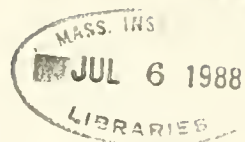


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A Stochastic Model of Resource Flexibility

Lode Li

*Sloan School of Management
Massachusetts Institute of Technology
Cambridge, MA 02139*

April 1987

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A Stochastic Model of Resource Flexibility

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Abstract

We study a stochastic control model of a two-stage production process with flexible workstations. The cumulative productions of both stages are governed by two Poisson counting processes with random intensities parameterized by production capacity and flexibility respectively. An optimal barrier policy is characterized by the critical values which indicate when one stage of production should be turned off, when one station should be switched to assist the other station and when to switch back to its regular task. Both the conditions for optimal barriers and the expected total discounted profit are explicitly determined. Thus, the economic value of resource flexibility can be measured by the difference in expected utility a firm can optimally realize by possessing it.

Key Words: Flexible Resource; Poisson Process; Intensity Control; Barrier Policy.

*This work has benefited considerably from discussions with Erhan Cinlar, Michael Harrison, and Morton Kamien. The author is greatly indebted for their many useful suggestions and comments.

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1. Formulation of the Model

Accompanying with recent advances of microprocessor-based manufacturing technologies, the flexible manufacturing systems of the present and near future reduce the changeovers between products to a matter of seconds. The questions of how to manage the flexible resources and how to quantify the value of flexibility become increasingly important. The presence of the flexible resources affect not only the market competition among firms but also the internal control of production processes. The economic value of the resource flexibility will not be accurately captured by ignoring either of the two. As a first step, the paper develops a stochastic control model to study the economic value of flexibility in the decision theoretic framework, that is, the difference in expected utility (profit) a firm can optimally realize by possessing it.

In general, the economics of multiple-use resources can only be discussed in the context of a model that has uncertainty, seasonality, or in general terms, some sort of variability. It is also necessary that one's model explicitly recognize the existence of more than one type of task or processing requirement within the firm. Thus, to consider any issue in the economic value of resource flexibility, we must consider a multi-stage process (some sort of network structure). A simple model of two-stage, two-task production process is considered as follows.

Consider a production process that consists of two work stations, 1 and 2, two tasks, producing an intermediate good 1 and a final good 2, and an inventory storing intermediate goods. Station 1 can produce good 1 at an average rate α , or it can produce good 2 at rate $\delta_1\alpha$ ($0 \leq \delta_1 \leq 1$). Similarly, station 2 can produce good 2 using good 1 at an average rate β , or produce good 1 at rate $\delta_2\beta$ ($0 \leq \delta_2 \leq 1$). Suppose the management monitoring the production rates within the range of their maximum rates, and can also switch either station from one type of production to the other. In the context of service industry, one may think of the model in terms of that service is completed only when two presequenced requirements are finished and the inventory is the customer wait. The cumulative input (production of good 1) and output (production of good 2) are represented by two increasing non-negative integer-valued stochastic processes $A = \{A(t), t \geq 0\}$ and $B = \{B(t), t \geq 0\}$, where $A(t)$ and $B(t)$ denote the amount of production 1 and the amount of production 2 in the time interval $(0, t]$. Then the inventory level at time t is

$$Z(t) = x + A(t) - B(t), \quad (1.1)$$

where x is the amount of inventory at time zero. We assume that

$$A, B \text{ are independent Poisson processes with random intensities } \{\alpha_t, t \geq 0\} \text{ and } \{\beta_t, t \geq 0\}. \quad (1.2)$$

That is, $\{A(t) - \int_0^t \alpha_s ds, t \geq 0\}$ and $\{B(t) - \int_0^t \beta_s ds, t \geq 0\}$ are martingales, and α_t and β_t can then be viewed as the actual production rates of products 1 and 2 respectively.

Given the capacities α and β , and the measurement of flexibility δ_1 and δ_2 (selected at time zero), a *feasible operating policy* is defined as a pair of stochastic processes (α_t, β_t) that jointly satisfy the following:

(1.3) (α_t) and (β_t) are left continuous and have right-hand limits.

(1.4) (α_t) and (β_t) are adapted with respect to Z ,

(1.5) $0 \leq \alpha_t \leq \alpha \cdot \mathbf{1}_{(0, \beta]}(\beta_t) + (\alpha + \delta_2 \beta) \cdot \mathbf{1}_{\{0\}}(\beta_t)$ and $0 \leq \beta_t \leq \beta \cdot \mathbf{1}_{(0, \alpha]}(\alpha_t) + (\beta + \delta_1 \alpha) \cdot \mathbf{1}_{\{0\}}(\alpha_t)$,
for all $t \geq 0$,

(1.6) $Z(t)$ is non-negative for all t .

Condition (1.4) implies that α_t and β_t are functions of $(Z(s), s \leq t)$. This says that the control that the firm exercises at time t is based on only the historical information before time t . In condition (1.5), for any set S , $\mathbf{1}_S(s)$ is an indicator function, i.e.,

$$\mathbf{1}_S(s) = \begin{cases} 1, & \text{if } s \in S; \\ 0, & \text{otherwise.} \end{cases}$$

For example,

$$\alpha \cdot \mathbf{1}_{(0, \beta]}(\beta_t) + (\alpha + \delta_2 \beta) \cdot \mathbf{1}_{\{0\}}(\beta_t) = \begin{cases} \alpha, & \text{if } 0 < \beta_t \leq \beta; \\ \alpha + \delta_2 \beta, & \text{if } \beta_t = 0. \end{cases}$$

Thus, condition (1.5) reflects the strategy space parameterized by the flexibility of the workstations. Together with conditions (1.3), it also ensures that (α_t) and (β_t) are integrable and predictable with respect to Z . The restriction (1.6) implies that the production of good 2 has no choice other than waiting if its demand of intermediate products can not be met from stock on hand. For the justification of the formulation, see Bremaud (1981) and Li (1986).

To complete our formulation, we specify the cost structure as follows. Each final product is worth p dollars. We simplify the market side where the firm serves as a supplier by assuming that the demand for final goods is infinite. The plant incurs a linear variable cost, c_1 dollars per unit of good 1 actually produced and c_2 dollars per unit of good 2 actually produced. The variable cost may comprise material cost and also labor cost if workers are paid piece-rate. The selling price p is assumed to cover the total variable cost of a final good, that is

$$p > c_1 + c_2. \tag{1.7}$$

A physical holding cost of h dollars per unit time is incurred for each unit of intermediate goods held in inventory. Assume that the firm earns interest at rate $r > 0$, compounded continuously, on the funds which are required for production operations, and the production is planned over an infinite time horizon. Therefore, given that the initial inventory is x , the expected profit is

$$\pi(x) \equiv E_x \left\{ \int_0^\infty e^{-rt} [p dB(t) - c_1 dA(t) - c_2 dB(t) - hZ(t) dt] \right\}, \tag{1.8}$$

where E_x denotes the expectation conditional on $Z(0) = x$.

Applying integration by parts theorem to (1.8), we have

$$\pi(x) = v(x) - \frac{h}{r}, \quad (1.9)$$

where

$$v(x) = E_x \left\{ \int_0^\infty e^{-rt} [qdB(t) - wdA(t)] \right\}, \quad (1.10)$$

$$q \equiv p - c_2 + \frac{h}{r}, \quad \text{and } w \equiv c_1 + \frac{h}{r}. \quad (1.11)$$

The problem of the management is to choose a pair of control processes (α, β) to maximize the expected profit, equivalently $v(x)$, such that assumption (1.1)-(1.2), and feasibility constraints (2.3)-(2.6) are satisfied. By the Poisson assumption, the objective function under a feasible policy can be further written as

$$v(x) = E_x \left\{ \int_0^\infty e^{-rt} (q\beta_t - w\alpha_t) dt \right\}, \quad (1.12)$$

where $q > w$ following from assumption (1.7).

2. The Barrier Policies

Instead of solving the general intensity control problem set up in Section 1, we investigate a class of feasible policies, namely, *barrier policies*, and establish some computational results.

By barrier policy, we require that station 1 works at its full capacity on its regular task, producing good 1, until inventory reaches some level b , at this point it either ceases production or switches to assist station 2 until the inventory has been depleted by b_1 units; similarly station 2 works at full capacity on its regular task, producing good 2, until the inventory is zero, then it either waits or switches to help station 1 until the inventory has been increased by b_2 units. There are three critical numbers, b , b_1 , and b_2 . The production of good 1 is turned off when the inventory level is b and resumes its regular rate when the production of good 2 (maybe with higher rate) brings the inventory down to $b - b_1$. On the other hand, the production of good 2 is ceased at zero inventory level and resumes when the production of good 1 (again at an increased rate) is accumulated to a level of b_2 . Obviously, only on-off strategies are allowed in a barrier policy, namely

$$\alpha_t \in \{0, \alpha, \alpha + \delta_2\beta\}, \quad \beta_t \in \{0, \beta, \beta + \delta_1\alpha\},$$

and the essence of control becomes an optimal stopping problem.

To characterize a generic barrier policy, let $E \equiv \{0, 1, \dots, b\}$ be the state space of the inventory process Z , $E_1 \equiv \{b - b_1, \dots, b\}$, and $E_2 \equiv \{0, 1, \dots, b_2\}$. To be general, we allow either work station switch back and forth between joining the other production process and staying resting when its

regular production process is ceased. An ordered sequence of disjoint sets, $K_{i1}, K'_{i1}, \dots, K_{im_i}, K'_{im_i}$, with $|K_{ik}| = y_{ik}$ and $|K'_{ik}| = y'_{ik}$, represents a division of set E_i , $i = 1, 2$, e.g., $K_{11} = \{b, b - 1, \dots, b - y_{11} + 1\}$, $K'_{11} = \{b - y_{11}, \dots, b - y_{11} - y'_{11} + 1\}$, \dots , etc. Given production of good i ($i \neq j$) is turned off, station i sits idel and station j works alone producing good j when $Z(t) \in K_{ik}$, and both stations work together when $Z(t) \in K'_{ik}$. For $i = 1, 2$, allowing $K_{i1} = \emptyset$ and/or $K'_{im_i} = \emptyset$, the choice of such a sequence represents all the possible order combinations of switching between using station j ($j \neq i$) alone and using both stations. For notation simplicity, denote $K_i \equiv \cup_{k=1}^{m_i} K_{ik}$, $K'_i \equiv \cup_{k=1}^{m_i} K'_{ik}$, and $y_i \equiv \sum_{k=1}^{m_i} y_{ik}$, for $i = 1, 2$. It follows that $\sum_{k=1}^{m_i} y'_{ik} = b_i - y_i$. The value $b_i - y_i$ represents the total units that production j ($j \neq i$) turns out with the help of station i .

We also define

$$T(K) \equiv \inf\{t \geq 0, Z(t) \in K\},$$

$$T(S, K) \equiv \inf\{t \geq S, Z(t) \in K\},$$

for any stopping time S and any subset $K \subseteq E$. Let

$$\hat{T}_0 \equiv 0, \quad T_n \equiv T(\hat{T}_{n-1}, b) \wedge T(\hat{T}_{n-1}, 0),$$

$$\hat{T}_n \equiv T(T(\hat{T}_{n-1}, b), b - b_1) \wedge T(T(\hat{T}_{n-1}, 0), b_2),$$

$$R_{1n}^0 \equiv T(\hat{T}_{n-1}, b), \quad R_{2n}^0 \equiv T(\hat{T}_{n-1}, 0),$$

$$S_{in}^k \equiv T(R_{in}^{k-1}, K_i), \quad R_{in}^k \equiv T(S_{in}^k, K'_i),$$

$$S_{1n}^{m_1+1} \equiv T(R_{1n}^0, b - b_1), \quad S_{2n}^{m_2+1} \equiv T(R_{2n}^0, b_2),$$

for $i = 1, 2$, $k = 1, \dots, m_i$ and $n = 1, 2, \dots$.

A barrier policy then can be written as, for $t \geq 0$,

$$\alpha_t = \alpha(1_{N_0}(t) + 1_{N_2}(t)) + (\alpha + \delta_2\beta)1_{N'_2}(t), \quad (2.1)$$

$$\beta_t = \beta(1_{N_0}(t) + 1_{N_1}(t)) + (\beta + \delta_1\alpha)1_{N'_1}(t). \quad (2.2)$$

where $N_0 \equiv \cup_{n=0}^{\infty} (\hat{T}_n, T_{n+1}]$, $N_i \equiv \cup_{n=1}^{\infty} \cup_{k=1}^{m_i} (S_{in}^k, R_{in}^k]$, and $N'_i \equiv \cup_{n=1}^{\infty} \cup_{k=0}^{m_i} (R_{in}^k, S_{in}^{k+1}]$ for $i = 1, 2$.

In fact, the inventory process Z under a barrier policy is constructed by switching among five Markov processes: a difference between two Poisson processes with intensities α and β respectively, two negative Poisson processes with intensities β and $\beta + \delta_1\alpha$ respectively, and two Poisson processes with intensities α and $\alpha + \delta_2\beta$ respectively.

Proposition 2.1. *The value function $v(x)$ under a barrier policy defined in (2.1) and (2.2) is of the form*

$$v(x) = \frac{1}{r}(q\beta - w\alpha) + \hat{v}(x), \quad (2.3)$$

where

$$\hat{v}(x) \equiv \alpha w u_1(x) + \alpha(w + \delta_1 q) u'_1(x) - \beta q u_2(x) - \beta(q + \delta_2 w) u'_2(x), \quad (2.4)$$

$$u_i(x) \equiv E_x \left\{ \int_0^\infty e^{-rt} 1_{N_i}(t) dt \right\}, \quad (2.5)$$

$$u'_i(x) \equiv E_x \left\{ \int_0^\infty e^{-rt} 1_{N'_i}(t) dt \right\}. \quad (2.6)$$

Proof. Note that

$$1_{N_0}(t) + 1_{N_1}(t) + 1_{N'_1}(t) + 1_{N_2}(t) + 1_{N'_2}(t) = 1. \quad (2.7)$$

The proposition follows by substituting (2.1) and (2.2) into (1.12), replacing 1_{N_0} using expression (2.7) and collecting terms. ■

Thus, the value function under a barrier policy obtains by computing u_i and u'_i . We take two steps in the computation. First compute the Laplace transforms of the hitting times $T(b)$ and $T(0)$. Then, show that u_i and u'_i can be expressed in terms of them.

Denote the Laplace transform of a hitting time by

$$\theta(x, y) \equiv E_x \left[e^{-rT(y)} \right].$$

Then

Proposition 2.2. For $0 \leq x \leq b$,

$$\theta(x, 0) = \frac{g(b-x)}{g(b)}, \quad \text{and} \quad \theta(x, b) = \frac{c(x)}{c(b)}, \quad (2.8)$$

where

$$\begin{aligned} g(x) &\equiv a_* \rho_1^{-x} - a^* \rho_2^{-x}, \\ c(x) &\equiv d_* \rho_1^x - d^* \rho_2^x. \end{aligned} \quad (2.9)$$

ρ_1 and ρ_2 are the two roots of the quadratic equation

$$\rho = \frac{\alpha}{\alpha + \beta + r} \cdot \rho^2 + \frac{\beta}{\alpha + \beta + r} \quad (2.10)$$

with $0 < \rho_1 < 1, \rho_2 > 1$, for $r > 0$,

$$\begin{aligned} a_* &\equiv 1 - (\xi_1 \rho_2^{-1})^{y_1} (\eta_1 \rho_2^{-1})^{b_1 - y_1}, & a^* &\equiv 1 - (\xi_1 \rho_1^{-1})^{y_1} (\eta_1 \rho_1^{-1})^{b_1 - y_1}, \\ d_* &\equiv 1 - (\xi_2 \rho_2)^{y_2} (\eta_2 \rho_2)^{b_2 - y_2}, & d^* &\equiv 1 - (\xi_2 \rho_1)^{y_2} (\eta_2 \rho_1)^{b_2 - y_2}, \end{aligned} \quad (2.11)$$

and

$$\xi_1 \equiv \frac{\beta}{\beta + r}, \quad \xi_2 \equiv \frac{\alpha}{\alpha + r}, \quad \eta_1 \equiv \frac{\beta + \delta_1 \alpha}{\beta + \delta_1 \alpha + r}, \quad \eta_2 \equiv \frac{\alpha + \delta_2 \beta}{\alpha + \delta_2 \beta + r}.$$

Proof. We shall only prove the first equation in (2.8) and the second follows from a similar argument.

Define a function ψ by

$$\psi(x, y, z) \equiv \frac{\rho_1^{x-z} - \rho_2^{x-z}}{\rho_1^{y-z} - \rho_2^{y-z}}, \quad \text{for } x, y, z \in E. \quad (2.12)$$

By Lemma 3.3.2.1 in Li (1984), we have

$$E_x[e^{-rT_1} \mathbf{1}_{\{Z(T_1)=0\}}] = \psi(x, 0, b). \text{ and } E_x[e^{-rT_1} \mathbf{1}_{\{Z(T_1)=b\}}] = \psi(x, b, 0),$$

where $T_1 \equiv T(0) \wedge T(b)$ as defined above. Thus,

$$\theta(x, 0) = \psi(x, 0, b) + \psi(x, b, 0)\theta(b, 0). \quad (2.13)$$

Particularly,

$$\theta(b - b_1, 0) = \psi(b - b_1, 0, b) + \psi(b - b_1, b, 0)\theta(b, 0). \quad (2.14)$$

Also note that

$$\theta(b, 0) = \xi_1^{y_1} \eta_1^{b_1 - y_1} \theta(b - b_1, 0). \quad (2.15)$$

Substituting (2.15) in (2.14) and using (2.12), we obtain

$$\begin{aligned} \theta(b, 0) &= \frac{\xi_1^{y_1} \eta_1^{b_1 - y_1} \psi(b - b_1, 0, b)}{1 - \xi_1^{y_1} \eta_1^{b_1 - y_1} \psi(b - b_1, b, 0)} \\ &= \frac{(1 - (\xi_1 \rho_2^{-1})^{y_1} (\eta_1 \rho_2^{-1})^{b_1 - y_1}) - (1 - (\xi_1 \rho_1^{-1})^{y_1} (\eta_1 \rho_1^{-1})^{b_1 - y_1})}{(1 - (\xi_1 \rho_2^{-1})^{y_1} (\eta_1 \rho_2^{-1})^{b_1 - y_1}) \rho_1^{-b} - (1 - (\xi_1 \rho_1^{-1})^{y_1} (\eta_1 \rho_1^{-1})^{b_1 - y_1}) \rho_2^{-b}} \\ &= \frac{g(0)}{g(b)}. \end{aligned} \quad (2.16)$$

The first equation in (2.8) for any $x \in E$ follows by substituting (2.16) and (2.12) into (2.13). ■

Note that the Laplace transforms of the hitting times $T(0)$ and $T(b)$ are independent of the switching orders when one production process is turned off and Z is in E_1 or E_2 because the sojourn times in E_1 or E_2 will not change as long as y_i ($|K_i|$), and b_i , $i = 1, 2$, are fixed.

Proposition 2.3.

$$\begin{aligned} u_1(x) &= \frac{\lambda_1}{r} \cdot \frac{\theta(x, b)}{1 - \xi_1^{y_1} \eta_1^{b_1 - y_1} \theta(b - b_1, b)}, \quad u'_1(x) = \frac{\lambda'_1}{r} \cdot \frac{\theta(x, b)}{1 - \xi_1^{y_1} \eta_1^{b_1 - y_1} \theta(b - b_1, b)}, \\ u_2(x) &= \frac{\lambda_2}{r} \cdot \frac{\theta(x, 0)}{1 - \xi_2^{y_2} \eta_2^{b_2 - y_2} \theta(b_2, 0)}, \quad u'_2(x) = \frac{\lambda'_2}{r} \cdot \frac{\theta(x, 0)}{1 - \xi_2^{y_2} \eta_2^{b_2 - y_2} \theta(b_2, 0)}, \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} \lambda_i &\equiv \sum_{k=1}^{m_i} \eta_i \sum_{j=1}^k y'_{i,j} \xi_i^{\sum_{j=1}^{k-1} y_{i,j}} (1 - \xi_i^{y_{i,k}}), \\ \lambda'_i &\equiv \sum_{k=1}^{m_1} \xi_i \sum_{j=1}^{k-1} y_{i,j} \eta_i^{\sum_{j=1}^{k-1} y'_{i,j}} (1 - \eta_i^{y'_{i,k}}), \end{aligned} \quad (2.18)$$

for $i = 1, 2$.

Proof. By the definition of a barrier policy ((2.1) and (2.2)), for each n and k , $R_{1n}^k - S_{1n}^k$ is the sojourn time in the states, $\{1, \dots, y_{1k}\}$, for a Poisson process with intensity β , and $S_{1n}^{k+1} - R_{1n}^k$ is the sojourn time in the states, $\{1, \dots, y_{1k}\}$, for a Poisson process with intensity $\beta + \delta_1 \alpha$. Therefore,

$$E[e^{-r(R_{1n}^k - S_{1n}^k)}] = \xi_1^{y_{1k}}, \quad \text{and} \quad E[e^{-r(S_{1n}^{k+1} - R_{1n}^k)}] = \eta_1^{y_{1k}}.$$

Also note that $R_{1,n+1}^0 - R_{1,n}^0$, the time between process Z 's two consecutive entry of the state b , $n = 1, 2, \dots$, are *i.i.d.* random variables. Hence,

$$\begin{aligned} u_1(x) &\equiv E_x\left[\int_0^\infty e^{-rt} 1_{N_1}(t) dt\right] = \frac{1}{r} \sum_{n=1}^\infty \sum_{k=1}^{m_1} E_x[e^{-rS_{1n}^k} - e^{-rR_{1n}^k}] \\ &= \frac{1}{r} \sum_{n=1}^\infty \sum_{k=1}^{m_1} E_x[1 - e^{-r(R_{1n}^k - S_{1n}^k)}] E_x[e^{-rR_{1n}^0} e^{-r \sum_{j=1}^k (S_{1n}^j - R_{1n}^{j-1})} e^{-r \sum_{j=1}^{k-1} (R_{1n}^j - S_{1n}^j)}] \\ &= \frac{1}{r} \sum_{k=1}^{m_1} (1 - \xi_1^{y_{1k}}) \eta_1^{y'_{11} + \dots + y'_{1k}} \xi_1^{y_{11} + \dots + y_{1,k-1}} \sum_{n=1}^\infty E_x[e^{-rR_{1n}^0}] \\ &= \frac{\lambda_1}{r} E_x[e^{-rT(b)}] \sum_{n=0}^\infty \{E_b[e^{-r(S_{11}^{n+1} - R_{11}^0)}] E_{b-b_1}[e^{-rT(b)}]\}^n \\ &= \frac{\lambda_1}{r} \cdot \frac{\theta(x, b)}{1 - \xi_1^{y_1} \eta_1^{b_1 - y_1} \theta(b - b_1, b)}. \end{aligned}$$

The rest of the proposition can be similarly proved. ■

Propositions 2.1, 2.2, and 2.3 give us the explicit form of the value function under a barrier policy.

3. When to Nap and When to Help

In a barrier policy, if production of good i is stopped, then station i can have a break while station j ($j \neq i$) is producing its assigned y_j units. The question is when these coffeebreaks should be, assuming that the parameters b, b_1, b_2, y_1 and y_2 are fixed. In other words, we want to determine what is the best order combination among all the possible choices, $(K_{i1}, K_{i1}^1, \dots, K_{im_i}, K_{im_i})$, $i = 1, 2$.

Let

$$\begin{aligned} \hat{m}_1 &= 2, \hat{K}_{11} = \emptyset, \hat{K}'_{11} = \{b, b-1, \dots, b-y_1+1\}, \hat{K}_{12} = \{b-y_1, \dots, b-b_1+1\}, \hat{K}'_{12} = \emptyset, \\ \hat{m}_2 &= 1, \hat{K}_{21} = \{0, 1, \dots, y_2-1\}, \hat{K}'_{21} = \{y_2, \dots, b_2-1\}. \end{aligned} \tag{3.1}$$

Proposition 3.1. For fixed b , b_i , and y_i , $i = 1, 2$, the barrier policy with the switching order choice (3.1) is the dominant strategy.

Proof. Denote by $\hat{\lambda}_i$ and $\hat{\lambda}'_i$, $i = 1, 2$, the functions defined in (2.18) under the order combination (3.1), i.e.,

$$\begin{aligned}\hat{\lambda}_1 &= \eta_1^{b_1 - y_1} (1 - \xi_1^{y_1}), & \hat{\lambda}'_1 &= 1 - \eta_1^{b_1 - y_1}, \\ \hat{\lambda}_2 &= 1 - \xi_2^{y_2}, & \hat{\lambda}'_2 &= \xi_2^{y_2} (1 - \eta_2^{b_2 - y_2}).\end{aligned}\tag{3.2}$$

Notice that in the value function, the order choice, K_{11}, K'_{11}, \dots , affects the values of λ_1 and λ'_1 only, whereas the order choice, K_{21}, K'_{21}, \dots , affects the values of λ_2 and λ'_2 only. Therefore, it suffices to prove

$$\begin{aligned}\alpha w \hat{\lambda}_1 + \alpha(w + \delta_1 q) \hat{\lambda}'_1 - \alpha w \lambda_1 - \alpha(w + \delta_1 q) \lambda'_1 &\geq 0, \\ \beta q \lambda_2 + \beta(q + \delta_2 w) \lambda'_2 - \beta q \hat{\lambda}_2 - \beta(q + \delta_2 w) \hat{\lambda}'_2 &\geq 0.\end{aligned}\tag{3.3}$$

Also note that for any order choice,

$$\lambda_i + \lambda'_i = 1 - \xi_i^{y_i} \eta_i^{b_i - y_i},\tag{3.4}$$

and

$$1 - \eta_i^{b_i - y_i} = \sum_{k=1}^{m_i} \eta_i^{\sum_{j=1}^{k-1} y'_{ij}} (1 - \eta_i^{y'_{ik}}),\tag{3.5}$$

since $\sum_{k=1}^{m_i} y_{ik} = y_i$ and $\sum_{k=1}^{m_i} y'_{ik} = b_i - y_i$. Hence,

$$\begin{aligned}&\alpha w \hat{\lambda}_1 + \alpha(w + \delta_1 q) \hat{\lambda}'_1 - \alpha w \lambda_1 - \alpha(w + \delta_1 q) \lambda'_1 \\ &= \alpha \delta_1 q (\hat{\lambda}'_1 - \lambda'_1) \\ &= \alpha \delta_1 q \sum_{k=1}^{m_1} (1 - \xi_1^{\sum_{j=1}^{k-1} y_{1j}}) \eta_1^{\sum_{j=1}^{k-1} y'_{1j}} (1 - \eta_1^{y'_{1k}}) > 0, \\ &\beta q \lambda_2 + \beta(q + \delta_2 w) \lambda'_2 - \beta q \hat{\lambda}_2 - \beta(q + \delta_2 w) \hat{\lambda}'_2 \\ &= \beta \delta_2 w (\lambda'_2 - \hat{\lambda}'_2) \\ &= \beta \delta_2 w \sum_{k=1}^{m_2} \xi_2^{\sum_{j=1}^{k-1} y_{2j}} \eta_2^{\sum_{j=1}^{k-1} y'_{2j}} (1 - \xi_2^{\sum_{j=k}^{m_2} y_{2j}}) (1 - \eta_2^{y'_{2k}}) > 0,\end{aligned}$$

if $\delta_1 > 0$ and $\delta_2 > 0$. ■

Suppose the management feels that the upstream station deserves a break when the work-in-process inventory reaches a upper limit and the downstream station deserves a break when the inventory is down to zero. Proposition 3.1 suggests a “principle of coffeebreak” for the management of the flexible workstations: when the inventory hits the upper limit, the upstream station (station 1) should first switch to help the downstream station (station 2) and then take a break; when the

inventory is empty, the downstream station should take a break first and then switch to help the upstream station.

Proposition 3.1 rules out many dominated strategies and hence significantly reduces the strategy space in our search for the optimal barrier policy in the following section.

4. The Optimal Barrier Policy

Using Proposition 3.1, the set of barrier policies of interest can be characterized by five critical values, b , b_i and y_i , $i = 1, 2$. That is, station 1 stops its regular task and switches to help station 2 when the inventory level reaches the upper limit b , station 1 starts a break when the inventory level is down to $b - b_1 + y_1$ until station 2 works alone to bring inventory down to $b - b_1$, and at this time, station 1 takes on its regular task and the normal production resumes. On the other hand, station 2 starts a break right away when the inventory level hits zero until station 1 brings the inventory up to y_2 , at this point, station 2 switches to help station 1 bringing the inventory level further up to b_2 , and then switches back to the production of good 2 (its regular task).

By Propositions 2.1 - 2.3 and (3.2), we can write the value function under a dominant barrier policy in terms of the five critical values,

$$\hat{v}(x) = \frac{\alpha w \eta_1^{b_1 - y_1} (1 - \xi_1^{y_1}) + \alpha (w + \delta_1 q) (1 - \eta_1^{b_1 - y_1})}{r} \cdot \frac{d_* \rho_1^x - d^* \rho_2^x}{a^* d_* \rho_1^b - a_* d^* \rho_2^b} - \frac{\beta q (1 - \xi_2^{y_2}) + \beta (q + \delta_2 w) \xi_2^{y_2} (1 - \eta_2^{b_2 - y_2})}{r} \frac{a_* \rho_1^{-(b-x)} - a^* \rho_2^{-(b-x)}}{a_* d^* \rho_1^{-b} - a^* d_* \rho_2^{-b}}, \quad (4.1)$$

where $a_*(b_1, y_1)$, $a^*(b_1, y_1)$, $d_*(b_2, y_2)$, and $d^*(b_2, y_2)$ are defined in (2.11).

Proposition 4.1. *In an optimal barrier policy, it must have $y_1 = y_2 = 0$. In other words, an optimal policy must fully utilize the flexible resources.*

Proof. Suppose under an optimal barrier policy, we have $\alpha_t = 0$, and $\beta_t = \beta$ in certain state x ($x > 0$). Let T be the first time the inventory level decreases by one unit, $S \equiv T(T, x)$, the state reaches x again after T , and $u(x - 1, x)$ be the expected present value starting in state $x - 1$ and following the optimal policy over the period $(T, S]$. By the strong Markov and renewal properties of the process, we have

$$\begin{aligned} v(x) &= \frac{E_x[e^{-rT}]}{1 - E_x[e^{-rT}]E_{x-1}[e^{-rS}]} (p - c_2 - \frac{E_x[\int_0^T e^{-rt} dt]}{E_x[e^{-rT}]} xh + u(x - 1, x)) \\ &= \frac{\xi_1}{1 - \xi_1 E_{x-1}[e^{-rS}]} (p - c_2 - \frac{xh}{\beta} + u(x - 1, x)). \end{aligned}$$

Note that $v(x) \geq 0$ if and only if $p - c_2 - xh/\beta + u(x - 1, x) \geq 0$. There must be the case that $p - c_2 - xh/\beta + u(x - 1, x) \geq 0$ since there is a barrier policy that can guarantee zero profit.

Now, choose an alternative policy which follows the optimal policy except in the state x where $\beta_t = \beta + \delta_1 \alpha$. Denote by $v'(x)$ the value function under the alternative policy. Then

$$v(x) = \frac{\eta_1}{1 - \eta_1 E_{x-1}[e^{-rS}]} \left(p - c_2 - \frac{xh}{\beta + \delta_1 \alpha} + u(x-1, x) \right) > v(x)$$

since $\eta_1 > \xi_1$ and $p - c_2 - xh/(\beta + \delta_1 \alpha) + u(x-1, x) > p - c_2 - xh/\beta + u(x-1, x) \geq 0$. This contradicts that the nominal policy is optimal. Hence, in an optimal barrier policy, $\alpha_t = 0$ implies that $\beta_t = \beta + \delta_1 \alpha$. We can similarly prove that $\beta_t = 0$ implies that $\alpha_t = \alpha + \delta_2 \beta$. ■

By Propositions 4.1, we can further reduce the set of undominated strategies to be the barrier policies with three critical numbers b_i , $i = 1, 2$, and b by assuming $y_i = 0$, $i = 1, 2$. Under a dominant barrier policy, the value function

$$\hat{v}(x) = \frac{\alpha(w + \delta_1 q)(1 - \eta_1^{b_1})}{r} \cdot \frac{d_* \rho_1^x - d^* \rho_2^x}{a^* d_* \rho_1^b - a_* d^* \rho_2^b} - \frac{\beta(q + \delta_2 w)(1 - \eta_2^{b_2})}{r} \cdot \frac{a_* \rho_1^{-(b-x)} - a^* \rho_2^{-(b-x)}}{a_* d^* \rho_1^{-b} - a^* d_* \rho_2^{-b}}, \quad (4.2)$$

where

$$a_*(b_1) \equiv 1 - (\eta_1 \rho_2^{-1})^{b_1}, \quad a^*(b_1) \equiv 1 - (\eta_1 \rho_1^{-1})^{b_1}, \quad d_*(b_2) \equiv 1 - (\eta_2 \rho_2)^{b_2}, \quad d^*(b_2) \equiv 1 - (\eta_2 \rho_1)^{b_2}. \quad (4.3)$$

Define

$$l(b; b_1, b_2) \equiv a_*(b_1) d^*(b_2) \rho_2^b - a^*(b_1) d_*(b_2) \rho_1^b. \quad (4.4)$$

By Fact 5. in Appendix, we know that for any b, b_1, b_2 such that $b \geq b_1 \wedge b_2$, $l(b; b_1, b_2) > 0$, $l(b - b_1 + 1; 1, b_2)$ is strictly decreasing in b_1 , and $l(b - b_2 + 1; b_1, 1)$ is strictly decreasing in b_2 . Thus, we can define \bar{b}_1 be such that

$$l(b - \bar{b}_1 + 1; 1, b_2) \geq 0, \quad \text{and} \quad l(b - b_1; 1, b_2) \leq 0, \quad (4.5)$$

and \bar{b}_2 be such that

$$l(b - \bar{b}_2 + 1; b_1, 1) \geq 0, \quad \text{and} \quad l(b - b_2; b_1, 1) \leq 0. \quad (4.6)$$

Lemma 4.1. *For any x and fixed b, b_2 , there is an optimal barrier b_1^* such that $v(x; b_1^*) = \max\{v(x; b_1^{**}), v(x, b)\}$ where $v(x; y)$ is the value function with $b_1 = y$, and b_1^{**} is uniquely determined by*

$$k_1(b_1^{**} + 1) \leq \frac{q + \delta_2 \omega}{\omega + \delta_1 q}, \quad \text{and} \quad k_1(b_1^{**}) \geq \frac{q + \delta_2 \omega}{\omega + \delta_1 q}. \quad (4.7)$$

$$\begin{aligned}
k_1(b_1) &\equiv \frac{\alpha(m_*(b_1)d^*(b_2)\rho_1^{-b} - m^*(b_1)d_*(b_2)\rho_2^{-b})}{\beta(1 - \eta_2^{b_2})a(b_1)}, \\
a(b_1) &\equiv a_*(b_1)a^*(b_1 - 1) - a_*(b_1 - 1)a^*(b_1), \\
m_*(b_1) &\equiv a_*(b_1 - 1)(1 - \eta_1^{b_1}) - a_*(b_1)(1 - \eta_1^{b_1 - 1}), \\
m^*(b_1) &\equiv a^*(b_1 - 1)(1 - \eta_1^{b_1}) - a^*(b_1)(1 - \eta_1^{b_1 - 1}),
\end{aligned} \tag{4.8}$$

and k_1 is strictly decreasing in b_1 for $b_1 \leq \bar{b}_1$ and is increasing in b_1 for $b_1 \geq \bar{b}_1$.

Proof. For fixed b and b_2 , compute

$$v(x; b_1) - v(x; b_1 - 1) = K_1 \left(k_1(b_1) - \frac{q + \delta_2 \omega}{\omega + \delta_1 q} \right),$$

where

$$K_1 \equiv - \frac{\rho_1^b \rho_2^b (\omega + \delta_1 q) \beta (1 - \eta_2^{b_2}) a(b_1) e(x)}{r l(b; b_1, b_2) l(b; b_1 - 1, b_2)} > 0$$

since $a(b_1) > 0$, $e(x) < 0$ (defined as in (2.9)), and $l(b; b_1, b_2) > 0$ by Facts 2., 4., and 5. in Appendix.

We also notice that

$$\begin{aligned}
k_1(b_1) - k_1(b_1 - 1) &= \frac{\alpha \eta_1^{3b_1 - 5} (1 - \eta_1) \rho_1^{-(b - b_1 + 2)} \rho_2^{-(b - b_1 + 2)} a_*(1) a^*(1) l(b - b_1 + 1; 1, b_2)}{\beta (1 - \eta_2^{b_2}) a(b_1) a(b_1 - 1)} \\
&\quad \cdot \left[\sum_{n=0}^{b_1 - 3} \eta_1^n ((1 - \rho_2^{-1}) \rho_1^{b_1 - 2 - n} - (1 - \rho_1^{-1}) \rho_2^{b_1 - 2 - n} - (\rho_1^{-1} - \rho_2^{-1})) \right] < 0
\end{aligned}$$

if and only if $l(b - b_1 + 1; 1, b_2) > 0$ since $a_*(1) > 0$, $a^*(1) < 0$, $a(b_1) > 0$, and $(1 - \rho_2^{-1}) \rho_1^x - (1 - \rho_1^{-1}) \rho_2^x > \rho_1^{-1} - \rho_2^{-1}$ for $x \geq 1$ by Facts 1., 3., and 4. in Appendix. Therefore, $k_1(b_1)$ is decreasing for $b_1 \leq \bar{b}_1$ and is increasing for $b_1 \geq \bar{b}_1$, and there are two possible local optimal points, b_1^{**} (defined in (4.7)) and b (the corner solution). ■

Lemma 4.2. For any x and fixed b , b_1 , there is an optimal barrier b_2^* such that $v(x; b_2^*) = \max\{v(x; b_2^{**}), v(x, 1)\}$ where $v(x; y)$ is the value function with $b_2 = y$. and b_2^{**} is uniquely determined by

$$k_2(b_2^* + 1) \leq \frac{q + \delta_2 \omega}{\omega + \delta_1 q}, \quad \text{and } k_2(b_2^*) \geq \frac{q + \delta_2 \omega}{\omega + \delta_1 q}, \tag{4.9}$$

where

$$\begin{aligned}
k_2(b_2) &\equiv \frac{\alpha(1 - \eta_1^{b_1})d(b_2)}{\beta((n^*(b_2)a_*(b_1)\rho_2^b - n_*(b_2)a^*(b_1)\rho_1^b))}, \\
d(b_2) &\equiv d_*(b_2 - 1)d^*(b_2) - d_*(b_2)d^*(b_2 - 1), \\
n^*(b_2) &\equiv d^*(b_2 - 1)(1 - \eta_2^{b_2}) - d^*(b_2)(1 - \eta_2^{b_2 - 1}), \\
n_*(b_2) &\equiv d_*(b_2 - 1)(1 - \eta_2^{b_2}) - d_*(b_2)(1 - \eta_2^{b_2 - 1}),
\end{aligned} \tag{4.10}$$

and k_2 is strictly increasing in b_2 for $b_2 \leq \bar{b}_2$ and is strictly increasing for $b_2 \geq \bar{b}_2$.

Proof. For fixed b and b_1 , compute

$$v(x; b_2) - v(x; b_2 - 1) = K_2 \left(k_2(b_2) - \frac{q + \delta_2 \omega}{\omega + \delta_1 q} \right),$$

where $v(x; y)$ is the value function with $b_2 = y$,

$$K_2 \equiv - \frac{\rho_1^b \rho_2^b (\omega + \delta_1 q) \beta (n^*(b_2) a_*(b_1) \rho_2^b - n_*(b_2) a^*(b_1) \rho_1^b) g(x)}{rl(b; b_1, b_2) l(b; b_1, b_2 - 1)} > 0,$$

since $g(x) > 0$ (defined as in (2.9)), $l(b; b_2, b_2) > 0$, and

$$n^*(b_2) a_*(b_1) \rho_2^b - n_*(b_2) a^*(b_1) \rho_1^b > 0$$

due to the facts $a_* > 0$, $a^* < 0$, $n^*(b_2) > 0$ and $n_*(b_2) > 0$ by Facts 2., 3., and 4. in Appendix.

We also have

$$\begin{aligned} k_2(b_2) - k_2(b_2 - 1) &= \frac{\alpha \eta_2^{3b_1 - 5} (1 - \eta_2) (1 - \eta_1^{b_1}) \rho_1^{2b_2 - 3} \rho_2^{2b_2 - 3} d_*(1) d^*(1) l(b - b_2 + 1; b_1, 1)}{\beta ((n^*(b_2) a_*(b_1) \rho_2^b - n_*(b_2) a^*(b_1) \rho_1^b) ((n^*(b_2 - 1) a_*(b_1) \rho_2^b - n_*(b_2 - 1) a^*(b_1) \rho_1^b))} \\ &\quad \cdot \left[\sum_{n=0}^{b_2 - 3} \eta_2^n (\rho_2 - \rho_1 - ((\rho_2 - 1) \rho_1^{-(b_2 - 2 - n)} - (\rho_1 - 1) \rho_2^{-(b_1 - 2 - n)}) \right] > 0 \end{aligned}$$

if and only if $l(b - b_1 + 1; 1, b_2) > 0$ since $d_*(1) < 0$, $d^*(1) > 0$, and $\rho_2 - \rho_1 < (\rho_2 - 1) \rho_1^{-(b_2 - 2 - n)} - (\rho_1 - 1) \rho_2^{-(b_1 - 2 - n)}$ for $x \geq 1$ by Facts 1., 3., and 4. in Appendix.

We conclude the proof by the same argument as in the proof of Lemma 4.1. ■

Lemma 4.3. For any x and fixed b_i , $i = 1, 2$, there is unique optimal barrier b^* determined by

$$k(b^* + 1) \geq \frac{q + \delta_2 \omega}{\omega + \delta_1 q}, \quad \text{and} \quad k(b^*) \leq \frac{q + \delta_2 \omega}{\omega + \delta_1 q}. \quad (4.11)$$

where

$$k(b) \equiv \frac{\alpha (1 - \eta_1^{b_1}) (a_*(b_1) d^*(b_2) (1 - \rho_2^{-1}) \rho_1^{-b} - a^*(b_1) d_*(b_2) (1 - \rho_1^{-1}) \rho_2^{-b})}{\beta (1 - \eta_2^{b_2}) a_*(b_1) a^*(b_1) (\rho_2^{-1} - \rho_1^{-1})}, \quad (4.10)$$

and k is a strictly increasing function of b .

Proof. It can be calculate that

$$v(x; b) - v(x; b - 1) = K \left(\frac{q + \delta_2 \omega}{\omega + \delta_1 q} - k(b) \right),$$

where $v(x; y)$ is defined in (4.2) with $b = y$,

$$K \equiv - \frac{\rho_1^b \rho_2^b c(x) (\omega + \delta_1 q) \beta a_*(b_1) a^*(b_1) (\rho_2^{-1} - \rho_1^{-1})}{rl(b; b_1, b_2) l(b - 1; b_1, b_2)} > 0,$$

since $a_* > 0$, $a^* < 0$, $e(x) < 0$ and $\rho_1 < \rho_2$.

■

Proposition 4.2. *There exists an optimal barrier policy with three critical numbers b^* , b_i^* , $i = 1, 2$, which jointly satisfy conditions in Lemma 4.1.-4.3.*

Proof. Given the monotonicity properties shown in Lemma 4.1 - 4.3, it can be shown that $b^*(b_1, b_2)$ determined by (4.11) is bounded independent of the choice of b_i , and hence, there always a fixed point such that conditions in Lemma 4.1.-4.3. are satisfied.

■

5. Conclusion

In this paper, we propose a stochastic control model to study the managerial issues regarding flexible resources. With respect to a class of feasible policies, namely barrier policies, we present general computational results, identify the dominating strategies in the real time allocation of the flexible workstations which can serve as a principle in the management of the flexible production system, and find the conditions which explicitly determine the optimal barrier policy. The importance of the barrier policies lies on the fact that the optimal barrier policy is an optimal Markovian policy in the sense that the state space consists of inventory and workstation status. However, the completeness of the optimal barrier policy (whether it is optimal among all the adapted policies) needs further investigation.

Appendix

We list the following facts as the reference of Section 4. and most of the proofs are immediate.

Lemma 5.1.

$$\begin{aligned} a_*(1) &\equiv 1 - \eta_1 \rho_2^{-1} = \frac{\alpha(\rho_2 + \delta_1)}{\beta + \delta_1 \alpha + r} (1 - \rho_2^{-1}) > 0, \\ a^*(1) &\equiv 1 - \eta_1 \rho_1^{-1} = \frac{\alpha(\rho_1 + \delta_1)}{\beta + \delta_1 \alpha + r} (1 - \rho_1^{-1}) < 0, \\ d_*(1) &\equiv 1 - \eta_2 \rho_2 = \frac{\beta(\rho_2^{-1} + \delta_2)}{\alpha + \delta_2 \beta + r} (1 - \rho_2) < 0, \\ d^*(1) &\equiv 1 - \eta_2 \rho_1 = \frac{\beta(\rho_1^{-1} + \delta_2)}{\alpha + \delta_2 \beta + r} (1 - \rho_1) > 0. \end{aligned}$$

Lemma 5.2. $a_*(\cdot)$ and $d^*(\cdot)$ are strictly increasing and $a^*(\cdot)$ and $d_*(\cdot)$ are strictly decreasing.

Lemma 5.3. $a(b_1), m_*(b_1), m^*(b_1), d(b_2), n_*(b_2)$ and $n^*(b_2)$ are positive and increasing for $b_i \geq 1$, $i = 1, 2$. Consequently, $a_*(b_1)/a^*(b_1)$, $(1 - \eta_1^{b_1})/a_*(b_1)$ and $(1 - \eta_1^{b_1})/a^*(b_1)$ are increasing in b_1 , and $d^*(b_2)/d_*(b_2)$, $(1 - \eta_2^{b_2})/d_*(b_2)$ and $(1 - \eta_2^{b_2})/d^*(b_2)$ are increasing in b_2 .

Proof.

$$\begin{aligned} a(b_1) &\equiv a_*(b_1)a^*(b_1 - 1) - a_*(b_1 - 1)a^*(b_1) \\ &= \eta_1^{b_1-1} a_*(1)a^*(1) \sum_{n=0}^{b_1-2} (\eta_1 \rho_1^{-1} \rho_2^{-1})^n (\rho_2^{-(b_1-1-n)} - \rho_1^{-(b_1-1-n)}) > 0, \\ m_*(b_1) &\equiv a_*(b_1 - 1)(1 - \eta_1^{b_1}) - a_*(b_1)(1 - \eta_1^{b_1-1}) \\ &= \eta_1^{b_1-1} (1 - \eta_1) a_*(1) \sum_{n=0}^{b_1-2} (\eta_1 \rho_2^{-1})^n (1 - \rho_2^{-(b_1-1-n)}) > 0, \\ m^*(b_1) &\equiv a^*(b_1 - 1)(1 - \eta_1^{b_1}) - a^*(b_1)(1 - \eta_1^{b_1-1}) \\ &= \eta_1^{b_1-1} (1 - \eta_1) a^*(1) \sum_{n=0}^{b_1-2} (\eta_1 \rho_1^{-1})^n (1 - \rho_1^{-(b_1-1-n)}) > 0, \\ d(b_2) &\equiv d_*(b_2 - 1)d^*(b_2) - d_*(b_2)d^*(b_2 - 1) \\ &= \eta_2^{b_2-1} d_*(1)d^*(1) \sum_{n=0}^{b_2-2} (\eta_2 \rho_1 \rho_2)^n (\rho_1^{b_2-1-n} - \rho_2^{b_2-1-n}) > 0, \\ n^*(b_2) &\equiv d^*(b_2 - 1)(1 - \eta_2^{b_2}) - d^*(b_2)(1 - \eta_2^{b_2-1}) \\ &= \eta_2^{b_2-1} (1 - \eta_2) d^*(1) \sum_{n=0}^{b_2-2} (\eta_2 \rho_1)^n (1 - \rho_1^{b_2-1-n}) > 0, \\ n_*(b_2) &\equiv d_*(b_2 - 1)(1 - \eta_2^{b_2}) - d_*(b_2)(1 - \eta_2^{b_2-1}) \\ &= \eta_2^{b_2-1} (1 - \eta_2) d_*(1) \sum_{n=0}^{b_2-2} (\eta_2 \rho_2)^n (1 - \rho_2^{b_2-1-n}) > 0. \end{aligned}$$

■

Lemma 5.4. $g(x) > 0$, $e(x) < 0$, $g(x)$ is increasing and $e(x)$ is decreasing in x .

Lemma 5.5. For any b, b_1, b_2 such that $b \geq b_1 \wedge b_2$, $l(b; b_1, b_2) > 0$, $l(b - b_1 + 1; 1, b_2)$ is strictly decreasing in b_1 , and $l(b - b_2 + 1; b_1, 1)$ is strictly decreasing in b_2 .

Proof. By Proposition 2.3.,

$$\begin{aligned} u'_1(x) &= E_x \left\{ \int_0^\infty e^{-rt} 1_{N'_1}(t) dt \right\} \\ &= \frac{1 - \eta_1^{b_1}}{r} \cdot \frac{d_*(b_2)\rho_1^x - d^*(b_2)\rho_2^x}{a^*(b_1)d_*(b_2)\rho_1^b - a_*(b_1)d^*(b_2)\rho_2^b} \\ &= -\frac{1 - \eta_1^{b_1}}{r} \cdot \frac{\rho_1^{-b}\rho_2^{-b}e(x)}{l(b; b_1, b_2)} > 0, \end{aligned}$$

which implies $l(b; b_1, b_2) > 0$ since $e(x) < 0$.

The second assertion follows from that

$$\begin{aligned} &l(b - b_1 + 1; 1, b_2) - l(b - b_1; 1, b_2) \\ &= a_*(1)d^*(b_2)(1 - \rho_2)\rho_2^{b-b_1+1} - a^*(1)d_*(b_2)(1 - \rho_1)\rho_1^{b-b_1+1} < 0. \end{aligned}$$

The third assertion can be similarly proved.

■

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