tschebyscheff behavior) in both pass and stop bands (Fig. 2b).

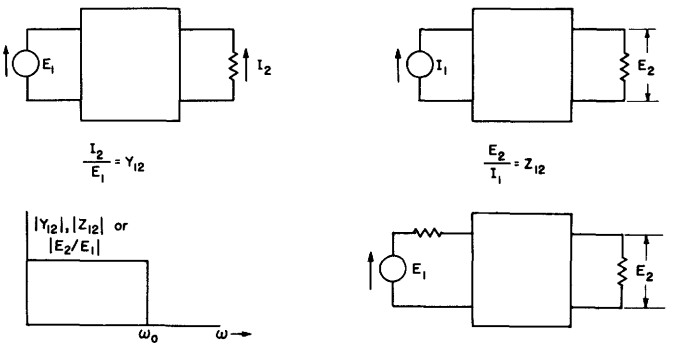

Fig. 1 Ideal low-pass characteristic and circuits to which it might apply.

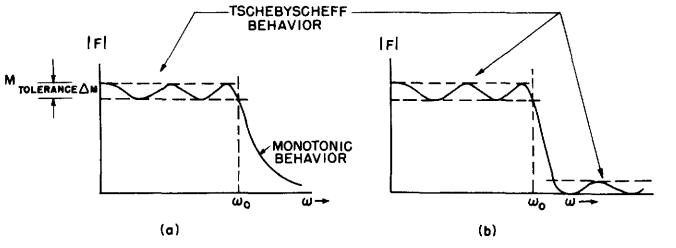

Fig. 2 Tschebyscheff behavior in filter characteristics.

A somewhat simpler approximation function than those used by Darlington was suggested by Butterworth (5). The characteristic corresponding to this function (Fig. 3) exhibits monotonic behavior in both pass and stop bands, the discrimination being higher the higher the order of the function.

The approximations mentioned above are treated in detail in the original reference cited and in simple form elsewhere (6, 7, 8, 9). The function, F, which has the magnitude variation shown in Fig. 2a or 3 when expressed in the form (1) possesses no internal

zeros and an array of poles as shown in Fig. 4a or 4b respectively for functions of the fifth order. The geometric figure on which the poles shown in Fig. 4a lie is an ellipse. The cut-off frequency is only slightly less than the value of ω at the intersection of the

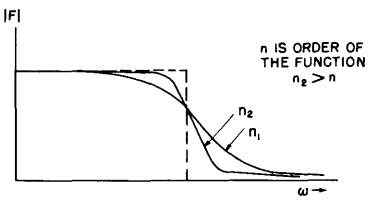

Fig. 3 Butterworth filter characteristic.

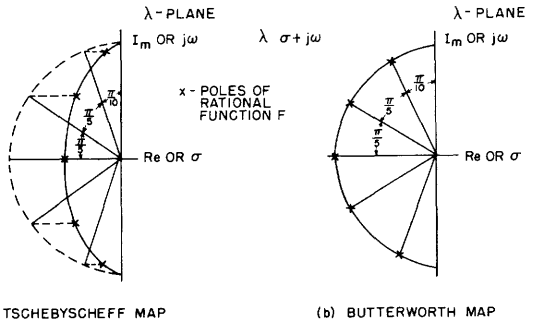

Fig. 4 Maps of poles for Tschebyscheff and Butterworth approximations.

$$F = \frac{1}{(\lambda - \lambda_{p1})(\lambda - \lambda_{p2})\ldots(\lambda - \lambda_{p5})}$$

ellipse and the axis of imaginaries. Simple relationships between the level and tolerance of $|F|$ of Fig. 2a and the size of the minor axis of the ellipse of Fig. 4a are given in the references cited. For the purposes of this report it is sufficient to state that as the minor axis of the ellipse is decreased (the poles brought nearer the imaginary axis always along lines parallel to the real axis) the level of F in the pass-band increases as does the tolerance. Increasing the number of poles* increases the level of $|F|$ for a fixed ellipse and the number of cycles of variation of $|F|$ in the pass band while it decreases the tolerance.

From Fig. 4 it is apparent that the Butterworth approximation is a degenerate form of the Tschebyscheff approximation in which the ellipse becomes a circle. One observes from Fig. 2a, 3 and 4 that small shifts in the position of the critical frequencies of a function cause marked changes in the characteristics of the function. This fact suggests the study of the relationship between changes in pole positions and changes in the function characteristics. This study and a method of successive approximations arising from it constitute the results given in this report.

The function corresponding to the characteristic of Fig. 2b has internal poles which are distributed on a figure slightly different from an ellipse. There are also internal zeros distributed along the imaginary axis beyond the cut-off frequency.

That networks having band-pass characteristics can be derived from those having low-pass characteristics is well known (6, 9, 10). It is enlightening to view the low-pass to band-pass transformation by comparing the maps of critical frequencies for corresponding low-pass and band-pass transfer functions. Band-pass characteristics and the map

*For odd numbers of poles, one always lies on the negative real axis; for even numbers of poles, none lies on the real axis.

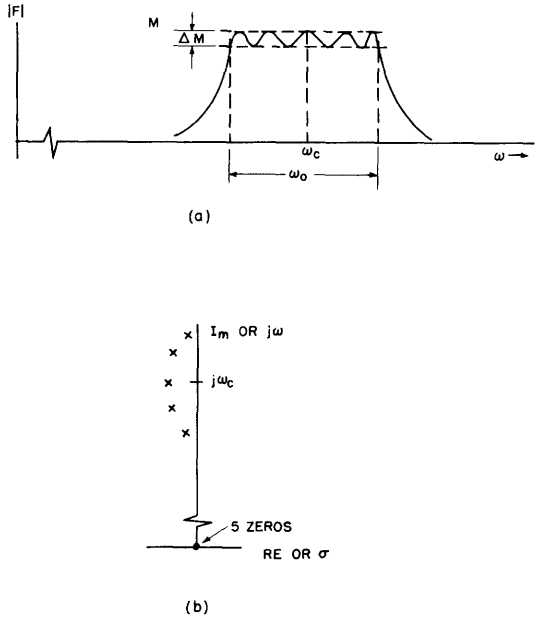

Fig. 5 Transfer characteristics and map of critical frequencies for band-pass network corresponding to Fig. 2a and Fig. 4a.

of critical frequencies corresponding to the low-pass case of Fig. 2a and Fig. 4a are shown in Fig. 5. For the case in which the mean frequency (ω_c) is large compared to the bandwidth (ω_o) the poles in Fig. 5b fall on an ellipse just half the size of the corresponding ellipse of Fig. 4a (see reference 6 for proof). If the mean frequency (ω_c) is not large compared to the bandwidth (ω_o), the poles of the band-pass transfer function no longer fall on an ellipse. Those nearest the origin are warped slightly closer both to the origin and to the imaginary axis.

That the magnitude and argument of a rational function are related to a potential distribution in a two-dimensional field problem is well known. This relationship can be exploited as an aid in solving the approximation problem. The relationship arises through the fact that the real and imaginary parts of the logarithm of a rational function are conjugate potential functions. This point may be easily appreciated by considering $\ln|F|$ from Eq. 1. The relation of Eq. 2

$$\ln|F| = \ln h + \sum_{i=1}^{m} \ln|\lambda - \lambda_{oi}| - \sum_{i=1}^{n} \ln|\lambda - \lambda_{pi}| \qquad (2)$$

to a potential distribution is evidenced through considering an infinite conducting plane (identified with the λ-plane) to which a unit current is supplied at some point in the plane. If the coordinates of the point at which current is injected are those of λ_o, the potential at any point λ is given by $C \ln(1/|\lambda - \lambda_o|)$. The conductivity of the conducting plane is determined by C. If a set of negative current sources are placed at points corresponding to the λ_{oi}'s of Eq. 2, and a set of positive current sources are placed at points corresponding to the λ_{pi}'s, the potential resulting along the imaginary axis of the λ-plane is an exact measure of $\ln|F|$.

Hansen and Lundstrom (11) used an electrolytic tank with probe current sources representing poles and zeros to set up two-dimensional potential distributions corresponding to rational functions. They proposed the tank as a convenient calculating device to get plots of the magnitude of rational functions for specified pole and zero positions. The author (12) used an electrolytic tank to solve approximation problems. The procedure was simply to find, by successive trials, a set of positions for the current sources leading to the desired kind of potential distribution along the imaginary axis. The method,

which is one of successive approximations, is simple and enlightening. With a small amount of adjustment one frequently can arrive at a good answer, and there is great flexibility in imposing arbitrary restrictions on the positions of poles and zeros of the function. A difficulty with the method lies in the fact that at times adjustments must be made to yield a compromise between conflicting requirements. For instance, a deviation from the desired characteristic at one frequency may suggest a shift in current sources of one nature, while the deviation at another frequency suggests a different kind of shift. In such a situation the designer finds himself essentially trying to solve a set of simultaneous equations by trial, and he may find the solution a slow task.

The method of solution of the approximation problem presented in this report is quite similar to the experimental solution by successive approximations. However, here the characteristics of successive trials are computed from curves. Successive shifts of pole and zero positions are made as the result of an algebraic computation. Each shift is chosen to remove the deviations of the preceding trial from the desired characteristic; appropriate compromises are made whenever it is impossible to reduce all deviations simultaneously.

2.2 Computation of Phase and Magnitude Characteristics for Specified Pole and Zero Positions

Normalized curves are used for the computation of the characteristics of the successive trials. The same curves may be used as a guide in the choice of the first approximation. The process of normalization is easily understood by considering the logarithm of F (F is defined in Eq. 1).

$$\ln F = \ln |F| + j \operatorname{Arg} F \tag{3}$$

$$\ln |F| = \ln h + \sum_{i=1}^{m} \ln |\lambda - \lambda_{oi}| - \sum_{i=1}^{n} \ln |\lambda - \lambda_{pi}| \tag{2}$$

$$\operatorname{Arg} F = \sum_{i=1}^{m} \operatorname{Arg}(\lambda - \lambda_{oi}) - \sum_{i=1}^{n} \operatorname{Arg}(\lambda - \lambda_{pi}) \tag{4}$$

The consideration of the logarithm of F is convenient in that it makes the contribution of each pole and zero separately evident. One observes that the sums in Eq. 2 and also in Eq. 4 are really made up of only two kinds of terms, terms corresponding to conjugate complex critical frequencies (poles or zeros) and terms corresponding to real critical frequencies. The distinction between zeros and poles is merely the distinction between plus and minus signs associated with the components in the sum.

For the complex critical frequencies, one may write the following from Eq. 2 and Eq. 4*.

$$\ln |F_c| = \ln |\lambda - \lambda_c| |\lambda - \overline{\lambda}_c| \tag{5}$$

and

$$\text{Arg } F_c = \text{Arg}(\lambda - \lambda_c)(\lambda - \overline{\lambda}_c) \tag{6}$$

in which

$$\lambda_c = \sigma_c + j\omega_c \quad . \tag{7}$$

If one divides F_c by ω_c^2 Eqs. 5 and 6 may be written

$$\ln \left| \frac{F_c}{\omega_c^2} \right| = \ln \left| \frac{\lambda}{\omega_c} - \frac{\sigma_c}{\omega_c} - j \right| \left| \frac{\lambda}{\omega_c} - \frac{\sigma_c}{\omega_c} + j \right| \tag{8}$$

and

$$\text{Arg } F_c = \text{Arg} \frac{F_c}{\omega_c^2} = \text{Arg} \left(\frac{\lambda}{\omega_c} - \frac{\sigma_c}{\omega_c} - j \right) \left(\frac{\lambda}{\omega_c} - \frac{\sigma_c}{\omega_c} + j \right) \quad . \tag{9}$$

Since one is ordinarily interested only in real frequencies, Eqs. 8 and 9 become

$$\ln \frac{|F_c|}{\omega_c^2} = \ln \left| \frac{j\omega}{\omega_c} - \frac{\sigma_c}{\omega_c} - j \right| \left| \frac{j\omega}{\omega_c} - \frac{\sigma_c}{\omega_c} + j \right| \tag{10}$$

and

$$\text{Arg } F_c = \text{Arg} \left(\frac{j\omega}{\omega_c} - \frac{\sigma_c}{\omega_c} - j \right) \left(\frac{j\omega}{\omega_c} - \frac{\sigma_c}{\omega_c} + j \right) \quad . \tag{11}$$

Clearly $\ln |F_c|$ and $\text{Arg } F_c$ are easily found as functions of ω for any λ_c if one has simply plots of $\ln |F_c| / \omega_c^2$ and $\text{Arg } F_c / \omega_c^2$ as functions of ω/ω_c for appropriate values of the parameter σ_c/ω_c. Figure 6 gives sketches of families of curves corresponding to Eqs. 10 and 11. An accurate set of curves of the same nature is given in the appendix**. Through use of the curves in the appendix one can readily obtain plots of $\ln |F_c|$ and $\text{Arg } F_c$ as functions of ω for any complex value of λ_c. Figure 6 is instructive in connection with the approximation problem. It is apparent that critical frequencies close to the imaginary axis in the complex frequency plane cause much more violent changes in the magnitude of the function for nearby real frequencies than do critical frequencies removed a considerable distance from the imaginary axis. Accordingly, if one is choosing a function which is to exhibit abrupt changes in magnitude at some range of frequencies, the suggestion is definite that critical frequencies with small displacement from the

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*In the following F_c will always represent the product of a pair of factors with complex conjugate zeros, λ_c and $\overline{\lambda}_c$.

**All logarithms in the curves are to the base 10 and all arguments are given in degrees. In the text, log refers to the base 10 and ln refers to the base ϵ; Arg is the angle in radians unless specifically stated otherwise.

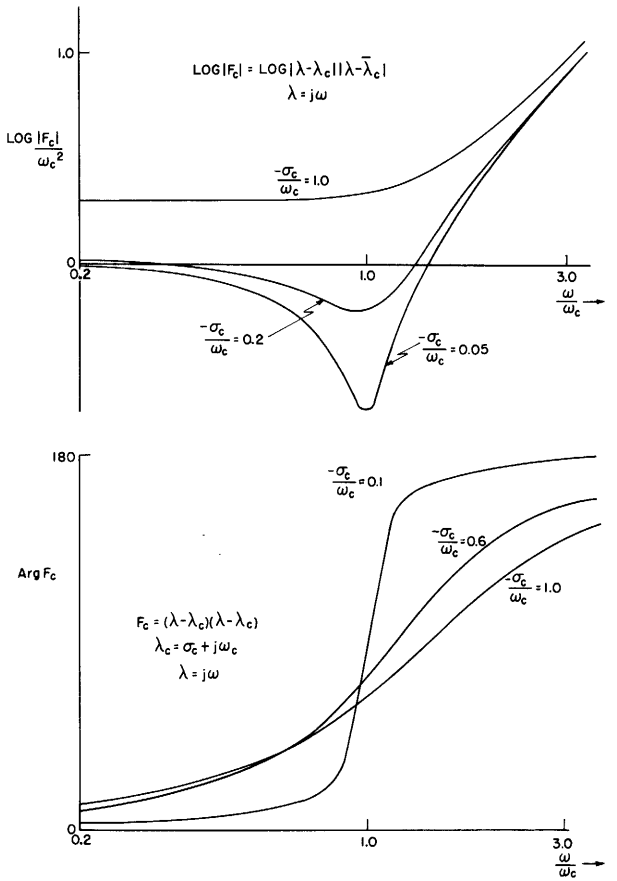

Fig. 6 Log $|F_c/\omega_c^2|$ and Arg F_c sketched as functions of ω/ω_c for a range of values of $-\sigma_c/\omega_c$.

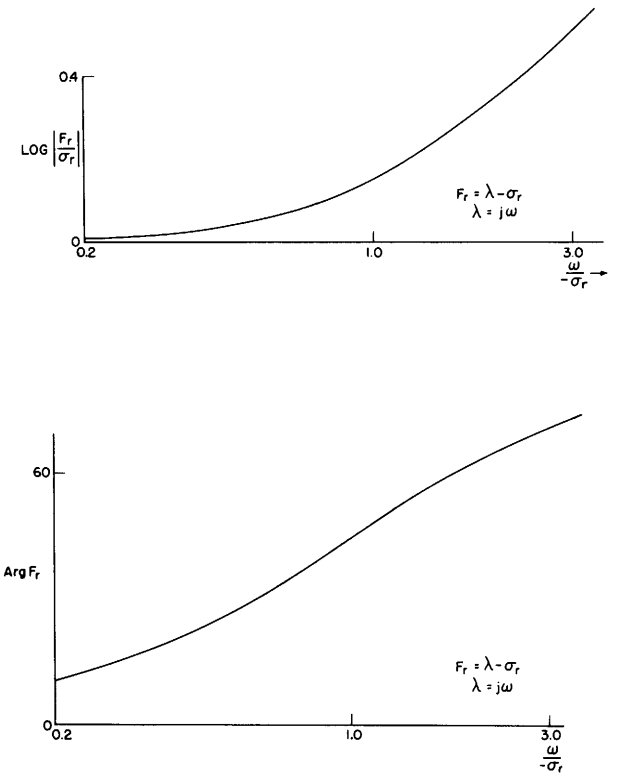

Fig. 7 Log $|F_r/\sigma_r|$ and Arg F_r sketched as functions of $\omega/-\sigma_r$.

imaginary axis be placed near the range of $j\omega$ where the change is desired. From Fig. 6 also it is clear that such critical frequencies cause rapid phase shift at the same time they cause an abrupt change in magnitude. This situation is a manifestation of the implicit relationship between magnitude and phase.

The second kind of terms mentioned in connection with Eqs. 2 and 4 are those representing a critical frequency falling on the real axis of the complex frequency plane*.

$$\ln |F_r| = \ln |\lambda - \sigma_r| \qquad \text{where } \sigma_r \text{ is real.} \tag{12}$$

$$\text{Arg } F_r = \text{Arg}(\lambda - \sigma_r) \tag{13}$$

If one divides F_r by $-\sigma_r$ and sets $\lambda = j\omega$, the result is

$$\ln \left| \frac{F_r}{\sigma_r} \right| = \ln \left| \frac{j\omega}{-\sigma_r} + 1 \right| \tag{14}$$

and

$$\text{Arg } \frac{F_r}{-\sigma_r} \; (= \text{Arg } F_r \text{ if } \sigma_r \text{ is negative}) = \text{Arg } \frac{j\omega}{-\sigma_r} + 1) \quad . \tag{15}$$

The sketch of Fig. 7 indicates the form of variation of $\ln|F_r|$ and Arg F_r for any real critical frequency, σ_r. Corresponding accurate plots for computational purposes are given in the appendix. The sketches of Figs. 6 and 7 are given for critical frequencies in the left half-plane. The same curves are useful for critical frequencies in the right half-plane since the logarithm of the magnitude of F_c or F_r is unchanged if the sign of σ_c or σ_r is changed and only the sign of the argument of F_c or F_r changes when the sign of σ_c or σ_r is changed.

2.3 An Illustrative Example

To illustrate the choice of the first approximation, a function of the form

$$F = \frac{1}{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_o} = \frac{1}{(\lambda - \sigma_1)(\lambda - \sigma_2 - j\omega_2)(\lambda - \sigma_2 + j\omega_2)} \tag{16}$$

will be chosen. The characteristics to be approximated are described in Fig. 8. It will be required that the approximating function have three poles, have approximately a linear phase shift over the range $0 \leqslant \omega \leqslant 1$ and have a logarithm of magnitude which decreases linearly (approximately) with frequency such that $|F|$ at $\omega = 1$ is half of $|F|$ at $\omega = 0$. The magnitude of the argument of F

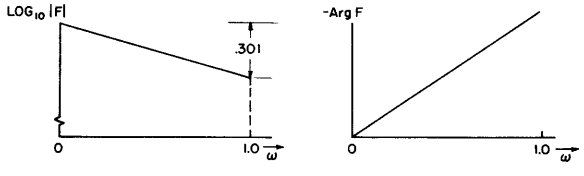

Fig. 8 Partial specification of requirements for illustrative example.

*In the following F_r will always represent a factor with a single real zero, σ_r.

at $\omega = 1$ is immaterial so long as its variation with frequency is linear. The level of $\log_{10}|F|$ is not specified. A large number of functions of the form represented in Eq. 16 will have characteristics of approximately the nature indicated in Fig. 8. Some of them approximate the magnitude characteristic well and the phase characteristic badly; some approximate the phase characteristic well and the magnitude characteristic badly; others fall between these classes approximating both phase and magnitude reasonably well. Before one can make a final choice of approximating function, some measure of quality of the approximation must be formulated. Such a measure of quality is given at a later point; at this point it is necessary only to choose any function of the form of Eq. 16 which has roughly the characteristics desired and to proceed from it as a starting point.

Knowledge of the functions used for Butterworth filter characteristics suggests that a function with poles distributed on a semi-circle as in Fig. 4b will give roughly the kind of characteristics desired. Hence a first approximation is

$$F(\lambda) = \frac{1}{(\lambda + 1.00)(\lambda + 0.50 + j0.866)(\lambda + 0.50 - j0.866)} \quad . \tag{17}$$

Figure 9, which is a plot of log $|F|$ and Arg F for Eq. 17, indicates that the first approximation shows a too slow decrease in magnitude with frequency. Hence some motion of the poles from their semi-circular distribution is desirable. The choice of the first

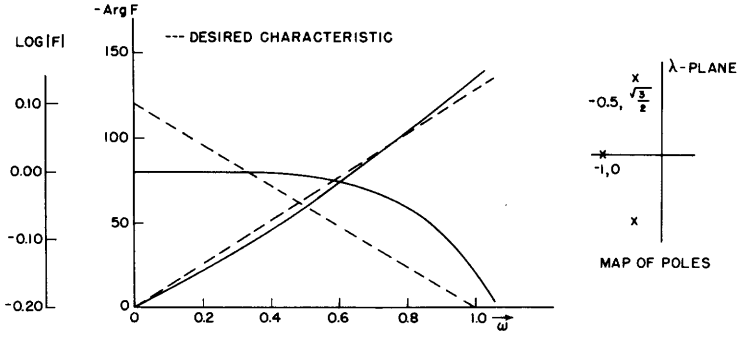

Fig. 9 Log $|F|$ and Arg F for the first approximation to the prescribed characteristics of Fig. 8.

approximation is quite arbitrary and different first approximations will lead to substantially the same end result. The closeness of the first approximation influences the length of the adjustment process which is the next step in the procedure; good first approximations decrease the number of adjustment steps required.

3.0 Successive Adjustments

Once the first approximation is chosen, attention is shifted to the problem of determining the changes in pole and zero positions required to effect the desired changes in the approximating function. One observes that the number of variable quantities which may be adjusted is equal to the total number of poles and zeros. In the illustrative example above these are three adjustable quantities, σ_1, σ_2 and ω_2. In that instance one wishes to choose the combination of values for $\Delta\sigma_1$, $\Delta\sigma_2$, and $\Delta\omega_2$ which together change the characteristics from those of Fig. 9 to characteristics much nearer those of Fig. 8.

A study of the influence of elementary shifts of the critical frequencies, $\Delta\sigma_1$ by itself, $\Delta\sigma_2$ by itself and $\Delta\omega_2$ by itself, is an enlightening guide to the choice of the combination. This fact suggests the study of

$$\frac{\partial \log|F|}{\partial\sigma_1}, \qquad \frac{\partial \text{Arg } F}{\partial\sigma_1}, \qquad \frac{\partial \log|F|}{\partial\sigma_2},$$

etc. since

$$\Delta \log|F| = \int_0^{\Delta\sigma_1} \frac{\partial \log|F|}{\partial\sigma_1}\, d\sigma_1 + \int_0^{\Delta\sigma_2} \frac{\partial \log|F|}{\partial\sigma_2}\, d\sigma_2 + \int_0^{\Delta\omega_2} \frac{\partial \log|F|}{\partial\omega_2}\, d\omega_2$$

$$\cong \text{(for small changes)} \frac{\partial \log|F|}{\partial\sigma_1}\Delta\sigma_1 + \frac{\partial \log|F|}{\partial\sigma_2}\Delta\sigma_2 + \frac{\partial \log|F|}{\partial\omega_2}\Delta\omega_2 . \qquad (18)$$

Similarly,

$$\Delta \text{ Arg } F \cong \text{(for small changes)} \frac{\partial \text{ Arg } F}{\partial\sigma_1}\Delta\sigma_1 + \frac{\partial \text{ Arg } F}{\partial\sigma_2}\Delta\sigma_2 + \frac{\partial \text{ Arg } F}{\partial\omega_2}\Delta\omega_2 . \qquad (19)$$

Fortunately, the use of the logarithm of F has separated the influence of individual critical frequencies as indicated by Eq. 2 and one has, for instance

$$\frac{\partial \log|F|}{\partial\sigma_1} = \frac{-\partial \log|\lambda - \sigma_1|}{\partial\sigma_1} . \qquad (20)$$

From Eq. 20 it is apparent that the derivative functions can be normalized and presented in curves just as was done for the logarithm and argument curves in Section 2.2. Further discussion of the use of the derivative curves will be given subsequent to the development of expressions for the derivative functions.

3.1 Normalization of the Derivative Functions

From the foregoing one appreciates that the functions which it is sufficient for all cases to consider are:

$$\frac{\partial \log|F_c|}{\partial\sigma_c}, \quad \frac{\partial \log|F_c|}{\partial\omega_c}, \quad \frac{\partial \log|F_r|}{\partial\sigma_r}, \quad \frac{\partial \text{ Arg } F_c}{\partial\sigma_c}, \quad \frac{\partial \text{ Arg } F_c}{\partial\omega_c} \text{ and } \frac{\partial \text{ Arg } F_r}{\partial\sigma_r}$$

(see Eqs. 5, 6, 7, 12 and 13). At this point expressions for $\partial \log|F_c|/\partial\sigma_c$ and $\partial \text{ Arg } F_c/\partial\sigma_c$ will be developed in detail and the expressions for the other derivative functions (which are developed by similar techniques) will be stated without proof.

$$\ln F_c(\lambda) = \ln|F_c(\lambda)| + j \text{ Arg } F_c(\lambda) . \qquad (21)$$

Since F_c is a rational function of λ with real coefficients, for $\lambda = j\omega$,

$$\overline{F_c(\lambda)}\Big|_{\lambda=j\omega} = F_c(-\lambda)\Big|_{\lambda=j\omega} \tag{22}$$

$$F_c(\lambda)\,\overline{F_c(\lambda)}\Big|_{\lambda=j\omega} = \left|F_c(\lambda)\right|^2_{\lambda=j\omega} = F_c(\lambda)\,F_c(-\lambda)\Big|_{\lambda=j\omega} \tag{23}$$

$$\ln\left|F_c(\lambda)\right|_{\lambda=j\omega} = \frac{1}{2}\ln F_c(\lambda)\,F_c(-\lambda)\Big|_{\lambda=j\omega} \tag{24}$$

$$\ln|F_c(j\omega)| = \frac{1}{2}\ln\left[(j\omega - \sigma_c - j\omega_c)(j\omega - \sigma_c + j\omega_c)(-j\omega - \sigma_c - j\omega_c)(-j\omega - \sigma_c + j\omega_c)\right] \tag{25}$$

$$\ln|F_c(j\omega)| = \frac{1}{2}\ln\left\{\left[-\sigma_c + j(\omega - \omega_c)\right]\left[-\sigma_c - j(\omega - \omega_c)\right]\left[-\sigma_c + j(\omega + \omega_c)\right]\left[-\sigma_c - j(\omega + \omega_c)\right]\right\}$$

$$= \frac{1}{2}\ln\left[\sigma_c^2 + (\omega - \omega_c)^2\right]\left[\sigma_c^2 + (\omega + \omega_c)^2\right] \tag{26}$$

$$\frac{\partial \ln|F_c|}{\partial \sigma_c} = \frac{\frac{\partial}{\partial \sigma_c}\left[\sigma_c^2 + (\omega - \omega_c)^2\right]\left[\sigma_c^2 + (\omega + \omega_c)^2\right]}{2\left[\sigma_c^2 + (\omega - \omega_c)^2\right]\left[\sigma_c^2 + (\omega + \omega_c)^2\right]}$$

$$= \frac{\sigma_c\left[\sigma_c^2 + (\omega + \omega_c)^2\right] + \sigma_c\left[\sigma_c^2 + (\omega - \omega_c)^2\right]}{\left[\sigma_c^2 + (\omega - \omega_c)^2\right]\left[\sigma_c^2 + (\omega + \omega_c)^2\right]}$$

$$= \frac{2\,\sigma_c\left[\sigma_c^2 + \omega_c^2 + \omega^2\right]}{|F_c|^2} \quad . \tag{27}$$

The derivative may be conveniently normalized.

$$\frac{\partial \ln|F_c|}{\partial \sigma_c} = \frac{\frac{2\,\sigma_c}{\omega_c^2}\left[\left(\frac{\sigma_c}{\omega_c}\right)^2 + 1 + \left(\frac{\omega}{\omega_c}\right)^2\right]}{\left|\frac{F_c}{\omega_c^2}\right|^2} \tag{28}$$

$$\omega_c\frac{\partial \ln|F_c|}{\partial \sigma_c} = \frac{\frac{2\,\sigma_c}{\omega_c}\left[\left(\frac{\sigma_c}{\omega_c}\right)^2 + 1 + \left(\frac{\omega}{\omega_c}\right)^2\right]}{\left|\frac{F_c}{\omega_c^2}\right|^2} \quad . \tag{29}$$

In terms of the logarithms to the base 10 one has

$$\omega_c \frac{\partial \log |F_c|}{\partial \sigma_c} = \frac{2}{2.305} \frac{\frac{\sigma_c}{\omega_c}\left[\left(\frac{\sigma_c}{\omega_c}\right)^2 + 1 + \left(\frac{\omega}{\omega_c}\right)^2\right]}{\left|\frac{F_c}{\omega_c^2}\right|^2} \quad . \tag{30}$$

A sketch of $\omega_c \partial \log |F_c|/\partial \sigma_c$ as a function of ω/ω_c for various values of the parameter σ_c/ω_c is given in Fig. 10. A set of accurate charts corresponding to Fig. 10 are given

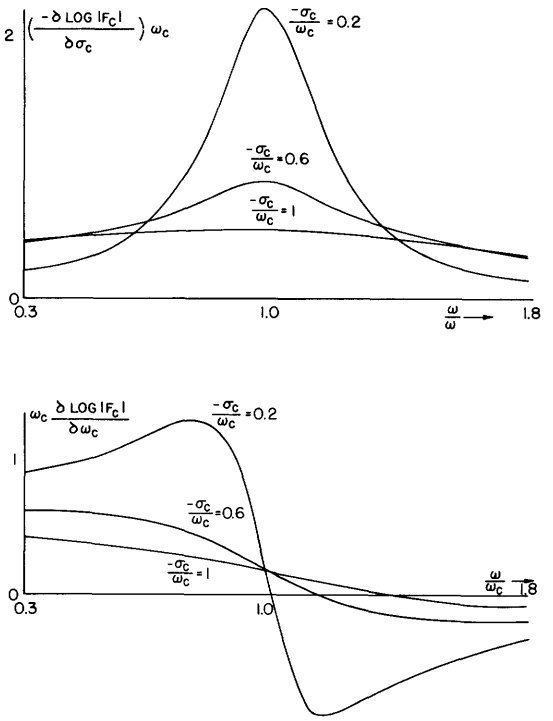

Fig. 10 Sketches of $\omega_c \dfrac{\partial \log |F_c|}{\partial \sigma_c}$ and $\omega_c \dfrac{\partial \log |F_c|}{\partial \omega_c}$ as functions of $\dfrac{\omega}{\omega_c}$ for various values of $\dfrac{\sigma_c}{\omega_c}$.

in the appendix. Figure 10 reveals that shifts in the displacement from the imaginary axis of poles or zeros influence the magnitude of the function most in a frequency range nearest the critical frequency being shifted. Through use of the charts in the appendix one can predict accurately the influence of a shift of a critical frequency. Alternately, one has suitable information for prescribing what kind of shift and how large a shift is necessary to obtain a required change in the magnitude characteristic.

Through a procedure similar to that presented from Eqs. 21 to 30 one can obtain

$$\omega_c \frac{\partial \log |F_c|}{\partial \omega_c} = \frac{2}{2.305} \frac{\left[1 + \left(\frac{\sigma_c}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\right)^2\right]}{\left|\frac{F_c}{\omega_c^2}\right|^2} \quad . \tag{31}$$

Equation 31 and the corresponding sketch in Fig. 10 indicates the influence on the

magnitude characteristic of a shift parallel to the imaginary axis of a conjugate pair of critical frequencies.

The starting point to evaluate $\partial \text{ Arg } F_c/\partial\sigma_c$ is again Eq. 21.

$$\ln F_c(\lambda) = \ln |F_c(\lambda)| + j \text{ Arg } F_c(\lambda) \tag{32}$$

If $\lambda = j\omega$,

$$\ln F_c(-\lambda) = \ln |F_c(\lambda)| - j \text{ Arg } F_c(\lambda) \ , \tag{33}$$

and

$$\ln \frac{1}{F_c(-\lambda)} = -\ln |F_c(\lambda)| + j \text{ Arg } F_c(\lambda) \ . \tag{34}$$

$$\text{Arg } F_c(\lambda)\Big|_{\lambda=j\omega} = \frac{1}{2} \ln \frac{F_c(\lambda)}{F_c(-\lambda)}\Bigg|_{\lambda=j\omega} \tag{35}$$

$$\text{Arg } F_c(\lambda)\Big|_{\lambda=j\omega} = \frac{1}{2} \ln \frac{(j\omega - \sigma_c - j\omega_c)(j\omega - \sigma_c + j\omega_c)}{(-j\omega - \sigma_c - j\omega_c)(-j\omega - \sigma_c + j\omega_c)} \tag{36}$$

$$\frac{\partial \text{ Arg } F_c}{\partial\sigma_c}\Bigg| = \frac{\partial}{\partial\sigma_c}\left[\frac{1}{2} \ln \frac{(\sigma_c + j\omega_c - j\omega)(\sigma_c - j\omega_c - j\omega)}{(\sigma_c + j\omega_c + j\omega)(\sigma_c - j\omega_c + j\omega)}\right] \ . \tag{37}$$

After a straightforward manipulation one has

$$\frac{\partial \text{ Arg } F_c}{\partial\sigma_c} = \frac{2j\omega\left[\sigma_c^2 - \omega_c^2 + \omega^2\right]}{\left(\sigma_c^2 + \omega_c^2 - \omega^2\right)^2 + 4\sigma_c^2\omega^2} \ . \tag{38}$$

Through normalizing the frequency with respect to ω_c and multiplying by 57.3 so that the units of Arg F_c are degrees for the curves, one has

$$\omega_c \frac{\partial \text{ Arg } F_c(\text{degrees})}{\partial\sigma_c} = 57.3 \times 2 \frac{\frac{\omega}{\omega_c}\left[\left(\frac{\sigma_c}{\omega_c}\right)^2 - 1 + \left(\frac{\omega}{\omega_c}\right)^2\right]}{\left|\frac{F_c}{\omega_c^2}\right|^2} \ . \tag{39}$$

Through a similar procedure

$$\omega_c \frac{\partial \text{ Arg } F_c(\text{degrees})}{\partial\omega_c} = \frac{57.3 \times 4 \frac{\omega}{\omega_c}\frac{\sigma_c}{\omega_c}}{\left|\frac{F_c}{\omega_c^2}\right|^2} \ . \tag{40}$$

Figure 11 shows sketches of $\omega_c(\partial \text{ Arg } F_c/\partial \sigma_c)$ and $\omega_c(\partial \text{ Arg } F_c/\partial \omega_c)$ for various values of the parameter σ_c/ω_c. Figures 10 and 11 give complete information on the influence on magnitude and phase characteristics for any shifts of a conjugate pair of complex critical frequencies.

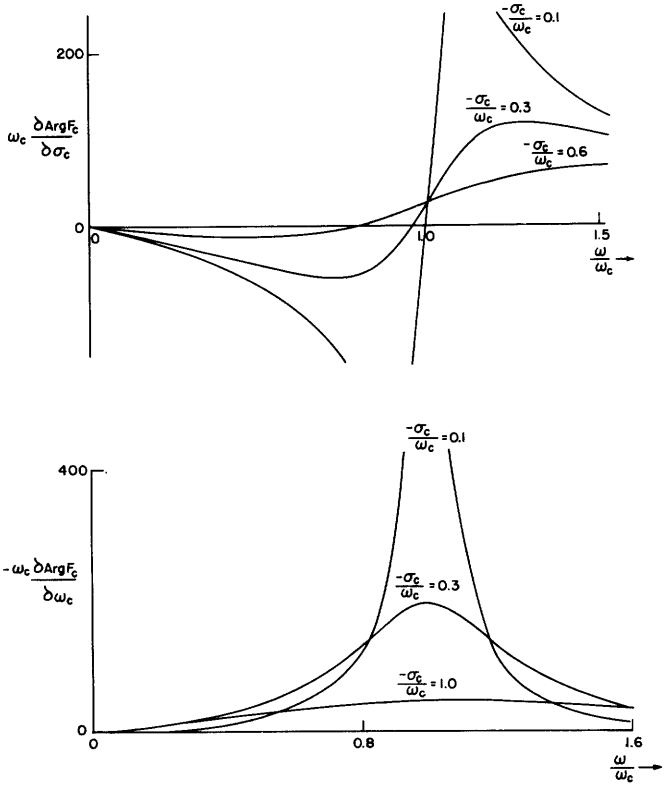

Fig. 11 Sketches of $\omega_c \dfrac{\partial \text{ Arg } F_c}{\partial \sigma_c}$ and $\omega_c \dfrac{\partial \text{ Arg } F_c}{\partial \omega_c}$ as functions of $\dfrac{\omega}{\omega_c}$ for various values of $\dfrac{\sigma_c}{\omega_c}$.

In an analogous manner to that just presented for conjugate critical frequencies one can determine the corresponding derivative functions for real critical frequencies.

$$-\sigma_r \frac{\partial \log |F_r|}{\partial \sigma_r} = \frac{-1}{2.305 \left[1 + \left(\dfrac{\omega}{\sigma_r}\right)^2\right]} \tag{41}$$

$$-\sigma_r \frac{\partial \text{ Arg } F_r}{\partial \sigma_r} = \frac{57.3 \dfrac{\omega_c}{-\sigma_r}}{1 + \left(\dfrac{\omega}{\sigma_r}\right)^2} \quad . \tag{42}$$

Figure 12 presents sketches of $-\sigma_r(\partial \text{ Arg } F_r/\partial \sigma_r)$ and $-\sigma_r(\partial \log |F_r|/\partial \sigma_r)$. Accurate plots for computational purposes are given in the appendix for functions corresponding to Fig. 11 and Fig. 12.

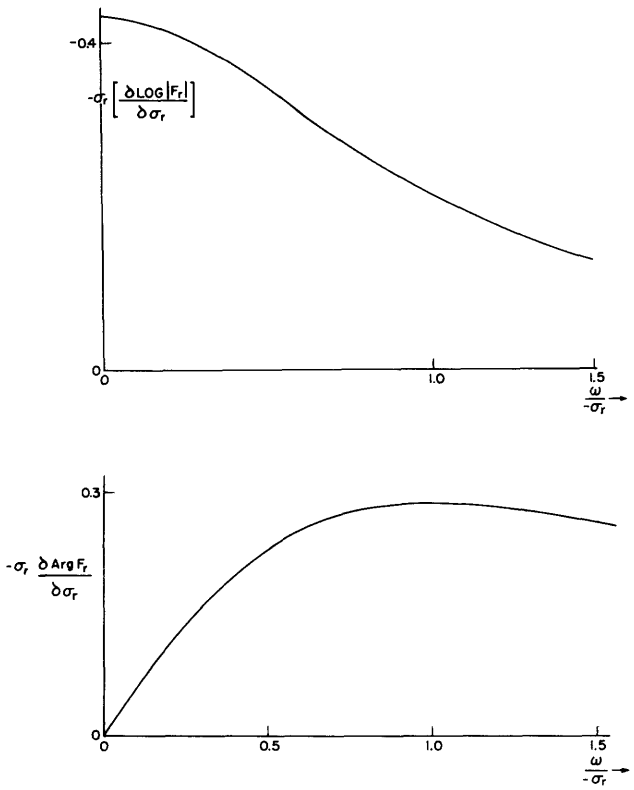

Fig. 12 Sketches of $-\sigma_r \dfrac{\partial \log|F_r|}{\partial \sigma_r}$ and $-\sigma_r \dfrac{\partial \text{ Arg } F_r}{\partial \sigma_r}$ as functions of $\dfrac{\omega}{-\sigma_r}$.

3.2 Illustration of the First Stage of the Adjustment Procedure

The preliminary step to stating what changes should be made in the pole and zero positions is the sketching of the derivative functions for the case at hand. A convenient medium to illustrate the process is the illustrative example of Section 2.3. One observes that the deviation of the magnitude characteristic from the desired is much more serious than that of the phase characteristic. Accordingly it is appropriate for the first adjustment to base the choice of changes on the deviation of the magnitude characteristic alone. Figure 13 shows plots of the desired change in the magnitude characteristic and the derivative functions. On the basis of Fig. 13 one estimates the size changes to be used to

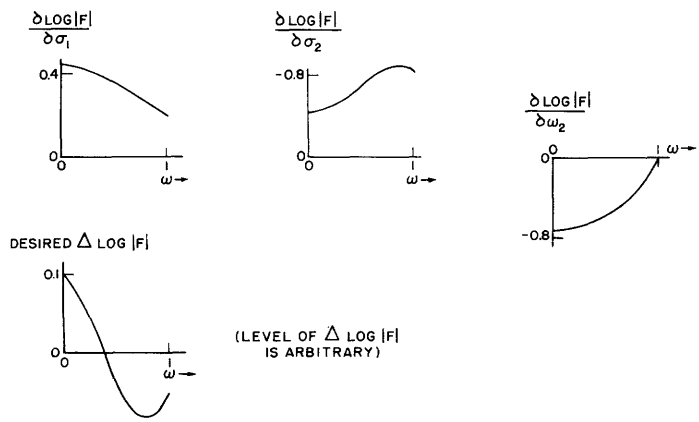

Fig. 13 Derivative curves and desired change in magnitude characteristic for the example of Fig. 9.

give the desired changes in the magnitude characteristic. For instance, it is clear that any positive increment in σ_1 will cause an improvement since the first estimate gives insufficient amplification in the low range. Rough calculations suggest the values $\Delta\sigma_1 = 0.2$, $\Delta\sigma_2 = -0.2$ and $\Delta\omega_2 = -0.1$. Figure 14 shows the second approximation corresponding to these changes. As long as it is obvious without a more precise method how to

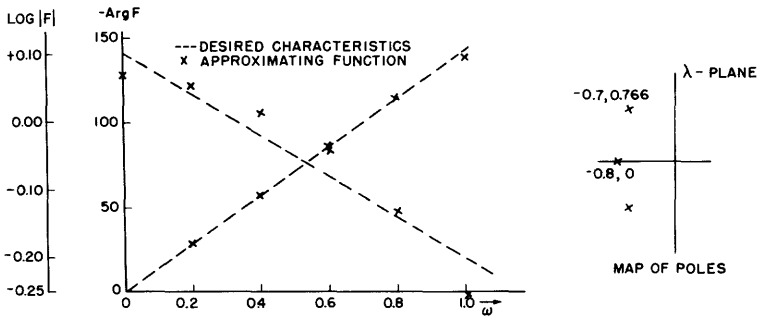

Fig. 14 Second approximation to the characteristics of Fig. 8.

obtain improvement in the characteristics, one continues to use this simple method of adjustment. Fortunately, for many problems, specifications allow considerable tolerance and one arrives at sufficiently close approximations without going to a more accurate method. Actually, as the present example illustrates, the simple procedure carefully applied leads to a very nice approximation. After a few more adjustments of the nature of that illustrated above, one arrives at the characteristics of Fig. 15.

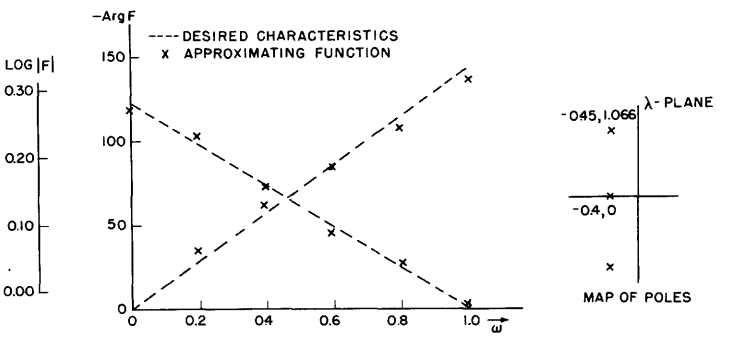

Fig. 15 Approximation to characteristics of Fig. 8 obtained by the first stage of the adjustment procedure.

The method of adjustment illustrated in the foregoing paragraphs becomes ineffective in complicated situations. For instance, the deviation of the characteristics at one frequency may indicate one shift of a pole while the deviation at a different frequency may indicate an entirely different shift. In such a situation it is difficult to prescribe appropriate shifts of the critical frequencies. If one tries to prescribe the shifts by the simple procedure illustrated, he finds himself in the predicament of one trying to solve a set of simultaneous equations one at a time. Incidentally, this same difficulty is encountered in solving the approximation problem with an electrolytic tank. A further defect of the

simple method of adjustment, and with the electrolytic tank as well, lies in the fact that one stops the procedure when he cannot obtain further improvement. He may be uncertain as to whether or not someone else can obtain a better approximation by stumbling upon a more appropriate combination of critical-frequency shifts.

At this point a more precise method of adjustment is illustrated which will handle the difficult cases. Moreover, it gives the designer a very strong signal when he has arrived at the "best" approximation (he will have had to define a criterion for "best", however). Finally, it will tell him whether he can substantially improve the approximation by asking for a different level of magnitude or an added increment of phase shift which is linear with frequency (different time delay).

3.3 Final Adjustments of the Positions of Poles and Zeros

The problem which must be solved more adequately in the final adjustment of the position of poles and zeros is the choice of $\Delta\sigma$'s and $\Delta\omega$'s in Eqs. 18 and 19. The means employed to prescribe the shifts must account for the effect of all shifts simultaneously.

At this point it is helpful to consider the derivative curves and to observe that changes of 10 to 20 percent in the pole positions cause rather small changes in the derivative. This fact indicates that in Eqs. 18 and 19 the derivatives may be considered constant for small changes. Equations 18 and 19

$$\Delta \log |F| \cong \frac{\partial \log |F|}{\partial \sigma_1} \Delta\sigma_1 + \frac{\partial \log |F|}{\partial \sigma_2} \Delta\sigma_2 + \frac{\partial \log |F|}{\partial \omega_2} \Delta\omega_2 \tag{18}$$

$$\Delta \text{ Arg } F \cong \frac{\partial \text{ Arg } F}{\partial \sigma_1} \Delta\sigma_1 + \frac{\partial \text{ Arg } F}{\partial \sigma_2} \Delta\sigma_2 + \frac{\partial \text{ Arg } F}{\partial \omega_2} \Delta\omega_2 \tag{19}$$

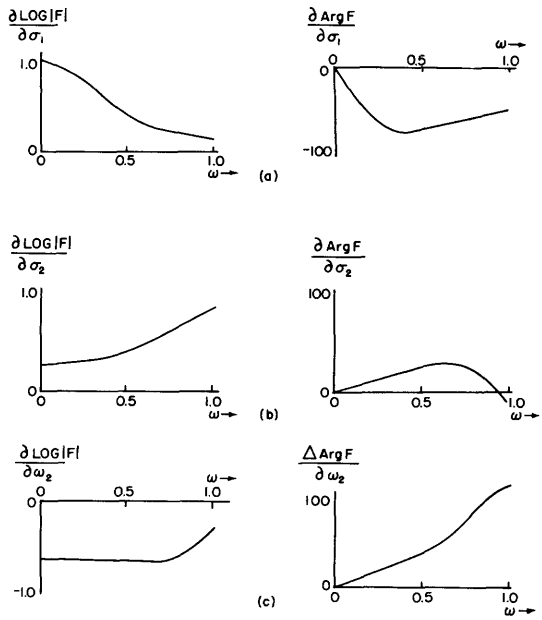

Fig. 16 Sketches of derivative functions for the pole positions of Fig. 15.

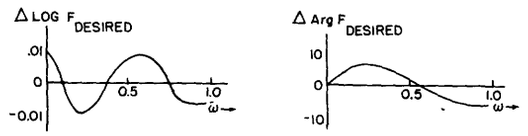

Fig. 17 Desired changes in magnitude and phase from Fig. 15.

suggest a mathematically convenient formulation of the problem of final adjustment which is most simply understood by reference to the problem of Fig. 15. In Fig. 16 are shown plots of the derivative functions corresponding to the critical frequency positions of Fig. 15. Figure 17 shows plots of the desired changes in the magnitude and phase from that illustrated in Fig. 15. (Note that a constant change in level or a linear change in argument may be prescribed in addition to the changes of Fig. 17.) The problem of the choice of $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\omega_2$ is the choice of these quantities such that, when the functions of (a) of Fig. 16 are multiplied by the proper $\Delta\sigma_1$, those of (b) are multiplied by the proper $\Delta\sigma_2$ and those of (c) are multiplied by the proper $\Delta\omega_2$ and the sums formed in accordance with Eqs. 18 and 19, these sums approximate the functions of Fig. 17 in the best possible manner. Clearly "the best possible manner" is in general not a perfect fit. One criterion of "the best possible manner" is that the integral of the square of the deviation subsequent to the chosen shifts be a minimum. The method of choosing $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\omega_2$ for such a criterion is well known. One forms a set of normal orthogonal functions from linear combinations of the derivative functions and evaluates the optimum $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\omega_2$ by the evaluation of appropriate integrals. However, the formation of the normal orthogonal functions is so laborious because of the evaluation of the integrals that the method is entirely impractical.

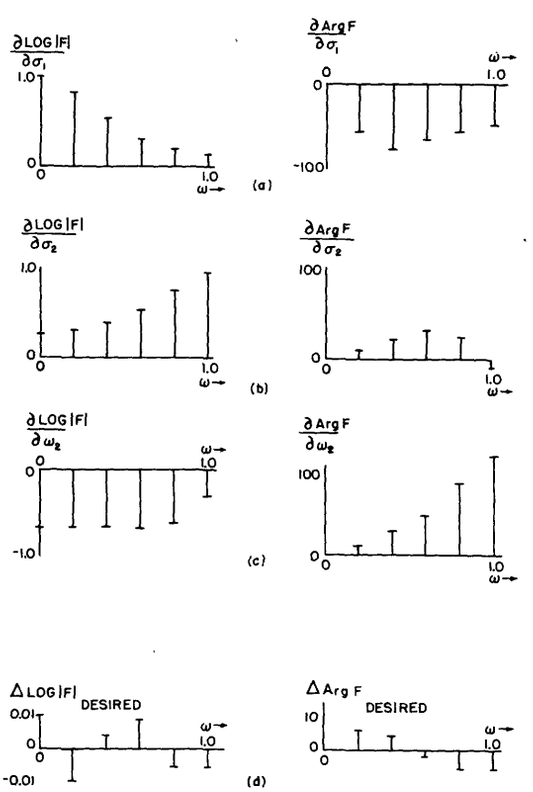

Fig. 18 Values of functions of Figs. 16 and 17 at selected matching points.

An alternate method which is based on the same kind of approach but is far simpler is that of approximating at a set of points rather than at every point from $\omega = 0$ to $\omega = 1$. The labor involved in forming the normal orthogonal functions arises because of the fact that one uses an infinite set of samples to get an average. It is easy to see by consideration of Figs. 16 and 17 that, if one finds the sums of Eqs. 18 and 19 matching well with Fig. 17 at five or six points on the range from $\omega = 0$ to $\omega = 1$, the match will be good elsewhere as the derivative functions and the desired deviations are smooth. The labor involved in matching at eleven points (six on the magnitude curve and five on the phase curve) is very much reduced. To formulate the procedure of solution suppose a set of points have been chosen from the range $\omega = 0$ to 1 and the values of the functions of Figs. 16 and 17 are indicated at those

points as shown in Fig. 18. One is under no compulsion to choose the same matching frequencies for the magnitude and argument curves though he is choosing $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\omega_2$ which apply to both at the same time. Ordinarily it is expedient to choose matching frequencies where the worst deviations occur or where the derivative curves are large. In any event one guides the choice of matching frequencies to fit the needs of the particular case.

The ordinates of Fig. 18 are of two different kinds; some are in terms of logarithms to the base 10 and some are in terms of degrees. Before proceeding one must put all of the ordinates on a common basis. In the problem at hand a deviation of 0.05 in $\log|F|$ will be said to be as serious as a deviation of 10 degrees in the argument. Accordingly all quantities referring to arguments should be divided by 200 to have the quantities on a uniform basis. One appreciates at this point that if the magnitude or phase at a particular frequency is more important than others it may be given a heavier weight. If in Fig. 18 the phase shift at $\omega = 1$ is to be more closely controlled than at other frequencies the ordinates of argument curves there might be divided by 100 instead of 200, for instance.

A compact notation for the functions of Fig. 18 (after weighting) is vector notation. The ordinates of the functions define vectors of eleven dimensions. Call F_D a vector in 11-space associated with the weighted ordinates of (d), $F_{\sigma 1}$ a vector in 11-space associated with the ordinates of (a), $F_{\sigma 2}$ a vector associated with (b) and $F_{\omega 2}$ a vector associated with (c). For the situation illustrated in Fig. 18, one has the following vectors.

$$\left.\begin{aligned}
F_D &= +0.010,\ -0.010,\ +0.004,\ +0.008,\ -0.006,\ -0.006,\ +0.030,\ +0.024,\ -0.004,\ -0.027,\ -0.027 \\
F_{\sigma 1} &= +1.08,\ +0.87,\ +0.54,\ +0.33,\ +0.22,\ +0.15,\ -0.29,\ -0.37,\ -0.33,\ -0.29,\ -0.25 \\
F_{\sigma 2} &= +0.29,\ +0.33,\ +0.40,\ +0.56,\ +0.76,\ +0.98,\ +0.059,\ +0.11,\ +0.17,\ +0.13,\ -0.055 \\
F_{\omega 2} &= -0.69,\ -0.69,\ -0.71,\ -0.73,\ -0.64,\ -0.33,\ +0.065,\ +0.15,\ +0.26,\ +0.44,\ +0.60
\end{aligned}\right\} \quad (43)$$

The problem to be solved, expressed in vector terminology, is to select $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\omega_2$ such that

$$F_D \cong \Delta\sigma_1 F_{\sigma 1} + \Delta\sigma_2 F_{\sigma 2} + \Delta\omega_2 F_{\omega 2} \ . \quad (44)$$

The Δ's found to solve the problem stated in vector terms are appropriate to be used as shifts of pole positions. Clearly the approximation can not be made exact in the general case when the number of dimensions is greater than the number of adjustable quantities as is the case here, eleven being larger than three. Accordingly, one must establish a measure of approximation and on the basis of it choose the Δ's. A mathematically convenient and physically practical approximation is that in which the sum of squares of deviations at the points considered is minimized. In vector terms the Δ's are chosen such that

$$\left[F_D - \Delta\sigma_1 F_{\sigma 1} + \Delta\sigma_2 F_{\sigma 2} + \Delta\omega_2 F_{\omega 2}\right] \cdot \left[\text{Same}\right] \text{ is minimized.} \quad (45)$$

The solution for the Δ's of Eq. 44 is most conveniently done by choosing a set of normal orthogonal vectors which are linearly dependent with $F_{\sigma 1}$, $F_{\sigma 2}$ and $F_{\omega 2}$ and first

approximating F_D with them. Finally one is led to the Δ's of Eq. 44. A simple set of normal orthogonal vectors is:

$$\left.\begin{aligned} F_{n1} &= a_{11} F_{\sigma 1} \\ F_{n2} &= a_{21} F_{\sigma 1} + a_{22} F_{\sigma 2} \\ F_{n3} &= a_{31} F_{\sigma 1} + a_{32} F_{\sigma 2} + a_{33} F_{\omega 2} \end{aligned}\right\} . \tag{46}$$

The a's are chosen to fulfill the following relationships:

$$\left.\begin{aligned} F_{n1} \cdot F_{n1} &= 1 \\ F_{n1} \cdot F_{n2} &= 0 \qquad F_{n2} \cdot F_{n2} = 1 \\ F_{n1} \cdot F_{n3} &= 0 \qquad F_{n2} \cdot F_{n3} = 0 \qquad F_{n3} \cdot F_{n3} = 1 \end{aligned}\right\} . \tag{47}$$

The solution of the equations above, or similar equations for a case of any number of poles and zeros, is fortunately simple. The first row of Eqs. 47 involves only a_{11} and one solves for a_{11} first. The second row (using the numerical values of a_{11}) involves a_{21} and a_{22}. The third row involves (using numerical values for a_{11}, a_{21} and a_{22}) a_{31}, a_{32} and a_{33}. It is simplest to solve the equations from top to bottom. If one considers any row of equations of Eqs. 47, he finds that only the last equation is quadratic. The equations preceding it in the row are linear. Further they can be solved one at a time. For the third row for instance, the first and second equations are of the form

$$\left.\begin{aligned} m_{11} a_{31} + m_{12} a_{32} + m_{13} a_{33} &= 0 \\ m_{22} a_{32} + m_{23} a_{32} &= 0 \end{aligned}\right\} . \tag{48}$$

Hence one can solve for a_{32} in terms of a_{33} and then for a_{31} in terms of a_{33}. Finally the third equation in the row is quadratic but is in terms of a_{33} only. By considering the equations in the proper order one completely avoids the solution of general simultaneous equations.

The simplicity indicated above applies to the determination of orthogonal vectors regardless of the number. The degeneracy of the equations of the form of Eq. 48 arises from the special manner in which the orthogonal vectors are obtained.

For the illustrative example at hand, one has the following:

$$\left.\begin{aligned} a_{11} &= 0.5902 \\ a_{21} &= -0.3608 \qquad a_{22} = 0.748 \\ a_{31} &= 1.063 \qquad a_{32} = 0.6356 \qquad a_{33} = 1.463 \end{aligned}\right\} . \tag{49}$$

$$\left.\begin{aligned} F_{n1} &= +0.6374,\ +0.5135,\ +0.3187,\ +0.1948,\ +0.1298,\ +0.0885,\ -0.1712,\ -0.2184,\ -0.1948,\ -0.1712,\ -0.1476 \\ F_{n2} &= -0.1144,\ -0.0200,\ +0.1335,\ +0.3177,\ +0.5010,\ +0.6870,\ +0.1331,\ +0.1958,\ +0.2248,\ +0.1862,\ +0.0355 \\ F_{n3} &= +0.3228,\ +0.1250,\ -0.2103,\ -0.3609,\ -0.2191,\ +0.2997,\ -0.1756,\ -0.1038,\ +0.1376,\ +0.4180,\ +0.5768 \end{aligned}\right\} \quad (50)$$

Once the normal orthogonal vectors are selected, one approximates F_D in terms of them.

$$F_D \cong c_1 F_{n1} + c_2 F_{n2} + c_3 F_{n3} \quad . \tag{51}$$

This approximation involves choice of the c's such that

$$\left[F_D - (c_1 F_{n1} + c_2 F_{n2} + c_3 F_{n3})\right] \cdot \left[\text{Same}\right] \tag{52}$$

is a minimum. This problem is particularly easy to solve since the vectors used are an orthogonal set. Once the c's of Eq. 51 are selected one can obtain directly to the optimum values for $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\omega_2$. Considering Quantity 52 and observing the orthogonality of the vectors one has

$$\left[\quad\right] \cdot \left[\quad\right] = F_D \cdot F_D - 2c_1 F_{n1} \cdot F_D + c_1^2 - 2c_2 F_{n2} \cdot F_D + c_2^2$$

$$- 2c_3 F_{n3} \cdot F_D + c_3^2 \quad . \tag{53}$$

To minimize Quantity 52 one determines the c's by setting the partial derivatives with respect to σ_1, σ_2 and ω_2 of Eq. 53 equal to zero. This step gives

$$\left.\begin{aligned} \frac{\partial}{\partial c_1}\left[\quad\right] \cdot \left[\quad\right] &= 2c_1 - 2F_{n1} \cdot F_D = 0 \\ \frac{\partial}{\partial c_2}\left[\quad\right] \cdot \left[\quad\right] &= 2c_2 - 2F_{n2} \cdot F_D = 0 \\ \frac{\partial}{\partial c_3}\left[\quad\right] \cdot \left[\quad\right] &= 2c_3 - 2F_{n3} \cdot F_D = 0 \end{aligned}\right\} \quad . \tag{54}$$

One observes that the second derivatives of Eqs. 54 with respect to the c's give 2 and accordingly c's chosen to satisfy Eqs. 54 give a minimum of Quantity 52. Consequently,

$$\left.\begin{aligned} c_1 &= F_{n1} \cdot F_D \\ c_2 &= F_{n2} \cdot F_D \\ c_3 &= F_{n3} \cdot F_D \end{aligned}\right\} \quad . \tag{55}$$

Finally, Eqs. 55 lead to the optimum pole shifts indicated in Eq. 56.

$$\left.\begin{aligned} \Delta\sigma_1 &= c_1 a_{11} + c_2 a_{21} + c_3 a_{31} \\ \Delta\sigma_2 &= \qquad c_2 a_{22} + c_3 a_{32} \\ \Delta\omega_2 &= \qquad\qquad c_3 a_{33} \end{aligned}\right\} \tag{56}$$

In connection with the illustrative example being considered one has

$$\left.\begin{aligned} c_1 &= 0.00175, \quad \Delta\sigma_1 = -0.039, \quad \sigma_1 = -0.439 \\ c_2 &= 0.00086, \quad \Delta\sigma_2 = -0.024, \quad \sigma_2 = -0.474 \\ c_3 &= 0.0374, \quad \Delta\omega_2 = -0.055, \quad \omega_2 = 1.011 \end{aligned}\right\} . \tag{57}$$

A plot of the characteristics of the approximating function given by Eq. 57 is a part of Fig. 19. One can see by comparison of the 0's and the x's how the final adjustment has changed the characteristics. Quantity 52 changed from 0.0033 to 0.0021. In this case the improvement has been quite small; this was to be expected since the example is simple and the first stage of adjustment is very effective in simple cases. However, one now can state definitely that it is impossible to obtain a better approximation to the "o" characteristics of Fig. 19 by further shifts of the poles. It is necessary to recall that the measure of quality of the approximations has been chosen and the approximation arrived at is best in the sense of the chosen measure. Different weightings of magnitude and phase characteristics would have resulted in somewhat different approximations.

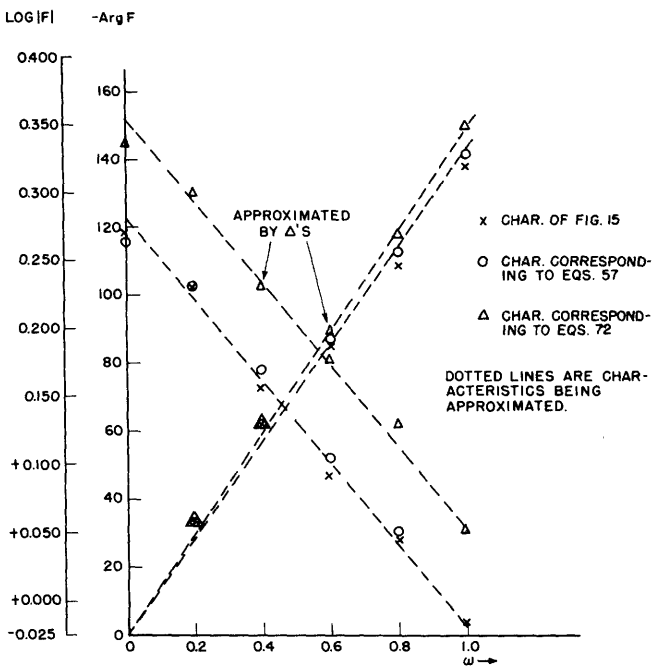

Fig. 19 Characteristics of approximating functions.

In cases where the first stage of approximation gives a far less satisfactory result than that of the accompanying illustrative example, it may be necessary to use more than one adjustment in the final state. This will be necessary only if the shifts required are so large that the derivative functions change a large amount. A simple method is given presently to compensate for changes in the derivative functions for large shifts. However, the compensation is not exact and one may wish to carry through the final stage of the adjustment procedure again with the correct values of the derivative functions to obtain the optimum result.

3.4 Further Topics in Connection with the Final Adjustment Procedure

Further useful conclusions can be drawn from the results presented above. The measure of seriousness of the deviations is $F_D \cdot F_D$ (or Quantity 52). As $F_D \cdot F_D$ (or Quantity 52) is decreased the quality of the approximation is increased. For the optimum choice of c's according to Eq. 55 one sees from Eq. 53 that the corresponding sum of deviations squared is reduced by the sum of the c's squared.

At times it may not be desirable to make the total shift which is specified by the c's. In the event that a shift is used which is k times the optimum shift one has the following sum of squares of deviations. (See Eqs. 53 and 55.)

$$c_{1f} = k\,c_1, \quad c_{2f} = k\,c_2, \quad \text{etc.} \tag{58}$$

$$\begin{aligned}\left[\ \ \right] \cdot \left[\ \ \right] &= F_D \cdot F_D - 2c_{1f}F_{n1} \cdot F_D + c_{1f}^2 - 2c_{2f}F_{n2} \cdot F_D + c_{2f}^2 - 2c_{3f}F_{n3} \cdot F_D + c_{3f}^2 \\ &= F_D \cdot F_D - 2c_1^2 k + k^2 c_1^2 - 2c_2^2 k + k^2 c_2^2 - 2c_3^2 k + k^2 c_3^2 \\ &= F_D \cdot F_D - (2k - k^2) \sum_{i=1}^{3} c_i^2 \quad .\end{aligned} \tag{59}$$

The second part of the last equality in Eq. 59 indicates the change in the square of deviations caused by the shift in poles. Consideration of Eq. 59 reveals that the maximum improvement occurs for k = 1 as it should. Any value of k between zero and two results in improvement. Larger values of k increase the square of deviations.

3.41 Compensation for Changes in the Derivative Functions

When one chooses the c's of Eq. 51, he assumes the F's are constant vectors. However when the c's lead to large changes in the pole or zero positions, one may find that the change in the approximation is not represented by the vector

$$c_1 F_{n1} + c_2 F_{n2} + \cdots c_m F_{nm}$$

but is instead represented by

$$c_1(F_{n1} + \Delta F_{n1}) + c_2(F_{n2} + \Delta F_{n2}) + \cdots c_m(F_{nm} + \Delta F_{nm})$$

since the value of the derivative functions has changed. If this is the case, the c's chosen do not lead to the best approximation and one needs to choose slightly different values. The problem then is to change the values of c to compensate for the change in the values of the F_n's. This means that one should consider

$$(c_1 + \Delta c_1)(F_{n1} + \Delta F_{n1}) + (c_2 + \Delta c_2)(F_{n2} + \Delta F_{n2}) + \cdots (c_m + \Delta c_m)(F_{nm} + \Delta F_{nm}) \cong F_D \quad . \tag{60}$$

In Eq. 60 $F_{ni} + \Delta F_{ni}$ is the mean value of the vector F_{ni} as the poles are shifted to the positions indicated by c_i's. Appropriate values of the Δc_i's may be obtained by neglecting the second order terms in Eq. 60. Accordingly,

$$\Delta c_1 F_{n1} + \Delta c_2 F_{n2} + \cdots \Delta c_m F_{nm} \cong F_D - \sum_{i=1}^{m} c_i F_{ni} - \sum_{i=1}^{m} c_i \Delta F_{ni} \quad . \tag{61}$$

Because of the orthogonality of the F_n's one can write immediately the appropriate values of Δc_j,

$$\Delta c_j = F_{nj} \cdot \left(- \sum_{i=1}^{m} c_i \Delta F_{ni} \right) \quad . \tag{62}$$

One recognizes that

$$\sum_{i=1}^{m} c_i \Delta F_{ni}$$

is the difference between the adjustment in characteristic actually obtained and that which would be obtained if the derivatives were constant. In difficult cases the compensation procedure just outlined reduces the number of stages of the final adjustment procedure and proportionately reduces the computational labor.

3.42 Choice of Level of Magnitude and Total Phase Shift for Best Approximation

In the choice of the desired changes in the magnitude and phase indicated in Fig. 17 an additional constant increase or decrease in level of magnitude or an added change in the phase shift which is linear with frequency may be applied without violating the requirements being approximated (Fig. 8). To approximate a different level of magnitude or a different linear phase shift is appropriate if these changed characteristics can be better approximated. The implicit relation between magnitude and phase actually enters this situation. For a prescribed level of magnitude (Fig. 8) over the frequency range of interest there is a most compatible linear phase shift over that range which can be best approximated.

To investigate the question of added magnitude it is appropriate to define F_m, a vector which added to F_D gives a unit added constant change in magnitude. For the illustrative example,

$$F_m = 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0. \tag{63}$$

F_p is a vector which added to F_D gives a unit added linear phase shift.

$$F_p = 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0 \quad . \tag{64}$$

The problem is to determine how much magnitude and how much phase shift to add. One should ask for the vector change $F_D + k_m F_m + k_p F_p$ in which the k's are chosen constants to make the best approximation possible. To study this problem it is first desirable to recognize that F_m and F_p are best approximated by

$$F_m \cong m_1 F_{n1} + m_2 F_{n2} + m_3 F_{n3} \tag{65}$$

and

$$F_p \cong p_1 F_{n1} + p_2 F_{n2} + p_3 F_{n3} \tag{66}$$

where

$$\left.\begin{aligned} m_1 &= F_{n1} \cdot F_m & p_1 &= F_{n1} \cdot F_p \\ m_2 &= F_{n2} \cdot F_m & p_2 &= F_{n2} \cdot F_p \\ m_3 &= F_{n3} \cdot F_m & p_3 &= F_{n3} \cdot F_p \end{aligned}\right\} . \tag{67}$$

For the present illustrative example

$$\begin{aligned} m_1 &= +1.88 & p_1 &= -0.523 \\ m_2 &= +1.50 & p_2 &= +0.427 \\ m_3 &= -0.0428 & p_3 &= +0.917 \end{aligned} \quad . \tag{68}$$

It is simple to verify that $F_D + k_m F_m + k_p F_p$ is best approximated by

$$\sum_{i=1}^{3} (c_i + k_m m_i + k_p p_i) F_{ni} \quad .$$

For any choice of k_m and k_p the sum of deviations squared (D_s) is

$$D_s = \left[F_p + k_m F_m + k_p F_p - \sum_{i=1}^{3} (c_i + k_m m_1 + k_p p_i) F_{ni} \right] \cdot \left[\text{Same} \right] \quad . \tag{69}$$

The variables in the above expression are k_m and k_p. Equation 69 contains a great deal of information. One can say, by considering it, how changes in magnitude and phase shift influence the quality of the approximation. For the illustrative example,

$$\frac{\partial D_s}{\partial k_m} = 0.375\, k_m + 0.782\, k_p - 0.00722 \tag{70}$$

$$\frac{\partial D_s}{\partial k_p} = 0.782\, k_m + 1.806\, k_p + 0.0106 \quad . \tag{71}$$

By setting Eqs. 70 and 71 to zero and solving one finds the optimum k_m and k_p. This

gives $k_m = 0.33$ and $k_p = -0.15$. However, these changes lead to shifts of the pole positions which are too great for the derivative functions to be approximately constant. If one used these shifts he would find the errors introduced by the changes in the derivative functions would consume the advantage indicated and it is necessary to proceed to the optimum by smaller steps. Less drastic changes may be made on the basis of Eqs. 70 and 71. One can find the most desirable changes in k_m for any specified change in k_p. As an example if one sets $k_p = -0.025$, he finds that the best k_m is 0.0715. This corresponds to the choice of poles indicated below.

$$\left.\begin{aligned} \sigma_1 &= -0.414 \\ \sigma_2 &= -0.418 \\ \omega_2 &= +0.973 \end{aligned}\right\} \qquad (72)$$

By integration of Eqs. 70 and 71 over the appropriate ranges one would predict the improvement in D_s to be about 0.0006. The characteristic corresponding to Eq. 72 is shown by Δ's in Fig. 19. The sum of squares of deviations for this choice is 0.0018 according to the calculation of the characteristics of Fig. 19 from the normalized curves in the appendix. The correlation between predicted and calculated deviations cannot be too great since small inaccuracies in reading the curves substantially influence the deviations which are themselves small.

The result obtained in Fig. 19 is not the absolute optimum. One can proceed to the optimum by further steps of the kind just completed, using the accurate values of the derivative functions for the pole positions of Eq. 72. In a practical problem one must balance the need for precision in the result against the labor involved in obtaining the result. If one is unable to obtain good elements whose behavior is close to ideal, it is as needless to carry the approximation problem to the finest point as it is to specify the dimensions of a house to thousandths of inches. As has been pointed out previously, the first stage of the adjustment procedure is sufficient for most problems; however, when one needs to obtain precise results, the method given is applicable as far as one needs to go.

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Appendix

Two rational functions are considered.

$$\text{(a)} \quad F_c = (\lambda - \lambda_c)(\lambda - \bar{\lambda}_c)$$

$$\lambda_c = \sigma_c + j\omega_c$$

$$\text{(b)} \quad F_r = \lambda - \sigma_r$$

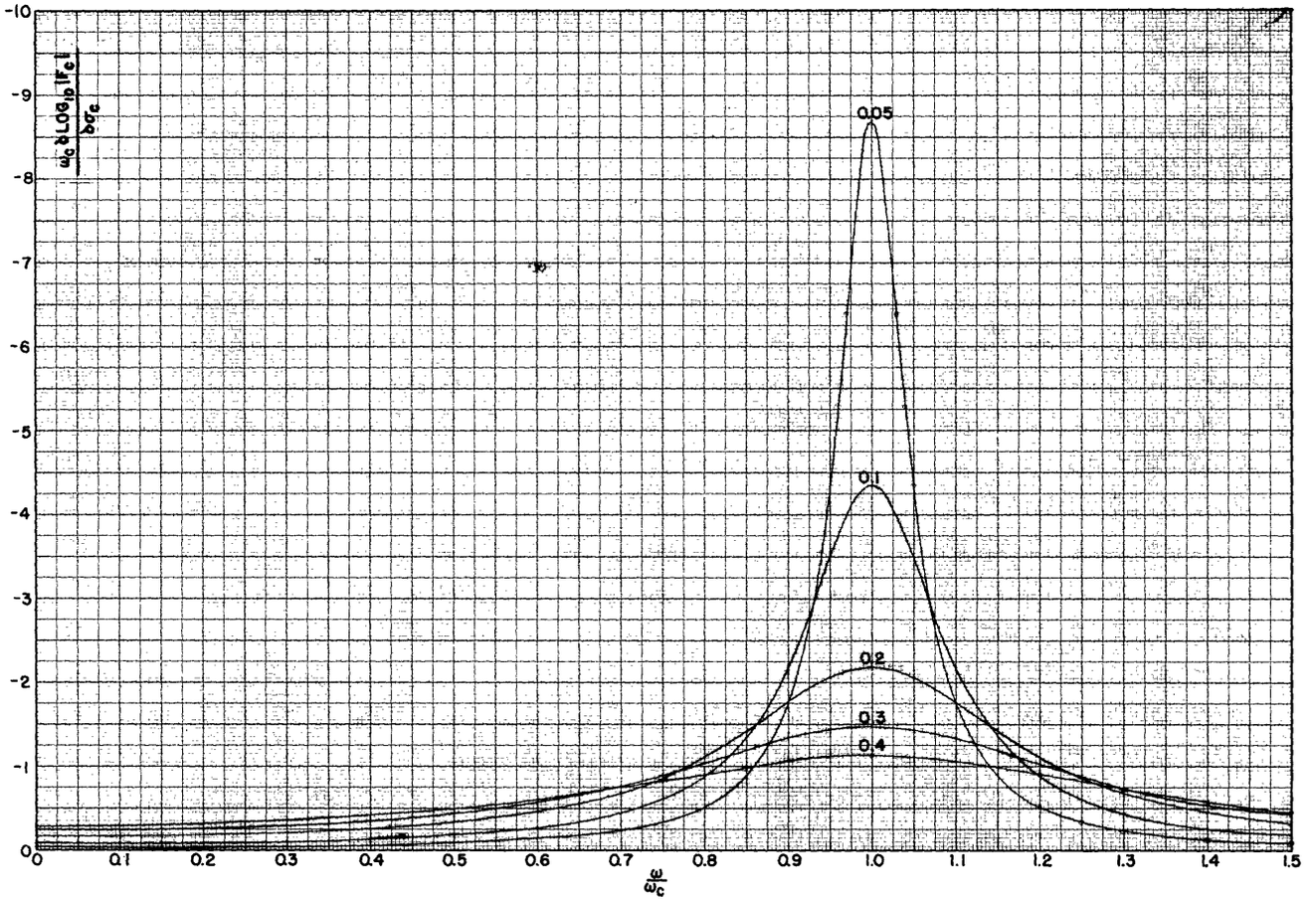
The functions $\log_{10}|F_c|$, $\log_{10}|F_r|$, Arg F_c, and Arg F_r (normalized) are shown in Set 3 and Set 4 plotted as a function of $\lambda = j\omega$. In connection with the function F_c, families of curves are given for a range of values of the parameter $-\sigma_c/\omega_c$. All logarithms are to the base 10, and all arguments are expressed in degrees.

The rates of change of the functions of Set 3 and Set 4 with shifts in the zero positions of F_c or F_r are given in Set 1 and Set 2.

Set 1

$-\omega_c \frac{\partial \log_{10}|F_c|}{\partial \sigma_c}$ and $\omega_c \frac{\partial \log_{10}|F_c|}{\partial \omega_c}$ as functions of $\frac{\omega}{\omega_c}$ for a range of values of the parameter $\frac{-\sigma_c}{\omega_c}$.

$-\sigma_r \frac{\partial \log_{10}|F_r|}{\partial \sigma_c}$ as a function of $\frac{\omega}{-\sigma_r}$.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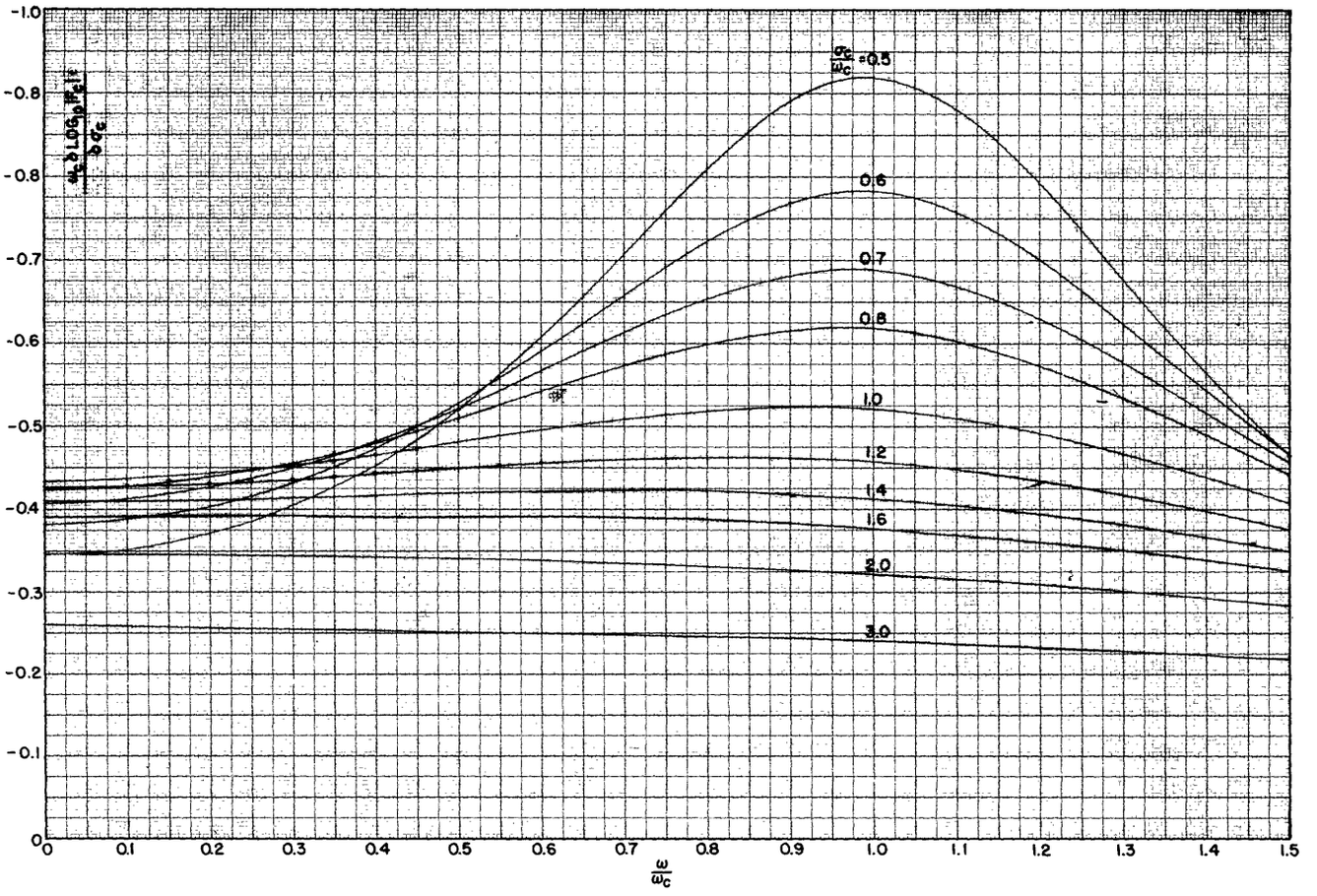
0.2
0.3
0.4
$\frac{\omega}{\omega_c}$


$\frac{\omega_c}{b\sigma_c} 2 \log_{10} |F_c|$
$\frac{\sigma_c}{\omega_c}$=0.5
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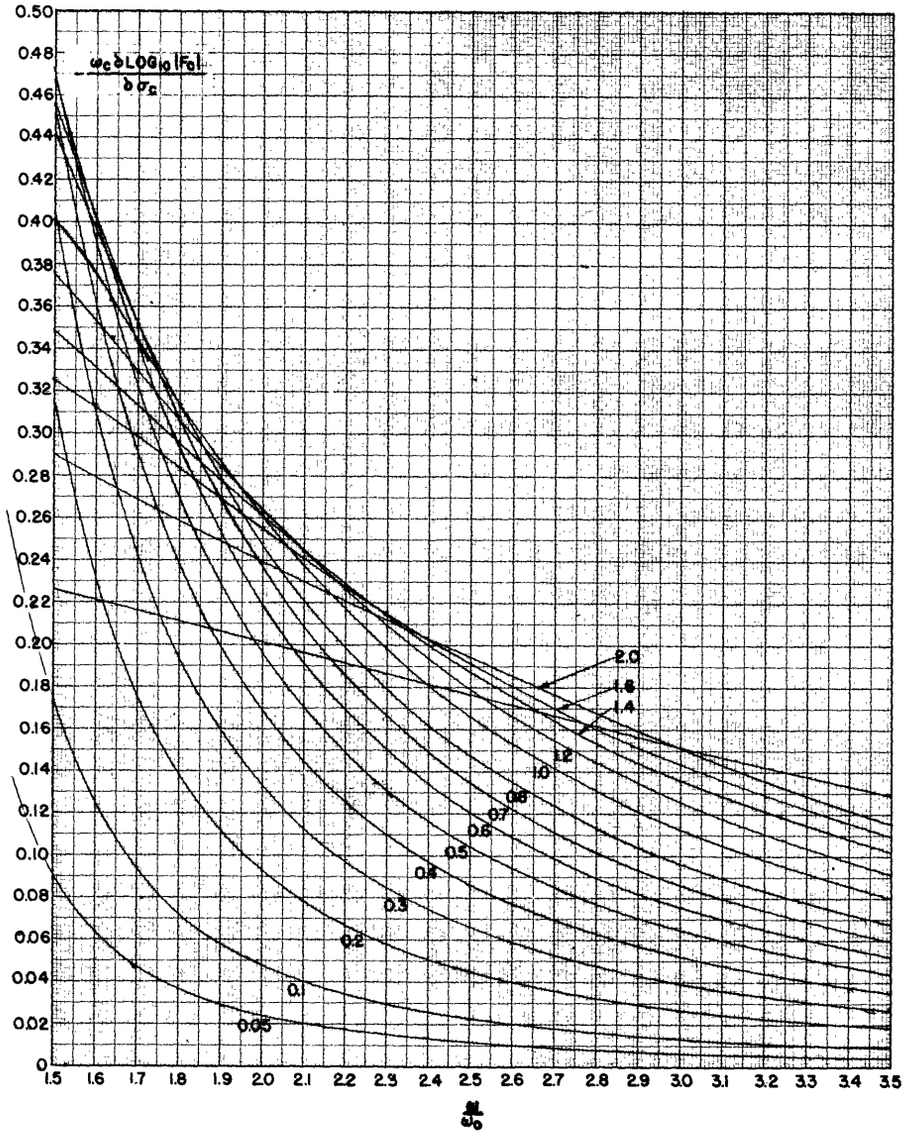
0.7
0.8
1.0
1.2
1.4
1.6
2.0
3.0
$\frac{\omega}{\omega_c}$


$\frac{\omega_c \delta \text{LOG}_{10}|F_0|}{\delta \sigma_\Sigma}$
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1.4
1.2
1.0
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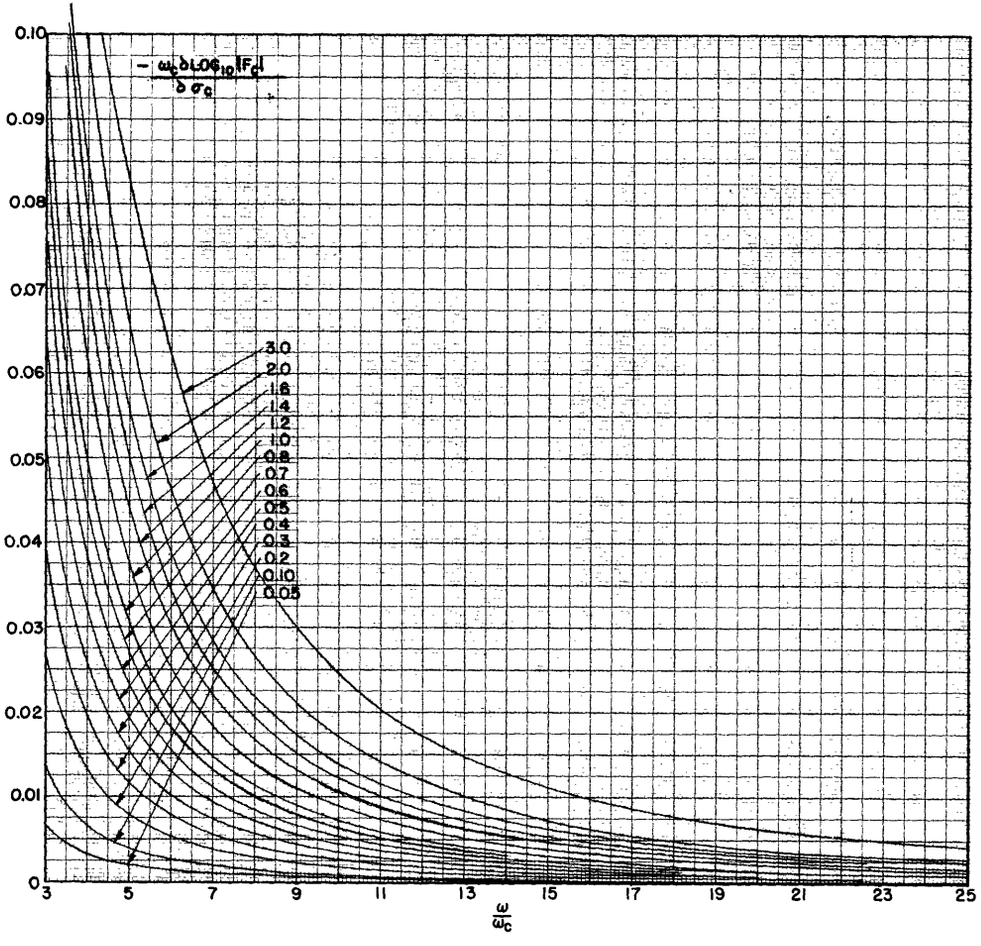
0.6
0.5
0.4
0.3
0.2
0.1
0.05
$\frac{\omega}{\omega_0}$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2.0
1.8
1.4
1.2
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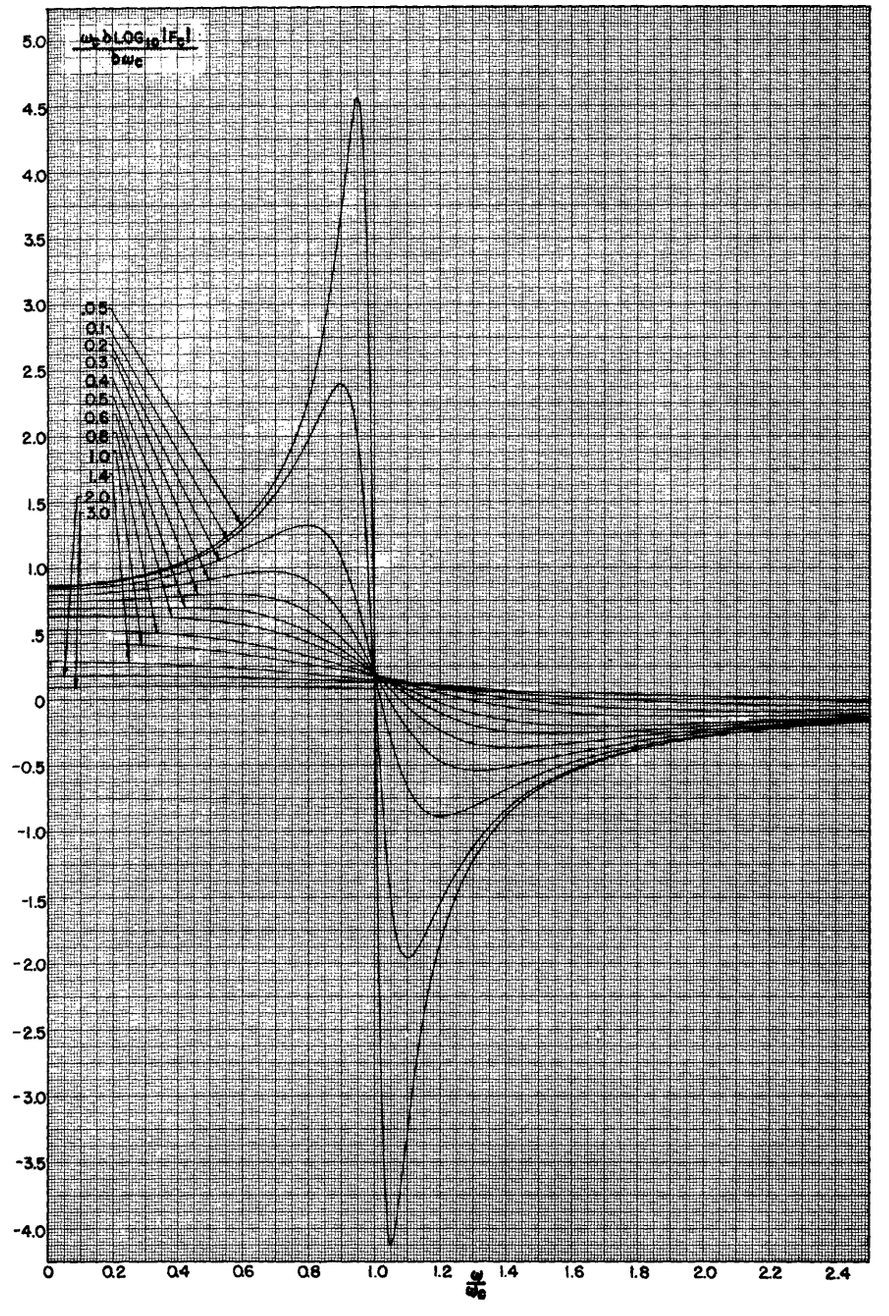
0.6
0.5
0.4
0.3
0.2
0.10
0.05
$\frac{\omega}{\omega_c}$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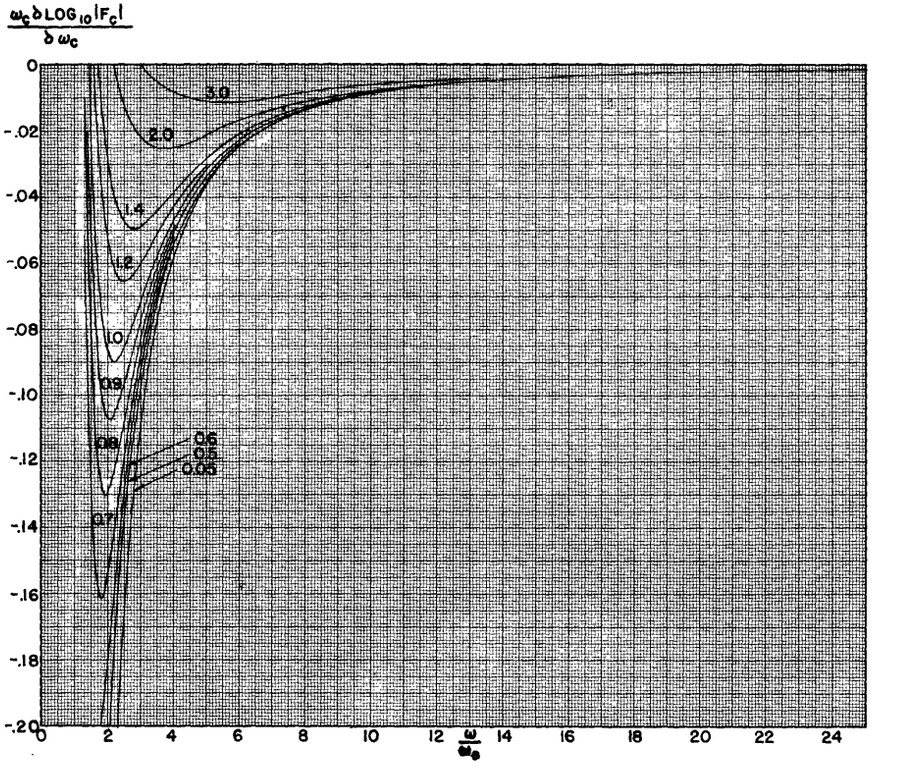
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
.05
0.1
0.2
0.3
0.4
0.5
0.6
0.8
1.0
1.4
2.0
3.0
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
$\frac{\omega}{\omega_c}$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-.06
-.08
-.10
-.12
-.14
-.16
-.18
-.20
3.0
2.0
1.4
1.2
1.0
0.9
0.8
0.7
0.6
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16
18
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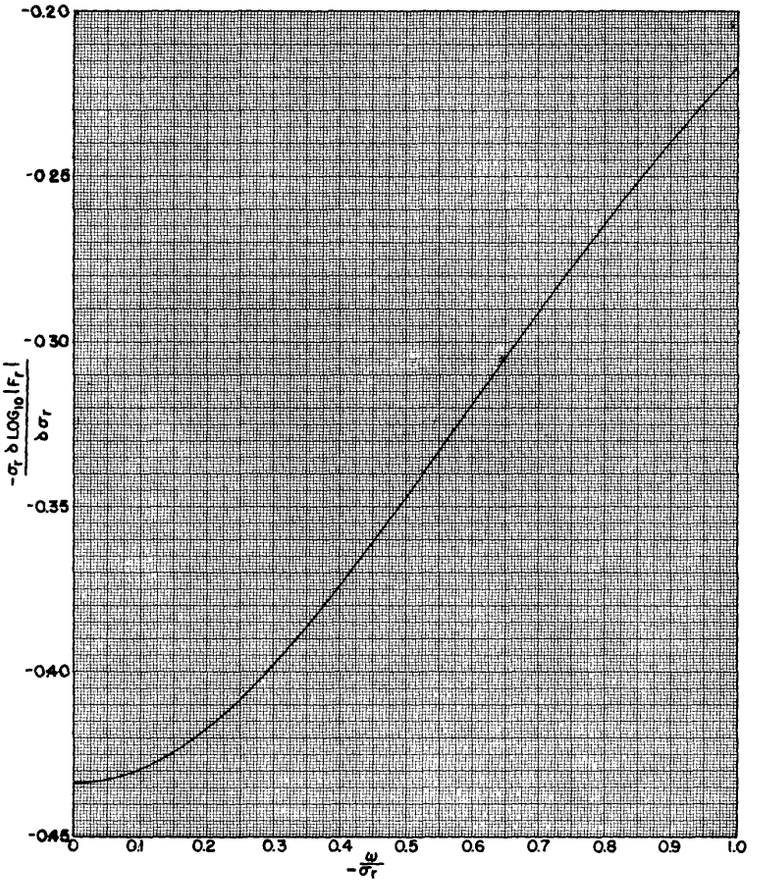
$\frac{\omega}{\omega_c}$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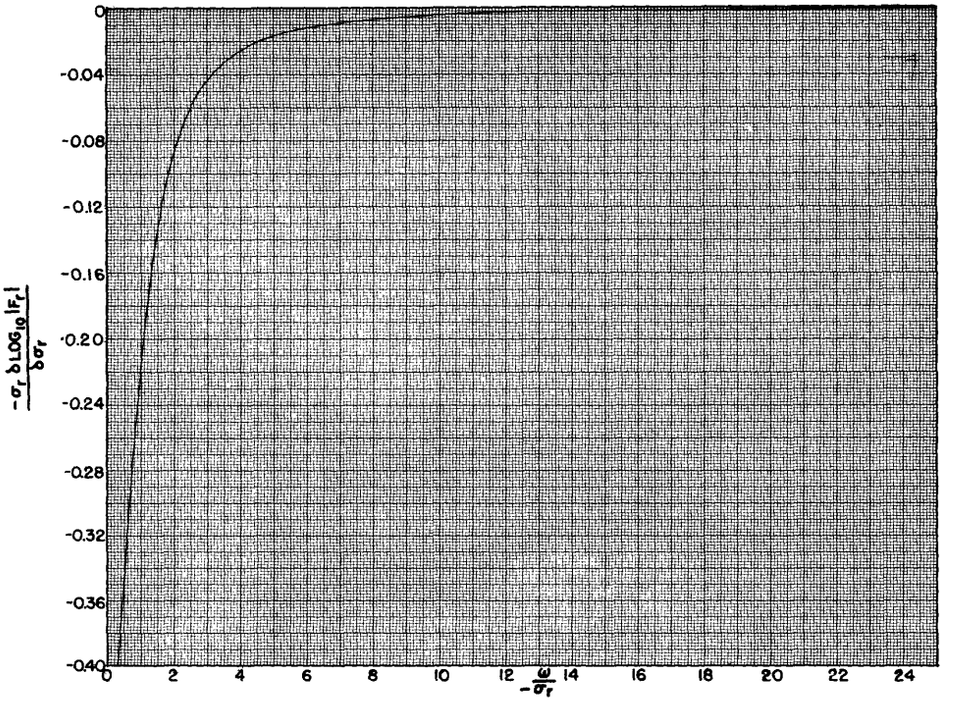
-0.25
-0.30
-0.35
-0.40
-0.45
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$-\frac{\sigma_r \partial \log_{10}|F_r|}{\partial \sigma_r}$
$-\frac{\omega}{\sigma_r}$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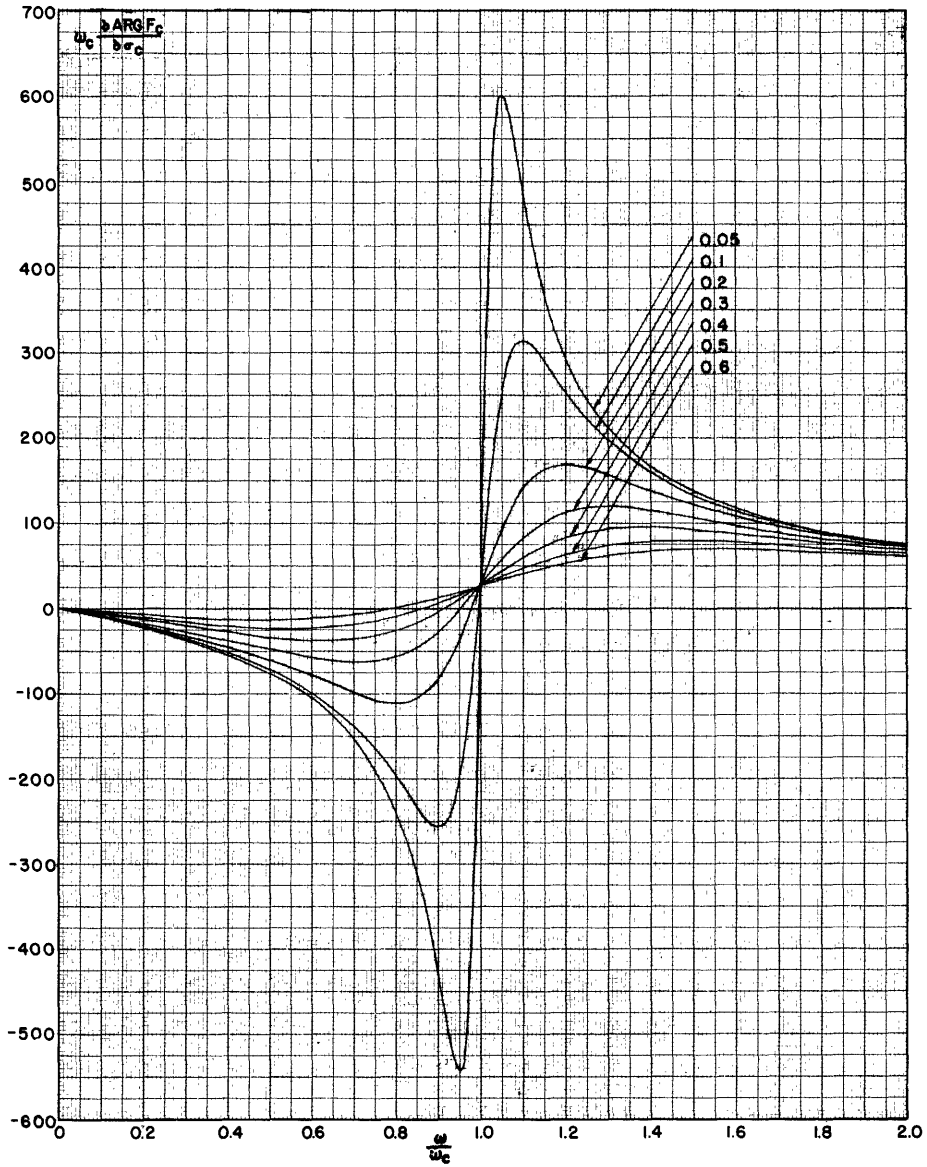
-0.32
-0.36
-0.40
0
2
4
6
8
10
12
14
16
18
20
22
24
$-\frac{\sigma_r \partial \log_{10}|F_r|}{\partial \sigma_r}$
$-\frac{\omega}{\sigma_r}$

Set 2

$\omega_c \frac{\partial \text{ Arg } F_c}{\partial \sigma_c}$ and $\omega_c \frac{\partial \text{ Arg } F_c}{\partial \omega_c}$ as functions of $\frac{\omega}{\omega_c}$ for a range of values

of the parameter $\frac{-\sigma_c}{\omega_c}$.

$-\sigma_r \frac{\partial \text{ Arg } F_r}{\partial \sigma_r}$ as a function of $\frac{\omega}{-\sigma_r}$.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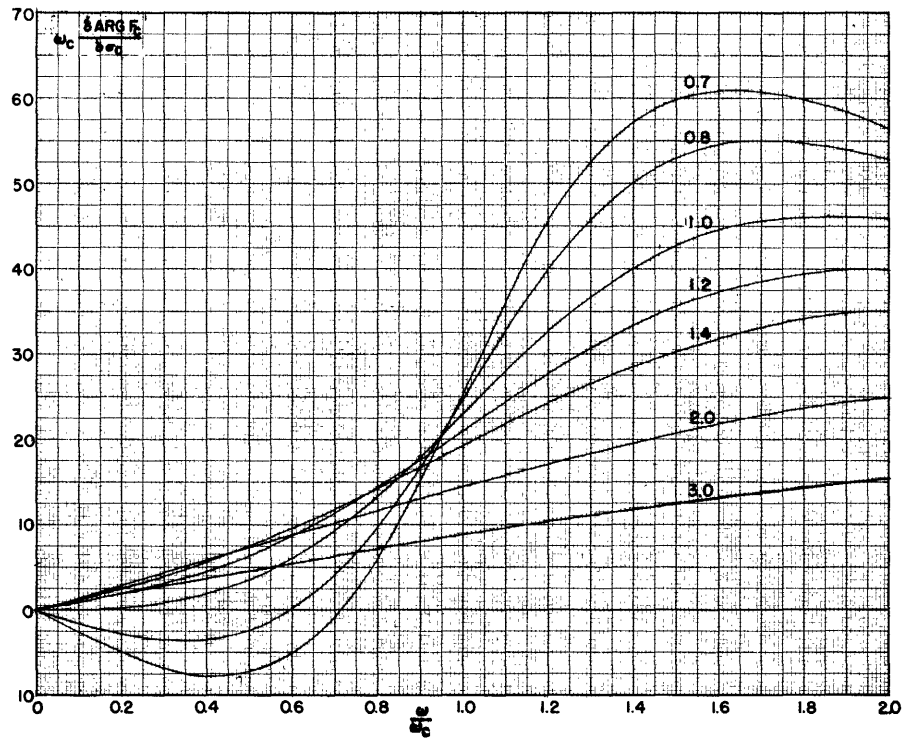
400
300
200
100
0
-100
-200
-300
-400
-500
-600
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
$\frac{\omega}{\omega_c}$
0.05
0.1
0.2
0.3
0.4
0.5
0.6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60
50
40
30
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10
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10
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
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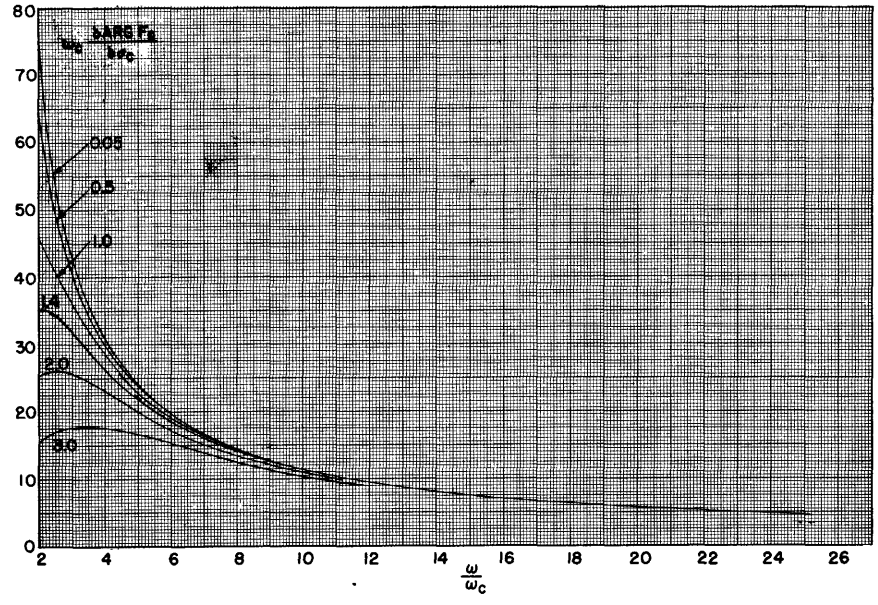
$\frac{\omega}{\omega_c}$
0.7
0.8
1.0
1.2
1.4
2.0
3.0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6
8
10
12
14
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18
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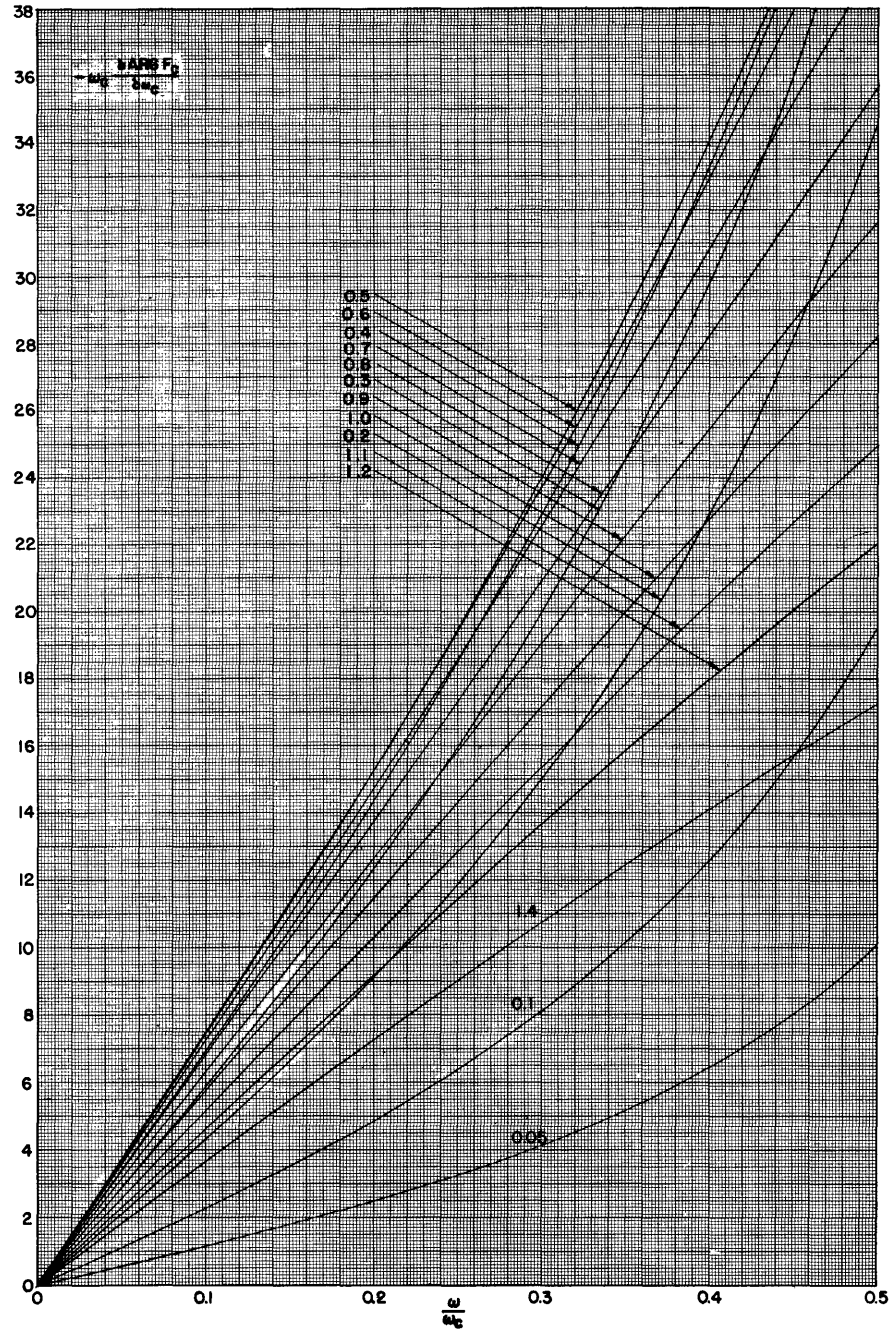
22
24
26
0.05
0.5
1.0
1.4
2.0
3.0
ω/ωc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0.3
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0.5
0.5
0.6
0.4
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0.9
1.0
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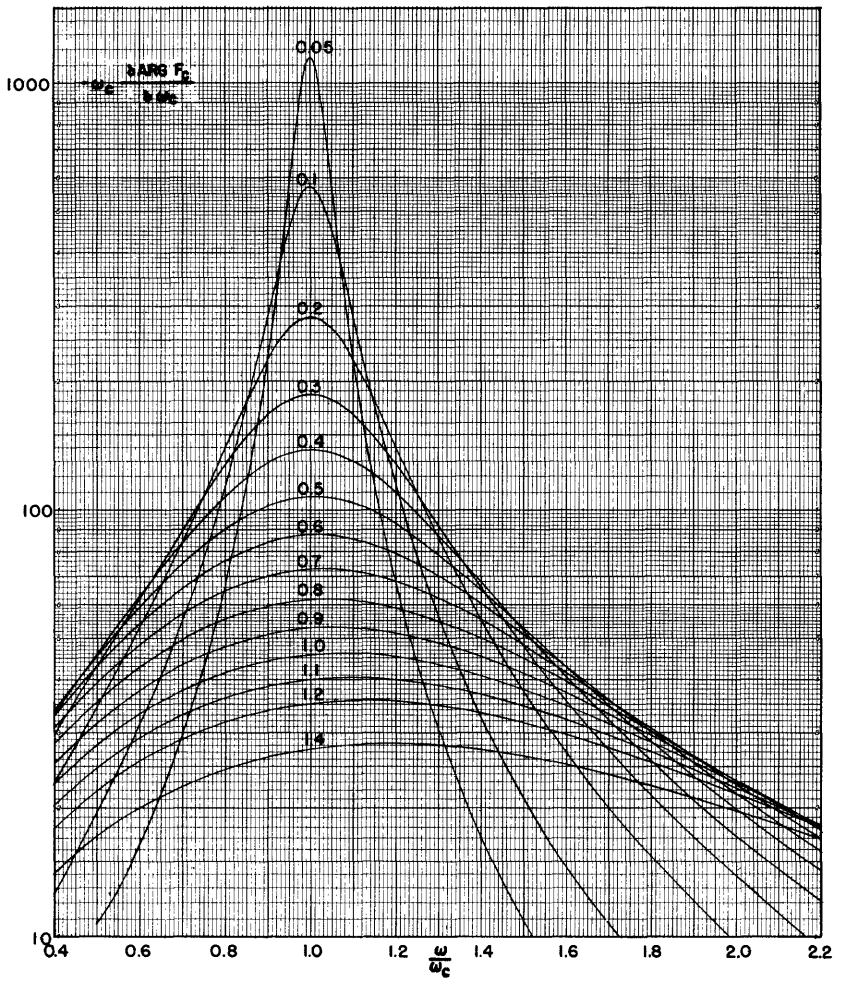
1.1
1.2
1.4
0.1
0.05
ω/ωc


$-\omega_c \frac{\delta ARG\ F_c}{\delta \omega_c}$
1000
100
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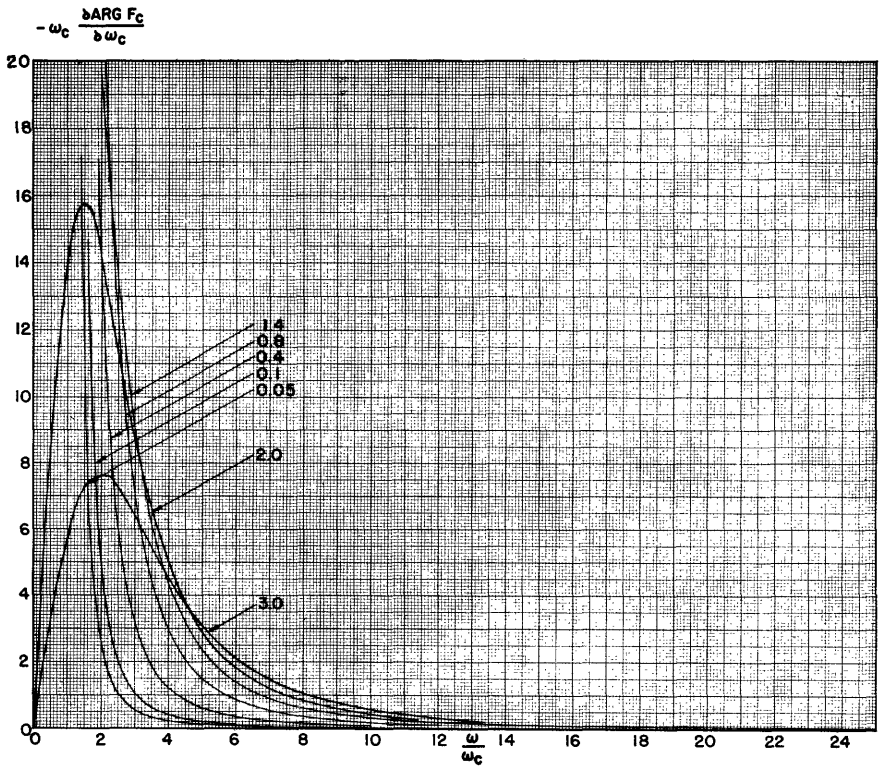
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.4
0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
$\frac{\omega}{\omega_c}$


$-\omega_c \frac{\delta ARG\ F_c}{\delta \omega_c}$
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18
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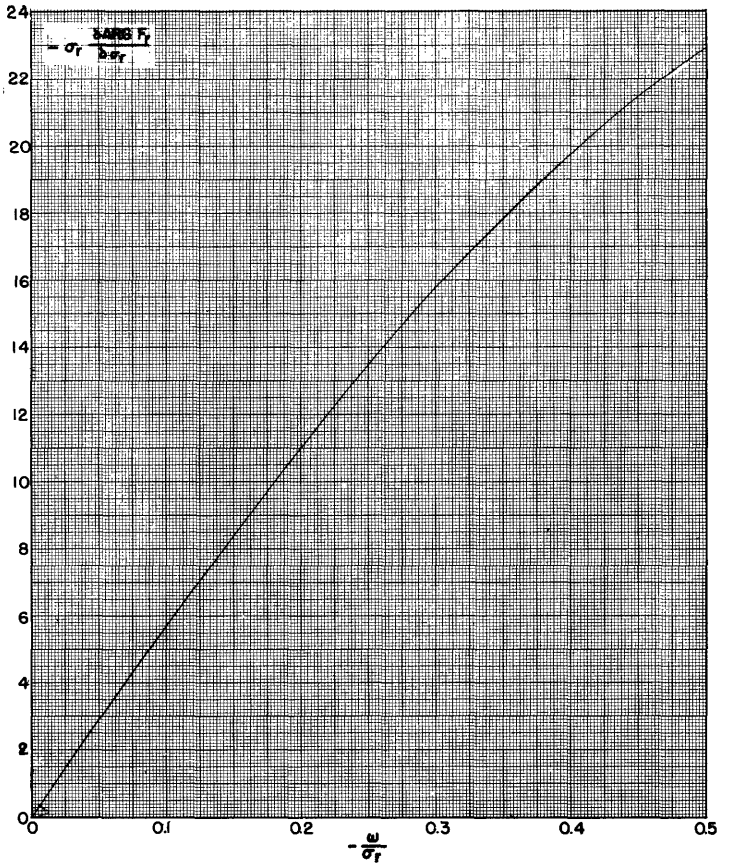
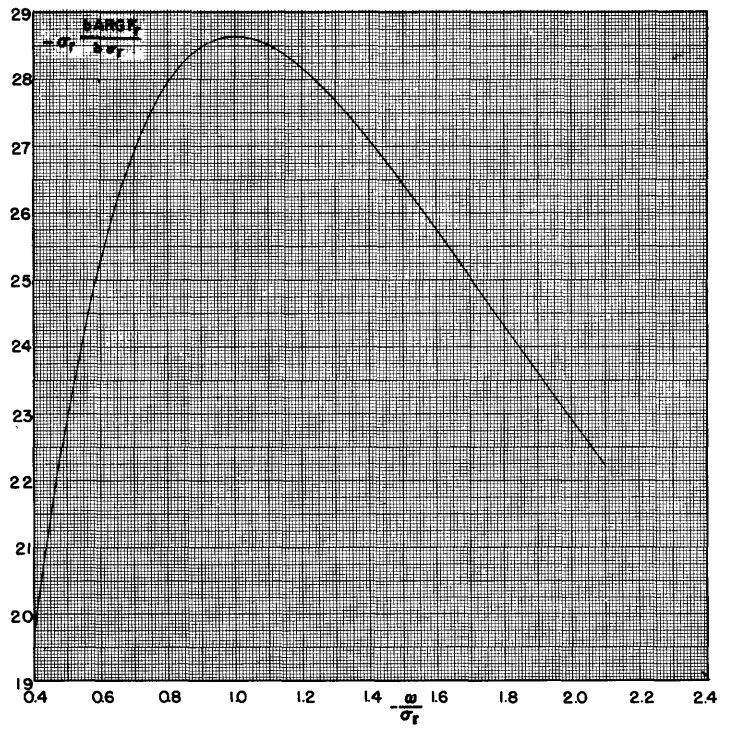
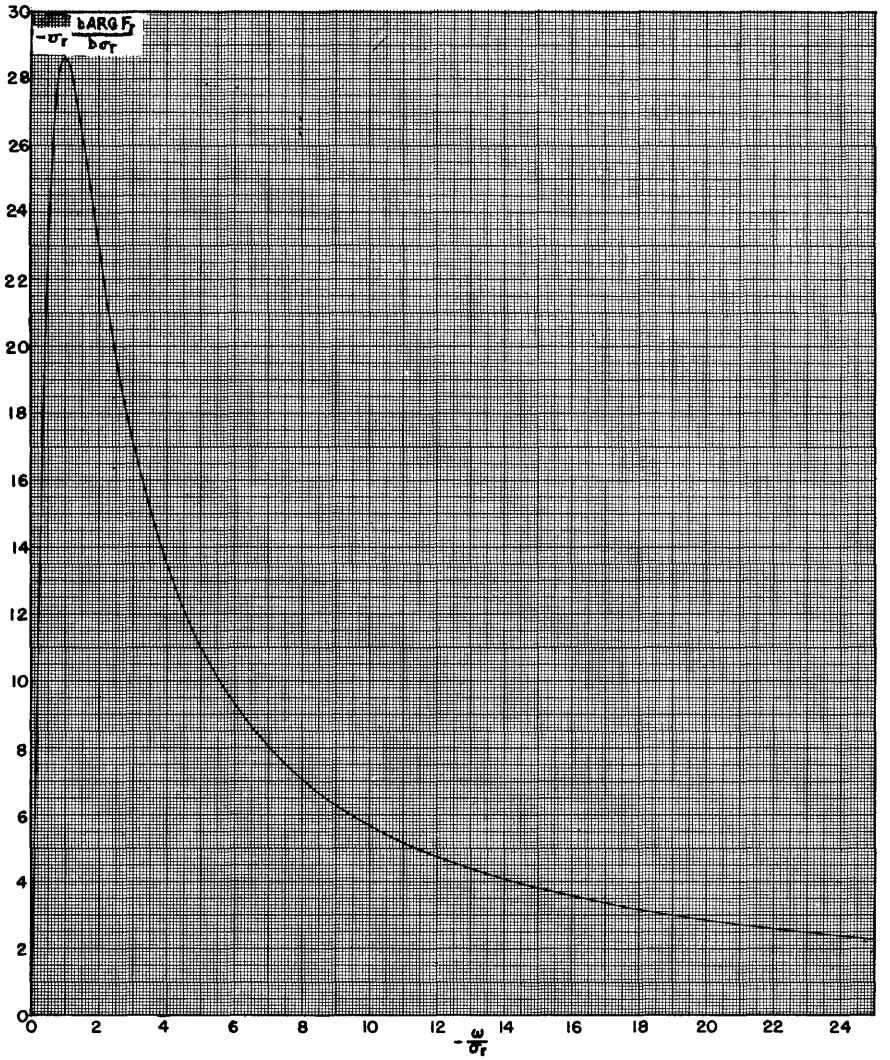
1.4
0.8
0.4
0.1
0.05
2.0
3.0
0 2 4 6 8 10 12 14 16 18 20 22 24
$\frac{\omega}{\omega_c}$


$-\sigma_r \frac{\partial \arg F_r}{\partial \sigma_r}$
24
22
20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
$-\frac{\omega}{\sigma_r}$

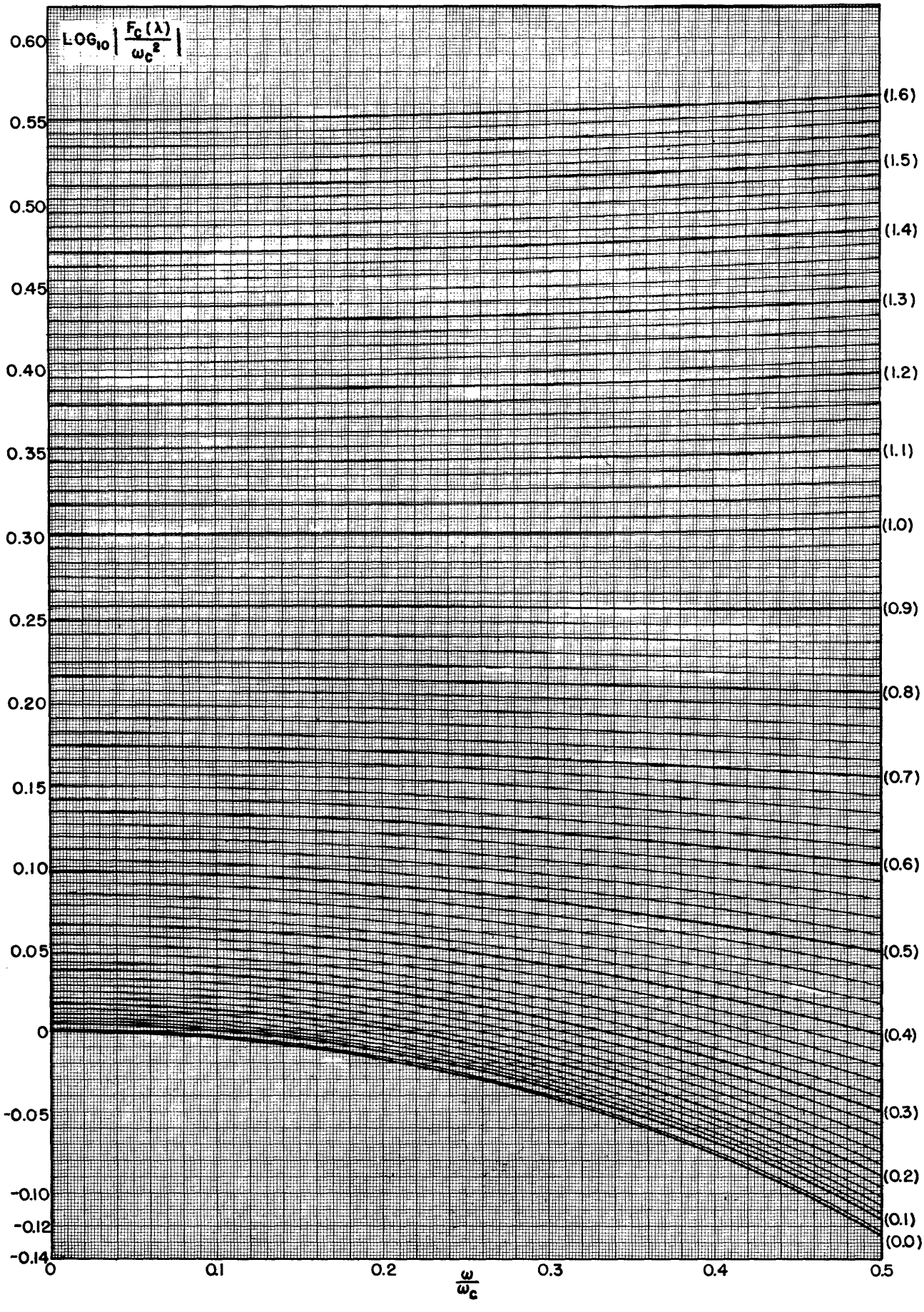

$-\sigma_r \frac{\partial \arg F_r}{\partial \sigma_r}$
29
28
27
26
25
24
23
22
21
20
19
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
$-\frac{\omega}{\sigma_r}$

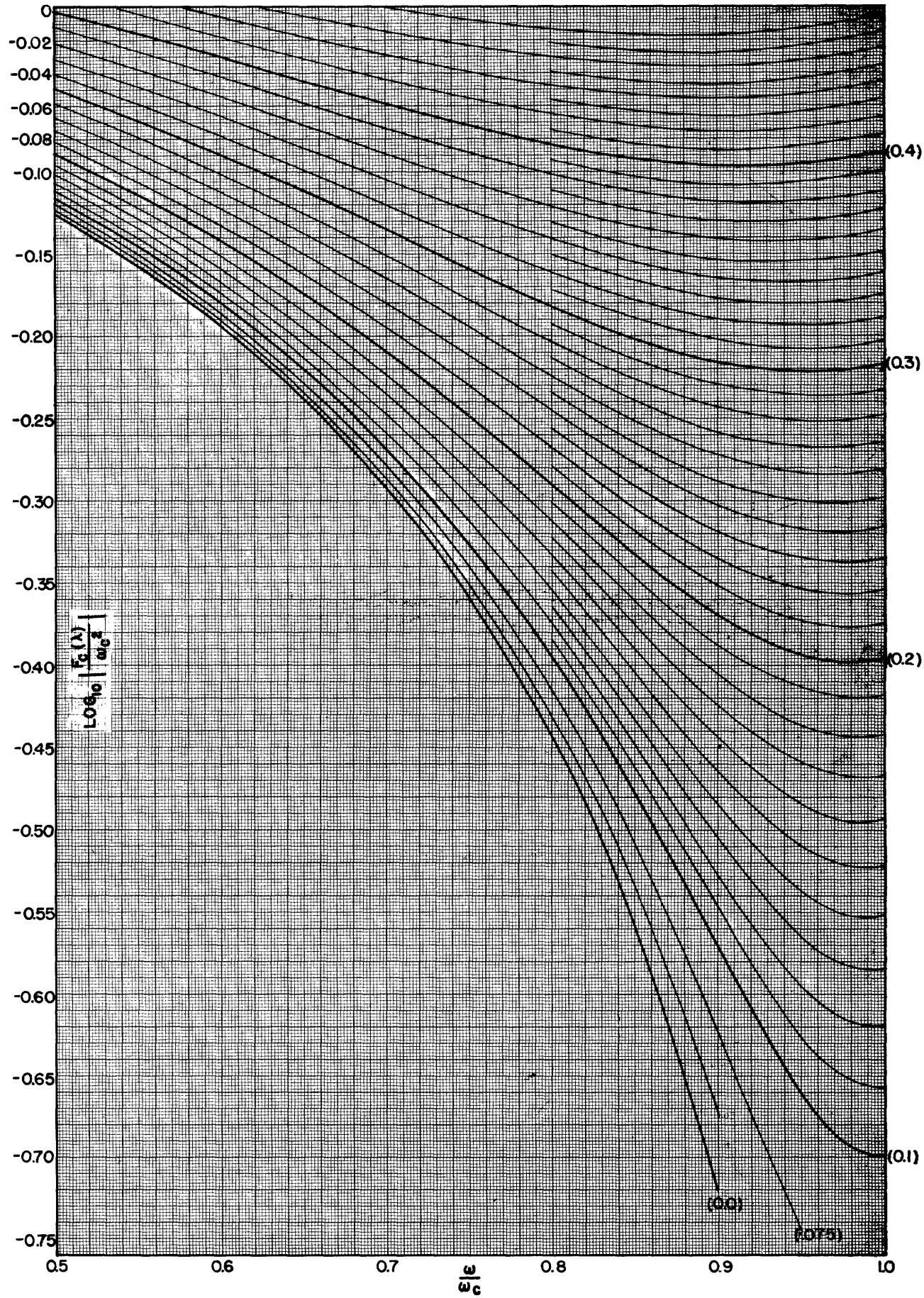

$-\sigma_r \frac{\partial \arg F_r}{\partial \sigma_r}$
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
0
2
4
6
8
10
12
14
16
18
20
22
24
$-\frac{\omega}{\sigma_r}$

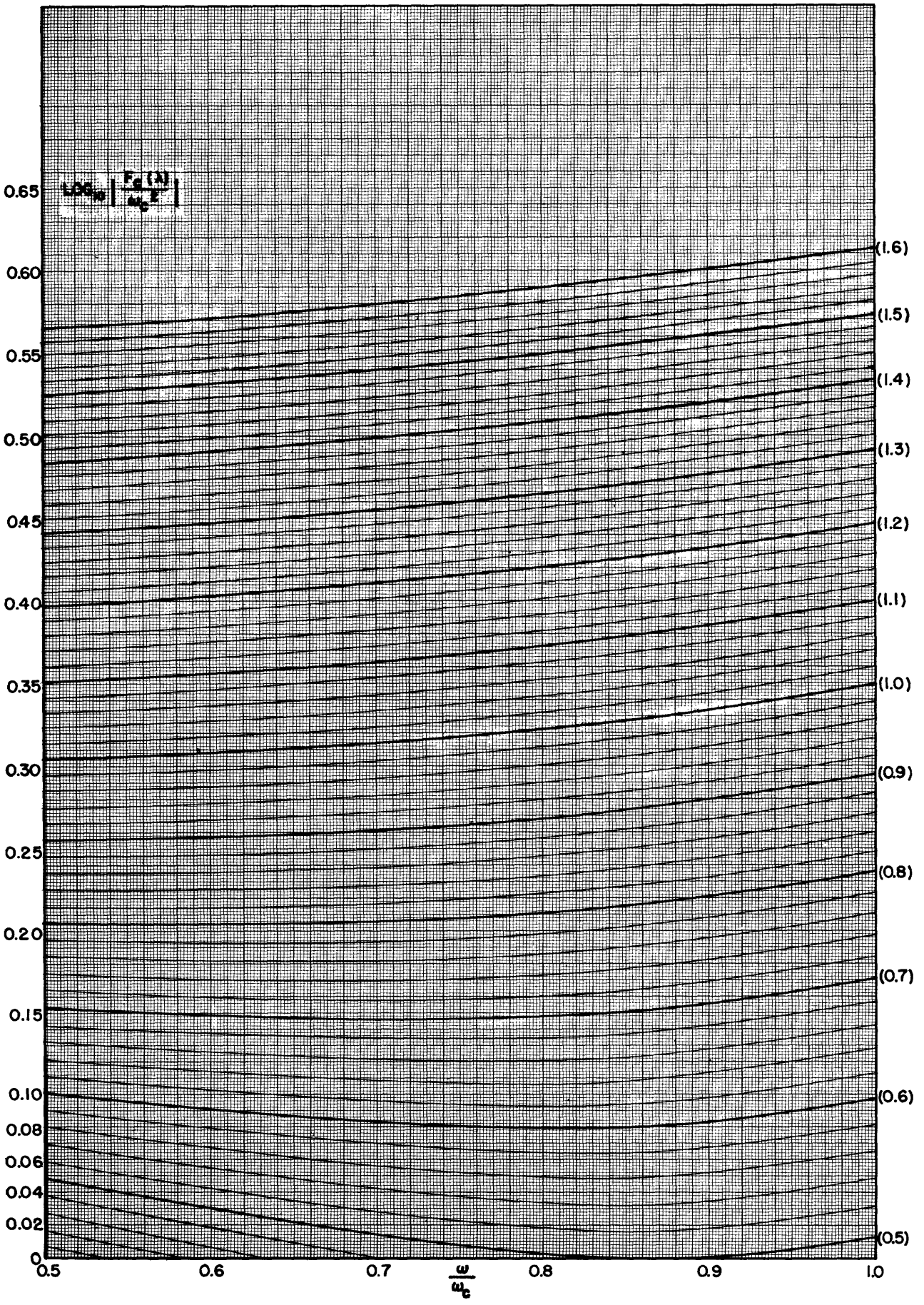
Set 3

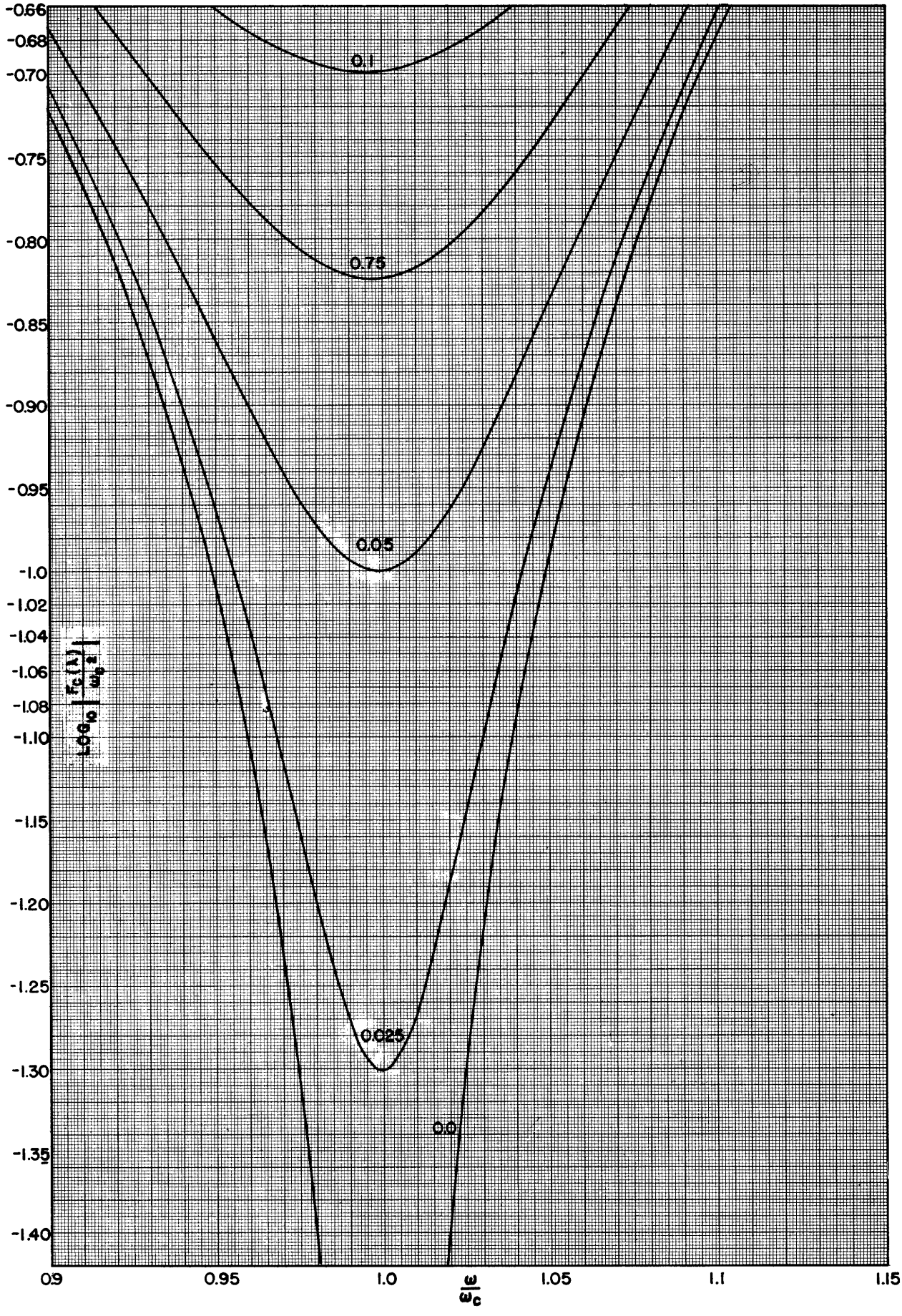
$\text{Log}_{10} \frac{|F_c|}{\omega_c^2}$ as a function of $\frac{\omega}{\omega_c}$ for a range of values of the parameter $\frac{-\sigma_c}{\omega_c}$.

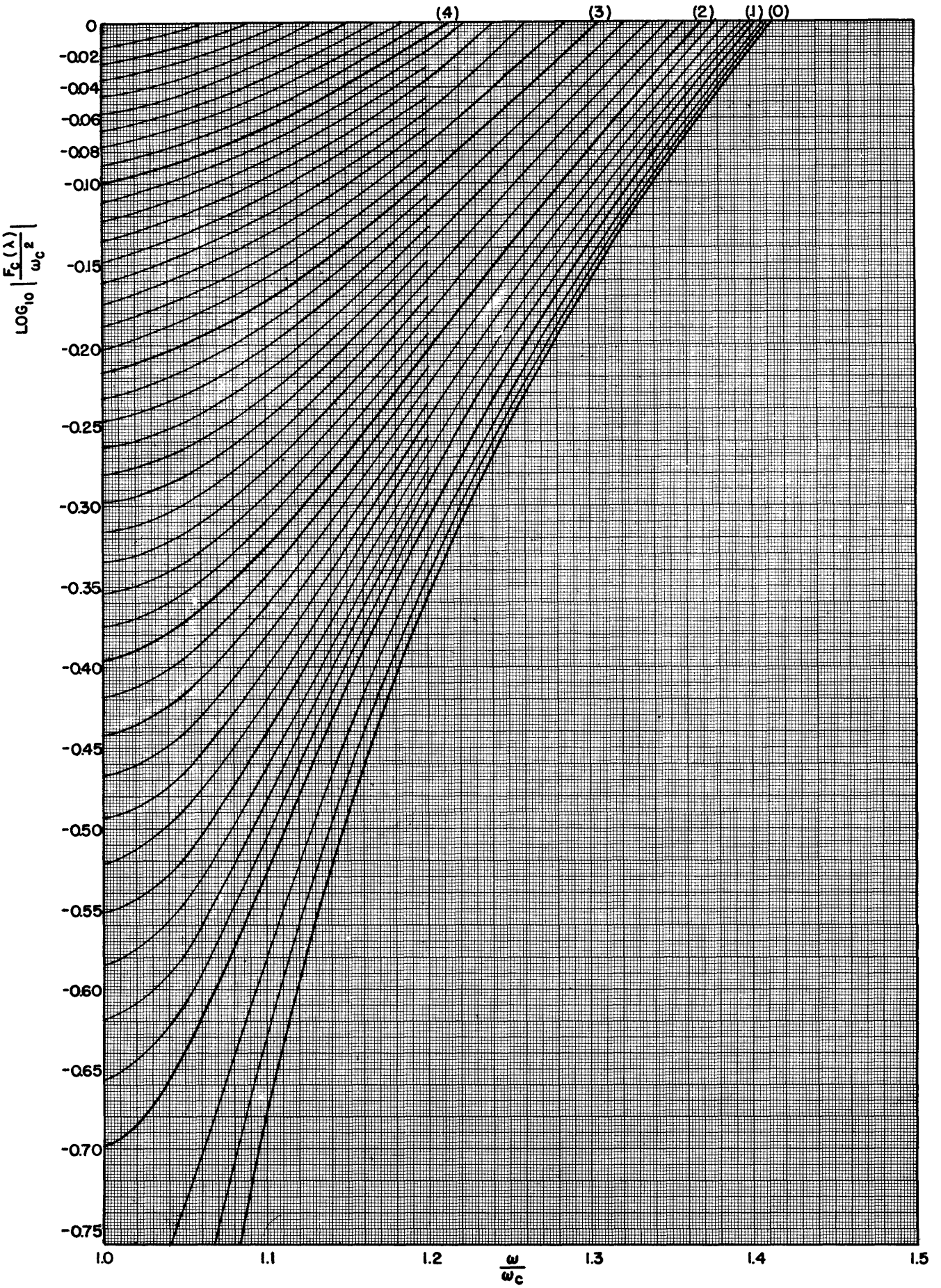
$\text{Log}_{10} \left| \frac{F_r}{\sigma_r} \right|$ as a function of $\frac{\omega}{-\sigma_r}$.

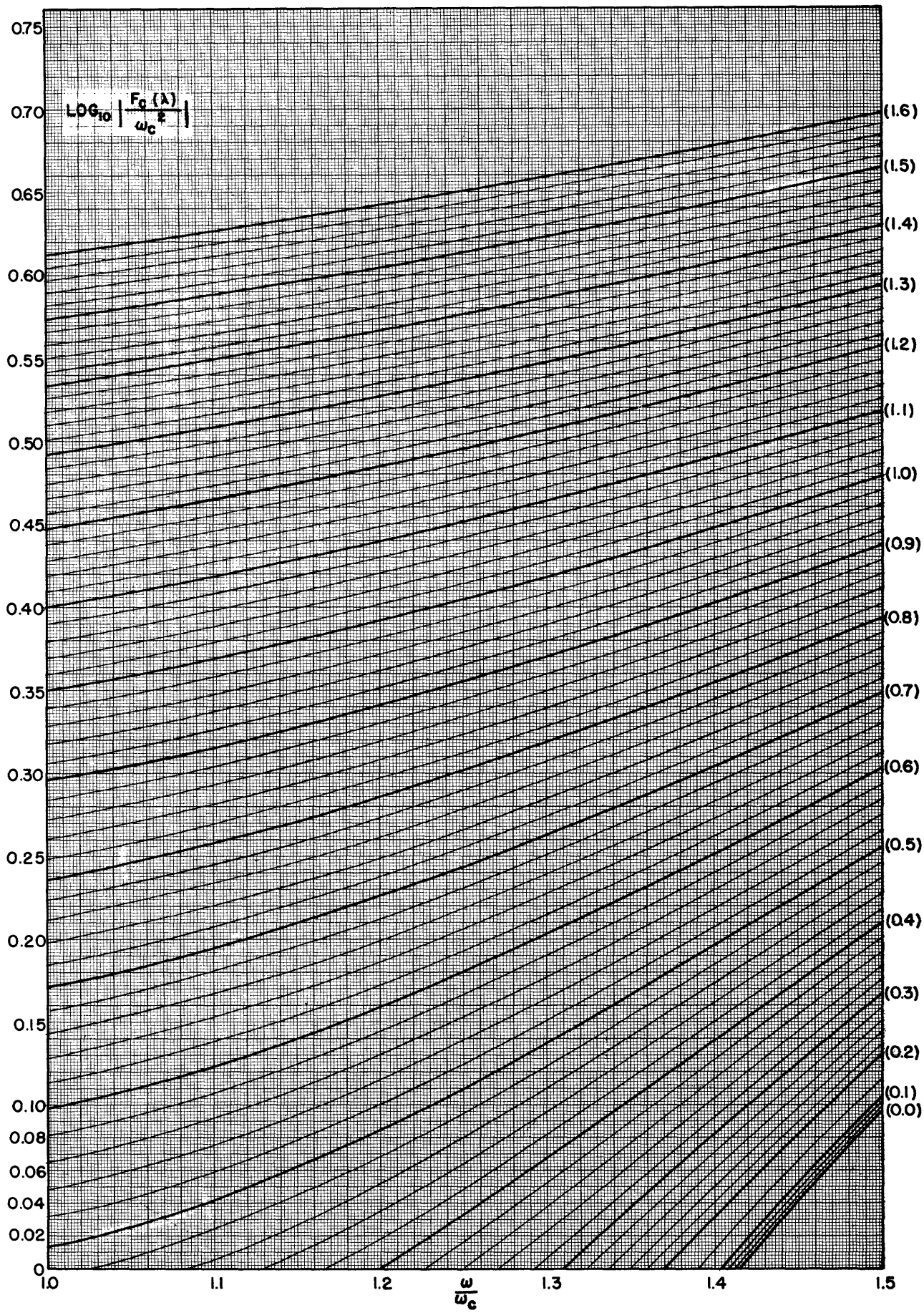

$\text{LOG}_{10}\left|\frac{F_c(\lambda)}{\omega_c^2}\right|$
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
-0.05
-0.10
-0.12
-0.14
(1.6)
(1.5)
(1.4)
(1.3)
(1.2)
(1.1)
(1.0)
(0.9)
(0.8)
(0.7)
(0.6)
(0.5)
(0.4)
(0.3)
(0.2)
(0.1)
(0.0)
0
0.1
0.2
0.3
0.4
0.5
$\frac{\omega}{\omega_c}$

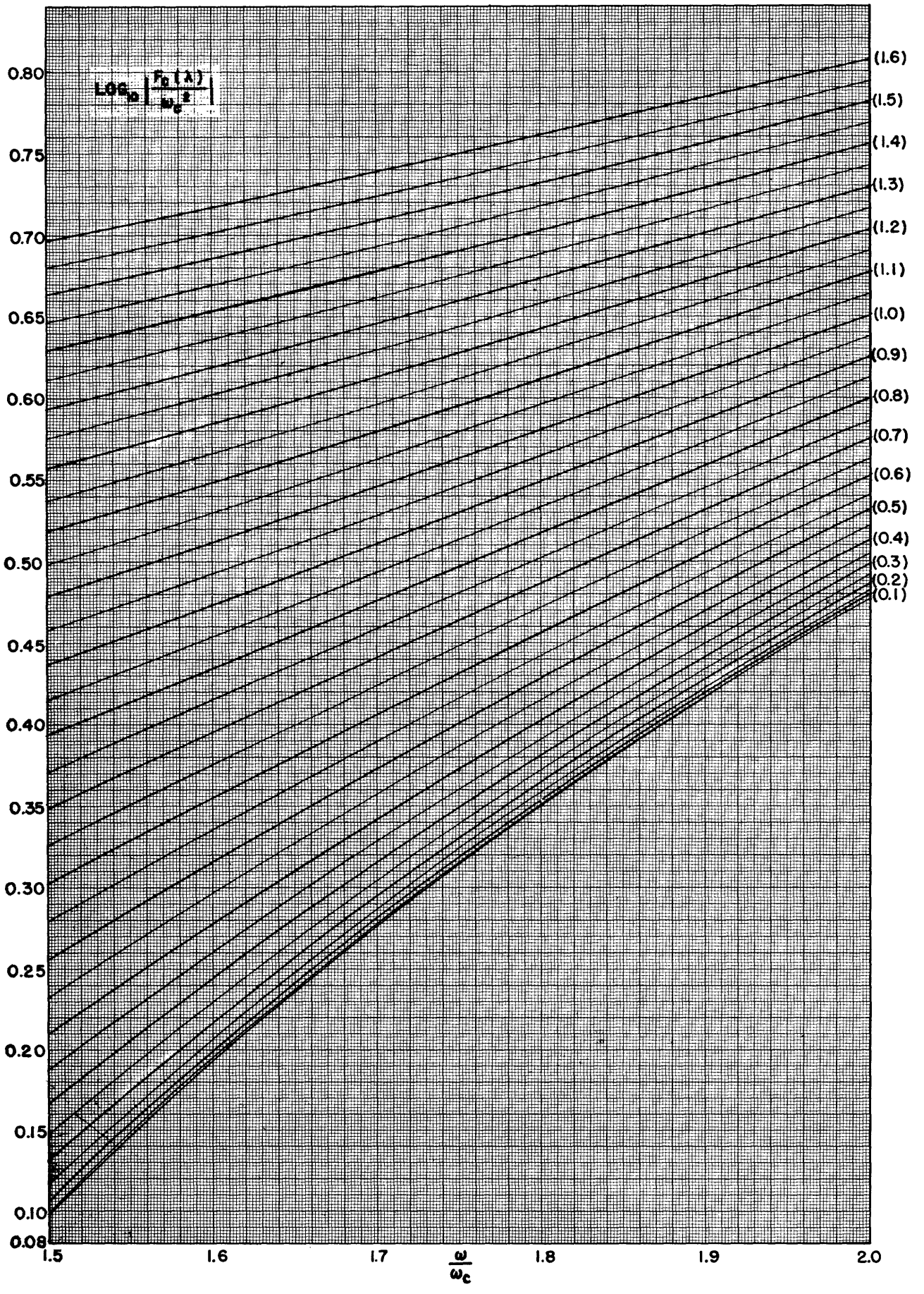

0
-0.02
-0.04
-0.06
-0.08
-0.10
-0.15
-0.20
-0.25
-0.30
-0.35
-0.40
-0.45
-0.50
-0.55
-0.60
-0.65
-0.70
-0.75
$\text{LOG}_{10}\left|\frac{F_c(\lambda)}{\omega_c^2}\right|$
(0.4)
(0.3)
(0.2)
(0.1)
(0.0)
(-0.075)
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{\omega}{\omega_c}$

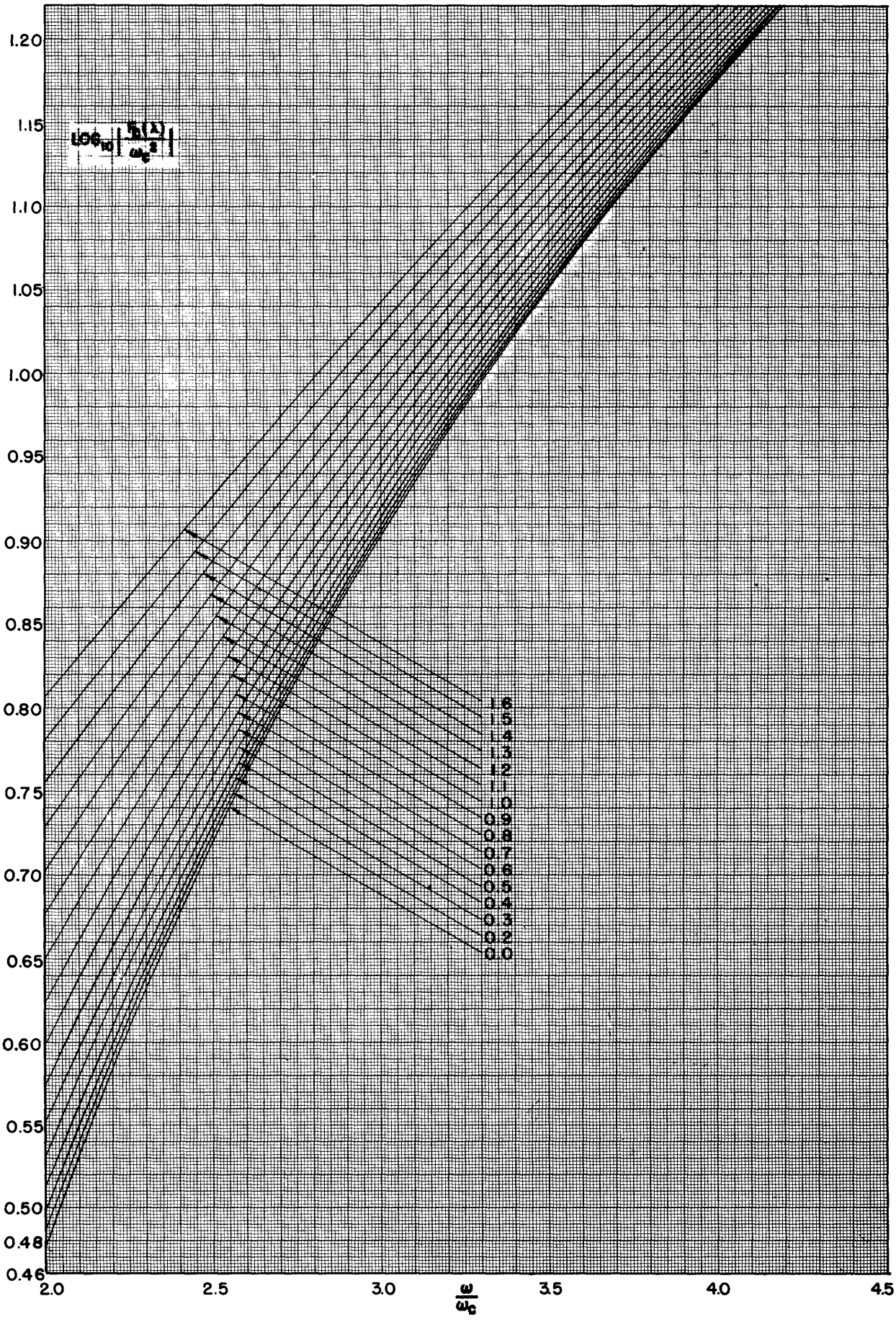

$\log_{10}\left|\frac{F_c(s)}{\omega_c^2}\right|$
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.08
0.06
0.04
0.02
0
(1.6)
(1.5)
(1.4)
(1.3)
(1.2)
(1.1)
(1.0)
(0.9)
(0.8)
(0.7)
(0.6)
(0.5)
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{\omega}{\omega_c}$

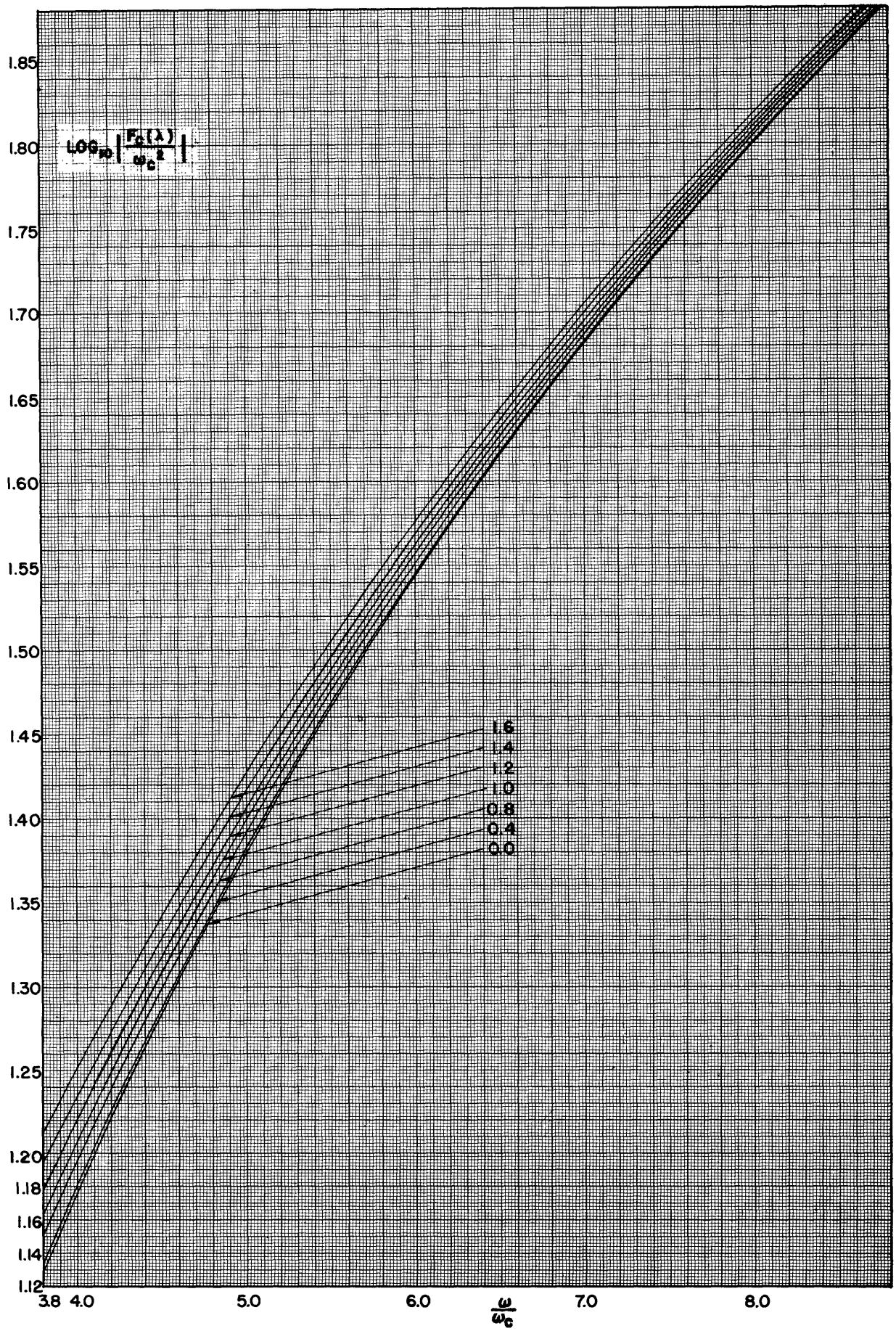

-0.66
-0.68
-0.70
-0.75
-0.80
-0.85
-0.90
-0.95
-1.0
-1.02
-1.04
-1.06
-1.08
-1.10
-1.15
-1.20
-1.25
-1.30
-1.35
-1.40
0.1
0.75
0.05
0.025
0.0
0.9
0.95
1.0
1.05
1.1
1.15
$\frac{\omega}{\omega_c}$
$\log_{10} \left| \frac{F_c(\lambda)}{\omega_c^{\alpha}} \right|$

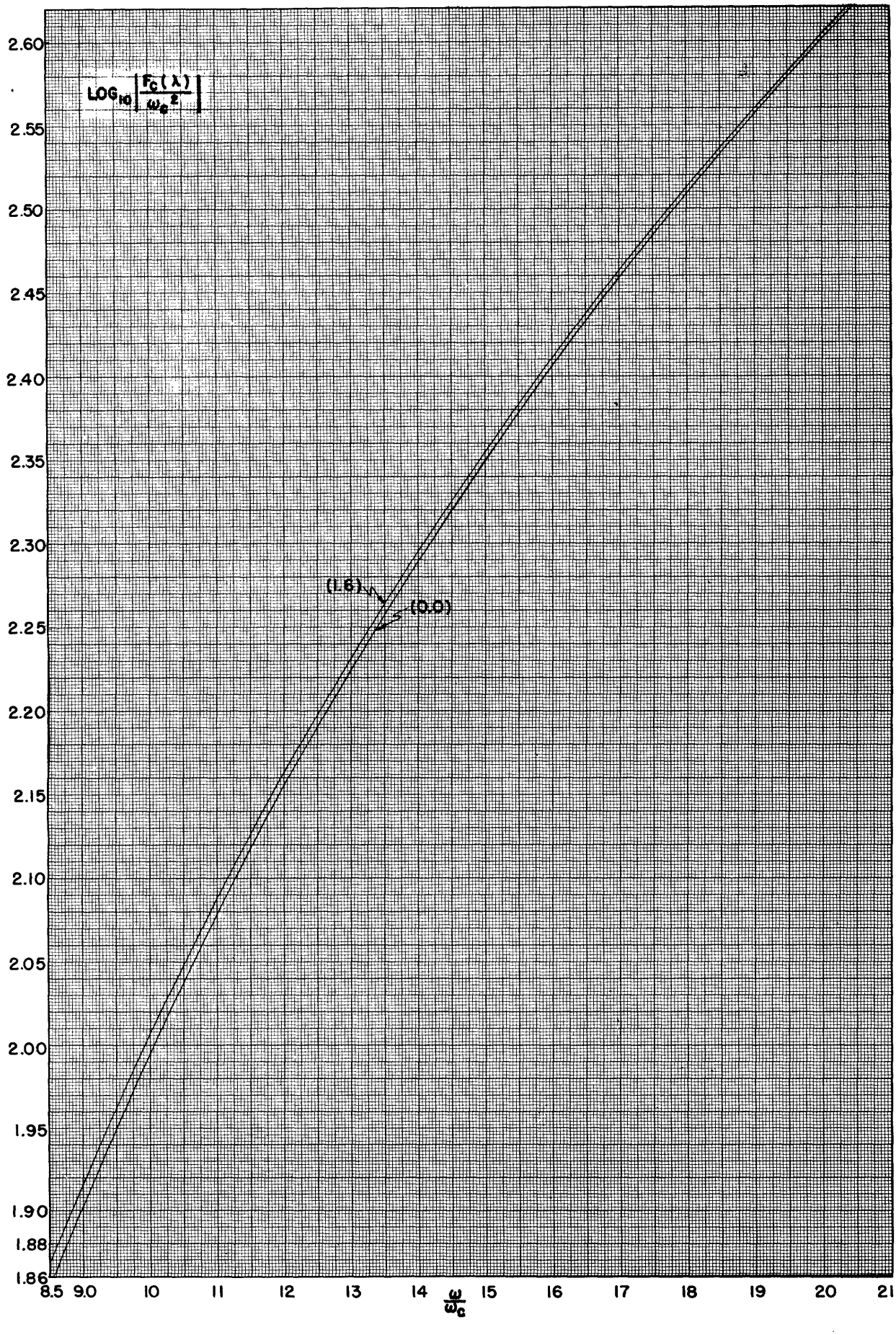

(4)
(3)
(2)
(1)
(0)
0
-0.02
-0.04
-0.06
-0.08
-0.10
-0.15
-0.20
-0.25
-0.30
-0.35
-0.40
-0.45
-0.50
-0.55
-0.60
-0.65
-0.70
-0.75
$\log_{10}\left|\frac{F_c(\lambda)}{\omega_c^2}\right|$
1.0
1.1
1.2
1.3
1.4
1.5
$\frac{\omega}{\omega_c}$

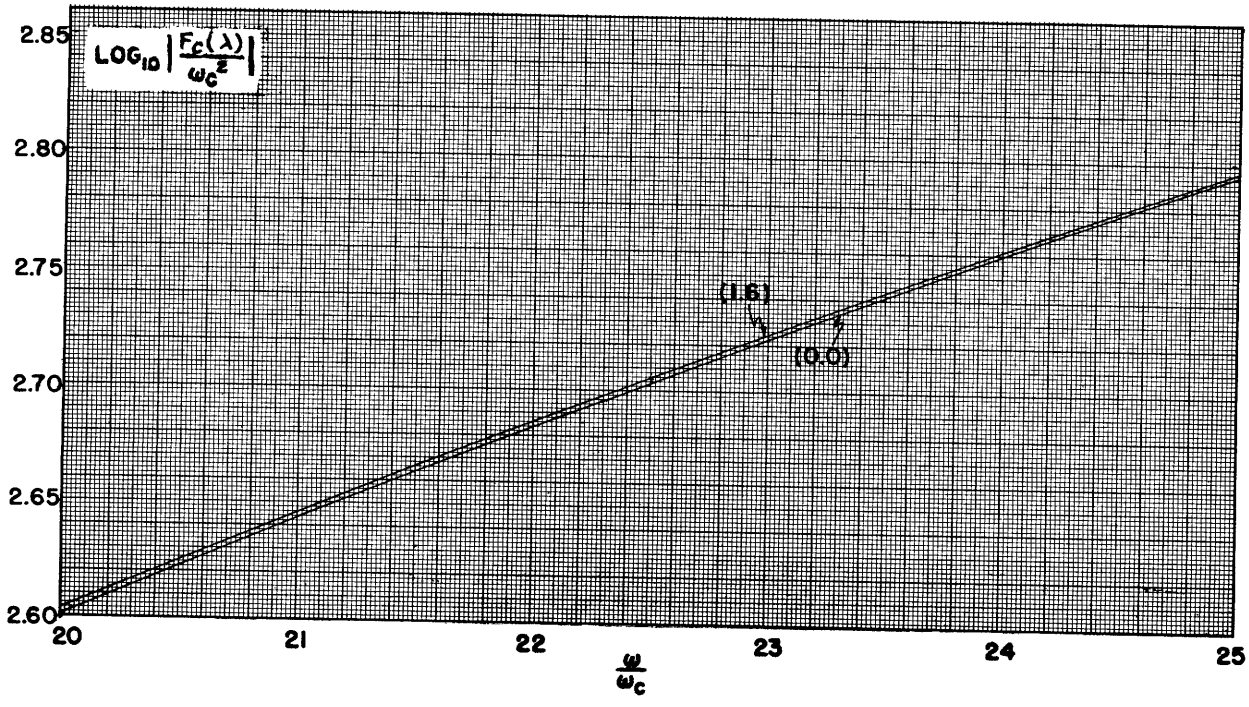

$\log_{10}\left|\frac{F_c(\lambda)}{\omega_c^{\lambda}}\right|$
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.08
0.06
0.04
0.02
0
(1.6)
(1.5)
(1.4)
(1.3)
(1.2)
(1.1)
(1.0)
(0.9)
(0.8)
(0.7)
(0.6)
(0.5)
(0.4)
(0.3)
(0.2)
(0.1)
(0.0)
1.0
1.1
1.2
1.3
1.4
1.5
$\frac{\omega}{\omega_c}$

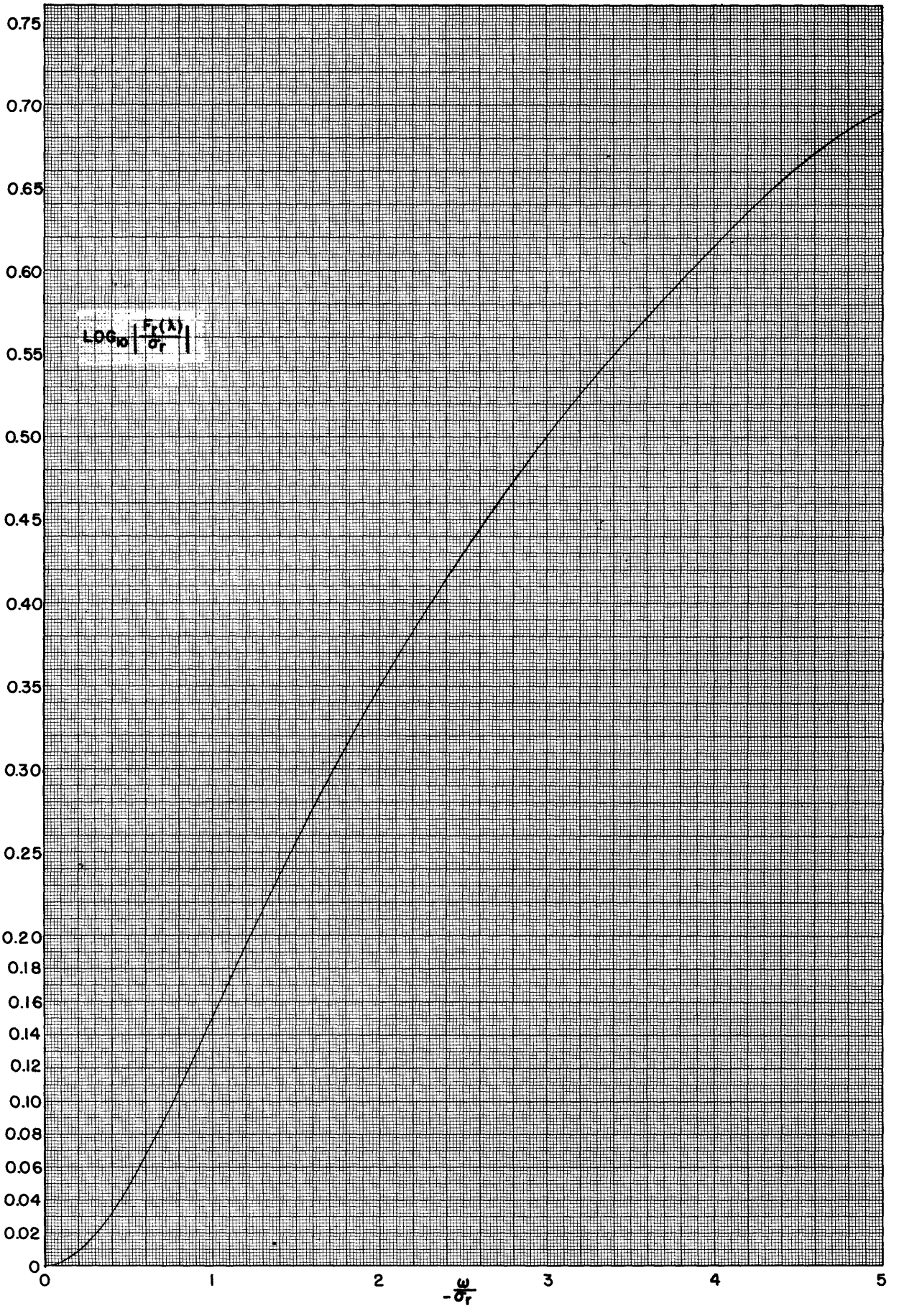

$\log_{10}\left|\frac{F_0(\lambda)}{\omega_c^s}\right|$
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.08
(1.6)
(1.5)
(1.4)
(1.3)
(1.2)
(1.1)
(1.0)
(0.9)
(0.8)
(0.7)
(0.6)
(0.5)
(0.4)
(0.3)
(0.2)
(0.1)
1.5
1.6
1.7
1.8
1.9
2.0
$\frac{\omega}{\omega_c}$

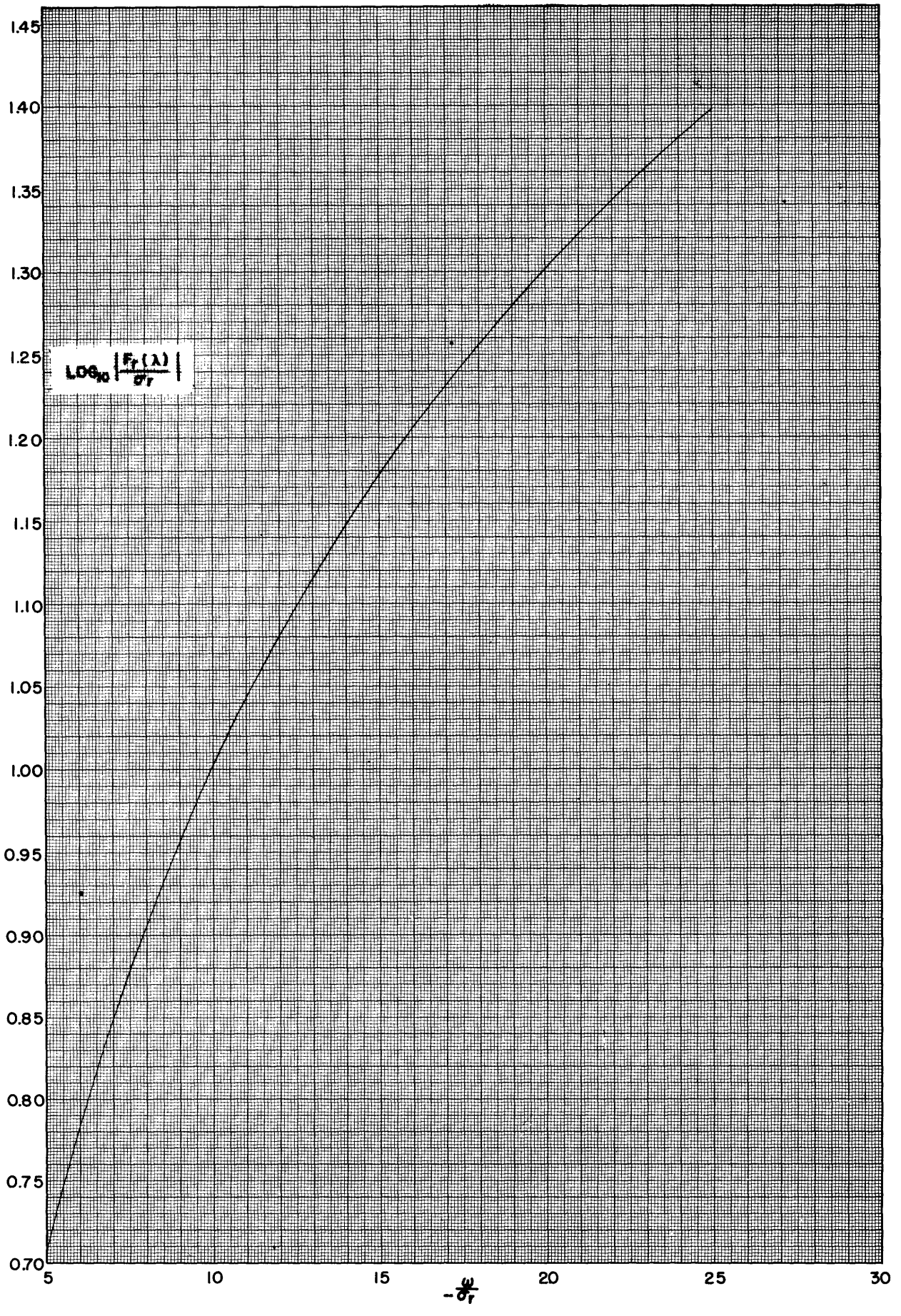

$\log_{10}\left|\frac{F_n(\lambda)}{\omega_c^2}\right|$
1.20
1.15
1.10
1.05
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.48
0.46
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.0
2.0
2.5
3.0
3.5
4.0
4.5
$\frac{\omega}{\omega_c}$


1.85
1.80
1.75
1.70
1.65
1.60
1.55
1.50
1.45
1.40
1.35
1.30
1.25
1.20
1.18
1.16
1.14
1.12
$\log_{10}\left|\frac{F_c(\lambda)}{\omega_c^2}\right|$
1.6
1.4
1.2
1.0
0.8
0.4
0.0
3.8
4.0
5.0
6.0
7.0
8.0
$\frac{\omega}{\omega_c}$


$\log_{10}\left|\frac{F_c(\lambda)}{\omega_c^2}\right|$
2.60
2.55
2.50
2.45
2.40
2.35
2.30
2.25
2.20
2.15
2.10
2.05
2.00
1.95
1.90
1.88
1.86
(1.6)
(0.0)
8.5
9.0
10
11
12
13
14
15
16
17
18
19
20
21
$\frac{\omega}{\omega_c}$


$\text{LOG}_{10}\left|\frac{F_c(\lambda)}{\omega_c^2}\right|$
2.85
2.80
2.75
2.70
2.65
2.60
(1,6)
(0,0)
20
21
22
23
24
25
$\frac{\omega}{\omega_c}$

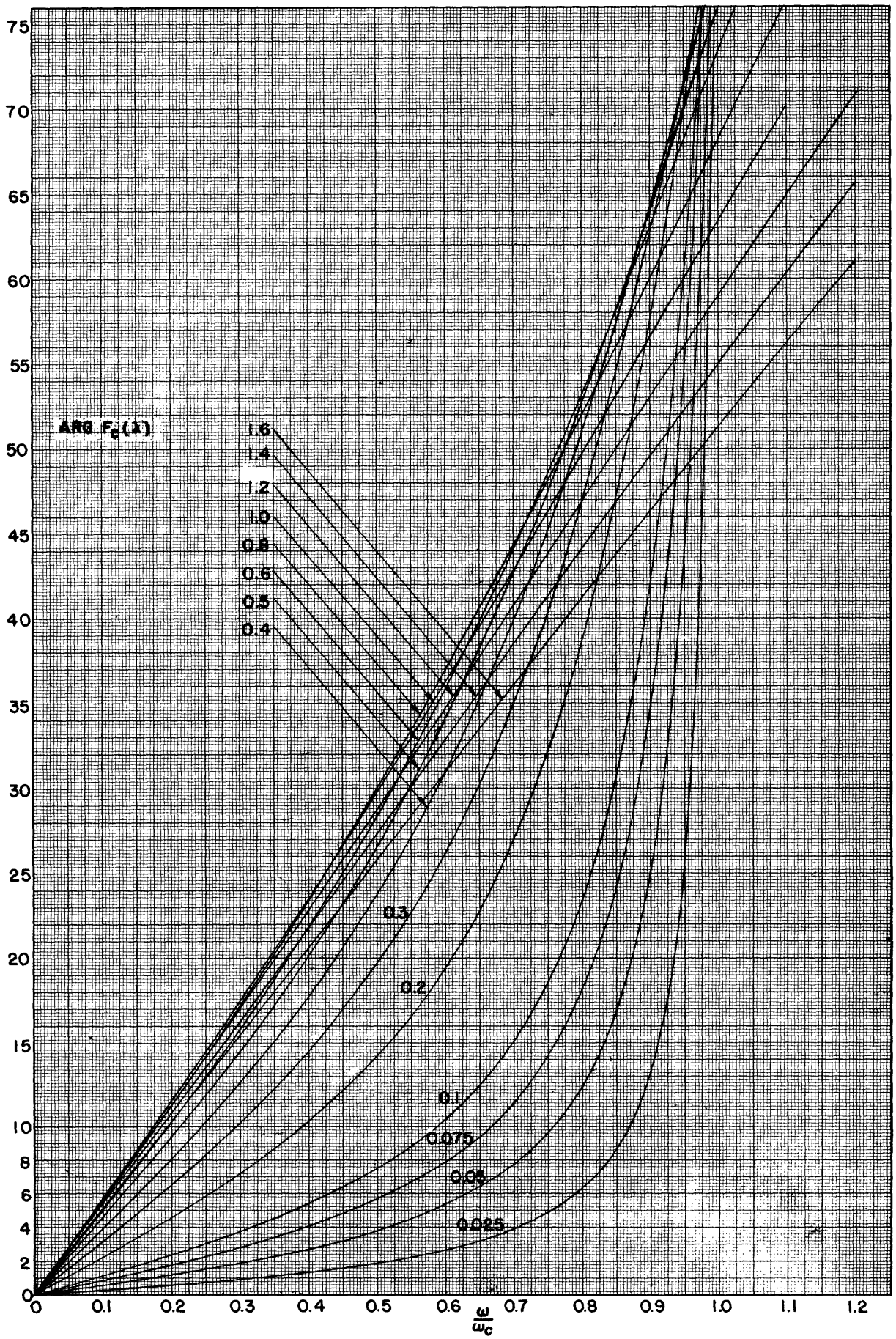

0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
$\log_{10}\left|\frac{F_r(x)}{\sigma_r}\right|$
0
1
2
3
4
5
$\frac{\omega}{-\sigma_r}$

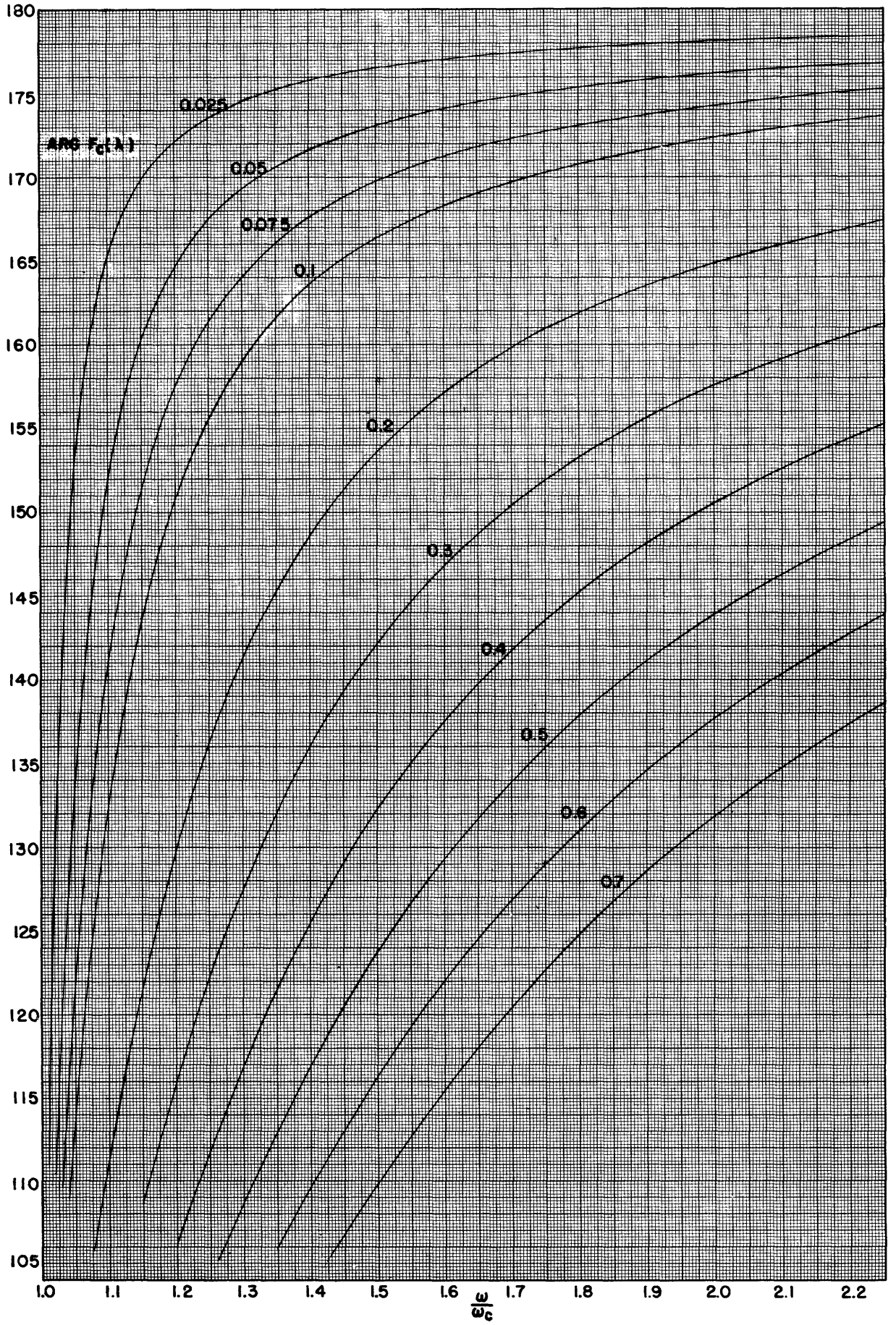

1.45
1.40
1.35
1.30
1.25
1.20
1.15
1.10
1.05
1.00
0.95
0.90
0.85
0.80
0.75
0.70
$\log_{10}\left|\frac{F_r(\lambda)}{\sigma_r}\right|$
5
10
15
20
25
30
$-\frac{\psi}{\sigma_r}$

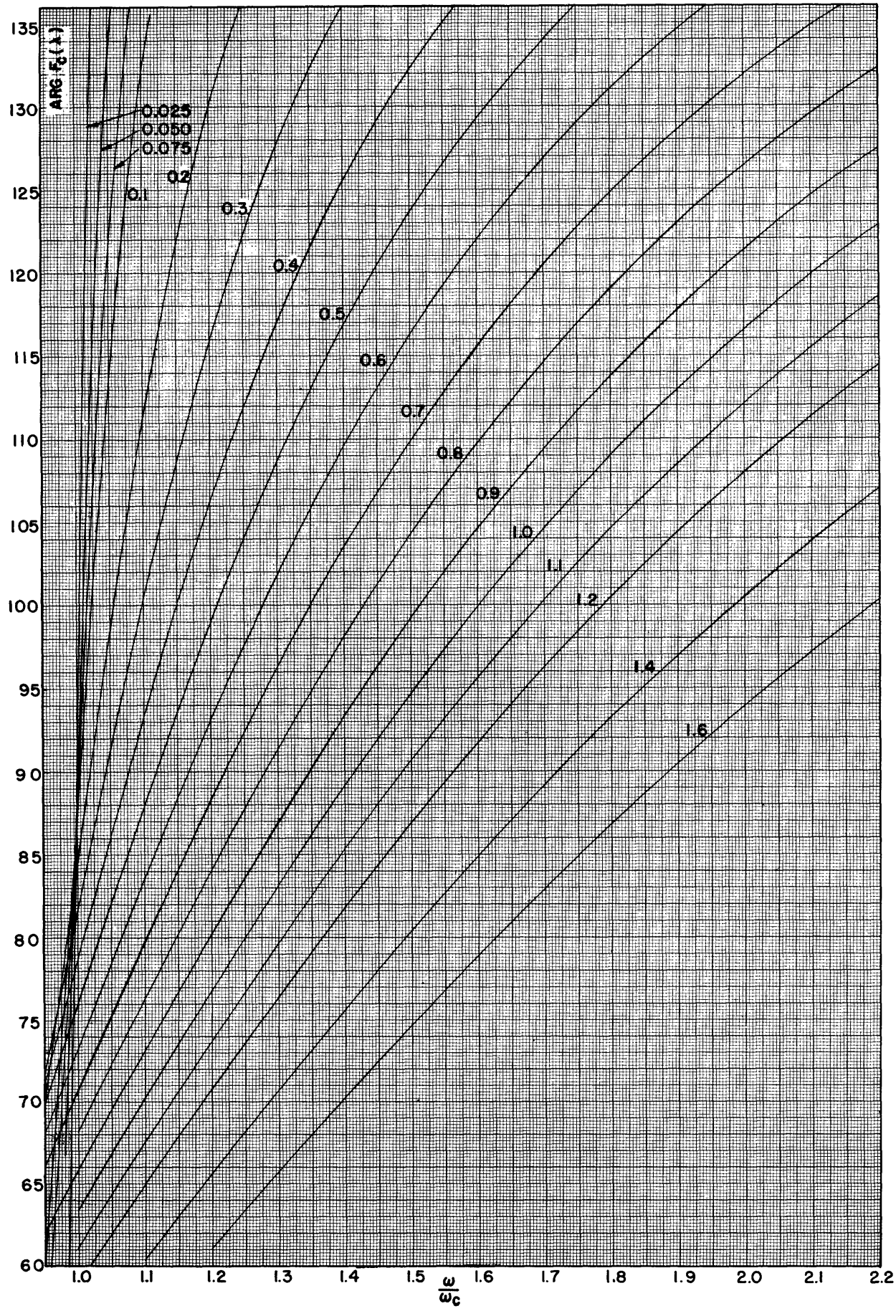
Set 4

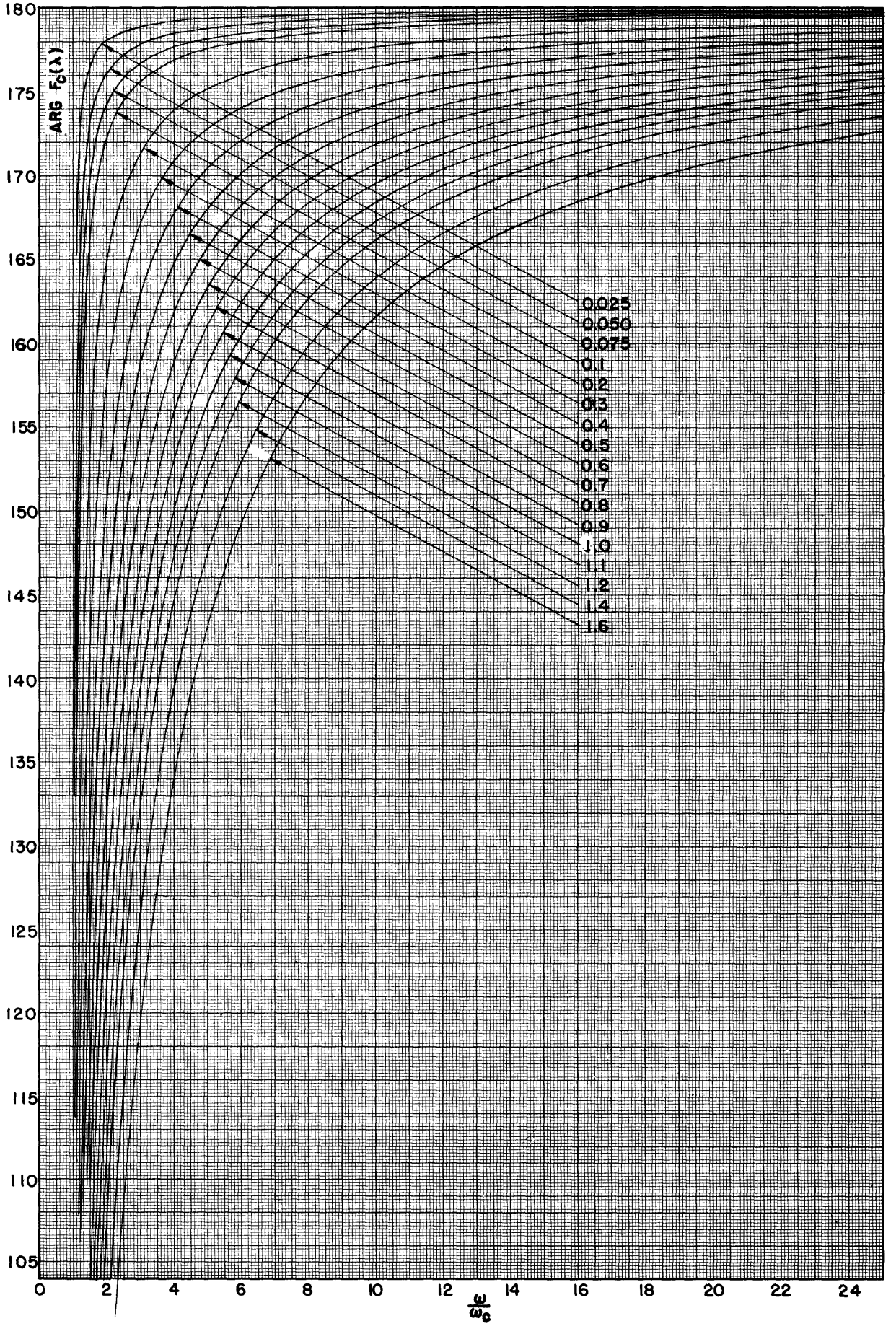
Arg F_c as a function of $\frac{\omega}{\omega_c}$ for a range of values of the parameter $\frac{-\sigma_c}{\omega_c}$.

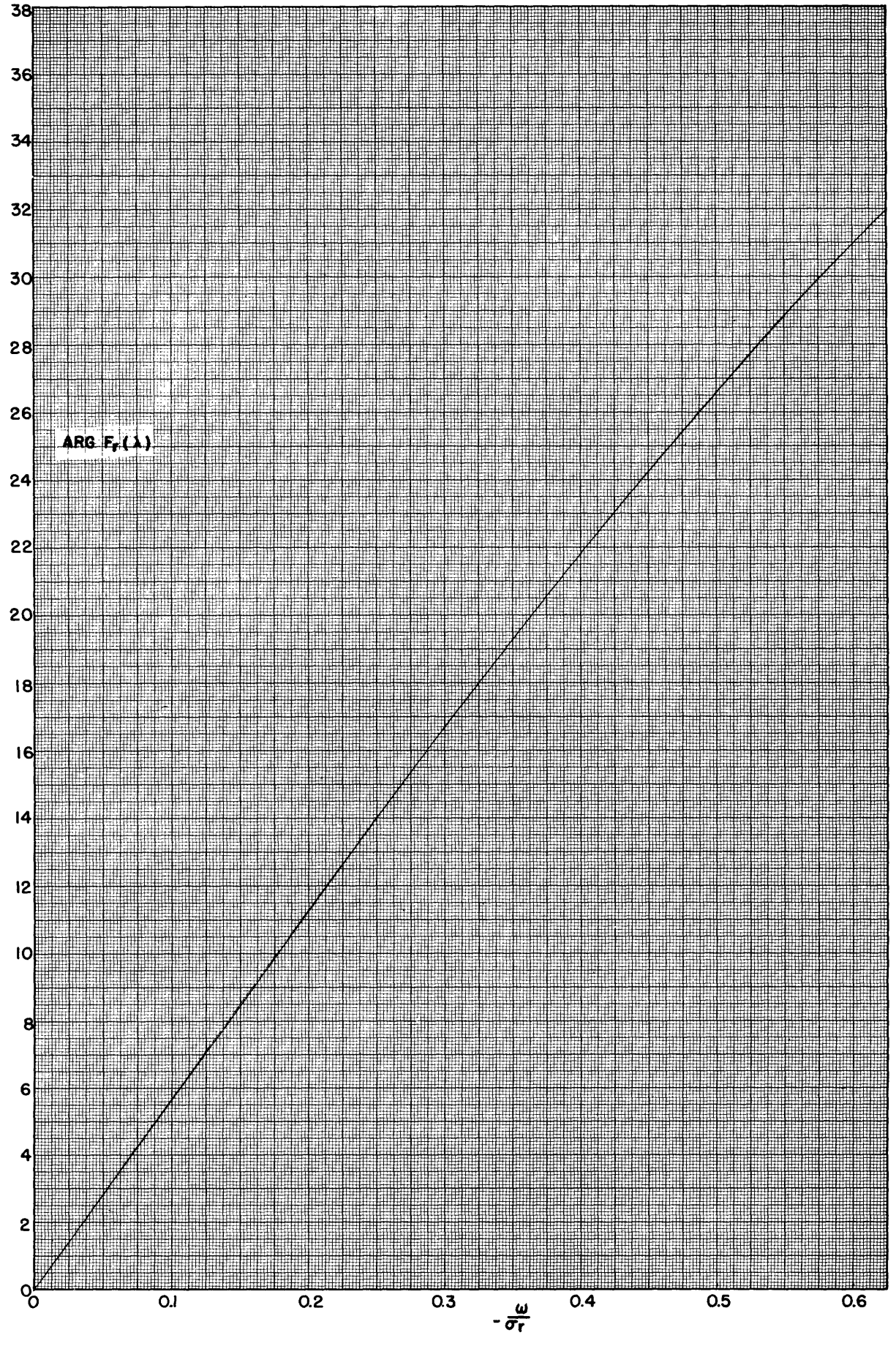
Arg F_r as a function of $\frac{\omega}{-\sigma_r}$.

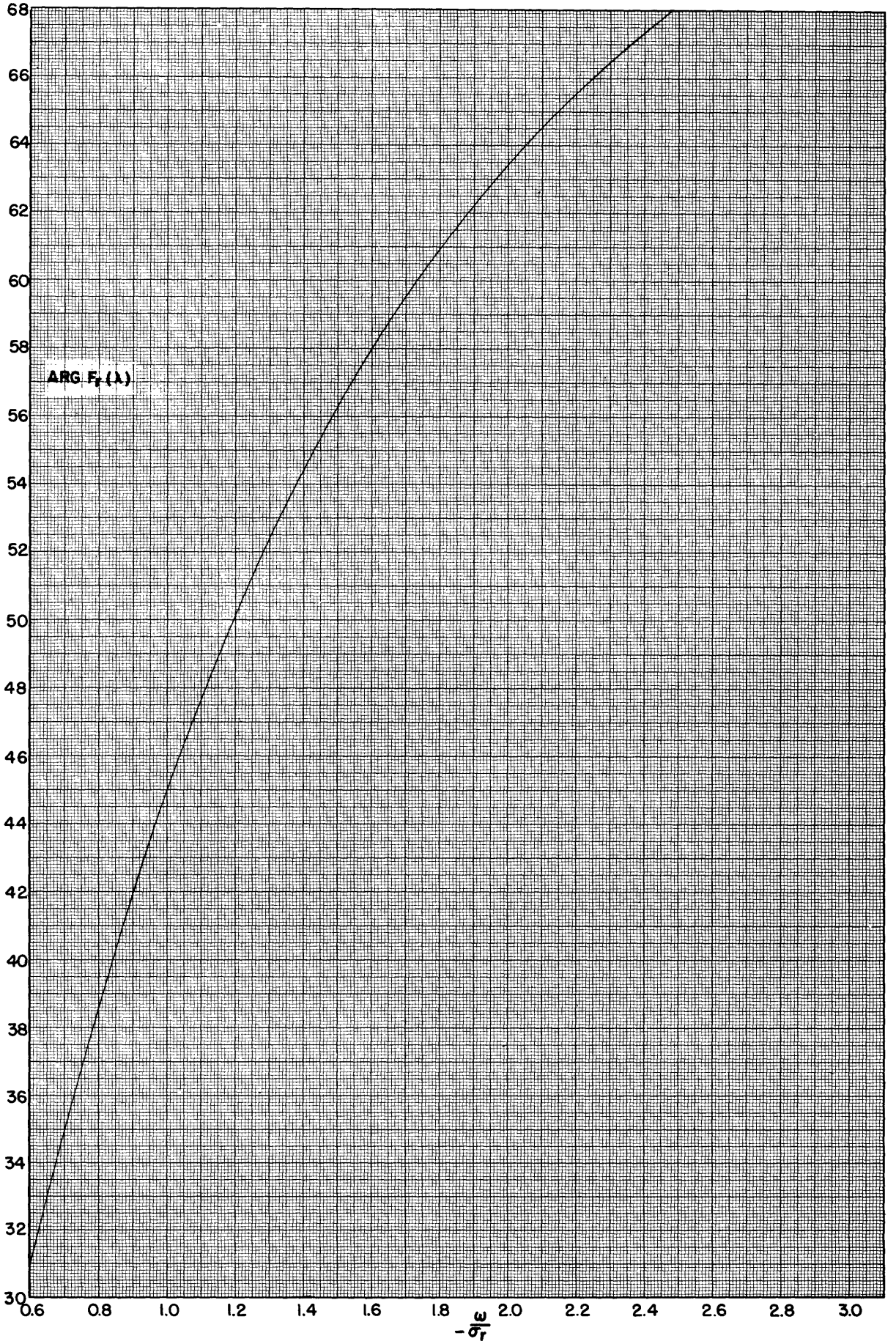

ARG $F_c(\lambda)$
1.6
1.4
1.2
1.0
0.8
0.6
0.5
0.4
0.3
0.2
0.1
0.075
0.05
0.025
75
70
65
60
55
50
45
40
35
30
25
20
15
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
$\frac{\omega}{\omega_c}$

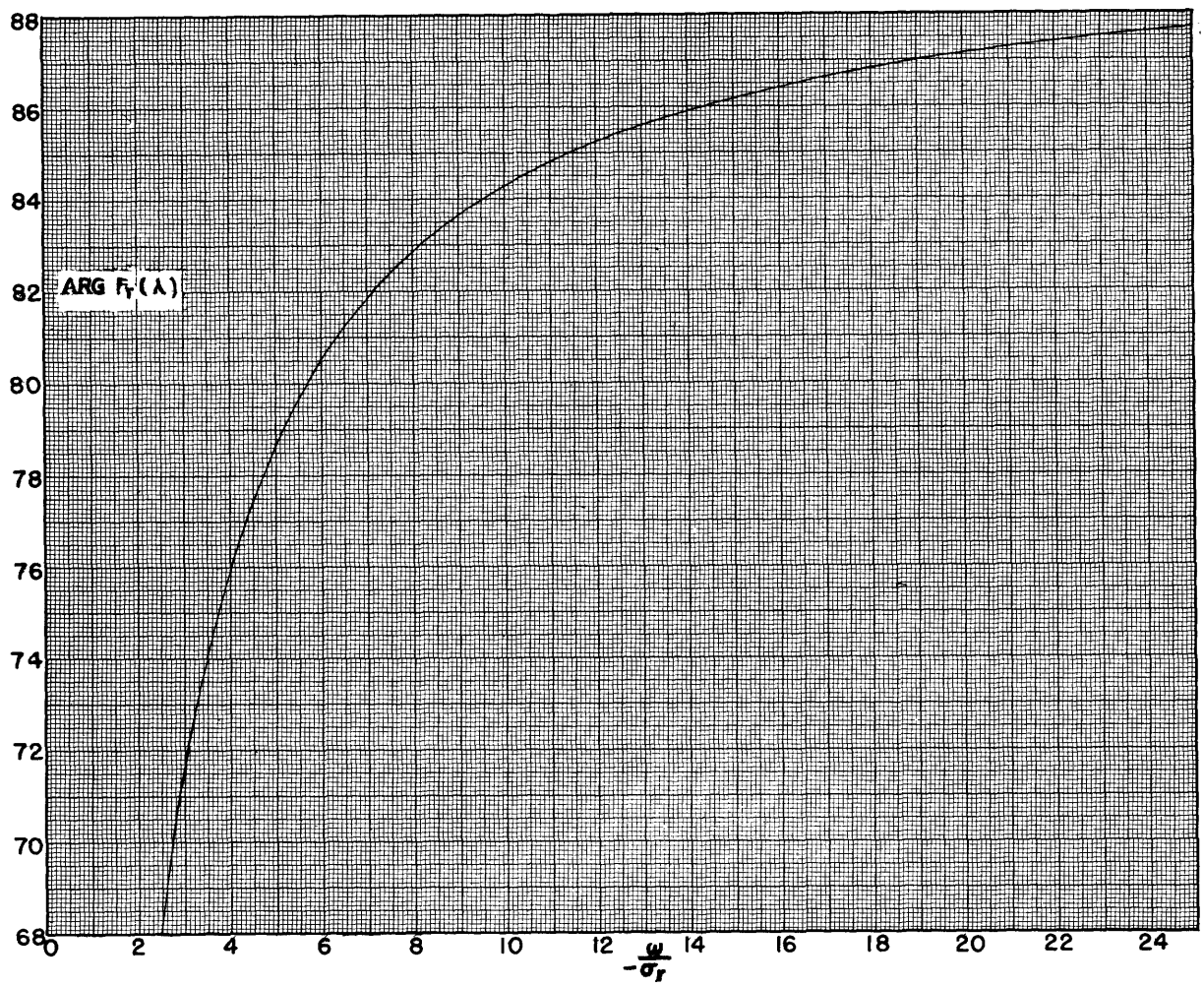

180
175
170
165
160
155
150
145
140
135
130
125
120
115
110
105
ARG $F_c(\lambda)$
0.025
0.05
0.075
0.1
0.2
0.3
0.4
0.5
0.6
0.7
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
$\frac{\omega}{\omega_c}$


ARG $F_c(j\lambda)$
135
130
125
120
115
110
105
100
95
90
85
80
75
70
65
60
0.025
0.050
0.075
0.2
0.1
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.4
1.6
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
$\frac{\omega}{\omega_c}$


ARG $F_C(\lambda)$
180
175
170
165
160
155
150
145
140
135
130
125
120
115
110
105
0.025
0.050
0.075
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.4
1.6
0
2
4
6
8
10
12
14
16
18
20
22
24
$\frac{\omega}{\omega_c}$


38
36
34
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
ARG $F_r(\lambda)$
0
0.1
0.2
0.3
0.4
0.5
0.6
$-\frac{\omega}{\sigma_r}$


68
66
64
62
60
58
56
54
52
50
48
46
44
42
40
38
36
34
32
30
ARG $F_r(\lambda)$
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
$-\frac{\omega}{\sigma_r}$


88
86
84
82
80
78
76
74
72
70
68
ARG $F_T(\lambda)$
0
2
4
6
8
10
12
14
16
18
20
22
24
$\frac{\omega}{-\sigma_r}$