WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

SUBCONTRACTING, COORDINATION, FLEXIBILITY, AND PRODUCTION SMOOTHING IN AGGREGATE PLANNING

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#2026-88

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April, 1988

Abstract

We propose a model in which subcontracting can be explicitly considered as a production planning strategy. The possible market and non-market subcontracting mechanisms and their transaction costs are discussed. We show that a set of feasible subcontracting mechanisms in which firms coordinate their production via subcontracts Pareto-dominates other mechanisms if the transaction costs are equal. We then study an example with quadratic cost functions and a *coordination* subcontract; linear decision rules for production, inventory, and subcontracting are derived. In the example, subcontracting reduces the variability in production and inventory. The same interpretation can be used for flexibility of manufacturing resources.

1. Introduction

Subcontracting is "the procurement of an item or service which is normally capable of economic production in the prime contractor's own facilities and which requires the prime contractor to make specifications available to the supplier" (J. S. Day (1956)). Despite a high degree of both vertical and horizontal integration, it is very doubtful whether any modern business enterprise can be self-sufficient in all its activities. Even ignoring the purchase of raw materials and the use of distribution systems to dispose of final products, most companies appear to depend upon other producers for performance of at least some work and services. Reflecting the extent of this dependence, "the extremely important role that subcontracting plays in industry has been recognized in the past few years as never before" (Sammet and Kelley (1980)). In particular, many observers (Williamson (1985), Piore and Sabel (1984)) point out that subcontracting is more extensively used in the industries and/or industrial districts (Japanese tool makers, Italian textile firms, etc.) where manufacturing technologies have greater flexibility. Because flexibility increases the feasibility of subcontracting while subcontracting helps to realize greater value of flexibility.

It has also been long recognized that subcontracting is an important option among strategies for aggregate production and capacity planning (Schroeder (1981)). In particular, subcontracting can be used as a substitute for other production smoothing strategies such as, holding inventory, adjustment of labor force size, pricing and promotions. Despite the enormous work on aggregate planning (Holt et al. (1960), Manne (1958), Sobel (1971), Thomas (1971), Hax and Meal (1975), Bitran and Hax (1977), etc.), little has been said about how to formally incorporate subcontracting into the production planning models. Unlike the alternatives such as inventory and adjustment of production capacity, the feasibility of subcontracting depends not only on the decision of the firm itself, but also on the willingness of other, possibly competing, firms who may have conflicting interests. The shift from in-plant-production to outside sources indicates a major shift in the economic institutions employed by firms to markets and/or relational contracting.

We want to emphasize that subcontracting discussed in this paper has more strict definition. We differentiate between "make-or-buy" decisions and "make-or-subcontract" decisions. From the viewpoint of operations management, the former that Williamson (1985) discusses in length under the title of Vertical Integration are more of design decisions to lay the efficient boundary of a firm. For example, Toyota relies on many small firms for the supply of parts on a permanent basis, and those small firms are frequently named as "subcontractors". However, managing the permanent subcontracting relations of this kind can be treated in the same way as that of managing the relations with raw material suppliers, which is not the concern of this paper. The "make-or-subcontract" decisions we refer to in this paper are tactic or operational decisions which depend on the dynamics of the prime producer as well as the subcontractors. Firms contract out parts or final products which are regularly produced in their own facilities on a temporary basis due to certain

"shocks" in the economic environment. Also, subcontracts discussed in this paper are either the spot-market or the spot-contracting transactions.

In this paper, we formulate a production planning model that explicitly considers subcontracting as a planning tool. We first use a dynamic programming approach to illustrate how the feasibility of a subcontract is determined and the possible ways to produce a feasible subcontract. We also show a particular set of subcontracts, namely, the *coordination* subcontracts, Pareto-dominates any other subcontract, assuming the transaction costs of subcontracting are equal. Here, a *coordination* subcontract is so defined that the joint production costs of the prime contractor and subcontractor are minimized via the subcontract. Moreover, the *coordination* subcontracts are socially more desirable than any other subcontract. Given a method of subcontracting, the optimal production plan can be determined in our conceptual model. We then show by an example that the *coordination* subcontracting can be easily included into the Holt, Modigliani, Muth and Simon (1960) planning model, and linear decision rules are obtained for production and subcontracting. The example also shows that subcontracting reduces the variability in production and inventory, and hence, contributes to production smoothing. Variability is smaller as the correlation of the production requirements and the subcontracting costs decrease.

2. A Conceptual Planning Model with Subcontracting

In this section, we introduce a dynamic programming model to discuss the issues regarding subcontracting. The main purpose of this section is to develop and illustrate concepts of feasibility, optimality and dominance relations of subcontracts. We adopt very general notation to incorporate all the decisions that have been discussed in the production planning literature but do not specify rigorous mathematical assumptions to assure the existence or uniqueness of a solution. In effect, we take a unique solution for granted. For simplicity, we also assume there are only two production organizations in the market, and the insights developed in this section can be extended to the situation with more firms.

Two production organizations, firm 1 and 2, are in the market, each faces a forecasted production requirement stream, $\{D_t^i, 1 \leq t \leq T\}$, $i = 1, 2$. For subcontracting to be possible, we assume that the technologies of the two firms are good substitutes, meaning, firm 1 can fulfill firm 2's production requirement with its existing technology and vice versa. As in the typical production planning setting, firms project the work force sizes, production rates, overtime, undertime and inventory levels in an aggregated production unit for each planing period t based on the current information and capacity constraints. For $i = 1, 2$, denote by (Γ_t^i, I_t^i) firm i's decision variables in period t, where I_t^i is a vector of the decision variables which directly affect the production cost of the next period and Γ_t^i is a vector of the decision variables which do not directly affect the next period cost. Typically, I_t^i may include inventory levels by the end of period t and work force size at

t; P_t^i may include production rate, overtime, and undertime, the effect of which on the next period cost is via the variables in I_t. Let $g_i(P_t^i, I_t^i, D_t^i, I_{t-1}^i) \geq 0$ be the set of constraints that the decision variables and state variables must satisfy, for example, the inventory balance constraint, capacity constraint, etc. Let the production cost of period t be $c_t^i(P_t^i, I_t^i; D_t^i, I_{t-1}^i)$ To discuss the production and subcontracting decisions in period t, we assume that firms know the minimal expected total production costs from $t+1$ on conditional on historical information about the production requirements, denoted by $C_{t+1}^i(I_t^1, I_t^2)$ for $i = 1, 2$. The discount factor is β $(0 < \beta \leq 1)$. We will use dynamic programming to show that C_{t+1}^i can be determined recursively. For simplicity, we assume a situation with complete information. That is, each firm has complete information about each other's cost functions and current and historic state variables (requirements, inventory data, work force sizes, etc.), and share the same belief regarding future requirements, given the available information (or use the same forecasting method) This assumption may be too stringent. However, complete information may be achieved through learning, signalling, and long-term relationships. Moreover, the subcontracting mechanism itself may provide incentives for firms to give truthful responses (see Hurwicz (1972), Myerson (1979), and also Section 4 for more discussion on the issue).

In essence, subcontracting is a reallocation of production requirements among firms We define a subcontract as a price and quantity pair (p_t, Q_t) where p_t (≥ 0) is the subcontracting price and Q_t is the subcontracting quantity from firm 1 to firm 2, i.e., firm 1 subcontracts quantity Q_t to firm 2 if $Q_t > 0$, and firm 2 subcontracts $-Q_t$ to firm 1 if $Q_t < 0$. We explicitly rule out the trivial subcontracting in which firm 1 subcontracts a certain part of its production to firm 2 and firm 2 gives the same amount back to firm 1. The costs of subcontracting we introduce later will eliminate this case as an optimal solution. Subcontracting may incur additional costs to both parties. Following Williamson, we categorize subcontracting costs into physical costs (which usually depend on the subcontracting quantity Q_t) and transaction costs. The variable costs, denoted by $T_i(Q_t)$ $(T_i(0) = 0)$, includes the shipment, handling, engineering and monitoring costs of the subcontracting quantity in addition to production costs. The transaction costs, denoted by F_i, includes time and manpower required for generating and implementing a subcontract (negotiation, bargaining, computing, drafting, and safeguarding against opportunism, etc.).

There are many possible ways to generate a subcontract, ranging from a simple market game to negotiation and bargaining for an agreement. We call a particular way of producing a subcontract a (subcontracting) *mechanism*. The following is an example of subcontracting mechanism. Suppose firm 1 intends to subcontract. It asks firm 2 to submit a bid p_t. Based on the subcontracting price

p_t, firm 1 decides how much to subcontract via the following program.

$$\min_{P_t^1, I_t^1, Q_t} c_t^1(P_t^1, I_t^1; D_t^1 - Q_t, I_{t-1}^1) + p_t Q_t + T_1(Q_t) + \beta C_{t+1}^1(I_t^1, I_t^2)$$

$$s.t. \quad g_1(P_t^1, I_t^1, D_t^1 - Q_t, I_{t-1}^1) \geq 0, \quad Q_t \geq 0$$

Denote the solution of the program by $P_t^1(p_t)$, $I_t^1(p_t)$, and $Q_t(p_t)$. Certainly, the solution is also a function of state variables, but we omit them for notational simplicity. Firm 2 then can set the subcontracting price p_t counting on the best response of firm 1 to the price. Thus, firm 2 solves

$$\min_{P_t^2, I_t^2, p_t} c_t^2(P_t^2, I_t^2; D_t^2 + Q_t(p_t), I_{t-1}^2) - p_t Q_t(p_t) + T_2(Q_t(p_t)) + \beta C_{t+1}^2(I_t^1(p_t), I_t^2)$$

$$s.t. \quad g_2(P_t^2, I_t^2, D_t^2 + Q_t(p_t), I_{t-1}^2) \geq 0, \quad p_t \geq 0.$$

Suppose p_t^* is the optimal in the above program. Then, a subcontract $(p_t^*, Q_t(p_t^*))$ is determined and the production plan (for period t) with subcontracting is determined accordingly. Of course, other mechanisms for subcontracting are possible. For example, firm 1 could name a price and then firm 2 could respond with the quantity it would accept at that price.

In general, for a subcontracting mechanism (indexed by S), we denote by p_t^S and Q_t^S the price and quantity specified in the subcontract generated by the mechanism and by P_t^{iS}, I_t^{iS} the corresponding optimal production decisions. The resulting costs for both parties are

$$C_t^{iS} \equiv c_t^i(P_t^{iS}, I_t^{iS}; D_t^i - \delta(i)Q_t^S, I_{t-1}^i) + \delta(i)p_t^S Q_t^S + T_i(Q_t^S) + \beta C_{t+1}^i(I_t^{1S}, I_t^{2S}),$$

where

$$\delta(i) \equiv \begin{cases} 1, & \text{if } i = 1; \\ -1, & \text{if } i = 2. \end{cases} \tag{2.1}$$

Note that C_t^{iS} exclude the transaction costs F_i since they do not affect the optimization and hence, can be treated separately.

Obviously, a subcontract is feasible if it yields each firm at most the cost it could be sure of without subcontracting. To establish the no subcontracting *status quo*, we solve the following two programs,

Program i:

$$\min_{P_t^i, I_t^i} c_t^i(P_t^i, I_t^i; D_t^i, I_{t-1}^i) + \beta C_{t+1}^i(I_t^1, I_t^2)$$

$$s.t. \quad g_i(P_t^i, I_t^i, D_t^i, I_{t-1}^i) \geq 0,$$

for $i = 1, 2$, and denote the minimal costs by C_t^{i*}.

Definition 2.1. A subcontract is *feasible* if $Q_t^{iS} \neq 0$ and $C_t^{iS} \leq C_t^{i*}$ for $i = 1, 2$

Definition 2.2. A subcontract is *transactionally feasible* if $Q_t^{iS} \neq 0$ and $C_t^{iS} + F_i \leq C_t^{i*}$ for $i = 1, 2$.

The feasibility, in fact, follows directly from the notion of *individual rationality* of game theory (see Luce and Raiffa (1957)). For a fixed subcontracting mechanism S, a production plan can be

derived via the following recursive computation. For each period t, first solve Programs 1 and 2 to set up the *status quo*, C_t^{i*}. Second, compute the subcontract and corresponding production decisions following the rules specified in the subcontracting mechanism and obtain C_t^{iS}. If the subcontract is transactionally feasible, then let $C_t^i = E[C_t^{iS}|D_s^i, s \leq t-1, i=1,2]$. Otherwise, let $C_t^i = E[C_t^{i*}|D_s^i, s \leq t-1, i=1,2]$. Go back to period $t-1$ and repeat the procedure.

In the above procedure, we have fixed the subcontracting mechanism. This mechanism may be transactionally infeasible. However, it does not imply firms cannot find a transactional feasible mechanism. On the other hand, the set of feasible subcontracting mechanisms can be very large and any given mechanism may not be the "best". We next establish certain dominance relations among subcontracting mechanisms. We say a mechanism S dominates mechanism S' if $C_t^{iS} \leq C_t^{iS'}$ for $i = 1, 2$. That is, mechanism S dominates S' if neither firm prefers S' to S.

We set the transaction costs aside for the moment. Let us look at a set of subcontracts generated as follows. First, solve a program,

Program 0:

$$\min_{Q_t, P_t^i, I_t^i, i=1,2} \sum_{i=1}^{2}[c_t^i(P_t^i, I_t^i; D_t^i - \delta(i)Q_t, I_{t-1}^i) + T_i(Q_t) + \beta C_{t+1}^i(I_t^1, I_t^2)]$$
$$s.t. \qquad g_i(P_t^i, I_t^i, D_t^i - \delta(i)Q_t, I_{t-1}^i) \geq 0, \quad i = 1, 2,$$

and let C_t^C be the optimal value, corresponding decision variables with superscript, C, be the optimal solution, and C_t^{iC*} be firm i's cost function evaluated at the optimal solution of the program, i.e., $C_t^{iC*} \equiv c_t^i(P_t^{iC}, I_t^{iC}; D_t^i - \delta(i)Q_t^C, I_{t-1}^i) + T_i(Q_t^C) + \beta C_{t+1}^i(I_t^{1C}, I_t^{2C})$. Certainly, $C_t^{1C*} + C_t^{2C*} = C_t^C$.

Lemma 2.1. *$C_t^C \leq C_t^{1*} + C_t^{2*}$ and $C_t^C \leq C_t^{1S} + C_t^{2S}$ for any subcontracting mechanism S.*

Proof. Note that the feasible solutions to Programs 1 and 2 plus any Q_t is a feasible solution to Program 0. In particular, $C_t^{1*} + C_t^{2*}$ is the value function of Program 0 evaluated at a feasible solution constituted by the optimal solutions to Programs 1 and 2 plus $Q_t = 0$, and hence, the first assertion in the lemma follows immediately. Similarly, $C_t^{1S} + C_t^{2S}$ is the value function of Program 0 (noticing in (2.1) that transfer value, $p_t^S Q_t^S$, is cancelled in the sum since $\delta(1) + \delta(2) = 0$) evaluated at a feasible solution to Program 0 which is constituted by the solution generated by mechanism S.

∎

Definition 2.3. A subcontract is called a *coordination* subcontract if the subcontracting price and quantity are determined by $Q_t = Q_t^C$ and

$$p_t = \frac{C_t^{1*} - C_t^{1C*} - \theta(C_t^{1*} + C_t^{2*} - C_t^C)}{Q_t^C} = -\frac{(1-\theta)(C_t^{1*} + C_t^{2*} - C_t^C) - (C_t^{2*} - C_t^{2C*})}{Q_t^C}.$$

for $0 \le \theta \le 1$ if $Q_t^C \neq 0$.

It can be easily verified the subcontracting price is nonnegative by noticing that $C_t^{1*} < C_t^{1C*}$ if $Q_t^C < 0$ and $C_t^{2*} < C_t^{2C*}$ if $Q_t^C > 0$ since it costs more to produce more. Also, it takes little manipulation to get

Lemma 2.2. *The total costs to firms 1 and 2 under a coordination subcontract with parameter θ are*

$$\begin{aligned} C_t^{1C} &= C_t^{1*} - \theta(C_t^{1*} + C_t^{2*} - C_t^C) \\ C_t^{2C} &= C_t^{2*} - (1-\theta)(C_t^{1*} + C_t^{2*} - C_t^C) \end{aligned} \tag{2.2}$$

In fact, to obtain a *coordination* subcontract, firms first determine subcontract quantity by operating with complete coordination as if they were two departments in one firm, and then find a price to divide the total cost savings, $C_t^{1*} + C_t^{2*} - C_t^C$.

Proposition 2.1. *A coordination subcontract is always feasible. For a subcontract generated by any mechanism, there exist a coordination subcontract which dominates that subcontract.*

Proof. The first part of the proposition follows directly from Lemma 2.1. and 2.2.

To prove the second part, note that $C_t^{1C} + C_t^{2C} = C_t^C$ by (2.2). For any subcontract mechanism S, by Lemma 2.1., there must exists a i such that $C_t^C \le C_t^{iS} + C_t^{j*}$, for $j \neq i$. Set θ^* such that $C_t^{iC} = C_t^{iS}$. This is doable since $C_t^{iC} = C_t^{i*} \ge C_t^{iS}$ if $\theta = 0$ and $C_t^{iC} = C_t^C - C_t^{j*} \le C_t^{iS}$ if $\theta = 1$. Then, under the coordination subcontract with parameter θ^*, $C_t^{jC} = C_t^C - C_t^{iC} = C_t^C - C_t^{iS} \le C_t^{jS}$, again by Lemma 2.1

∎

From the above discussion, we can see that the *coordination* subcontracts constitute an important set of subcontracts. First of all, it is a set of undominated subcontracts if transaction costs are equal. Secondly, unlike other subcontracting mechanism, it results in an unique subcontracting outcome which is socially more desirable since it minimizes the total cost of production. Moreover, we can use *coordination* subcontracting to test whether the set of transactionally feasible subcontracts is empty or not.

Proposition 2.2. *Suppose the transaction costs of coordination subcontracts are less than or equal to that of any other subcontract. Then, the set of transactionally feasible subcontracts is not empty if and only if $Q_t^C \neq 0$ and $C_t^{1*} + C_t^{2*} \ge C_t^C + F_1 + F_2$.*

Proof. The "if" part directly follows Proposition 2.1. To get the "only if" part, we simply include the transaction costs in the cost savings and divide $C_t^{1*} + C_t^{2*} - C_t^C - F_1 - F_2$ among firms via a price similar to that in Definition 2.3.

∎

We now proceed to a discussion about the transaction costs related to different subcontracting mechanisms. The physical costs of subcontracting usually do not depend on the way in which a subcontract is generated. That is why we treat the variable costs of subcontracting as mechanism-independent in the above discussion. But the transaction costs will depend on how a subcontract is produced. The general consensus is that market mechanisms have low transaction costs. That is against the assumption in Proposition 3.2., and hence, we can not ignore the market mechanisms in general. Let us see why the *coordination* subcontracts may have high transaction costs. A lot of effort is devoted to multi-plant planning models and computationally, they are not more difficult than single plant models. As we can see, in our conceptual model, Program 0 is usually not drastically more difficult than Programs 1 and 2. Thus, the computational part of the transaction cost is relatively moderate for the *coordination* subcontracts. However, the parameter θ in the *coordination* subcontracts is hard to specify. That might take long negotiation and bargaining, and firms might have to use outcomes of other mechanisms as *status quo* to nail it down. This is the major cause of high transaction costs of *coordination* subcontracts.

In many situations, long-term relationships between parties, reputation effects, or regulations may reduce the difficulties of dividing the benefits from subcontracting and lower the transaction costs of *coordination* subcontracting. Williamson (1985) mentions that successful Japanese manufacturers place more emphasis on building an intimate personal relations between contracting parties than on drafting and safeguarding a detailed contract. This explains why Japanese rely more extensively on subcontracting than is true in the United States. The discussion in this section predicts and proves their success.

3. Subcontracting, Coordination, Flexibility and Production Smoothing – An Example

In the preceding discussion, we can see, conceptually, there is no difficulty in considering subcontracting in any planning model. We now use a simple example to show how subcontracting smooths production.

Holt, Modigliani, Muth and Simon (1960) propose a quadratic approximation of a firm's production cost function and derive a linear decision rule for aggregate planning. We adopt their framework in our discussion because the quadratic approximation avoids combinatorial difficulties, and it is simple to analyze, but the qualitative insights obtained in the framework prevail in more realistic models. Another advantage of quadratic costs is that the optimal solution for the uncertainty case can be obtained directly from the solution of the certainty case as long as the cost function is positive definite. Holt, Modigliani, Muth and Simon (1960) discuss and prove this "certainty equivalence" property in their Chapter 6. Also, there is evidence that linear rules are optimal or nearly optimal not only for quadratic cost functions but also for more general cost functions (Schneeweiss 1971, 1974).

To avoid complex notation, we further assume that firms maintain the work force size relatively constant and labor adjustment costs are negligible. Therefore, following Holt et al. (1960), the stage production costs for firm i, $i = 1, 2$, are

$$
\begin{aligned}
c_t^i(P_t^i, I_t^i; D_t^i, I_{t-1}^i) = & C_{1i}(P_t^i - C_{2i})^2 + C_{3i}P_t^i \\
& + C_{4i}(I_t^i - C_{5i} - C_{6i}D_t^i)^2 + C_{7i},
\end{aligned}
\tag{3.1}
$$

where P_t^i is the production rate for period t, I_t^i is the inventory level by the end of the period t, D_t^i is the production requirement in period t, and C_{ki}, $k = 1, \ldots, 7$, are constant cost parameters. We can solve two single firm problems

$$
\begin{aligned}
\min_{P_t^i, I_t^i, t=1,\ldots,T} \quad & \sum_{t=1}^{T} c_t^i(P_t^i, I_t^i; D_t^i, I_{t-1}^i) \\
s.t. \quad & I_{t-1}^i + P_t^i - D_t^i = I_t^i, \quad t = 1, \ldots, T
\end{aligned}
$$

by the same approach as in Holt, Modigliani, Muth and Simon (1960). In particular,

Lemma 3.1. *For $i = 1, 2$,*

$$
\begin{aligned}
I_1^i = & \frac{1}{C_{1i}(2 - \lambda_i) + C_{4i}}\Big[\sum_{t=0}^{\infty} \lambda_i^t (C_{1i}(D_{t+2}^i - D_{t+1}^i) + C_{4i}C_{6i}D_{t+1}^i) \\
& + C_{1i}(I_0^i - C_{5i})\Big] + C_{5i},
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
P_1^i = & \frac{1}{C_{1i}(2 - \lambda_i) + C_{4i}}\Big[\sum_{t=0}^{\infty} \lambda_i^t (C_{1i}(D_{t+2}^i - D_{t+1}^i) + C_{4i}C_{6i}D_{t+1}^i) \\
& - (C_{1i}(1 - \lambda_i) + C_{4i})(I_0^i - C_{5i})\Big] + D_1^i + C_{2i} - \frac{C_{3i}}{2C_{1i}},
\end{aligned}
\tag{3.3}
$$

if $T \to \infty$, where λ_i $(0 < \lambda_i < 1)$ is the smaller root of quadratic equation,

$$
C_{1i}\lambda_i^2 - (2C_{1i} + C_{4i})\lambda_i - C_{1i} = 0. \tag{3.4}
$$

If subcontracting strategies are considered, then the transaction costs of subcontracting will add more combinatorial difficulties to the analysis, as illustrated in Section 2. On the other hand, the problem becomes relatively simple if the transaction costs are neglected. For illustration, we assume $F_i = 0$, $i = 1, 2$. This can be a good approximation when the two firms have a good long-term relationship. By Proposition 2.1. and 2.2., *coordination* subcontracts are not only always transactionally feasible, but also Pareto-dominate any other subcontracts. Therefore, we can reasonably expect that firms follow a certain *coordination* subcontracting mechanism. By the dynamic programming procedure described above, for each period $(t-1)$, the expected future costs are, $C_t^i = E[C_t^{iC}|D_s^i, s \le t-1, i = 1, 2]$. Thus, the expected future cost in Program 0 (for $t-1$) is

$$
C_t^1 + C_t^2 = E[C_t^{1C} + C_t^{2C}|D_s^i, s \le t-1, i = 1, 2] = E[C_t^C|D_s^i, s < t, i = 1, 2].
$$

Hence, by the "certainty equivalence" argument, in order to find the optimal production plan, we only need to solve the following program.

$$\min_{Q_t, P_t^i, I_t^i, i=1,2, t=1,\dots,T} \sum_{t=1}^{T} [c_t^1(P_t^1, I_t^1; D_t^1 - Q_t, I_{t-1}^1) + c_t^2(P_t^2, I_t^2; D_t^2 + Q_t, I_{t-1}^2) + T_1(Q_t) + T_2(Q_t)]$$

$$s.t. \quad I_{t-1}^1 + P_t^1 - D_t^1 + Q_t = I_t^1, \tag{3.5}$$

$$I_{t-1}^2 + P_t^2 - D_t^2 - Q_t = I_t^2, \quad t = 1, \dots T \tag{3.6}$$

provided $T_1(Q_t) + T_2(Q_t)$ can also be approximated by a quadratic function centered at 0, i.e., assuming that $T_1(Q) + T_2(Q) = C_0 Q^2$, $i = 1, 2$. In comparison with the single firm problem (without the consideration of subcontracting), this program has twice the number of decision variables plus Q_t's; the explicit linear decision rule can still be obtained fairly easily.

The first order conditions yield

$$C_{11}(P_{t+1}^1 - P_t^1) - C_{41}(\bar{I}_t^1 - C_{61}(D_t^1 - Q_t)) = 0, \tag{3.7}$$

$$C_{12}(P_{t+1}^2 - P_t^2) - C_{42}(\bar{I}_t^2 - C_{62}(D_t^2 + Q_t)) = 0, \tag{3.8}$$

$$\begin{aligned} & C_{11}\bar{P}_t^1 - C_{41}C_{61}(\bar{I}_t^1 - C_{61}(D_t^1 - Q_t)) - C_{12}\bar{P}_t^2 \\ & \quad + C_{42}C_{62}(\bar{I}_t^2 - C_{62}(D_t^2 + Q_t)) - C_0 Q_t = 0, \end{aligned} \tag{3.9}$$

where $\bar{P}_t^i \equiv P_t^i - \bar{C}_{2i}$, $\bar{C}_{2i} \equiv C_{2i} - C_{3i}/2C_{1i}$, and $\bar{I}_t^i \equiv I_t^i - C_{5i}$. Rewrite (3.9) to be

$$Q_t = \frac{1}{C}[C_{41}(C_{61})^2 D_t^1 - C_{42}(C_{62})^2 D_t^2 + C_{11}\bar{P}_t^1 - C_{12}\bar{P}_t^2 - C_{41}C_{61}\bar{I}_t^1 + C_{42}C_{62}\bar{I}_t^2], \tag{3.10}$$

where $C \equiv C_{41}(C_{61})^2 + C_{42}(C_{62})^2 + C_0$. Substituting (3.10) into (3.5) and (3.6) and solving them for $\bar{P}_t^i$ in terms of $\bar{I}_t^i$ and $\bar{I}_{t-1}^i$, we have

$$\begin{aligned} \bar{P}_t^1 = & \frac{1}{C + C_{11} + C_{12}}[(C_{42}(C_{62})^2 + C_{12})D_t + C_0 D_t^1 + (C + C_{12})(\bar{I}_t^1 - \bar{I}_{t-1}^1 - \bar{C}_{21}) \\ & + C_{12}(\bar{I}_t^2 - \bar{I}_{t-1}^2 - \bar{C}_{22}) + C_{41}C_{61}\bar{I}_t^1 - C_{42}C_{62}\bar{I}_t^2]. \end{aligned} \tag{3.11}$$

$$\begin{aligned} \bar{P}_t^2 = & \frac{1}{C + C_{11} + C_{12}}[(C_{41}(C_{61})^2 + C_{11})D_t + C_0 D_t^2 + (C + C_{11})(\bar{I}_t^2 - \bar{I}_{t-1}^2 - \bar{C}_{22}) \\ & + C_{11}(\bar{I}_t^1 - \bar{I}_{t-1}^1 - \bar{C}_{21}) - C_{41}C_{61}\bar{I}_t^1 + C_{42}C_{62}\bar{I}_t^2]. \end{aligned} \tag{3.12}$$

where $D_t \equiv D_t^1 + D_t^2$, the total production requirement in period t and $C \equiv C_{41}(C_{61})^2 + C_{42}(C_{62})^2 + C_{11} + C_{12}$. Using equations (3.10), (3.11) and (3.12), we can obtain from (3.7) and (3.8) a system of difference equations of $\bar{I}_t^i$ as following.

$$a_1\bar{I}_{t+1}^1 - (2a_1 + b_1)\bar{I}_t^1 + a_1\bar{I}_{t-1}^1 + d_1\bar{I}_{t+1}^2 - (d_1 + d_2 + e)\bar{I}_t^2 + d_2\bar{I}_{t-1}^2 + A_t^1 = 0, \tag{3.13}$$

$$a_2\bar{I}_{t+1}^2 - (2a_2 + b_2)\bar{I}_t^2 + a_2\bar{I}_{t-1}^2 + d_2\bar{I}_{t+1}^1 - (d_1 + d_2 + e)\bar{I}_t^1 + d_1\bar{I}_{t-1}^1 + A_t^2 = 0. \tag{3.14}$$

where

$$
\begin{aligned}
a_i &\equiv C_{1i}(C_{41}(C_{61})^2 + C_{42}(C_{62})^2 + C_0 + C_{1j} + C_{4i}C_{6i}), \\
b_i &\equiv C_{4i}(C_{4j}(C_{6j})^2 + C_0 + C_{11} + C_{12}), \\
d_i &\equiv C_{1i}(C_{1j} - C_{4j}C_{6j}), \\
e &\equiv C_{41}C_{61}C_{42}C_{62}, \\
A_t^i &\equiv (C_{4j}(C_{6j})^2 + C_{1j})(C_{1i}(D_{t+1} - D_t) + C_{4i}C_{6i}D_t) \\
&\quad + C_0(C_{1i}(D_{t+1}^i - D_t^i) + C_{4i}C_{6i}D_t^i),
\end{aligned}
\tag{3.15}
$$

for $i = 1, 2$, $j \neq i$

Solving the linear difference equations (3.13) and (3.14), we can obtain the decision rules for I_1^1 and I_1^2, and the decision rules for production and subcontracting follow directly. For general methods of solving linear difference equations, see Miller (1968)

We now look at the situation in which firms have symmetric cost functions. Let $C_k = C_{ki}$ for $i = 1, 2$, $k = 1, \ldots, 7$, $a = a_i$, $b = b_i$, $d = d_i$ and $I_0 = I_0^i$, for $i = 1, 2$. Then, the system of equations (3.13) and (3.14) is easy to solve. Taking the sum and difference of (3.13) and (3.14), we obtain the following two system of difference equations,

$$(a + d)U_{t+1} - (2(a + d) + b + e)U_t + (a + d)U_{t-1} + A_t^1 + A_t^2 = 0, \tag{3.16}$$

$$(a - d)V_{t+1} - (2(a - d) + b - e)V_t + (a - d)V_{t-1} + A_t^1 - A_t^2 = 0, \tag{3.17}$$

where $U_t \equiv \bar{I}_t^1 + \bar{I}_t^2$ and $V_t \equiv \bar{I}_t^1 - I_t^2$ Each of (3.16) and (3.17) is a single variable linear difference equations which can be solved easily Observing that

$$
\begin{aligned}
a + d &= C_1(2(C_4(C_6)^2 + C_1) + C_0), \\
b + e &= C_4(2(C_4(C_6)^2 + C_1) + C_0), \\
a - d &= C_1(2(C_4(C_6)^2 + C_4C_6) + C_0), \\
b - e &= C_4(2C_1 + C_0),
\end{aligned}
$$

and equations (3.11), (3.12) and (3.10), we have the following.

Let λ and λ^* be the smaller roots of

$$C_1\lambda^2 - (2C_1 + C_4)\lambda - C_1 = 0, \tag{3.18}$$

$$C_1(\lambda^*)^2 - (2C_1 + C_4 + \frac{2(C_1 - C_4C_6(C_6 + 1))}{2(C_4(C_6)^2 + C_4C_6) + C_0})\lambda^* - C_1 = 0, \tag{3.19}$$

respectively. Note that $\lambda = \lambda^*$ when $C_0 = \infty$.

Lemma 3.2. *For $i = 1, 2$,*

$$I_1^{iC} = \frac{1}{C_1(2-\lambda)+C_4}[\sum_{t=0}^{\infty} \lambda^t (C_1(\bar{D}_{t+2}^i - D_{t+1}^i) + C_4 C_6 D_{t+1}^i)$$

$$+ C_1(I_0 - C_5)] + C_5, \tag{3.20}$$

$$P_1^{iC} = \frac{1}{C_1(2-\lambda)+C_4}[\sum_{t=0}^{\infty} \lambda^t (C_1(\hat{D}_{t+2}^i - \hat{D}_{t+1}^i) + C_4 C_6 \hat{D}_{t+1}^i)$$

$$- (C_1(1-\lambda)+C_4)(I_0 - C_5)] + \alpha D_t^i + (1-\alpha) D_t^j + C_2 - \frac{C_3}{2C_1}, \tag{3.21}$$

$$Q_1^C = \frac{1}{2C_4(C_6)^2 + C_0}[(C_4(C_6)^2 + C_1(2\alpha - 1))\Delta D_1$$

$$+ \frac{1}{C_1(2-\lambda)+C_4} \sum_{t=0}^{\infty} \lambda^t (C_1(\Delta \check{D}_{t+2} - \Delta \check{D}_{t+1}) + C_4 C_6 \Delta \check{D}_{t+1})] \tag{3.22}$$

if $T \to \infty$, where λ is the smaller root of (3.18), for $i = 1, 2$, $j \neq i$,

$$\alpha \equiv \frac{C_4(C_6)^2 + C_1 + C_0}{2(C_4(C_6)^2 + C_1) + C_0}, \tag{3.23}$$

$$D_t^i \equiv \beta_t D_t^i + (1-\beta_t) D_t^j, \quad \beta_t \equiv \frac{1}{2}(1+\delta_t), \tag{3.24}$$

$$\delta_1 \equiv \frac{C_0(C_1(2-\lambda)+C_4)}{2C_1C_4(C_6(C_6+1)(2-\lambda^*)+1) + C_0(C_1(2-\lambda^*)+C_4)},$$

$$\delta_t \equiv \delta_1 \cdot \left(\frac{\lambda^*}{\lambda}\right)^t \cdot \frac{C_1 - \lambda^*(C_1 - C_4C_6)}{C_1 - \lambda(C_1 - C_4C_6)}, \qquad \text{for } t \geq 2,$$

$$\hat{D}_t^i \equiv \eta \bar{D}_t^i + (1-\eta) D_t^j, \quad \eta \equiv \frac{2C_4(C_6)^2 + C_1 + C_0 + C_4C_6}{2(C_4(C_6)^2 + C_1) + C_0}, \tag{3.25}$$

or equivalently,

$$\hat{D}_t^i \equiv \gamma_t D_t^i + (1-\gamma_t) D_t^j, \quad \gamma_t \equiv \beta_t \eta + (1-\beta_t)(1-\eta), \tag{3.26}$$

$$\check{D}_t^i \equiv C_1 \hat{D}_t^i - C_4 C_6 D_t^i, \tag{3.27}$$

$\Delta D_t \equiv D_t^1 - D_t^2$, $\Delta D_t \equiv \bar{D}_t^1 - D_t^2$, and $\Delta \check{D}_t \equiv \check{D}_t^1 - \check{D}_t^2$.

Notice that in the above lemma, we intentionally write the decision rules of production and inventory of firm i in the very same linear form as those it uses in a single-firm planning problem as in (3.3) and (3.4) except that instead of using only its own demand forecasts, firm i uses a weighted sum of its own demand and the other firm's demand. The weights on its own demands increases as the cost of subcontracting (C_0) increases, the decision rules converge to those of a single firm as C_0 approaches infinity, and at the other extreme, firms simply use the arithmetic average demand of the two in their decision rules when $C_0 = 0$.

Lemma 3.3. *The weights, α, β_t and γ_t, for $t \geq 1$, increase from one half to one as C_0 increases from zero to infinity.*

Assume that the demands $\{D_t^1, D_t^2, t \geq 1\}$ are independent from period to period, $E[D_t^i] = \mu$, $Var[D_t^i] = \sigma^2$, and $Cov[D_t^1, D_t^2] = \rho\sigma^2$ for $i = 1, 2$ and $t \geq 1$, where ρ is the correlation coefficient of the two demand streams, $-1 \leq \rho \leq 1$.

Proposition 3.1. *For firms having symmetric cost functions,*

$$E[I_1^{iC}] = E[I_1^i], \quad E[P_1^{iC}] = E[P_1^i], \quad E[Q_1^C] = 0,$$

$$Var[I_1^{iC}] \leq Var[I_1^i], \quad Var[P_1^{iC}] \leq Var[P_1^i],$$

and the equalities hold only when $\rho = 1$ or $C_0 = \infty$. That is, subcontracting reduces the variability of inventory and production as long as the subcontracting cost is not too high or the demands of the firms are not perfectly (positively) correlated. Moreover, the variances of inventory and production decrease and the variance of the subcontracting quantity increases as the correlation coefficient of demand, ρ, decreases and as the subcontracting cost, C_0, decreases.

Proof. The first result is obvious by simply noticing that the rules with and without subcontracting are the same except for using different demand forecasts, but demands for the two firms have the same expected value.

To prove the second result, first note that

$$\begin{aligned} Var[\bar{D}_t^i] &= Var[\beta_t D_t^i + (1-\beta_t)D_t^j] \\ &= \beta_t^2 Var[D_t^i] + (1-\beta_t)^2 Var[D_t^j] + 2\beta_t(1-\beta_t)Cov[D_t^i, D_t^j] \\ &= \sigma^2 - 2\beta_t(1-\beta_t)(1-\rho)\sigma^2. \end{aligned} \tag{2.28}$$

Second, we rewrite

$$\begin{aligned} &\sum_{t=0}^{\infty} \lambda^t (C_1(\bar{D}_{t+2}^i - D_{t+1}^i) + C_4 C_6 D_{t+1}^i) \\ &= (C_4 C_6 - C_1)D_1 + \sum_{t=0}^{\infty} \lambda^t (C_1 + \lambda(C_4 C_6 - C_1))D_{t+2}^i. \end{aligned}$$

Note that we assume independence between periods and that I_1^i and I_1^{iC} has the same linear form (independent of ρ and C_0) and have demands D_t^i and $\bar{D}_t^i$ respectively. By (2.28) and Lemma 3.3., $Var[\bar{D}_t^i]$ is increasing in ρ and C_0 since $\beta_t = 1/2$ when $C_0 = 0$ and β_t is increasing in C_0, and $Var[\bar{D}_t^i] = Var[D_t^i]$ if and only if $\rho = 1$ or $C_0 = \infty$ ($\beta_t = 1$). And $Var[I_1^{iC}]$ depends on ρ and C_0 solely via $Var[\bar{D}_t^i]$. The conclusion with respect to the production decision rule can be similarly proved.

To calculate the variance of Q_1^{C}, note that

$$\Delta \bar{D}_t = (2\beta_t - 1)\Delta D_t, \quad \Delta \check{D}_t = (C_1(2\gamma_t - 1) - C_4 C_0(2\beta_t - 1))\Delta D_t$$

Therefore, $Var[Q_1^C]$ must be some positive term multiplied by $Var[\Delta D_t]$, but

$$Var[\Delta D_t] = Var[D_t^i] + Var[D_t^j] - 2Cov[D_t^i, D_t^j] = 2(1 - \rho)\sigma^2,$$

which decreases as ρ increases, and the positive term is an decreasing function of C_0.

∎

From the proposition, we can see that subcontracting can absorb the variability as do other planning strategies such as inventory. As a result, both production and inventory variabilities are reduced. If we do a more comprehensive exercise to include work force planning, then, we can show that subcontracting helps to stabilize the work force size as well. It is particularly costly to ignore subcontracting as a planning strategy when the firms' production requirements are negatively correlated and when it is relatively easy to underwrite and implement a subcontract. It is usually the case that demands are negatively correlated if the products of firms are substitutes.

The model we have examined has the second interpretation to access the value of flexible technology. As we mentioned in the beginning, we assume that the technologies of the two firms are good substitutes for the physical possibility of subcontracting. There could be two cases. In the first case, the output of firms requires the similar technology that firms currently have. In the other case, the presence of flexible manufacturing technology makes firms able to fulfil production requirements which need traditionally different technologies. Therefore, facing demands with traditionally different technology requirements, it is infeasible to have subcontracting with the dedicated technologies but it is feasible with the flexible technologies. In the context of intrafirm production planning with two departments having demands that require traditionally different technologies, it is infeasible to coordinate the production requirements of two departments with the dedicated technologies but it is feasible to do so with the flexible technologies. Having this in mind, the single-firm problems discussed above represent the optimal plan with the dedicated technologies, while the multi-firm problem represents the optimal plan with perfect flexible technologies, for both inter- and intra-firm situations. We can conclude that the flexibility is also a substitute for strategies such as inventory and subcontracting to reduce the variability of production.

Piore and Sabel (1984) present a brilliant study of a new system of industrial production, a system of flexible specialization, as opposed to standardized mass production. American mini steel mills, French and Italian textile firms, and Japanese tool makers are exemplary of this new industrial order. Piore and Sabel (1984) point out that technological innovation, constant subcontracting rearrangements and the search for new products are the structuring elements of the systems of

flexible specialization. In the particular examples of French and Italian textile firms, the variability of demand results in the constant rearrangement of subcontracting patterns: every prime contractor could become a subcontractor, every subcontractor a prime contractor. The evidence indicates the close relations between subcontracting and flexibility and between subcontracting and coordination of interorganization production.

4. Discussion

Variability in production requirements has added more complexity for researchers and practitioners in production planning and scheduling. There are mainly two ways to achieve *production smoothing*. The first is through the use of inventory, which represents production in advance. The second is through modification of the demand pattern itself. Subcontracting as a production planning strategy falls in the second category. There are other important strategies in the second category, pricing, promotion or rejection. Subcontracting plays an important role when the other strategies fail to eliminate the variability completely (it is usually impossible).

There are many ways in which a subcontract can be produced. There are two categories of subcontracting mechanisms: market mechanisms and relational contracting (non market) mechanisms. *Coordination* subcontracting falls in the second category. When the number of contracting parties is small, market mechanisms usually result in less desirable outcomes but have low transaction costs. *Coordination* subcontracts, on the other hand, yield the most total cost savings and Pareto-dominate all possible subcontracts, but may have high transaction costs. Considering subcontracting in production planning, we need a careful study of the forms of subcontracting mechanisms and their costs. Nevertheless, *coordination* subcontracting can at least test whether the set of feasible subcontracts is nonempty and project the "right" subcontracting quantities.

To have subcontracting as a planning tool, we need more information. Firms do not only need to forecast their own production requirements but also need the other parties demand information. When the products of firms are good substitutes, this is not difficult to do. In particular, sometimes, it is easier to predict total industry demand "correctly" than to do it for each individual firm. Firms also need each other's cost data which is crucial in determining a *coordination* subcontract. The question here is how firms obtain other firms' private information if they don't have it in the first place. In subcontracting via markets, the situation can be formulated as a game with incomplete information. In subcontracting via relational contracting, one may look for an optimal subcontract within a subset of feasible subcontracts which induces truth-telling behavior, namely, incentive compatible subcontracts (Hurwicz (1972) and Myerson (1979)). In general, firms would incur informational losses in those situations. As we argued before, learning and reputation effects will lead to the consensus on information if subcontracting is carried out on a long-term basis.

Subcontracting requires additional computation. For example, we need to solve a multi-firm

plan in order to find a *coordination* subcontract. When the number of parties in contract is small, this increase in computation is moderate. As in our example, we can see that firms use the very same linear rule as in single-firm problem and simply replace the requirements by the weighted sum of the requirements of the two firms. When the number of parties increases, the computational burden of a *coordination* subcontract will be heavy, and some simple market mechanism may be preferable.

Pricing is a very important strategy which one certainly should consider in organizing production. Kamien, Li and Samet (1988) study a one-shot pricing game with subcontracting. There, we identify that firms may use subcontracting as means of collusion to achieve a price more desirable to both firms. Considering pricing as a strategy in production planning is certainly an important item on our future research agenda.

In sum, subcontracting is one way to reallocate production requirements among productive organizations and to achieve production smoothing. Subcontracting usually lowers the total production costs, and hence, is socially desirable. To incorporate subcontracting into production planning, firms are required to expand their scope of planning to consider other agents in the economy.

There are many other issues concerning subcontracting practice we have not discussed in detail in this paper. Two of them are of most importance. The first one is "sourcing", namely, how to to select (qualify) subcontractors. The second is "managing", namely, how to manage a subcontract once it is signed. These two aspects have tremendous impact on subcontracting costs. The criteria here could be quality, on-time delivery, conformance to engineering specification, etc. Regarding the economic environment, there might be uncontrollable uncertainties, private information (problems of adverse selection), and/or unobservable actions (problems of moral hazard). How to deal with sourcing and managing subcontracts with the possible applications of economic theories such as principal-agent and mechanism design deserves further research.

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