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SOME MODELS OF ORGANIZATION RESPONSE TO BUDGETED MULTIPLE GOALS

by

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1. Introduction

This paper represents a further attempt to construct models to predict the behavior of individuals who are faced with effort allocation decisions among several areas which compete for his total effort. While we have chosen to impute to these individuals certain kinds of maximizing intent the resulting optimal behavior in fact resembles satisficing[1] in some of these models. In the models to be presented here, as well as in our earlier paper [24], the models are constructed utilizing knowledge or theory of human behavior which may be found in or inferred from the psychological literature where such knowledge or theory exists.[2] Inasmuch as psychologists have concerned themselves little with testing the way in which aspiration levels or goals affect task performance where only one performance area is involved, and even less where multiple tasks exist, much of the hypothesizing is based upon what "reasonable" behavior would be in light of what evidence and theory is available.

Each of the models is constructed upon a set of assumptions about what an individual wants to do--e.g. minimize the risk of unacceptable performance--and proceeds to an optimal pattern of effort allocation for him should he be in fact working to achieve his desire. The optimal behavior patterns so deduced may be used to formulate hypotheses for experimental test. If, for example, in a supervisor-subordinate relationship, the supervisor presents a set of goals to his subordinate, the latter may strive to achieve as many of

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[1]This term was coined by H. A. Simon. For a thorough treatment, see his [21]; also see Charnes and Cooper [5] for similar chance-constrained formulations.

[2]It should be pointed out that maximizing behavior assumptions are not foreign to the psychological literature. See Lewin, Dembo, Festinger and Sears [15], Davidson, Siegel and Suppes [9], Mosteller and Nogee [18] and Edwards [11]. More closely related to this paper, Radner [19] has presented a linear programming formulation of the effects of two competing activities on optimal resource allocation for profit maximization.

the goals as possible. Alternatively, he may strive to increase his chance of achieving all of them <u>or</u> he may strive to achieve as many as possible but with particular emphasis on certain goal areas he perceives as critical and in which he is willing to accept only a limited risk of non-attainment. Observation of the goal structure as well as questioning of the subordinate as to his perception of it and observational and interview determination of the effort allocation will serve, with the aid of the models, to provide insight into the actual decision rules through which he responds to a set of goals.

It is not suggested that an individual's non-job-oriented personal goals do not serve as a determinant of his behavior. These goals may be provided for explicitly by arraying them alongside of the supervisor's goals. Alternatively, they may be provided for implicitly in terms of the limitation on overall effort which the subordinate is willing to allocate among the performance areas for which goals have been set by his supervisor. In addition, the individual's personal goals are expected to affect the function of the supervisor's goals he chooses to strive to attain. To utilize a concept in aspiration level theory[1] people whose personal goals reflect a pattern of "hope of success" insight be inclined toward a function of externally provided goals in which he attempts to attain as many goals which he knows to be important as possible. A person whose pattern was "fear of failure" might alternatively be expected to attempt to reduce his risk of turning in unacceptable performance--i.e. attempt to make the probability of attainment of all of the goals set for him as high as possible--even though in so doing he knowingly allocates more effort to unimportant performance areas than they merit.

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[1]See, for example, Becker and Siegel [3], Becker and Parsons [2], and Rotter [20].

It has been frequently suggested that goals should be set by a supervisor only with the participation of the subordinate.[1] Recent evidence obtained by French, Kay and Meyer [12] indicates that the presence or absence of a specific goal in an area appears to affect performance improvement while significant effects on performance improvement depending upon whether or not goals were set participatively could not be shown. Evidence obtained by one of the authors ([22] and [23]) indicates that amount of public commitment[2] obtained to a presented goal appears to affect performance, but that the magnitude of the goal is also of importance. One is therefore led, on the basis of what evidence is available, to investigate how the number of goals set and their magnitudes (and what these magnitudes represent in terms of the amount of effort required to assure a given probability of attainment) affect performance. Should participation in the goal setting process be shown to affect performance in certain situations, the effects can certainly be expressed in terms of additional effort the individual will make available to the areas in which goals have been set or to the relative weights an individual might attach to goal attainment in areas in which goals are set participatively as opposed to areas in which they are set unilaterally (by his supervisor). In either case the effects of participation could be incorporated without change in the models of this paper. In fact, the models could be used as an aid to more precise evaluation of the effects of participation on goal attainment.[3]

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[1]See, for example, McGregor [17], Argyris [1].

[2]In the sense used by Deutsch and Gerard [10].

[3]Evidence presented by Hoppe [13] from study of children indicates that if explicit rewards are provided for goal attainment the goals set by the children are lowered. Recognizing difficulties in extrapolating these results to adults in an organizational setting, a caveat is suggested that, in tests of participation, measures of performance must be carefully chosen to guard against a confounding of observed performance improvement with a (possible) lowering of the goals with which performance is compared in the participative setting.

In the current paper, we present first a model of expected performance maximization where performance is assumed to be a weighted sum of performance in several areas. The performance in each area is assumed to be a stochastic function of the effort allocated to the area and a constraint is imposed on total effort. We then proceed to show that an equivalent problem can be constructed with explicit goals in each of the areas and rewards for attainment of them. Thus, if a supervisor would wish to have a subordinate maximize expected performance but explicit goals are more meaningful to the subordinate, the transition from one criterion to the other is readily obtained.

Having established this equivalence, we investigate the situation in the case of an individual who wishes either to maximize expected performance or to maximize his expected reward through goal attainment, but subject to a set of limitations on the risk he is willing to take on the non-attainment of some minimum acceptable level in certain (or all) of the performance areas. We show that this problem can be readily transformed into the expected performance or expected reward maximization where the individual goal areas are not so constrained. The transformation involved is equivalent to placing a lower bound on the amount of effort allocated to each of the constrained performance areas and optimizing over the set of pseudovariables which represent the value of the variable decreased by their lower bounds. This process is analogous to that used in dealing with lower bounds in the bounded variables problem of linear programming.[1]

Finally, we present a model in which the individual wishes to maximize the probability of the joint occurrence of goal attainment in all of the performance areas. This problem is analogous to that of a manufacturer who,

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[1]See Charnes and Lemke [7] and Charnes and Cooper [4].

producing a product on special order or contract, wishes to maximize the probability of meeting all of the specifications the customer lays down for acceptance of the item. Although beyond the scope of this paper, this model is directly relevant to the question of producer's risk in quality control and may be understood as a form of risk aversion if interpreted in terms of economic theory.

In each of the models constructed, a computational algorithm is provided for obtaining the optimal solution. Since each of the models starts from a different assumption as to the motivation of behavior, the observed performances and effort allocations can be compared with the computed predictions of the models (and others) in order to assess the prediction value of the competing behavioral assumptions.[1]

[1]For a discussion of the kinds of organizational goals which are reasonable to consider and a justification of investigation of models of maximizing behavior with respect to various goals, see Marschak [16].

2. Performance as a Stochastic Function of Effort

In order that we may construct a framework within which to study the effects on effort allocation of differing performance criteria, it is necessary to make some assumptions about the response of performance in each of the performance areas to the amount of effort applied. It is desired that this response be stochastic--i.e. that increase in effort will result in greater expected performance and greater probability of performance at or above a particular level--rather than a deterministic response that can be predicted with certainty. Further, in the interest of realism, it is desirable that no finite amount of effort can assure performance at a given level (above some minimal level that can be achieved without effort expenditure). Finally, it is desired that the function portray diminishing returns to additional effort as effort is increased.

A distribution which possesses all of the desirable attributes and will be used here is a modification of that presented by Charnes and Cooper in [ ].

Assume that the probability that performance in the j^{th} performance area, s_j, is greater than or equal to some amount a_{ij} may be expressed as a function of the form:

(1a) $$P(s_j \geq a_{ij}) = k_{ij}(1 - e^{-\alpha_j \rho_j})$$

where

$$i = 1,\ldots,\infty$$

$$j = 1,\ldots,n$$

$$k_{ij} \geq k_{\ell j} \Longrightarrow i \leq \ell$$

and

(1b) $$P(s_j \geq a_{oj}) = 1$$

where

$$a_{ij} > a_{oj} \text{ for } i > 0$$

and where ρ_j is the amount of effort allocated to the j^{th} performance variable, $\alpha_j > 0$ a given constant, and $0 \leq k_{ij} < 1$. It is seen that:

$$(2) \qquad k_{ij} \equiv \lim_{\rho_j \to \infty} P(s_j \geq a_{ij})$$

that is, k_{ij} is the limiting probability of attainment of the i^{th} performance level as effort is increased without bound. It is assumed that k_{ij} is a function of past performance and technology and hence independent of any decision in the current period. Also, as $\rho_j \to 0$, the probability of attaining any level above some minimum level a_{oj} (which can be attained with no managerial effort allocated to the j^{th} variable) approaches zero. The positive constant α_j may be interpreted as the relative "sensitivity" of the j^{th} variable to increased effort, i.e., the rate at which the limiting probability is approached as ρ_j is increased.

For use in a short-run model, this formulation would appear inherently sound. It includes both a technological constraint on possible performance independent of the manager's activity and a variable portion over which he does have control. Product interdependence can be introduced to a limited extent through a constraint on overall managerial effort, namely

$$(3) \qquad \sum_j \rho_j \leq \rho$$

The distribution function defined above may be used in the construction of models of many types of goal-oriented managerial behavior, one of which we have presented in [24]. In order that it may be utilized in the construction of models dealing with the expected values of performance in a particular area we note from (1a) and (1b) that:

$$(4) \qquad P(s_j = a_{ij}) = P(s_j \geq a_{ij}) - P(s_j \geq a_{i+1,j})$$

$$= (k_{ij} - k_{i+1,j})(1 - e^{-\alpha_j \rho_j}) \quad \text{where } i \geq 1$$

and

$$(5) \qquad P(s_j = a_{1j}) = 1 - k_{1j}(1 - e^{-\alpha_j \rho_j}) \qquad \text{where } i = 0$$

A Profit Maximization Model

A possible objective for a decision on effort allocation might be profit maximization in the classical sense, where it may be assumed that each unit of performance has a price and a variable cost, their difference representing the contribution of one unit of performance to fixed cost and profit. Let us assume that management wishes to maximize expected profit, represented by a sum of the expected performances in n areas weighted by the relative contributions of a unit of performance in each of the areas. I.e., assume that management wishes to:

$$\text{Max} \quad E(\pi) = \sum_{j=1}^{n} \beta_j E(s_j)$$

$$(6) \qquad \text{subject to} \sum_{j=1}^{n} \rho_j \leq \rho$$

$$\rho_j \geq 0$$

Returning to Equations (4) and (5) we obtain for the expected values of performance in the j^{th} area:

$$(7) \quad E(s_j) = \sum_{i=1}^{\infty} (k_{ij} - k_{i+1,j})(1 - e^{-\alpha_j \rho_j}) a_{ij} + [1 - k_{1j}(1 - e^{-\alpha_j \rho_j})] a_{oj}.$$

Let us define S_j as the limit of the expected value of performance in the j^{th} area as $\rho_j \longrightarrow \infty$, or:

(8) $$\tilde{s}_j = \sum_{i=1}^{\infty} (k_{ij} - k_{i+1,j})\, a_{ij} + (1 - k_{ij}) a_{oj}$$

Then (7) becomes

(9) $$E(s_j) = \tilde{s}_j - e^{-\alpha_j \rho_j} [\tilde{s}_j - a_{oj}]$$

and

(10) $$E(\pi) = \sum_{j=1}^{n} \beta_j E(s_j) = \sum_{j=1}^{n} \left\{ \beta_j \tilde{s}_j - \beta_j e^{-\alpha_j \rho_j} [\tilde{s}_j - a_{oj}] \right\}$$

Performing, a further substitution,

(11) $$\eta_j = \beta_j [\tilde{s}_j - a_{oj}]$$

And noting that $\sum_{j=1}^{n} \beta_j \tilde{s}_j$ is independent of the effort allocation process, we can express management's profit maximization problem as:

(12) $$\text{Min} \quad \sum_{j=1}^{n} \eta_j e^{-\alpha_j \rho_j}$$
$$\text{subject to } \sum \rho_j = \rho$$
$$\rho_j \geq 0$$

noting that the strict monotonicity properties of the $e^{-\alpha_j \rho_j}$ insure that the optimal allocation will allocate all of the available effort, ρ.

The profit minimization problem under the specified conditions has thus been reduced to the same convex programming problem we have presented in [ ]. The solution, adapted from an earlier solution presented by Charnes and Cooper [6], follows from the Kuhn-Tucker theorem whose [14] conditions become:

(13) $$\alpha_j e^{-\alpha_j \rho_j} \eta_j = \phi \qquad j \in J$$
$$\alpha_j \eta_j = \phi - \psi_j \qquad j \notin J , \ \psi_j \geq 0$$

where $J = \left\{ j \mid \rho_j > 0 \right\}$.

Now let

(14) $$\hat{\gamma}_j = \ell n(\alpha_j \eta_j)$$

so that (13) becomes:

$$\hat{\gamma}_j = \alpha_j \rho_j = \ell n \phi \qquad j \in J$$

$$\hat{\gamma}_j \leqq \ell n \phi \qquad j \notin J$$

Eliminating the algebra, which has been presented in the earlier paper, it obtains that:

(15) $$\ell n \phi = \frac{1}{\Sigma \frac{1}{\alpha_j}} \left[\sum_{j \in J} \frac{\hat{\gamma}_j}{\alpha_j} - \rho \right]$$

so that conditions for the optimal higher management choice of those areas to which effort should be allocated--i.e., necessary and sufficient conditions for the selection of indices $r \in J$ --become:

(16) $$\min_{r \in J} \hat{\gamma}_r > \frac{1}{\sum_{j \in J} \frac{1}{\alpha_j}} \left[\sum_{j \in J} \frac{\hat{\gamma}_j}{\alpha_j} - \rho \right] \geqq \max_{s \notin J} \hat{\gamma}_s$$

and optimal ρ^*_r, $r \in J$ and ψ^*_s, $s \notin J$ become

(17) $$\rho^*_r = \frac{1}{\alpha_r} \left[\hat{\gamma}_r - \frac{1}{\sum_{j \in J} \frac{1}{\alpha_j}} \left(\sum_{j \in J} \frac{\hat{\gamma}_j}{\alpha_j} - \rho \right) \right]$$

and $$\psi^*_s = \phi - \alpha_j \eta_j$$

4. Reward Maximization

This model has been the subject of our previous explorations [24] but will be sketched here for completeness. Utilizing the same performance distribution function described by equations (4) and (5), we again assume that attainment of a particular level of performance is a function of the effort allocated to the area. However, we assume that the reward associated with attainment is a function of the number of areas in which a particular "critical" level of performance is achieved, weighted by some set of relative values which could be assigned to the various areas.

This type of reward function can be related to the psychologist's conception of aspiration level where the latter is defined as that level of performance at or above which an individual perceives "success" and below which he perceives "failure." Thus, a student enrolled in several courses might consider the grade of B to represent success in mathematics but a grade of A success in English and C success in thermodynamics. However, the relative value to him of success in each of these areas might be determined by the presumed importance of each to his major field, mechanical engineering, which would be in the order: thermodynamics, mathematics, English. If numerical values could be determined for these relative importance parameters, as well as the probability parameters, our function could be constructed. The student's problem would be to allocate his effort in such a way as to maximize his expected success over the various subject areas in which he can achieve it.

We will, however, construct our model utilizing the terminology of the budgeting process as practiced in many industrial organizations. Here, for each supervisor above a certain level in the organization, a series of budgets (or standards) are devised which serve as "acceptable" levels of performance

in certain areas. For example, there might be budgets for overhead cost, scrap, overtime cost and production. There is, presumably, a difference in importance attaching to the attainment of each one of these budgets or goals and hence in the amount of reward or punishment (explicit or otherwise) a supervisor can expect depending upon which goals he attains or does not attain.

Clearly if a subordinate is to be judged against a standard, some result must be expected from the judgment. That these rewards and punishments are not usually meted out period by period is not a serious criticism of this approach inasmuch as these cumulative effects should be in some sense significant (although perhaps subject to a form of revenue discounting, later attainments being seen as more significant than early ones). We shall adhere to the usual practice of judgment--i.e., that black (or positive) variances are "good," red (or negative) variances "bad"--where the degree of redness and blackness is not considered formally. It is assumed that a reward r_j is administered for attainment, a punishment p_j for non-attainment, of the j^{th} product budget. A subordinate's expected reward may be expressed, then, as:

$$R_t = R_t^o + \sum_{j=1}^{n} r_j E(z_j^+) - \sum_{j=1}^{n} p_j E(z_j^-) \tag{18}$$

where z_j^+ (resp. z_j^-) is 1 (resp. 0) if $s_{jt} \geq b_{jt}$ and 0 (resp. 1) otherwise, s_{jt} and b_{jt} are, respectively, the actual performance and budgeted performance for the j^{th} area in the t^{th} period, R_t is the subordinate's expected reward in the t^{th} period and r_j and p_j the amounts which are added to or subtracted from the base reward R_t^o in the event of attainment or non-attainment, respectively, of budgeted performance in the j^{th} area to obtain the reward for the period. A more thorough discussion of this function may be found in [24].

If we now look only at the t^{th} period, we may drop the t subscript. We then assume that management can select for b_j one of the levels of attainment a_{ij} for which a probability of attainment has been defined in equations (1a) and (1b), say $a_{b_j j}$. The probability of budget attainment may then be expressed as[1]

$$(19) \qquad P(s_j \geq a_{b_j j}) = \begin{cases} k_{b_j j}\,(1 - e^{-\alpha_j \rho_j}) & , \quad b_j = 1,\ldots,\infty \\ 1 & , \quad b_j = 0\ . \end{cases}$$

Thus $b_j = 0$ is seen to be the index of a budget attainable with or without effort allocation. All other budgets are attainable with some probability less than 1, depending upon the amount of effort allocated to their attainment.

Returning to equation (18) we note that:

$$(20a) \qquad E(z_j^+) = (1)\ P(s_j \geq a_{b_j j}) + (0)P(s_j < a_{b_j j})$$

and

$$(20b) \qquad E(z_j^-) = (0)\ P(s_j \geq a_{b_j j}) + (1)P(s_j < a_{b_j j})$$

so that eliminating that subscript, the expected one-period reward function becomes:

$$(21) \qquad E(R) = R_o + \sum_{j=1}^{n} r_j P(s_j \geq a_{b_j j}) - \sum_{j=1}^{n} p_j\ P(s < a_{b_j j}).$$

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[1]Because we are using a discrete distribution, $a_{b_j j}$ may be considered to represent, in fact, a range about the stated budget which is considered to be just attaining the budget. The usual application of the "principle of exceptions" in fact allows such a range (or threshold) before an "exception" is recognized.

We shall number the performance areas such that:

$$(22)\qquad \begin{aligned} &\text{where} && b_j > 0\ , && j = 1,\ldots,m \\ &\text{and where} && b_j = 0\ , && j = m+1,\ldots,n \end{aligned}$$

Clearly those products for which $b_j = 0$ simply contribute to the fixed reward and the rational manager will not allocate effort to them provided there is at least one product for which $b_j > 0$, which shall be assumed also.[1] Eliminating fixed rewards, the manager, if he acts rationally, will attempt to:

$$(23)\qquad \begin{aligned} &\text{Minimize} && C = \sum_{j=1}^{m} (r_j + p_j)\, k_{b_j j}\, e^{-\alpha_j \rho_j} \\ &\text{subject to} && \sum_{j=1}^{m} \rho_j = \rho \end{aligned}$$

Thus he attempts to obtain the maximum reward for the effort he expends. For simplicity, let:

$$(24)\qquad h_j = (r_j + p_j)\, k_{b_j j} \qquad j = 1,\ldots,m.$$

Then the problem can be stated:

$$(25)\qquad \begin{aligned} &\text{Minimize} && C = \sum_{j=1}^{m} h_j\, e^{-\alpha_j \rho_j} \\ &\text{subject to} && \sum_{j=1}^{m} \rho_j = \rho \\ &&& \rho_j \geq 0 \end{aligned}$$

[1] Otherwise, at least in the short run, the manager cannot improve his reward by expending effort, so the allocation problem would be trivial.

It will, of course, be immediately obvious that this problem is of the same form as the profit maximization problem. Its solution is identical in form but to provide a convenient distinction, we have substituted $\hat{f}_j = \ln(\alpha_j h_j)$ for $\hat{\gamma}_j$, ν_s for ψ_s and added primes on the ρ^* and J. The necessary and sufficient condition for the selection of indices $r \epsilon J'$ becomes:

(26) $$\min_{r \epsilon J'} \hat{f}_r > \frac{1}{\sum_{j \epsilon J'} \frac{1}{\alpha_j}} \left[\sum_{j \epsilon J'} \frac{\hat{f}_j}{\alpha_j} - \rho \right] \geq \max_{s \notin J'} \hat{f}_s$$

and the optimal $\rho_r^{*'}$, $r \epsilon J'$ and ν_s^*, $s \notin J'$ become

$$\rho_r^{*'} = \frac{1}{\alpha_r} \left[\hat{f}_r - \frac{1}{\sum_{j \epsilon J'} \frac{1}{\alpha_j}} \left(\sum_{j \epsilon J'} \frac{\hat{f}_j}{\alpha_j} - \rho \right) \right]$$

(27)

and

$$\nu_s^{*'} = \mu - \alpha_s h_s .$$

Suppose management desires to maximize expected profit (where profit can again be defined as a weighted sum of the s_j's) and wishes to give a subordinate a set of goals and rewards for attainment such that he will allocate his effort so that profit will be maximized. Provided the subordinate acts to maximize his expected reward, a sufficient condition assuming the profit and reward functions defined as above is that the b_j, r_j and p_j are so chosen that

$$b_j = 0 \qquad \text{for all } j \notin J$$

(28)

and

$$b_j > 0 \quad \text{and} \quad h_j = \eta_j \quad \text{for all } j \epsilon J$$

This condition is not necessary, however. In order that the solution to (25) will provide the same effort allocation as the solution to (6) we require the weaker condition which we shall state and prove as a theorem.[1]

<u>Theorem</u>: Necessary and sufficient conditions for $\rho_r^{*'} = \rho_r^{*}$, $r \in J$ are choice of b_j, r_j and p_j such that

(i) $$b_j > 0 \qquad \text{for all } j \in J$$

(ii) $$\hat{f}_r = \hat{\gamma}_r + \hat{g}\,, \qquad r \in J$$

where $\hat{g}$ is an arbitrary constant.

(iii) $$\hat{f}_{\substack{s \\ s \notin J}} \leq \frac{1}{\left(\sum_{j \in J} \frac{1}{\alpha_j}\right)} \left[\sum_{j \in J} \frac{\hat{\gamma}_j}{\alpha_j} - \rho\right] + \hat{g}$$

and

(iv) $$\min_{j \in J} \hat{f}_r > \frac{1}{\left(\sum_{j \in J} \frac{1}{\alpha_j}\right)} \left[\sum_{j \in J} \frac{\hat{f}_j}{\alpha_j} - \rho\right]$$

<u>Proof</u>

Condition (i) is clearly sufficient for J to be included in the set of the first m j's in the cut effected by (22). For necessity, suppose that for some $k \in J$, $b_k = 0$. Then $\rho_k^{*'} = 0$; but by definition of J, $\rho_k^{*} > 0$ so that $\rho_k^{*} \neq \rho_k^{*'}$, a contradiction.

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[1]For brevity, the terminology developed in earlier sections of the paper will be introduced without further (formal) definition.

Condition (ii) implies

$$\frac{1}{(\sum_{j\in J}\frac{1}{\alpha_j})}\left[\sum_{j\in J}\frac{\hat{\gamma}_j}{\alpha_j}-\rho\right]+\hat{g}=\frac{1}{(\sum_{j\in J}\frac{1}{\alpha_j})}\left[\sum\frac{\hat{f}_j}{\alpha_j}-\rho\right]$$

so that conditions (iii) and (iv) together imply $J' = J$. Also

$$\rho_r^{*'}=\frac{1}{\alpha_r}\left[\hat{f}_r-\frac{1}{(\sum_{j\in J}\frac{1}{\alpha_j})}(\sum_{j\in J}\frac{\hat{f}_j}{\alpha_j}-\rho)\right]$$

$$=\frac{1}{\alpha_r}\left[\hat{\gamma}_r+\hat{g}-\frac{1}{(\sum_{j\in J}\frac{1}{\alpha_j})}(\sum_{j\in J}\frac{\hat{\gamma}_j}{\alpha_j}+\hat{g}\sum_{j\in J}\frac{1}{\alpha_j}-\rho)\right]$$

$$=\frac{1}{\alpha_r}\left[\hat{\gamma}_r-\frac{1}{(\sum_{j\in J}\frac{1}{\alpha_j})}(\sum_{j\in J}\frac{\hat{\gamma}_j}{\alpha_j}-\rho)\right]$$

$$=\rho_r^{*}$$

Conversely, $\rho_r^{*'} = \rho_r^{*}$ implies $J' = J$, for otherwise either for some $k\in J$, $\rho_k^{*} > 0$ and $\rho_k^{*'} = 0$ or for some $k\in J'$, $\rho_k^{*'} > 0$ and $\rho_k^{*} = 0$. Together, $\rho_r^{*'} = \rho_r^{*}$ and $J' = J$ imply

$$\left[\hat{f}_r-\frac{1}{(\sum_{j\in J}\frac{1}{\alpha_j})}(\sum_{j\in J}\frac{\hat{f}_j}{\alpha_j}-\rho)\right]=\left[\hat{\gamma}_r-\frac{1}{(\sum_{j\in J}\frac{1}{\alpha_j})}(\sum_{j\in J}\frac{\hat{\gamma}_j}{\alpha_j}-\rho\right.$$

$$\hat{f}_r-\hat{\gamma}_r=\frac{1}{(\sum_{j\in J}\frac{1}{\alpha_j})}\sum_{j\in J}(\frac{\hat{f}_j-\hat{\gamma}_j}{\alpha_j}) .$$

Hence $\hat{f}_r$ and $\hat{\gamma}_r$ differ by the same constant, say $\hat{g}$, for all r. For the necessity of condition (iii) we introduce without proof the following lemma:[1]

Lemma

If
$$\hat{f}_{m+1} > \frac{1}{(\sum_{j=1}^{m} \frac{1}{\alpha_j})} \left[\sum_{j=1}^{m} \frac{\hat{f}_j}{\alpha_j} - \rho \right],$$

then
$$\hat{f}_{m+1} > \frac{1}{(\sum_{j=1}^{m+1} \frac{1}{\alpha_j})} \left[\sum_{j=1}^{m+1} \frac{\hat{f}_j}{\alpha_j} - \rho \right].$$

Now then, suppose $\exists\, f_t$, $t \notin J$ such that

$$\hat{f}_t > \frac{1}{(\sum_{j\in J} \frac{1}{\alpha_j})} \left[\sum_{j\in J} \frac{\hat{\gamma}_j}{\alpha_j} - \rho \right] + \hat{g} \quad .$$

By condition (ii) and the lemma,

$$\hat{f}_t > \frac{1}{(\sum_{j\in J} \frac{1}{\alpha_t})} \left[\sum_{j\in J} \frac{\hat{f}_j}{\alpha_j} + \frac{\hat{f}_t}{\alpha_t} - \rho \right]$$

implying $t \in J'$, or $J' \neq J$, a contradiction.

Finally, suppose, for some $q \in J$,

$$\hat{f}_q \leq \frac{1}{(\sum_{j\in J} - \frac{1}{\alpha_j})} \left[\sum_{j\in J} \frac{\hat{f}_j}{\alpha_j} - \rho \right]$$

By condition (ii)

$$\hat{\gamma}_q \leq \frac{1}{(\sum_{j\notin J} \frac{1}{\alpha_j})} \left[\sum_{j\in J} \frac{\hat{\gamma}_j}{\alpha_j} - \rho \right]$$

implying $q \notin J$, a contradiction. Q.E.D.

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[1]Proved in [24].

Constraints on Risk in Critical Areas

It is frequently of interest to maximize the weighted sum of expected values (or its equivalent problem of maximization for reward attainment) subject to constraints on minimum acceptable performance in a subset of "critical" performance areas. We may thus wish to solve the problem:

$$\text{Max} \quad E(\pi) = E\left(\sum_{j=1}^{n} \beta_j s_j\right)$$

$$(29) \quad \text{subject to} \quad P(s_j \geq a_{d_j j}) \geq \xi_j, \qquad j = 1,\ldots,m$$

$$\sum_{j=1}^{n} \rho_j = \rho$$

$$\rho_j \geq 0 \quad , \qquad j = 1,\ldots,n$$

where the performance areas are numbered so that the first m j's are constrained. The constraints may be interpreted as a statement of the maximum risk $(1 - \xi_j)$ that will be accepted for performance below the acceptable level, $a_{d_j j}$. The problem may be restated with the functional and all constraints in terms of the ρ_j as:

$$\text{Min.} \quad \sum_{j=1}^{n} \eta_j e^{-\alpha_j \rho_j}$$

subject to

$$(30) \quad k_{d_j j} (1 - e^{-\alpha_j \rho_j}) \geq \xi_j, \quad j = 1,\ldots,m$$

$$\sum_{j=1}^{n} \rho_j = \rho$$

$$\rho_j \geq 0 \quad , \quad j = 1,\ldots,n$$

We define, for the constrained performance areas, the minimum level of effort which will just attain the required performance in that area, $\bar{\rho}_j$, by the equations:

$$(31) \qquad k_{d_j j}\left(1 - e^{-\alpha_j \bar{\rho}_j}\right) = \xi_j, \qquad j = 1,\ldots,m$$

The minimum values of ρ_j in the other areas are 0, so that, preserving consistency, we may define

$$(32) \qquad \bar{\rho}_j = 0 \qquad , \qquad j = m+1,\ldots,n$$

By means of the following transformations,

$$(33) \qquad \begin{aligned} \rho'_j &= \rho_j - \bar{\rho}_j \\ \eta'_j &= \eta_j \, e^{\gamma_j \bar{\rho}_j} \qquad\qquad j = 1,\ldots,n \\ \rho' &= \rho - \sum_{j=1}^{n} \bar{\rho}_j \end{aligned}$$

we transform the problem into

$$\text{Min} \quad \sum_{j=1}^{n} \eta'_j \, e^{-\alpha_j \rho'_j}$$

$$\text{subject to} \quad \sum_{j=1}^{n} \rho'_j = \rho'$$

$$\rho'_j \geq 0 \qquad , \qquad j = 1,\ldots,n$$

where the non-negativity conditions on the first m ρ_j's are clearly both necessary and sufficient for satisfaction of the attainment constraints in the specially selected areas. This problem is then of the same form as the expected performance maximization and so may be solved by the method of section 3.

5. Maximization of the Probability of Joint Goal Attainment

In dealing with certain kinds of effort allocation problems, one may be interested not in maximizing expected performance or expected reward, but in the probability of the joint occurrence of goal attainment or acceptable performance over the set of performance areas. This is equivalent to the minimization of the risk of not achieving acceptable performance where the latter is defined as performance up to an acceptable level in all areas. If we again assume independent probabilities of performance attainment (subject to the functional dependence imposed by a constraint on overall effort), we may express the functional as:

$$\Omega = \prod_{j=1}^{m} P(s_j \geq a_{b_j j}) = \prod_j k_{b_j j} (1 - e^{-\alpha_j \rho_j}) = (\prod_j k_{b_j j}) \prod_j (1 - e^{-\alpha_j \rho_j}) \tag{35}$$

For convenience, we liminate the constant term and transform the functional by

$$\ell = -\ln \left(\frac{\Omega}{\prod_j k_{b_j j}}\right) \tag{36}$$

so that the problem may be stated as:

$$\text{Min.} \qquad \ell = -\sum_{j=1}^{m} \ln (1 - e^{-\alpha_j \rho_j})$$

$$\text{subject to} \qquad \rho - \sum_{j=1}^{m} \rho_j \geq 0. \tag{37}$$

The Kuhn-Tucker conditions become:[1]

$$(38) \qquad - \frac{\alpha_j e^{-\alpha_j \rho_j}}{1 - e^{-\alpha_j \rho_j}} = \mu \quad \text{where } \rho - \sum_{j=1}^{m} = 0,\ \mu \geq 0,\ j = 1, \ldots, m$$

which yield

$$\alpha_j e^{-\alpha_j \rho_j} = \mu - \mu e^{-\alpha_j \rho_j}$$

$$(39) \qquad -\alpha_j \rho_j = \ln \left(\frac{\mu}{\mu + \alpha_j}\right)$$

$$\rho = \Sigma \rho_j = \sum_j \frac{1}{\alpha_j} \ln \left(\frac{\mu + \alpha_j}{\mu}\right)$$

On examining the original statement of the functional, Ω, it may be remarked that unless the minimum risk that is sought is small, the criterion would probably be meaningless and another adopted. The area of interest for this problem, then, is one for which the maximum value of Ω is close to probability 1, indicating that each of the $(e^{-\alpha_j \rho_j})$ will be small--certainly less than, say, 0.1 and in all likelihood, much smaller. Thus the ratio $\frac{\mu}{\mu + \alpha}$ is small or:

$$(40) \qquad \mu << \alpha,$$

providing a possibility of approximation of μ. For an approximation, we note that

$$(41) \qquad \rho = \sum_j \frac{1}{\alpha_j} \ln \left(\frac{\mu + \alpha_j}{\mu}\right) \geq \sum_j \frac{1}{\alpha_j} \ln \left(\frac{\alpha_j}{\mu}\right) .$$

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[1] Note that for $\mu = 0$, no solution obtains for finite ρ_j so that the condition on $\Sigma \rho_j$ must be satisfied as an equality.

and let $\tilde{\mu}$ be defined by

(42) $$\rho \equiv \sum_j \frac{1}{\alpha_j} \ln \left(\frac{\alpha_j}{\tilde{\mu}}\right)$$

Since μ is determined by $\rho = \sum_j \frac{1}{\alpha_j} \ln \left(1+\frac{\alpha_j}{\mu}\right)$, smaller values of $\frac{\alpha_j}{\mu}$ are involved than in $\rho = \sum \frac{1}{\alpha_j} \ln \left(\frac{\alpha_j}{\mu}\right)$, hence μ must be larger than $\tilde{\mu}$.

Thus $\tilde{\mu}$ is a lower bound for μ. An explicit solution is easily obtainable for $\tilde{\mu}$, for

(43) $$\rho = \sum_j \frac{1}{\alpha_j} \ln \alpha_j - \ln(\tilde{\mu}) \sum_j \frac{1}{\alpha_j}$$

$$\tilde{\mu} = e^{\frac{1}{\sum_j \frac{1}{\alpha_j}} \left[\rho - \sum_j \frac{1}{\alpha_j} \ln \alpha_j\right]}$$

We note further that:

(44) $$\rho = \sum_j \frac{1}{\alpha_j} \ln \left(\frac{\mu+\alpha_j}{\mu}\right) \leq \sum_j \frac{1}{\alpha_j} \ln \left(\frac{\alpha_j}{\mu} e^{\frac{\mu}{\alpha_j}}\right)$$

since $\mu + \alpha_j \leq \alpha_j e^{\frac{\mu}{\alpha_j}}$.

We next rewrite the preceding inequality on ρ as

(45) $$-\left[\rho - \sum_j \frac{1}{\alpha_j} \ln \alpha_j\right] \geq \ln \mu \sum_j \frac{1}{\alpha_j} - \mu \sum_j \frac{1}{\alpha_j^2}$$

Dividing through by $\sum_j \frac{1}{\alpha_j}$, the left side becomes the expression for $\ell n\, \tilde{\mu}$. Thus

(46) $$\ell n\, \tilde{\mu} \geq \ell n\, \mu - \mu \frac{\sum_j \frac{1}{\alpha_j^2}}{\sum_j \frac{1}{\alpha_j}}$$

Writing

(47) $$\beta \equiv \frac{\sum_j \frac{1}{\alpha_j^2}}{\sum_j \frac{1}{\alpha_j}} ,$$

(48) $$\mu \geq \mu\, e^{-\beta\mu} \geq \mu\, (1 - \beta\mu)$$

Let $\bar{\mu}$ and $\bar{\bar{\mu}}$ be defined by $\tilde{\mu} \equiv \bar{\bar{\mu}}\, (1 - \beta\bar{\bar{\mu}}) \equiv \bar{\mu}\, e^{-\beta\bar{\bar{\mu}}}$.

Clearly

(49) $$\bar{\bar{\mu}} \geq \bar{\mu}$$

so that:

(50) $$\bar{\mu}\, e^{-\beta\, \bar{\mu}} \geq \tilde{\mu} = \bar{\mu}\, e^{-\beta\, \bar{\bar{\mu}}} \geq \mu\, e^{-\beta\, \mu}$$

The function $y = x\, e^{-\beta x}$ is strictly increasing function of x for $0 \leq x \leq 1/\beta$.

Thus if $1/\beta \geq \mu$, $\bar{\mu} \geq 0$, then from $\bar{\mu}\, e^{-\beta\mu} \geq \bar{\mu}\, e^{-\beta\mu}$,

it follows that $\bar{\mu} \geq \mu$.

Hence

(51) $$\bar{\bar{\mu}} \geq \bar{\mu} \geq \mu.$$

Solving

$$\bar{\bar{\mu}}\,(1 - \beta\bar{\bar{\mu}}) = \tilde{\mu} \tag{52}$$

we obtain[1]

$$\bar{\bar{\mu}} = \frac{1 - \sqrt{1 - 4\beta\tilde{\mu}}}{2\beta} \tag{53}$$

which yields the implicit constraint $\tilde{\mu} \leq 1/4\beta$.

Solving

$$\bar{\mu}\, e^{-\beta\bar{\bar{\mu}}} = \tilde{\mu}, \tag{54}$$

we obtain

$$\tilde{\mu} \leq \mu \leq \bar{\mu} = \tilde{\mu}\, e^{\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \beta\mu}\right)} \tag{55}$$

The error of the estimate, μ, is thus seen to be bounded via

$$1 \leq \frac{\mu}{\tilde{\mu}} \leq e^{\left(\frac{1}{2}\ \sqrt{\frac{1}{2} - \beta\mu}\right)} \tag{56}$$

Recalling the functional in the original problem:

$$\Omega \equiv \prod_j P\left\{s_j \geq a_{b_j j}\right\} = \prod_j k_{b_j j}\,(1 - e^{-\alpha_j\rho_j}) \tag{57}$$

Let

$$w \equiv \frac{\Omega}{\prod_j k_{b_j j}} = \prod_j (1 - e^{-\alpha_j\rho_j}),$$

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[1]Since by (49) either root satisfies $\bar{\bar{\mu}} \geq \bar{\mu}$, the smaller as the closer approximation to μ is preferred.

wherein we suppose optimal values for the ρ_j.

Thus

$$1 - e^{-\alpha_j \rho_j} \leq 1 - \frac{\mu}{\mu+\alpha_j} = \frac{\alpha_j}{\mu+\alpha_j} . \tag{59}$$

Then

$$w = \prod_j \left(\frac{\alpha_j}{\mu+\alpha_j} \right) = \prod_j \left(\frac{1}{1+\frac{\mu}{\alpha_j}} \right) \geq \prod_j \left(1 - \frac{\mu}{\alpha_j}\right) \geq 1 - \mu \sum_j \frac{1}{\alpha_j} . \tag{60}$$

Now

$$\frac{1}{w} = \prod_j \left(1 + \frac{\mu}{\alpha_j}\right) \geq 1 + \mu \sum_j \frac{1}{\alpha_j} . \tag{61}$$

Thus

$$\frac{1}{1 + \mu \sum_j \frac{1}{\alpha_j}} \geq w. \tag{62}$$

Hence

$$\frac{1}{1 + \tilde{\mu} \sum_j \frac{1}{\alpha_j}} \geq \frac{1}{1 + \mu \sum_j \frac{1}{\alpha_j}} \geq w \geq 1 - \mu \sum_j \frac{1}{\alpha_j} \geq 1 - \bar{\mu} \sum_j \frac{1}{\alpha_j} \tag{63}$$

We have thus established lower and upper bounds on the estimate of μ (and hence estimated ρ_j) and w, which provides limits on the estimated probability of acceptable performance.

Conclusion

The models which have been presented naturally suffer from difficulties of oversimplification and dependence upon a single assumed type of density function. By proceeding in this fashion, however, we gain the advantages of the existence of computational algorithms which may be used to facilitate hypothesis testing. In those cases in which a fairly simple algorithm exists--such as the sequential choice of areas for effort allocation as provided for in some of the models--it may be assumed that the subordinate can arrive at optimal solutions to the object he seeks without special technical training.

Some of the optimal allocation rules, however, are quite complex. It is therefore likely that although the subordinate may be seeking such an objective he may not be capable of allocating his effort in an optimal way. This suggests, of course, additional areas for experimental study, as, e.g., in the misdirection of effort relative to perceived or held goals due to technical inadequacy to infer optimal effort allocations. In such cases it is possible that the role of the supervisor can be helpful in a suggestive or directive manner; possibly to the extent of translating the subordinate's goals into another set of goals with which he is adequately prepared to deal. Such work, of course, would tie in with the experimental work of Churchill and Cooper [8] of the effect of an auditor and his actions on audited personnel.

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