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R. E. SCOTT

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R. E. Scott

Abstract

An analog device has been developed which can solve the approximation problem of network synthesis with sufficient accuracy for most network problems. The complex frequency plane is represented by a conducting sheet of Teledeltos paper. The zeros and poles of the rational function are represented by positive and negative currents introduced into the plane. The voltage along the $j\omega$ axis in the plane represents the logarithm of the magnitude of the rational function. This voltage is scanned by a commutator at 10 cycles per second and displayed upon the face of a cathode ray tube. The locations of the poles and zeros for a desired function are obtained by a cut and try process.

A comparison of calculated and observed results shows that the accuracy of the device is 1-5 percent for the logarithm of the magnitude of the network function, and 3-15 percent for the phase, depending upon the particular function which is being solved. This accuracy is adequate for most network problems. A complete analysis is included of the errors introduced by the finite size of the plane, the finite size of the current and voltage probes, and the finite series impedance of the current sources.

AN ANALOG DEVICE FOR SOLVING THE APPROXIMATION PROBLEM OF NETWORK SYNTHESIS

INTRODUCTION

In this paper an analog device is described which provides a means of solving experimentally the approximation problem of network synthesis. For many years the theory of synthesis of linear, passive, lumped-parameter systems has been known (1, 2). Practical applications of the general theory have lagged behind because of the difficulty of making suitable approximations. At the present time the necessary approximations are usually made by means of laborious cut-and-try methods. The same general procedure is used with the analog, (10, 11) but the time required is reduced from days to minutes. Thus, much of the tedious labor is removed from network synthesis and the designer can focus his attention on the more essential elements of a problem.

The Nature of Network Synthesis

Network synthesis can be described in general as the problem of designing an electrical network which will produce a desired output response when a given input stimulus is applied to it. A typical situation is illustrated in Fig. 1.

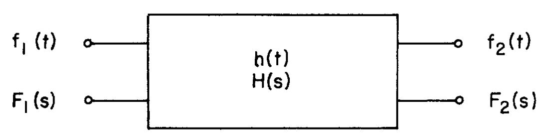


Fig. 1 The general synthesis problem.

In Fig. 1 the symbols are defined as follows:

$f_1(t)$	the input stimulus to the network
$f_2(t)$	the output response from the network
$h(t)$	the output response for a unit impulse input
$F_1(s)$	the frequency response of the input stimulus
$F_2(s)$	the frequency response of the output response
$H(s)$	the frequency response of the network
$s = \sigma + j\omega$	the complex frequency variable
σ	the damping factor
ω	the angular frequency

In the time domain the network is characterized by its impulse response, h(t). This function is related to the input, $f_1(t)$, and the output, $f_2(t)$, by the relatively complicated convolution integral

$$f_2(t) = \int_0^\infty f_1(t)\, h(t - T)\, dT \quad . \tag{1}$$

In the frequency domain the network is characterized by its frequency response H(s). This function is the quotient of the input $F_1(s)$, and the output $F_2(s)$.

$$F_2(s) = F_1(s)\,H(s) \quad . \tag{2}$$

The relative simplicity of Eq. 2 in comparison with Eq. 1 has led to the development of network synthesis in terms of frequency responses, rather than in terms of time responses. If the data for a problem are given in the frequency domain, the procedure is straightforward. If they are given in the time domain the first step is usually to transform them to the frequency domain by taking a Fourier transform or its equivalent. These two cases will be discussed separately.

Data Given in the Frequency Domain

In a great many problems of network synthesis the data are given directly in the frequency domain. Often they take the form of curves of the desired response as a function of the frequency ω. These curves may arise as theoretically desirable forms, e.g. the flat amplitude response of a low-pass filter, or they may be derived from experimental results, e.g. frequency compensation for an amplifier. The synthesis procedure when the data are in this form consists of two steps.

1. The approximation of the given curve by a function of ω which can be realized by linear, passive, lumped-parameter elements
2. The calculation of the network elements from this function

It is the first of these two steps which is facilitated by the use of the potential analog which is described in this paper. There are relatively direct methods (3, 4) for accomplishing the second step, and they will not be discussed here.

Data in the Time Domain

When the data are given in the time domain the designer has the choice of working directly in terms of time functions (5, 6) or of transforming the data to the frequency domain. This latter procedure is usually the most satisfactory, at least within the limitations of the present techniques for time domain synthesis. It may be carried out by Fourier series when the functions are periodic, and by Fourier integrals (or Laplace transforms) when they are not. The complete process is then

1. The conversion of the given data to the frequency domain
2. The approximation by a realizable function of ω
3. An inverse transformation to the time domain to check on the tolerances of the approximation process
4. The calculation of the network elements from the approximate function

The approximation problem remains an essential element of network synthesis regardless of the original form of the data.

The Role of the Approximation Problem in Network Synthesis

The approximation problem in network synthesis arises from the necessity of expressing the network function H(s) in a form which can be realized by the physical network elements, resistors, inductors, and capacitors. To be physically realizable (7), H(s) must be a rational function of the form

$$H(s) = \frac{A(s - s_1)(s - s_3)\ldots}{(s - s_2)(s - s_4)\ldots} \tag{3}$$

where A is a constant and s is the complex frequency variable.

The complex frequencies s_1, s_3, ... which make the numerator of H(s) zero, are called the zeros of the function, and the complex frequencies s_2, s_4, ... which make the denominator zero are called the poles of the network. The function H(s) is determined by the locations of its poles and zeros in the complex frequency plane, to within a constant multiplier.

If H(s) is to represent the driving-point impedance of a two-terminal network the poles and zeros must satisfy the following additional requirements.

1. The poles and zeros must lie in complex conjugate pairs (or on the real axis).
2. The poles and zeros must lie in the left half-plane (in the limit they may lie on the imaginary axis).
3. Poles on the imaginary axis must be simple and the residues must be real and positive.
4. The phase angle of the function along the imaginary axis must remain in the range $-90°$ to $+90°$ i.e. the poles and zeros on either axis must alternate.

If H(s) is the transfer impedance of a four terminal network, the above requirements may be relaxed a little.

1. The poles and zeros must lie in complex conjugate pairs or on the real axis as before.
2. The poles must lie in the left half-plane as before, but the zeros may be anywhere.
3. Poles on the imaginary axis must be simple.

The network response obtained as the ratio of the required output response to the given input response (Eq. 2) will not in general have the form of the rational function of Eq. 3. The approximation problem consists in finding a function of the required form which will approximate the desired function to a required degree of accuracy, and at the same time meet the requirements which have just been stated. Essentially the problem is one of finding a set of poles and zeros which meet the required conditions, and which approximate the desired function for $s = j\omega$. An experimental method of obtaining the poles and zeros will now be described.

The Use of Potential Analogs for Locating Poles and Zeros

The approximation problem can be solved analytically, but the amount of work required is prohibitive. Fortunately, an experimental method is available for locating poles and zeros which will approximate any H(s). This method is based upon a potential analog which satisfies an equation similar to Eq. 3. If the logarithm of Eq. 3 is taken, the result is

$$\ln H(s) = \ln A + \sum_{n\,odd} |s - s_n| - \sum_{n\,even} |s - s_n| + j\left[\sum_{n\,odd} \arg(s - s_n) - \sum_{n\,even} \arg(s - s_n)\right]$$

$$= G + j\phi \tag{4}$$

$$\text{where } G = \ln A + \sum_{n\,odd} |s - s_n| - \sum_{n\,even} |s - s_n| \tag{5}$$

$$\phi = \sum_{n\,odd} \arg(s - s_n) - \sum_{n\,even} \arg(s - s_n) \tag{6}$$

G is the gain function of the network and ϕ is the phase function of the network.

From potential theory (see proof below) it is known that the voltage in a uniform conducting sheet has the same form as the gain function in Eq. 5, if positive currents are introduced at points corresponding to the zeros, and negative currents at points corresponding to the poles. This analog may be used directly to approximate a given gain function. The procedure is:

1. The required gain function is plotted.
2. Currents are introduced into the analog plane to represent the poles and zeros of the network.
3. The poles and zeros are arranged to give a voltage along the imaginary axis which has the same form as the required gain function.

If this procedure is followed the poles and zeros of the network will be obtained. To realize the full benefit of the analog, a number of additional requirements must be met.

1. The phase as well as the gain function must be available for observation.
2. The voltage along the imaginary axis must be plotted continuously, so that the effect of changing a pole position will be instantly observable.
3. The magnitude of the errors introduced from all sources must be known and kept within reasonable limits.

These conditions are met by the analog device which is described on the following page.

THE THEORY OF THE ANALOG

Prior to a description of the analog device a brief outline of the theoretical background will be given. In the course of this derivation a method of obtaining the phase will be presented, and also a method of overcoming the errors introduced by the finite size of the analog plane.

The Derivation of the Analog Formulas

The analog formulas will be derived by considering a small region of area A, bounded by a closed curve C, in a conducting sheet of infinite extent. This situation is pictured in Fig. 2.

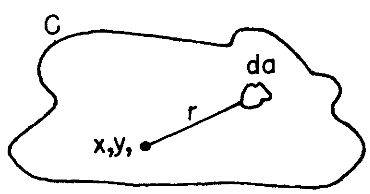


Fig. 2. A closed region in a conducting plane.

If there are no sources within the region, the integral of the normal component of the current density around the contour C is zero. If there are sources present the integral is equal to the total current originating in the region.

$$\int_C J.n\,ds = I_t \tag{7}$$

For convenience it will be supposed that all the current sources within the region C have the same current I, and that they are distributed about the region in a continuous fashion with a density m. Outside the region C there are no current sources, and the boundary is remote enough so that it produces no effect within C. Then

$$\int_A \text{div } J\,da = \int_A m.I\,da \tag{8}$$

or

$$\text{div } J = mI \tag{9}$$

But

$$J = \frac{E}{\rho} = -\frac{\text{grad }\phi}{\rho}$$

where ρ = surface resistivity and ϕ = potential.

Hence

$$\text{div grad }\phi = -m\rho I$$

or

$$\nabla^2\phi = -m\rho I \quad . \tag{10}$$

This is Poisson's equation. The solution for the conditions assumed here is (8).

$$\phi(x,y) = \frac{\rho I}{2\pi} \int_A m \ln\frac{r_o}{r}\,da \quad . \tag{11}$$

where x, y, is any point in the region C

r is the distance from x, y, to the element of area da

and r_o is a constant which sets the zero potential level.

Equation 11 may be written

$$\phi(x,y) = \frac{\rho I}{2\pi} \int_A (m \ln r \, da) - A \tag{12}$$

where A is a constant depending on the zero level only.

Instead of continuous distribution of currents we may now assume a finite distribution, with positive currents at the points s_1, s_3, ... corresponding to the locations of the zeros, and negative currents at the points s_2, s_4, ... corresponding to the locations of the poles. The point x, y, will be represented by the complex variable s, and the distance r, from any point s_n will be $|s - s_n|$. The current density m will be an impulse function, and the value of mda in the neighborhood of a zero will be unity, in the neighborhood of a pole, negative unity. Accordingly Eq. 12 becomes

$$\phi(s) = \frac{\rho I}{2\pi} \left[\sum_{n\,odd} \ln |s - s_n| - \sum_{n\,even} \ln |s - s_n| \right] - A \tag{13}$$

Equation 13 is identical in form with Eq. 5 for the gain function of a network. The multiplying constant $\frac{\rho I}{2\pi}$ merely determines the scale factor of the result; the additive constant A determines the zero level of the voltage, i.e. the impedance level of the gain function. Practical methods of determining these constants are discussed in Sect. 4.

Thus it has been shown that the voltage at any point in a conducting plane has the same form as the gain function, providing that positive and negative currents are introduced at the points corresponding to the zeros and poles of the gain function. The analog has been established under the restrictive assumptions of infinitely small probe size, and an infinitely large conducting medium. The errors introduced by variations from these ideal conditions are discussed in Sect. 5.

The Phase Function

The logarithm of the impedance function is a complex function equal to

Gain + j Phase.

It is an analytic function and accordingly the Cauchy-Riemann equations hold. Thus

$$\frac{\partial(\text{Gain})}{\partial \sigma} = \frac{\partial(\text{Phase})}{\partial \omega} \tag{14}$$

and

$$\text{Phase}(\omega) = \int_0^{\omega} \frac{\partial \,\text{Gain}}{\partial \sigma} \, d\omega \, . \tag{15}$$

In the potential analog the gain is represented by the potential, and the phase is represented by the conjugate potential function, i.e. the current flow. If the phase is required at the point $s_1 = \sigma_1 + j\omega_1$ it can be obtained by integrating Eq. 15 along the line $\sigma = \sigma_1$ from $\omega = 0$ to $\omega = \omega_1$. As an alternative the phase is the total current flow across the line joining $s = \sigma_1 + j\omega_1$ to the point $s = \sigma_1$. This method of obtaining the phase omits the

90° phase shift which is produced by a pole or a zero at $\sigma = \sigma_1$ (at the origin, if the phase is measured at real frequencies). In practice, it is a simple matter to take this into account.

At real frequencies the formula for the phase becomes

$$(\text{Phase})(\omega_1) = \int_{\partial}^{\omega_1} \frac{\partial(\ln|Z|)}{\partial\sigma} \, d\omega$$

$$= K \int_{0}^{\omega_1} \rho J_{\sigma} \, d\omega$$

where K is a proportionality constant between gain function and the voltage and J_{σ} is the current density normal to the jω axis.

$$\text{Phase}\,(\omega_1) = K I_{\sigma} \tag{16}$$

where I_{σ} is the total current flowing across the jω axis between the origin and the point at which the phase is being measured.

In the physical analog it is simpler to measure the voltage gradients at right angles to the jω axis and to integrate them to obtain the phase than it is to measure the total current flowing across the axis. For conceptual purposes however, the proportionality of the phase and the total current is particularly attractive.

Symmetry Conditions in the Potential Analog

Symmetry conditions in the potential analog can be used to simplify the construction of a physical device. Perhaps more important is the way in which they simplify the conceptual background of the subject.

Symmetry About the Real Axis

Because of the conjugate nature of the poles and zeros, network impedances are symmetrical about the real axis. This means that in the potential analog the sources and sinks will be likewise distributed symmetrically with respect to the real axis. Thus, no current will flow across the real axis, and the plane may be cut along it without altering the voltage distribution in any way. Either half-plane by itself contains all the information which is required for the analog.

Poles and zeros right on the axis introduce a further consideration. The current from them would normally split half-and-half between the two half-planes, and accordingly if only a single half-plane is used in the analog, currents introduced along the axis should be reduced to one-half of their normal value.

Symmetry About the Imaginary Axis

There is no symmetry about the imaginary axis in the original problem. But symmetry can be created if the original problem is set up and simultaneously with it a problem which is its mirror image in the imaginary axis. Along the imaginary axis the voltage produced by each of these problems will be identical and the measured voltage will simply double. In addition, the axis represents a saddle point for the sum of the two solutions, and this means that the measuring probes do not have to be so accurately positioned.

If the phase is being measured by means of the rate of change of voltage in a direction at right angles to the imaginary axis, some further changes are necessary. The required derivative is everywhere zero if the method of the preceding paragraphs is used. (Indeed this is the point of using the method.) Reversing the currents in the probes representing the image problem gives a voltage gradient which is simply double the required one.

These results can be proved mathematically in the following fashion. Let the original impedance function be

$$Z(s) = \sum \frac{s - s_k}{s - s_j} \quad \text{as in Eq. 4.}$$

The logarithm of the magnitude of the impedance function is then the voltage measured along the axis

$$V = \sum \ln \left| \frac{j\omega - s_k}{j\omega - s_j} \right| \tag{17}$$

and the phase function is

$$\sum \arg \frac{j\omega - s_k}{j\omega - s_j} \ . \tag{18}$$

Magnitude

For the measurement of the logarithm of the magnitude of the impedance function, the image solution is added. This changes Z(s) to

$$Z(s)' = \sum \frac{s - s_k}{s - s_j} \, \frac{s + \overline{s_k}}{s + \overline{s_j}} \tag{19}$$

where the bar denotes the conjugate complex number.

Now the logarithm of the magnitude of the impedance function along the $j\omega$ axis is

$$\ln Z(j\omega)' = \sum \ln \left| \frac{j\omega - s_k}{j\omega - s_j} \right|^2$$

$$= 2 \sum \ln \left| \frac{j\omega - s_k}{j\omega - s_j} \right| \tag{20}$$

and the phase is 0. The result shows that the magnitude measurement is simply doubled and the phase is zero.

Phase

For the phase measurements the problem is set up in the same way except that the poles and zeros in the image solution are interchanged. Thus Z(s) becomes

$$Z(s)'' = \sum \frac{s - s_k}{s - s_j} \; \frac{s + \overline{s_j}}{s + \overline{s_k}} \; . \qquad (21)$$

The logarithm of the magnitude of the impedance function along the jω axis is now zero and the phase is

$$\sum 2 \arg \frac{j\omega - s_k}{j\omega - s_j} \; . \qquad (22)$$

The result shows that the magnitude is zero, and that the phase is simply doubled.

An interesting corollary follows from these results. When the magnitude function is measured with the added image solution, the current across the jω axis is zero and the plane could be slit along it without changing the solution in any way. Accordingly, the logarithm of the magnitude of the impedance function is proportional to the voltage along the open circuited jω axis.

When the phase is being measured, the voltage along the jω axis is zero and a conductor could be placed along it without altering the solution in any way. The current in this conductor at any point is the integral of the current density from the end of the conductor to the point. Thus, the phase is proportional to the current which flows along the short-circuited jω axis.

These two concepts are very helpful in presenting an intuitive picture of the gain and phase functions of a network from the position of its poles and zeros.

A Conformal Transformation to Minimize Errors Due to the Finite Size of the Plane

An analysis of the errors incurred because of the finite size of the plane is given in Sect. 5. In the device which is being described here these errors are minimized by the use of a logarithmic transformation of the conducting plane(12).

The transformation

$$W = \ln s = \ln r + j(\theta + 2n\pi)$$

where $s = re^{j\theta}$ and $W = U + jV$ maps the whole s plane into a strip of width 2π in the W plane. The whole W plane thus corresponds to an infinite number of sheets in the s plane. A sketch of the transformation is given in Fig. 3.

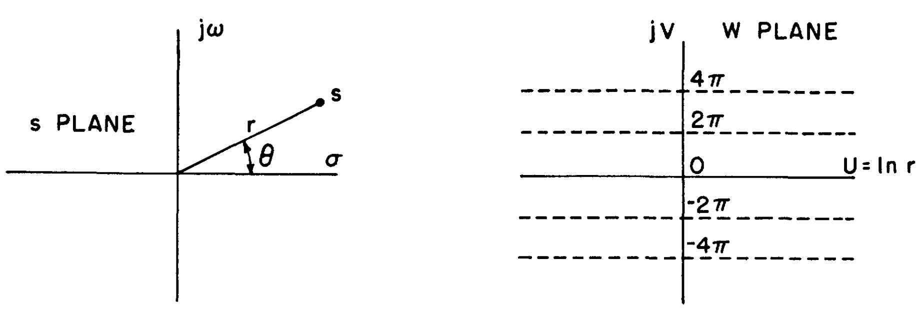


Fig. 3
The transformation
W = ln s.

The problem of limiting the analog to a single sheet in the s plane may be solved in the following manner.

1. Imagine that each sheet in the s plane has a distribution of poles and zeros. The configurations will be identical and they will all be symmetrical about the real axis because of the nature of impedance functions.
2. Make the branch cut along the positive, real axis.

If these steps are followed, no current will flow from one Riemann sheet to the next. Accordingly, a single sheet may be set up to represent the analog. In the W plane, this corresponds to two half-planes separated along the $V = 0$ axis, and with a symmetrical distribution of poles and zeros repeated every distance of 2π in the V direction. Because of the symmetry, no current will flow across any line $V = n\pi$. Thus, additional cuts may be made along any line $V = n\pi$ without altering the distribution of potential, and a single strip of width π will represent the complete analog. (These additional cuts are not branch cuts.)

Advantages of the Analog

1. The finite-size problem is solved by the logarithmic nature of the coordinates.
2. A linear spacing of the probes may be used for both the gain and the phase. The curves are then plotted against ln (frequency).
3. The logarithmic coordinates are convenient for many problems. They correspond to the gain-phase analysis of circuits and log-db analysis of servomechanisms.
4. Calibration is simple using the ln-db techniques.
5. The accuracy is uniform over the useful region of the plane.
6. If a small area near each end is left unused, the errors due to the finite size of the plane will be negligible.

The advantages of the logarithmic transformation led to the construction of the physical analog in the logarithmic plane. In order to facilitate the operation of the device a conformal map of the original s-plane coordinates was printed on the conducting material in the logarithmic plane.

General Description of the Analog Device

The essential elements of the analog device are shown in the block diagram of Fig. 4. The conducting plane is a rectangular strip of Teledeltos paper which represents a semi-infinite strip in the logarithmic plane. Theoretically it should extend to infinity at both ends, but very little error is introduced by using a finite length and placing a conductor at each end, one to represent zero, and one to represent infinity in the logarithmic plane. The $j\omega$ axis from the s plane maps into a line along the center of the strip from one end to the other. The positive, real axis in the s plane maps as the bottom edge of the strip in Fig. 4.

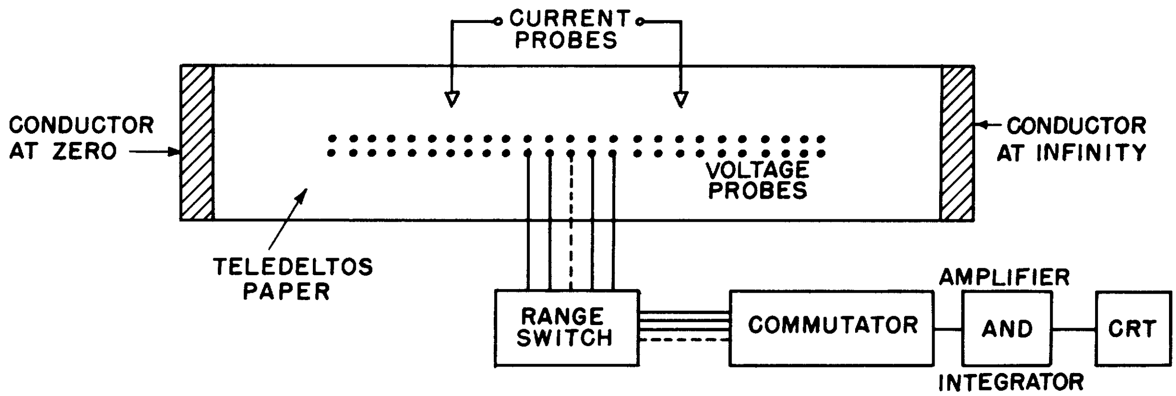


Fig. 4 Block diagram of the analog device.

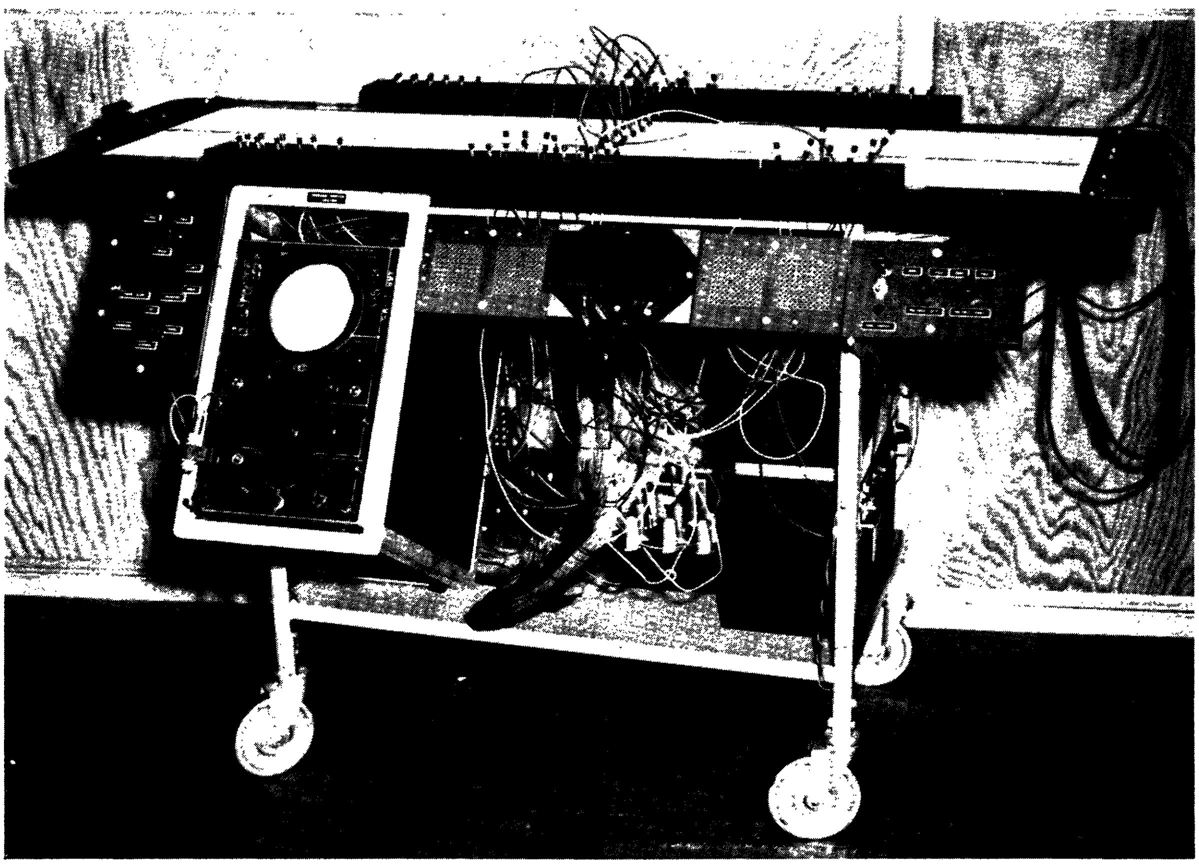


Fig. 5 The complete machine.

Currents are introduced into the conducting plane by means of current probes. These probes are simply steel needles mounted in polystyrene holders. Ideally the sources feeding the current probes should be pure current sources. In practice this ideal is approximated by using 300 kilohms in series with each one, and a d-c source of either plus or minus 300 volts.

The voltage along the frequency axis is picked up by means of two rows of permanent probes. These probes are phonograph needles, mounted from beneath the plane so that only their tips come through the paper. Two rows are used so that both the gain and the phase function may be obtained. To obtain the former the average voltage of the probes in the two rows is used, and to obtain the latter the difference in voltage is used. There are 300 probes in each of the two rows and the commutator is capable of handling only one-third of these at a time. Hence a range switch is provided to select the three sections of the plane in turn. The output voltage from the commutator is amplified and displayed on the cathode ray tube for the gain function. For the phase function an extra stage of integration is required. The complete machine is shown in Fig. 5.

The Method of Operation

The method of operation of the machine is fairly simple. The desired form of the gain or the phase function is pasted upon the cathode ray tube and the poles and zeros are moved around from place to place in the conducting sheet until the picture on the cathode ray tube agrees with the desired one to within the desired tolerances.

In this manner the approximation problem of network synthesis can be solved in a few moments. The corresponding analytical procedure would require several days in many cases. Above all it should be noticed that problems requiring a large number of poles and zeros are handled in the analog with almost the same facility as the simplest problem. The same thing is not true at all of the analytical approach.

The Elements of the Machine

The Teledeltos Paper

Teledeltos paper is a carbon-impregnated paper with a surface resistivity of about 4000 ohm-meters per meter. It has a current-carrying capacity of about 9 ma per linear inch. (It burns at about 40 ma per linear inch.) The paper is fairly uniform. Deviations from uniformity and the errors introduced into the analog are discussed in Sect. 5. In the analog a strip of Teledeltos paper 55 inches long and 13.6 inches wide was used for the conducting plane. This size corresponds to a length of 10 inches per decade. This was chosen as a convenient value. It allows a plane of 5 decades in length without exceeding the size which could be handled on the printing presses in the Boston area. The three decades in the center of the plane represent the working space; one decade at each end is left free to minimize the errors due to the finite size of the plane.

The steps which were followed in printing the coordinate graph on the Teledeltos paper were

1. A template was made up which represented the curves σ = const and ω = const in the W plane.
2. A draftsman used this to produce one decade of the final graph.
3. This was photographed and duplicated to produce the required five decades.
4. Graphs were then printed on the paper by an offset process by the Buck Printing Company of Boston.

The Design of the Templates

The first step in the design of the template is to show that a single template will represent all the lines σ = const and ω = const in the W plane. For this purpose, the transformation W = ln s is reproduced in some detail in Fig. 6.

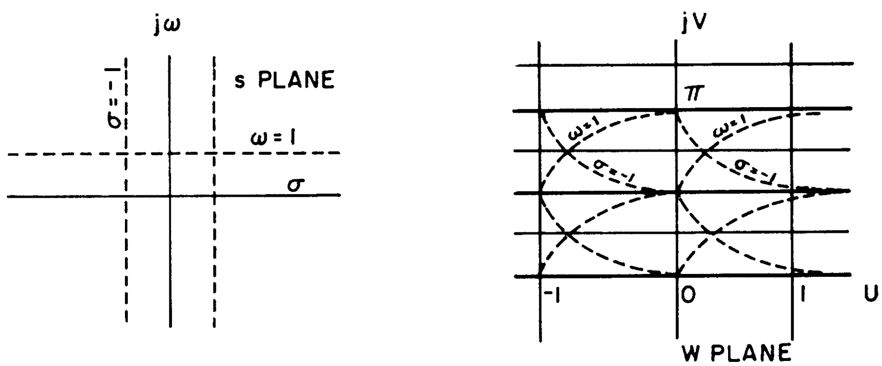


Fig. 6 The transformation W = ln s.

If

$$W = \ln s = \ln r + j\phi = 2.306 \ln_{10} r + j\phi$$
$$= U + j V$$

then

$$s = e^{W} = e^{U} \cos V + j e^{U} \sin V$$
$$= \sigma + j \omega$$

from which

$$\sigma = e^{U} \cos V \qquad (23)$$
$$\omega = e^{U} \sin V . \qquad (24)$$

It will be observed from the last two equations that the curves σ = const and ω = const have the same form except for a shift of $\pi/2 - V$ in the value of V. This means that the same template can be used for both curves. It is necessary merely to turn it over and shift its axis by 90° when passing from one curve to the other.

Finally it must be shown that all the curves of either family are of the same shape. For this purpose consider two curves

$$e^{U} \cos V = c_1 \qquad (25)$$
$$e^{U'} \sin V' = c_2 . \qquad (26)$$

The distance between the two curves on the real axis where V = 0 is

$$U - U' = \ln \frac{c_1}{c_2} \quad . \tag{27}$$

If this value is added to the value of U for curve one in Eq. 25, the curve will become identical to curve two of Eq. 26. Thus

$$e^{U+\ln c_1/c_2} \cos V = c_1 \tag{28}$$

becomes

$$e^U \times \frac{c_1}{c_2} \cos V = c_1 \tag{29}$$

or

$$e^U \cos V = c_2 \quad . \tag{30}$$

This equation is identical with the equation of the second curve, 26. This result shows that the second curve is exactly the same as the first except that it is displaced in the U direction by a distance ln c_1/c_2.

The final conclusion is that a single template will represent all the curves in the W plane. The equation of this curve may be taken for the simple case of $c_1 = 1$. Then

$$e^U \cos V = 1 \tag{31}$$

$$U = \log_e \frac{1}{\cos V} \quad . \tag{32}$$

The curve of V against U can be calculated from this equation. The values must still, however, be converted to inches.

The scale factor is

$$10 \text{ inches} = 1 \text{ unit of } \log_{10}$$
$$= 2.3026 \text{ units of } \log_e \, .$$

Thus 1 unit of $\log_e$ = 10/2.3026 = 4.343 inches and the value of V in inches = 4.343 × (value in radians). (33)

$$\text{The value of U in inches} = 4.343 \times \log_e \frac{1}{\cos V}$$
$$= 4.343 \times 2.3026 \log_{10} \frac{1}{\cos V}$$
$$= 10 \log_{10} \frac{1}{\cos V} \quad . \tag{34}$$

To make the template, a graph of Eq. 34 was glued to a sheet of brass of 40-mil thickness. The brass was then cut out to this form, but slightly larger than required. The final shape was obtained by careful filing.

The next step in the process was the construction of the graph from the template. A draftsman used the template to produce one decade of the completed graph. For ease in setting the template, a second template was constructed of a ten-inch logarithmic scale. This was used to determine the points along the axis where the curves were to start.

The final step was carried out by the Buck Printing Company of Boston. By a photographic process they made an offset plate which was five decades long, and from this they printed the final graphs on the Teledeltos paper. This graph is plainly visible in the picture of the machine in Fig. 5.

The Current Probes

The purpose of the current probes is to introduce positive and negative currents of constant magnitude into the conducting plane at specified positions. The method of constructing the current probes was evolved after considerable experimentation. The probe holders are made from a 7/8 inch length of 1/4 inch polystyrene rod. To make them easier to hold, the center 1/2 inch is turned down to 3/16 inch diameter. A hole, 0.028 inch in diameter, is then drilled longitudinally through the rod. This diameter is sufficient to pass the steel needles which are to be used as the probes. One end of the hole is drilled out 0.035 inch diameter to a depth of 1/4 inch. Then a 0.034 inch sleeve is spot-welded to the needle. It is 1/4 inch long, and its position on the needle is 1/4 inch from the sharp end of the needle. In order to make a successful spot weld it is necessary to use a nickel sleeve.

The needle is fitted into the holder and a half-inch length of 0.1 inch nickel tubing is spot-welded to the free end, thus securing the needle firmly in its mount. The wire which connects to the needle is soft-soldered to the free end of the 0.1 inch tubing and an insulating shield is glued to the outside. Figure 7 shows the complete assembly.

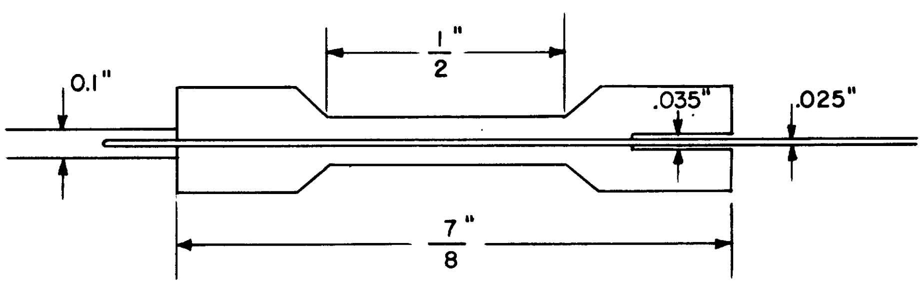


Fig. 7 The current probe.

The allowable current density in the Teledeltos paper is 9 ma for no heating, 40 ma for burning.

Steel needles are used to introduce currents into the plane. They have a diameter of 0.025 inch. The corresponding total currents which they can carry are

$$I = \pi \times 0.025 \times 9 = 0.8 \text{ ma for no heating}$$

$$= \pi \times 0.025 \times 40 = 3.5 \text{ ma for ignition.}$$

From these results, the current chosen for the current sources was 0.835 ma. This is about the maximum which can be used without running into nonlinear effects in the vicinity of the probes. However, it is worth noting that values greater than this can be used without introducing significant errors except in the neighborhood of the probes. This conclusion follows from the fact that the current from a probe is determined by the current source and will not be changed by a change in the surface resistance adjacent to the probe. Any other fields which exist in the plane will be distributed in this vicinity only.

Poles and zeros at zero produce a uniform current flowing the full length of the plane in the logarithmic analog. For convenience in calibration, a switch is provided to introduce poles and zeros of any order at the origin.

Multiple order poles and zeros require currents in the analog plane which are multiples of the basic current. This follows from the fact that taking the logarithm of an impedance function of the form

$$Z(s) = (s - s_k)^n$$

gives

$$\ln Z(s) = n \ln \left| s - s_k \right| + j\, n \arg (s - s_k) \ . \qquad (35)$$

Both the gain and the phase are multiplied by the order of the pole or zero. The corresponding result in the analog plane is obtained by multiplying the current emerging from the pole or zero by the order of the pole or zero.

Poles and zeros on the real axis have been split in two by the conformal mapping. That is, half of the current which would flow from them in the s plane flows in the upper half-plane and half flows in the lower half-plane. Only the upper half-plane is mapped into the logarithmic strip and so the current must be reduced by half for the probes on the axis. To accomplish this the series resistance for these probes is doubled.

It has been explained that the accuracy of the device is increased if the mirror image problem in the $j\omega$ axis is solved simultaneously with the original problem. It was also pointed out that the mirror image probes would have to have all their currents reversed in order to obtain the phase. In the analog there are two boxes of probes. One is used for the original solution and one for the image solution. The currents from the second box are automatically reversed when the device is switched from magnitude to phase.

The Voltage Probes

The logarithm of the magnitude of the impedance function is the voltage along the $V = 90°$ axis in the W plane. A linear spacing of the voltage probes is used along this axis and, in conjunction with a linear sweep on the cathode ray tube, a plot of the logarithm of the magnitude of the impedance function against the logarithm of the frequency is obtained. Because of the necessity of making phase measurements, two rows of probes are used instead of one. They are on either side of the axis at a distance of 0.05 inch from it.

The physical size of the probes was set by the size of phonograph needles which could be conveniently obtained. The diameter of the needles used is about 0.025 inch. The number of probes which could be used per decade of the plane was set partly by the physical size of the probes, and partly by the fact that the commutator could handle only 100 lines at a time. For convenience, it was desirable that the commutator should scan one decade at a time. This set the number of probes per decade at 100 in each row, and the spacing between them at 1/10 inch since 1 decade is 10 inches in the analog plane. This value was actually the fundamental design parameter for the whole machine. The practical problem of setting the probes in the plane made the 1/10 inch spacing very attractive. In addition, with the size of probes used, the error caused by the finite size of the probes is less than 1 percent (see below).

The usual procedure in operating the machine is to take the log magnitude function to be the average of the voltages on the two rows of probes. A switch is provided, however, which allows the operator to view the voltage on either set independently. If necessary, the plane may be shifted until the 90° axis lies along one of the rows of probes. In this case, the error introduced by the two-row system of measurements can be reduced to zero. A discussion of the magnitude of this error is given below.

It has been shown that the derivative of the phase with respect to the frequency is equal to the derivative of the logarithm of the magnitude of the function with respect to the damping ratio σ. This relation held in the s plane. In the logarithmic plane, the derivative of the phase with respect to the real axis variable U, is equal to the derivative of the logarithm of the magnitude of the impedance function with respect to the imaginary axis variable V. The integral of this quantity with respect to the variable U is the phase as a function of U, (i.e. as function of log frequency). This integration is performed by means of an analog computor and the result is displayed on the cathode ray tube as phase vs. log frequency.

The slope of this curve is also of some interest and a separate switch is provided by which it may be viewed. It is related to the delay of the network; for

$$\frac{d\phi}{dU} = \frac{d\phi}{d\omega}\frac{d\omega}{dU} = 2\pi f\frac{d\phi}{d\omega} = \omega\frac{d\phi}{d\omega} \qquad (36)$$

and $d\phi/d\omega$ is the delay function of the network. The quantity which is measured on the analog plane is the derivative of the logarithm of the magnitude of the impedance function with respect to the imaginary axis variable V. In order to measure this, two rows of probes are placed one on either side of the imaginary axis, and the voltage difference between them is measured. These discrete readings of the derivative are made into a continuous step curve by the commutator action, and the phase function is the integral of this curve.

The voltage probes were constructed prior to the current probes. Steel phonograph needles were used instead of the steel needles, and they were held in place by means of a pressed fit in bakelite, rather than by a spot-welded sleeve. The main disadvantage of this method of construction was the difficulty of soldering the lead-off wire to the butt end of the phonograph needles. This was an extremely exacting and laborious task and it was not found possible to entrust it to a technician. A cross section of the mounting detail is shown in Fig. 8.

The phonograph needles are 5/8 inch long and about 0.045 inch in diameter for the part which is embedded in the bakelite. (Because of the taper they are only 0.025 inch where they enter the paper.) The hole in the bakelite was drilled through with a No. 59 drill (0.041 inch) and then the ends were reamed out with a 56 drill (0.046 inch) to a depth of 3/64 inch. This was necessary to prevent splitting the bakelite when the needles were forced into place.

The bakelite strip was milled to size, and then drilled while in place on the milling machine. This allowed the tolerance on the setting of any one hole to be held to about 0.002 inch. The forced fit was also carried out on the milling machine, so that the tolerance on the vertical setting of the probes is also a few thousandths of an inch.

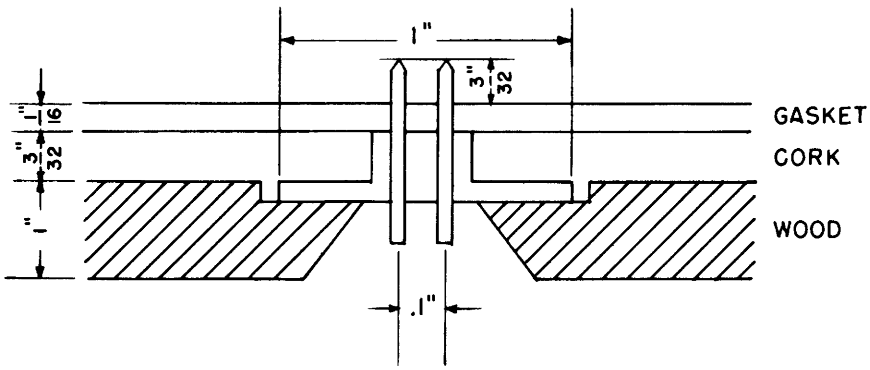


Fig. 8 A cross section of the voltage probes.

The plane itself is of wood. The slot was cut on a circular saw, and the ledge to receive the bakelite member was chiseled out by hand. In the final assembly, the bakelite was glued to the wood. Before this step the solder connections were made from the separate probes to the terminal board below. This was necessary because of the delicacy of the probe to lead-off wire junction. When it is recalled that there are 600 of these junctions, and that they are spaced 0.1 inch apart in two rows, it will be appreciated that this order of procedure was indeed necessary.

The Range Switch

The range switch is necessary because there are 600 voltage pick-up probes, and the commutator can handle only 200 at a time. For calibration purposes, it is convenient to be able to look at a decade at a time, and so the division has been made in this fashion. The range switch consists of three sets of 200 female banana plugs which are connected to the probes in the respective decades of the plane, and a single set of 200 male banana plugs which is connected to the commutator. The general arrangement may be seen in the picture of the machine.

The female receptors are made from 3/32 inch bakelite strips, 7 inches × 4 3/4 inches. Around the outside of each is a supporting shield of 5/32 inch brass, 1 inch wide, which adds necessary rigidity. The banana plugs are mounted in two square 10-by-10 arrays, with a center-to-center spacing of 1/4 inch between them. The male member is identical with the ones which have been described except that a square box has been built to cover the connections which are made to it. This provides protection and at the same time a convenient handle when the switch is moved from one position to another. The bakelite pieces for the three female and the one male member were set up in a jig and drilled at the same time. This procedure was necessary to assure that they would be identical and fit properly when they were completed.

The switch operates very satisfactorily, but it has been found advisable to add vaseline to the contacts to reduce the pressure required to insert and to remove the two hundred probes. When this procedure is followed, the switch may be moved from position to position with relative ease.

The Commutator

The purpose of the commutator is to scan the 200 probes in each decade at a linear rate. The output of the commutator is then applied to the vertical deflection plates of the cathode ray tube. A linear sweep is applied to the horizontal plates, and the curve which results on the face of the scope is a plot of the voltage on the probes against the distance along the axis, or, in other words, the logarithm of the magnitude of the impedance function against the logarithm of the frequency.

The commutator is a mechanical one. It consists of a fixed member which contains the 200 contacts and two slip rings, and a rotating member which carries two copper brushes that make connection between the slip rings and the individual contacts. The individual contacts are connected to the probes through the range switch, and the slip rings are connected to the cathode ray tube through a buffer amplifier.

The picture in Fig. 9 gives the arrangement of the contacts in the stationary member. The circular contacts are brass fillet-head screws which have been countersunk in the bakelite support and faced off smooth in a lathe.

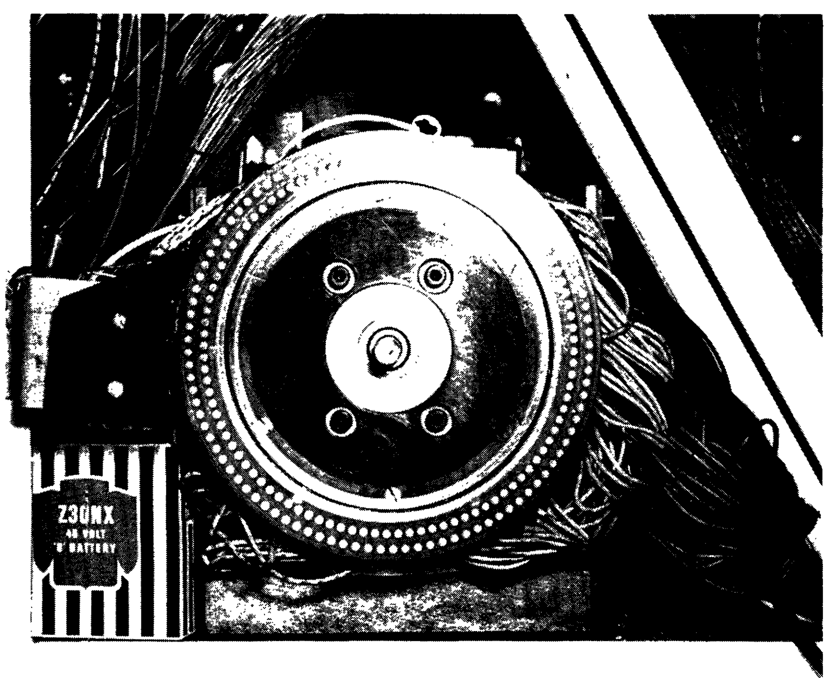


Fig. 9 The commutator.

The diameter of the screws is 1/16 inch. The diameter of the heads, as faced off, is 7/64 inch. The space between the contacts is 1/16 inch on the outer row, and 3/64 inch on the inner row. The radius at which the outer contacts are placed is 3 inches. The radius for the inner contacts is 2-3/4 inches.

The moving contacts were more difficult to design than the fixed ones. In their final form they consist of a "U" shaped strip of 15 mil beryllium copper. The depth of the "U" is 1/2 inch for both sets of brushes, and the width is 3/4 inch for the outside one, and 3/8 inch for the inside one. The actual rubbing surfaces are coated with silver, which is soldered on to the copper. The area of the rubbing surface is just slightly larger than the area of a fixed contact. Thus wear is minimized and a very short dead time between contacts is achieved.

In order to prevent chatter it was necessary to make the spring constant of the brushes quite high. The resonant frequency is about 2000 cycles per second. At the speed at which the commutator is used (10 cycles per second, with 100 contacts) the fundamental vibration component occurs at 1000 cycles per second. With this ratio of natural to applied frequency no damping has been found necessary, but when the speed is raised by any appreciable amount a rubber damper must be added to obtain satisfactory operation.

It has been found advantageous to lubricate the commutator with vaseline. This practice reduces the friction and helps to prevent wear of the bakelite. At the present time the commutator will operate satisfactorily for 10-15 hours with no attention. After this time, however, it needs cleaning and adjusting.

For protection a terminal board was mounted on the commutator, and permanent connections were made from the fixed contacts to the terminal board. The whole unit is removable, however, in order that it may be serviced with the greatest possible ease.

The addition of an extra contact to the rim of the rotating member provides a synchronizing pulse which is used to start the linear sweep on the cathode ray tube, and the linear sweep of the phase-slope control (see the buffer amplifier below).

The Buffer Amplifier and Integrator Unit

The buffer amplifier and integrator unit was designed to receive the voltages from the two channels of the commutator, and to deliver as outputs, the magnitude function, the slope of the phase function, and the phase. The magnitude function is the average of the two input voltages, the slope of the phase function is the difference of the two voltages, and the phase is the integral of this difference voltage. A block diagram of the system is shown in Fig. 10. The complete diagram appears in Fig. 11.

Before the details are explained, the overall operation will be discussed. The system is a novel one, in that it uses a-c amplifiers throughout and at the same time retains the full duty cycle. This effect is accomplished by performing the required operations on the ac components and on the dc components separately, and then adding the two components at the output. For the magnitude this involves merely the addition of a constant term at the output. This operation is the equivalent of setting the zero db line on the cathode ray tube, and can conveniently be carried out on the cathode ray tube.

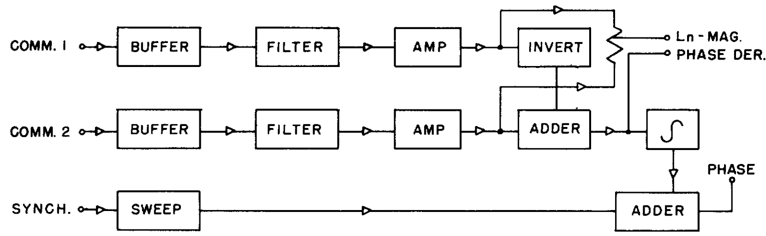


Fig. 10 Block diagram of the buffer amplifier and integrator unit.

For the phase the procedure is more complicated, since an integrated constant or a sweep wave must be added. The amount of sweep wave which must be added is determined from the fact that the phase curve must be flat at very low and at very high frequencies.

In terms of the block diagram, the operation is as follows. In the magnitude position the inputs from the two commutators are identical. They are fed through buffer amplifiers and filters and then to a mixer tube in which the average of the two voltages is taken. This voltage is proportional to the logarithm of the magnitude of the impedance function. In the phase position the inputs from the two commutators are slightly different. They pass through buffer amplifiers and filters as before, but one of them is then inverted and added to the other, thus giving the difference between them. This difference is the derivative of the phase function. Finally the derivative of the phase function is integrated and the necessary sweep wave is added, giving the phase. Each of the elements in the block diagram will now be discussed in detail.

The Buffer Amplifiers

The buffer amplifiers are cathode-followers which are shown as tubes 1 and 2 in the complete circuit diagram, Fig. 11. The input impedance of the circuit is about 10 megohms. The internal impedance of the probe source to which they are connected is about 15,000 ohms, and so there is very little loading on the circuit. The coupling capacitor in the input is likewise critical. If it is too small the circuit will lack the necessary low frequency response to pass the 10 cycle per second fundamental which is delivered by the commutator. If it is too large the commutator hash is accentuated. A value of 0.05 ufd has been found experimentally to be a suitable compromise.

The Filter

The filter is designed to remove the commutator hash. The majority of the undesirable signal occurs at the fundamental commutator ripple frequency of 1000 cycles per second. The filter is composed to two rejector circuits in series, both of which are tuned to approximately 1000 cycles per second, and an input capacitor which cuts

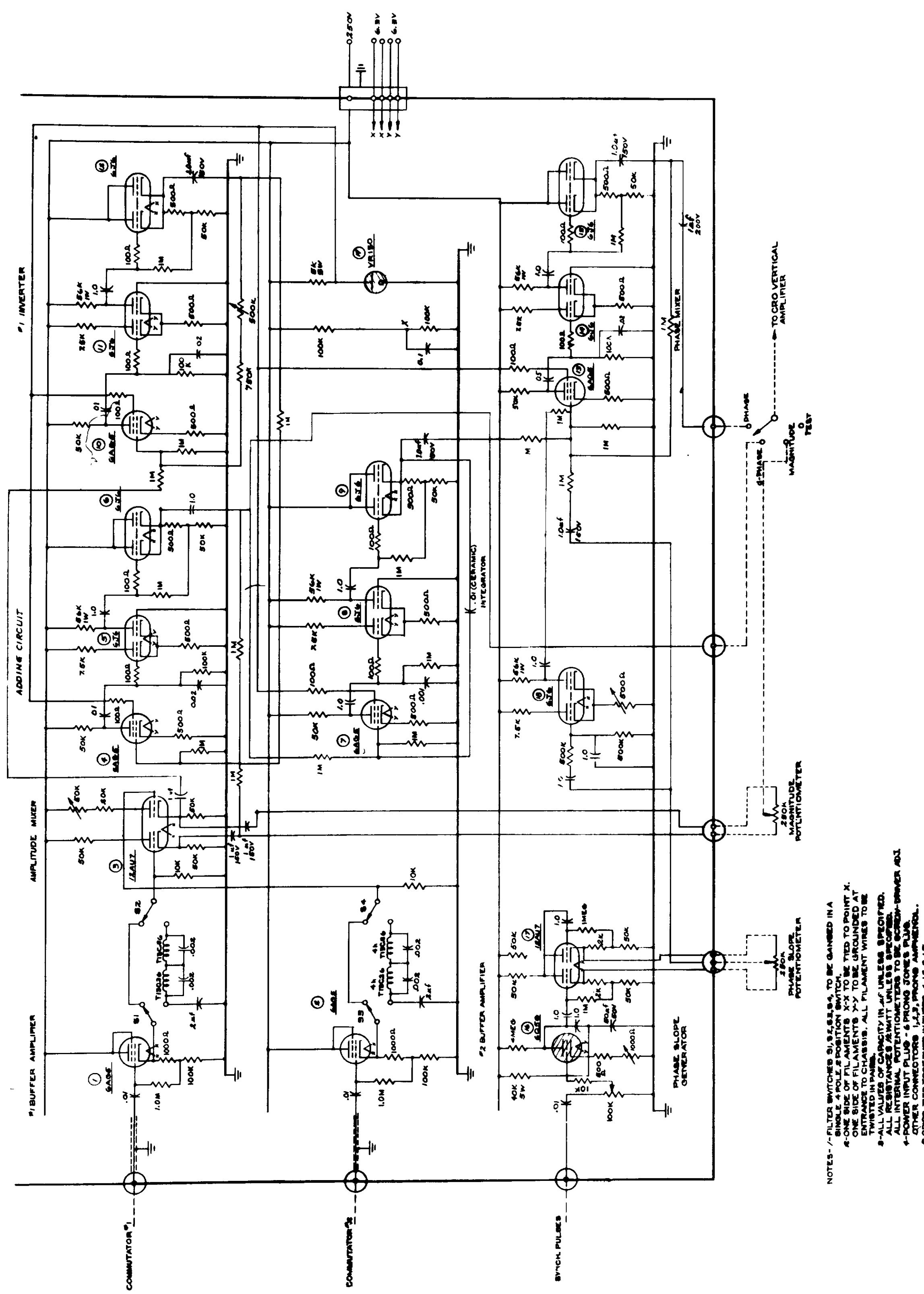


Fig. 11 Circuit diagram.

down the high frequency response in general. The purpose of using two tuned circuits in series was to obtain the required impedance level with coils of relatively poor Q.

A switch has been provided which removes the filter from the circuit for testing purposes.

The Mixer Stage

A 12AU7 tube (No. 3) is used to mix the voltages from the two channels of the commutator. The tube is a double triode with separate cathodes, and each half is used as a cathode follower. The voltages which appear at the respective cathodes are the filtered voltages from the two separate channels of the commutator. By varying the setting of a potentiometer connected between these cathodes, the output voltage can be made the average of the two cathode voltages with any weighting factor that is desired. Thus the operator can view at will the output from either row of probes, or any combination of these outputs. Normally the outputs from the two rows of probes are equal and the potentiometer is in the center position to give a simple average.

The Inverter

An active inverter is used in order to provide the maximum of stability and independence of tube parameters. The feedback scheme has been described elsewhere (9) and will not be discussed in detail here.

The amplifier for the inverter comprises tubes 10, 11, and 12 in the complete circuit diagram. The first tube is a high gain pentode, the second is a double triode circuit that is equivalent to a cathode follower feeding a grounded grid stage, and the third is a cathode follower output. The requirements on the amplifier are stringent. It must have a high gain and a wide bandwidth, but it must not break into oscillations when large amounts of feedback are applied. In addition the output impedance must be low, and the output must be 180° out of phase with the input. All these requirements are satisfied by the design used here. The amplification is about – 1000, the output impedance is about 100 ohms, and the bandwidth with feedback extends from about 1/10 cycle per second to about 200 Kc.

In order to prevent oscillations when the feedback is applied it is necessary to reduce the bandwidth of the 6J6 stage to a few hundred cycles, from about 75 cycles per second to about 300 cycles per second. The overall bandwidth, however, remains large as shown by the data of the preceding paragraph.

The Adder

The principle of the adder is similar to that of the inverter. In the complete circuit diagram the adder is composed of tubes 4, 5, and 6. The two inputs are

1. The voltage from one of the rows of probes.

2. An inverted form of the voltage from the other row of probes.

Thus the output of the adder is the difference between the voltages on the two sets of probes, a function which is proportional to the derivative of the phase curve (except for a possible constant term).

The Integrator

The principle of the integrator is not unlike that of the active adder and the active inverter. With a value of R = 1 megohm, and C = 0.01 ufd, and the amplifier which has been discussed above the finite-gain and bandwidth errors are of the order of 1 percent. This result has been checked experimentally with a square wave input.

In the complete circuit diagram the integrator is made up of tubes 7, 8, and 9. The input to the integrator is the voltage representing the slope of the phase curve. The output is proportional to the phase, except for the addition of the integral of a constant as explained at the beginning of this section.

The Phase Slope Generator

The phase slope generator is a standard thyratron sweep circuit. It is synchronized with the commutator by means of the same positive pulse which is used to start the linear sweep on the cathode ray tube. The thyratron is tube No. 16 in the complete circuit diagram. Tube No. 17 is an amplifier and inverter which produces sweep waves of opposite phase at its cathodes. The front panel control "phase slope" is a potentiometer between these cathodes, which can select either phase, and any amount of the sweep wave. The output from the potentiometer goes directly to the output phase mixer circuit, and also through a low-frequency compensation circuit to the grid of tube No. 18. The output from tube No. 18 is added to the phase in the phase mixer.

The Phase Mixer Circuit

The phase mixer circuit is composed of tubes 13, 14, and 15 in the complete circuit diagram. It is an adder circuit identical to the one which has been discussed previously. The inputs to this circuit are:

1. The phase minus the integral of any d-c components in the derivative curve.

2. A compensated sweep wave which replaces these lost components.

The output of this circuit is the phase.

General Arrangement

The general arrangement of the machine can be seen in the picture in Fig. 5. Front panel switches are provided for viewing the Gain, the Phase, or the derivative of the phase. In addition a front panel switch allows multiple poles and zeros to be introduced at the origin.

Controls are also provided on the front panel for the magnitude of the sweep wave which is added to the phase, and for mixing the outputs from the two rows of probes in any desired proportion when the Gain function is being viewed.

Two regulated power supplies supply the plus and minus three hundred volts which are required for the current probes. The buffer amplifier and integrator unit is supplied with power from the plus three hundred volt supply.

The Calibration and Operation of the Machine

The calibration procedures for the machine depend upon a knowledge of the characteristics of the so-called "log-db" type of curve. The output curves from the machine are in the form of Gain or Phase vs. log-frequency. These curves have been discussed extensively in the literature (13), but a short summary will be given here of the results which are pertinent to the operation of the machine.

The "log-db" theory

Any impedance function Z(s) can be written in the form

$$Z(s) = \frac{A\,(T_1 s + 1)\,(\ldots)\,(T_2^2\ s^2 + 2T_2\zeta_2 s + 1)\,(\ldots}{(T_3 s + 1)\,(\ldots)\,(T_4^2\ s^2 + 2T_4\zeta_4 s + 1)(\ldots} \quad . \tag{37}$$

Terms of the form (Ts + 1) are produced by single poles (denominator) or by single zeros (numerator) on the negative real axis. The quadratic terms are produced by pairs of conjugate poles or zeros. (If ζ is greater than one the quadratic term is factored into two linear terms). If the logarithm of Z(s) is taken the result is

$$\begin{aligned} \ln\left|Z(s)\right| &= \ln\left|T_1 s + 1\right| + \ldots \ln\left|T_2^2\ s^2 + 2T_2\zeta_2\ s + 1\right| + \ldots \\ &-\ln\left|T_3\ s + 1\right| - \ldots \ln\left|T_4^2\ s^2 + 2T_4\zeta_4\ s + 1\right| - \ldots \end{aligned} \tag{38}$$

$$\begin{aligned} \arg\,(Z(s)) &= \operatorname{Arg}(T_1\ s + 1) + \ldots \operatorname{Arg}\,(T_2^2\ s^2 + 2T_2\zeta_2 s + 1) + \ldots \\ &-\arg\,(T_3 s + 1) - \ldots \arg\,(T_4^2\ s^2 + 2T_4\zeta_4\ s + 1) - \ldots \quad . \end{aligned} \tag{39}$$

It is apparent from Eqs. 38 and 39 that the logarithm of the magnitude of the impedance function and the phase of the impedance function are both obtained as the algebraic sum of the contributions from terms of the form (Ts + 1) and $(T^2 s^2 + 2T\zeta s + 1)$. In the "log-db" method of analysis, the logarithm of the magnitude and the phase of these standard forms are recorded graphically. The form of the curve for (Ts + 1) is shown in Fig. 12.

The curve for the logarithm of the magnitude is asymptotic to the zero db line at low frequencies, and to a +6 db per octave line at high frequencies. The asymptotes meet at a frequency of 1/T, and at this point the actual curve is 3 db above the asymptotes. The curve for the phase is asymptotic to the zero phase line at low frequencies and to the 90° phase line at high frequencies. It is symmetrical about the frequency 1/T.

In the analog plane the term (1 + Ts) is represented by a zero on the negative real axis. In the W = ln s plane this point is at the edge of an infinite strip. The current flows from the probe along the strip to a conductor at infinity. The voltage along the

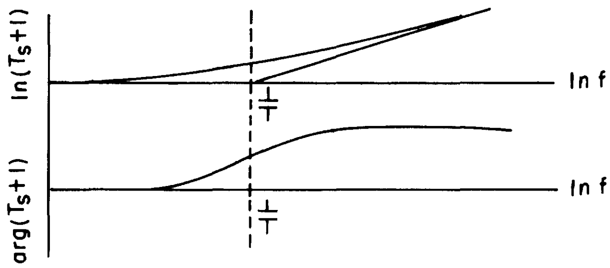


Fig. 12 The "log-db" curves for the factor $(T_s + 1)$.

axis is zero to the left of the probe, and it approaches a constant slope (of 6 db per octave) when the current flow in the sheet becomes uniform at high frequencies. Thus the 6 db-slopes of the "log-db" theory are identified with a uniform current flow of one unit in the analog.

Terms of the form $(T^2 s^2 + 2T\zeta s + 1)$ have zero db log-magnitude asymptotes at low frequencies, and +12 db per octave asymptotes at high frequencies. The junction of the asymptotes is again the frequency 1/T. The exact behavior in the neighborhood of this point depends on the value of ζ. Families of curves are plotted for various values of ζ (ref. 10). The phase curves are asymptotic to the zero phase line at low frequencies and to the 180° phase line at high frequencies. They are symmetrical about the frequency 1/T, and have different slopes for different values of ζ.

In the analog a term of the form $(T^2 s^2 + 2T\zeta s + 1)$ is represented by a pair of conjugate zeros at the points

$$s = -\frac{\zeta}{T} \pm j\frac{\sqrt{\zeta^2 - 1}}{T} \tag{40}$$

$$= -\frac{1}{T} \pm e^{j \tan^{-1}\sqrt{\frac{\zeta^2 - 1}{\zeta}}} . \tag{41}$$

Equation 41 shows that the radius at which the zeros occur is independent of the value of ζ, and that the angle at which they occur is independent of the value of T. In the W plane, the position of the zeros along the strip is fixed by T, and the distance of the probes from the axis is fixed by ζ. The current spreads out from these probes and flows along the strip towards infinity. The voltage along the V = 90° axis is zero to the left of the probes, and it approaches a constant slope (of 12 db per octave) when the current flow becomes uniform in the sheet. Thus the 12-db slopes of the "log-db" theory are identified with a uniform current flow from a pair of conjugate probes in the analog. The distance between the probes and the 90° axis is associated with the damping factor ζ.

Calibration of the Log-Magnitude Plots

The calibration of the frequency scale in the log-magnitude plot presents no difficulty. The commutator scans one decade at a time in the frequency scale, from a relative frequency of 1 to a relative frequency of 10. The frequency scale therefore covers just one decade.

The calibration of the log-magnitude scale is accomplished by using the results above. In particular it has been shown that a single pole or zero has a "log-db" response which is asymptotic to $\pm$ 6 db at high frequencies. If the pole or zero is located at the origin this slope exists throughout the whole frequency spectrum. The corresponding result is achieved in the analog by introducing a uniform current at the zero end of the W plane and allowing it to flow to the other end which represents infinity. The curve which is obtained may be used to calibrate the scale on the cathode ray tube. Examples of 0, $\pm$ 6, and $\pm$ 12 db slopes which were obtained in this way are shown in Fig. 13.

The infinite slopes shown in Fig. 13 provide a calibration for the vertical scale, but the problem of setting the zero db line remains. If there are no poles or zeros at zero, the asymptotic behavior at low frequencies is zero, and this fact determines the zero db line to within the constant A of Eq. 37. When poles or zeros are present at zero they can easily be removed for the purpose of calibration.

The limited range of the commutator also poses the problem of matching the curves obtained over two or three decades. The calibration is carried out as before in the decade nearest to zero, and the other decades are matched to it by adjusting the levels at the boundaries. An example of this process is shown in Fig. 14.

The Calibration of the Phase

The frequency scale for the phase is the same as the frequency scale for the log-magnitude, and requires no further calibration. The vertical scale for the phase plots is calibrated by measuring the phase of a single pair of conjugate poles, or zeros. The distance between the low and the high frequency asymptotes is 180° for conjugate probes of either polarity. A calibration curve for the phase is shown in Fig. 15.

In the absence of poles or zeros at zero the phase curve has a low frequency asymptote of zero. The addition of poles or zeros at zero does not change the phase as it is measured on the machine. Therefore poles or zeros at the origin must be added into the calibration. A pole at the origin changes the calibration of the low frequency asymptote to −90°, and a zero at the origin changes it to +90°. Multiple poles or zeros change the calibration by multiples of 90°.

The Experimental Results

The experimental results which are presented in this section show the accuracy of the machine, and the scope of its operation. The first results are simply reproductions of the standard "log-db" curves for the quadratic factor $(T^2 s^2 + 2T\zeta s + 1)^{-1}$ with various values of the damping ratio ζ. The accuracy of the machine can be judged by the

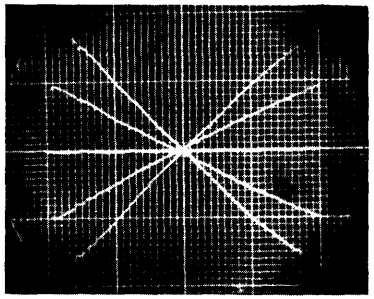


Fig. 13

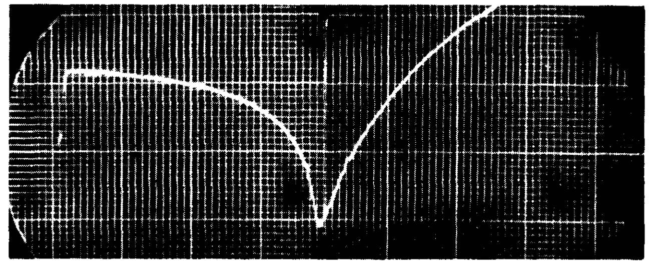


Fig. 14

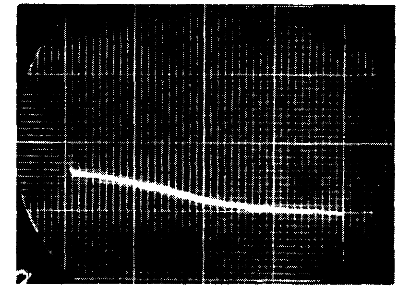
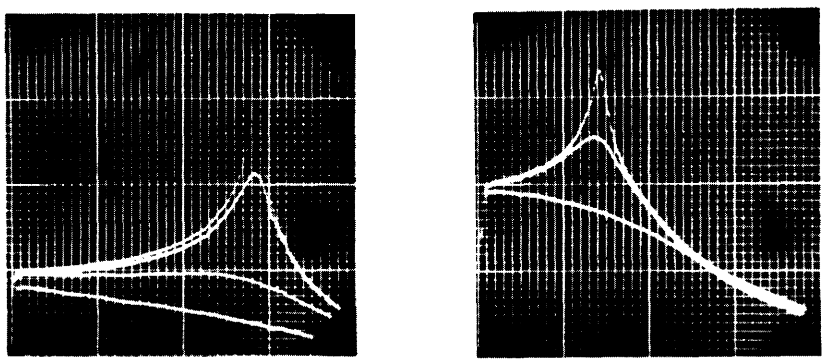


Fig. 15



a b

Fig. 16 a. $\zeta = 0.01, 0.05, 0.5, 1.0$
b. $\zeta = 0.02, 0.1, 0.71$

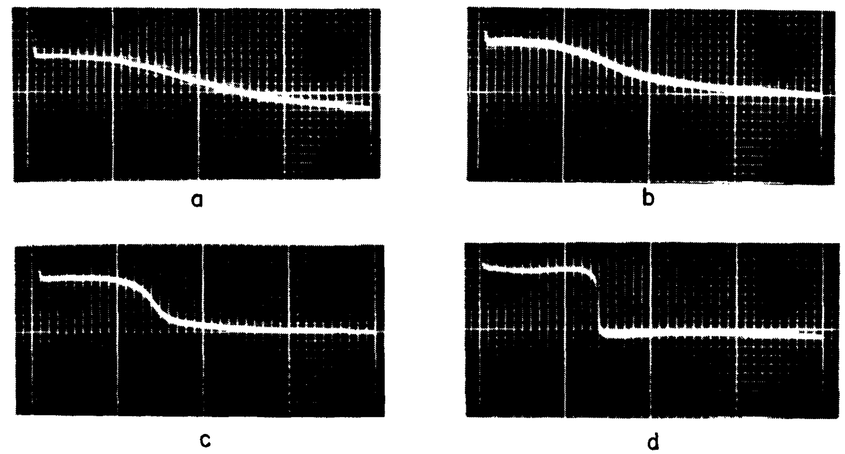


Fig. 17 a. $\zeta = 1.0$ b. $\zeta = 0.5$
c. $\zeta = 0.05$ d. $\zeta = 0.01$

Fig. 13 Calibration curves.
Fig. 14 Matching the boundaries of two decades.
Fig. 15 The calibration of the phase.
Fig. 16 Log-magnitude values of the factor $(T^2 s^2 + 2T \zeta s + 1)^{-1}$.
Fig. 17 Phase plots of the quadratic factor $(T^2 s^2 + 2T \zeta s + 1)^{-1}$.

accuracy with which it will reproduce these curves. The magnitude functions are shown in Fig. 16 and the phase functions in Fig. 17.

The experimental values for $\zeta = 0.5$ are plotted in the graph of Fig. 18 along with the calculated values.

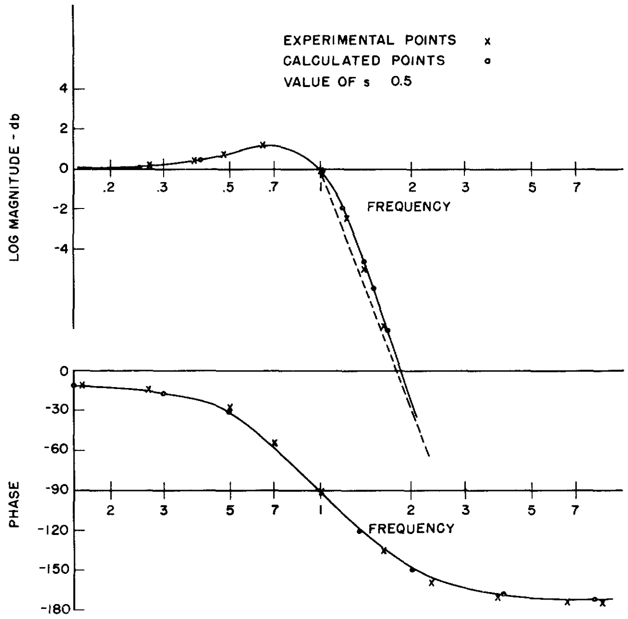


Fig. 18 The quadratic factor $1/s^2 + 2\zeta s + 1$.

The agreement between the experimental and the calculated values is within 1 percent for the log-magnitude, and 3 percent for the phase. The experimental values were obtained from the negatives of the photographs by means of a projector-enlarger.

A Butterworth Filter

The prototype, low-pass Butterworth filter is represented by the transfer impedance Z_{12} in the formula

$$\left| Z_{12} \right|^2 = \frac{1}{1 + s^{2n}} \quad . \tag{42}$$

The zeros of this function are all at infinity, and the poles are distributed uniformly around the circumference of a unit circle in the s plane. For n = 4 the locations of the poles are shown on Fig. 19.

In order to realize the filter by a physical network, Z_{12} must be identified with the poles in the left half-plane. (This usage corresponds to the symmetry conditions about the $j\omega$ axis which were discussed earlier.) The log-magnitude and the phase plots for this filter are shown in Fig. 20.

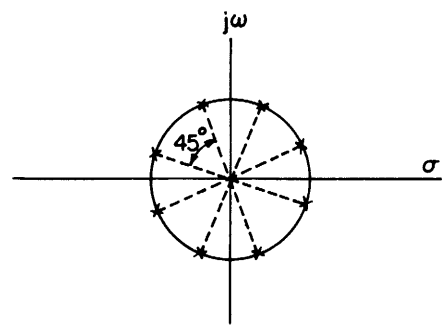


Fig. 19

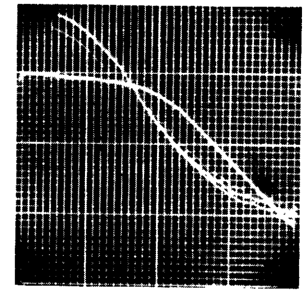


Fig. 20

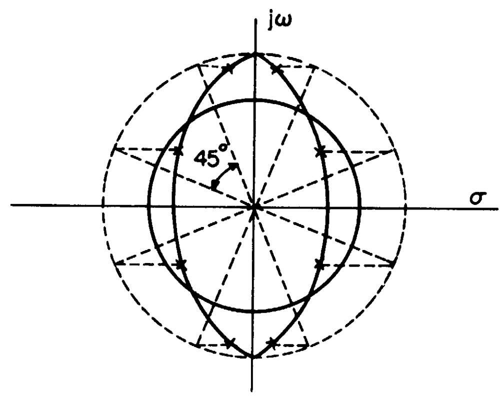


Fig. 21

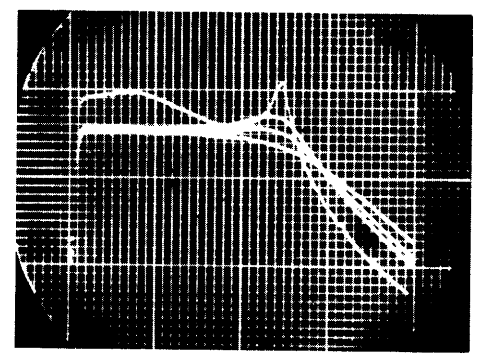


Fig. 22

Fig. 19 Butterworth filter (n = 4) in the s plane.

Fig. 20 Log-magnitude and phase vs. log-frequency for a Butterworth filter.

Fig. 21 Tschebyscheff filter (n = 4) in the s plane.

Fig. 22 Log-magnitude vs. log-frequency for Tschebyscheff filters. Sinh a = 1.0, 0.7, 0.4, 0.2.

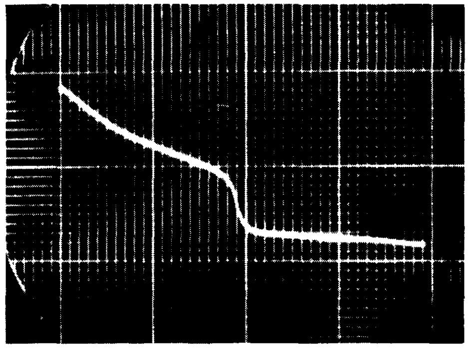


Fig. 23

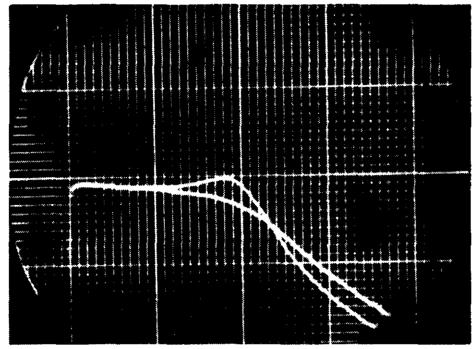


Fig. 24

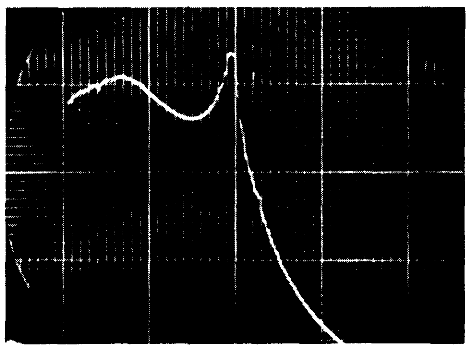


Fig. 25

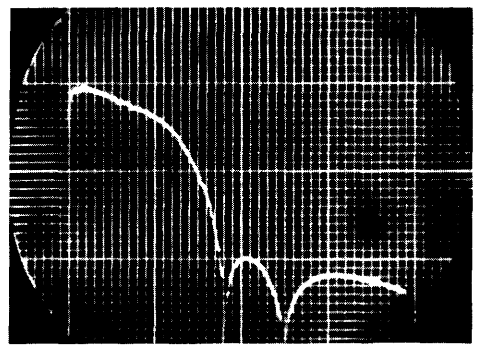


Fig. 26

Fig. 23 The phase of a Tschebyscheff filter.

Fig. 24 Butterworth and Tschebyscheff filters of the order n = 4.

Fig. 25 A modified Tschebyscheff filter.

Fig. 26 A Norton filter.

Tschebyscheff Filters

The prototype of the low-pass Tschebyscheff filter has a transfer impedance Z_{12} given in the formula

$$\left|Z_{12}\right|^2 = \frac{1}{1 + \epsilon^2 V_n^2(\omega)} \tag{43}$$

where V_n is the Tschebyscheff polynomial $\cos(n \cos^{-1} x)$ and 2ϵ is the peak-to-peak ripple in the pass band. The zeros of this filter are again at infinity, and the poles lie on an ellipse whose major and minor axes are cosh a, and sinh a, respectively, where a is determined by the allowable ripple ϵ according to the formula

$$e^a = \left[\frac{\sqrt{\epsilon^2 - 1} + 1}{\epsilon}\right]^{\frac{1}{n}} \quad . \tag{44}$$

For a given value of a, the locations of the roots are obtained from the locations of the Butterworth roots by the following simple procedure.

1. Find the roots of the Butterworth filter for the same value of n.
2. Superimpose an ellipse of major and minor axes cosh a and sinh a, respectively.
3. Extend the radial lines through the Butterworth poles to cut a circle of radius cosh a.
4. Draw horizontal lines from these intersections to cut the ellipse. The intersections of the horizontal lines and the ellipse are the locations of the Tschebyscheff poles.

An example of this procedure is illustrated in Fig. 21 for n = 4.

The characteristics of Tschebyscheff filters for n = 4 and for various values of "a" are shown in Fig. 22.

From Fig. 22 it can be seen that the log-magnitude curve for a Tschebyscheff filter cuts off more rapidly as the amount of ripple in the pass band is increased. The phase, on the other hand, becomes less linear. In Fig. 23 the phase of a Tschebyscheff filter is shown.

The order of the filter, n determines the rate of cut-off for the Butterworth filter. It determines the number of oscillations in the pass band of the Tschebyscheff filter, and the ultimate rate at which it cuts off. However, the value of ϵ affects the cut-off of Tschebyscheff filter in the neighborhood of the cut-off frequency. This effect can be seen clearly in the photographs of Fig. 24, which show Tschebyscheff and Butterworth filters of the order n = 4.

Filters of a More General Type

A Tschebyscheff filter has a ripple in the pass band which is constant in amplitude over the whole band. For some applications this effect may not be necessary. For example, it is possible to obtain a filter which cuts off more sharply if the amplitude of the ripples at the edge of the pass band is allowed to increase. This is accomplished by moving the conjugate poles, which are the second closest to the axis, in towards the axis until they are at the same distance from it as the closest pair. This effect is shown in Fig. 25 for Tschebyscheff filter of the order n = 4.

The filter shown in Fig. 25 would be very difficult to handle analytically. On the analog machine it is no more difficult than the original filter.

A second way in which the prototype filters can be modified is illustrated in Fig. 26. Zeros are added in the cut-off region to increase the rate of cut-off of the filter. This type of filter is often called a Norton filter.

The examples which have been given illustrate the operation of the machine. In a synthesis problem, of course, the picture on the cathode ray tube is given, and the poles and zeros are moved around until the desired form is obtained. The rapidity with which the machine displays the solution makes this process both short and simple.

Analysis of the Errors in the Device

A comparison of the results obtained upon the machine and the calculated results in the previous section showed that in one particular case, at least, the error in the Gain was of the order of 1 percent and the error in the Phase of the order of 3 percent. Unfortunately, the percentage error varies with the problem which is being solved, and in order to predict the accuracy of the machine for a given problem it is necessary to examine the sources of the error in some detail. The principal sources of error are

1. Nonuniformities in the conducting paper
2. The finite size of the plane
3. The current probes – the effect of conformal mapping
   – their finite size
   – the finite series resistance
   – the holes they leave in the paper
4. The voltage probes – their finite size
   – the accuracy of positioning
   – the granularity error in the output
5. Miscellaneous errors.

The Uniformity of the Paper

Investigations into the uniformity of the paper may be divided into two parts. The first is the measurement of gradual changes such as those caused by a change in the spacing of the rollers while the paper was being made. The second is the measurement of random changes in the resistivity such as those caused by impurities and inhomogeneities in the material.

To accomplish the first purpose, the resistivities of about 35 samples from a single roll were measured. The samples were 55 inches long (the length used in the analog) and they were taken in succession from a single roll of the low-resistance paper. The resistivity is plotted in the graph of Fig. 27 vs. the distance along the roll. The results show that there is a noticeable change from one end of the roll to the other. However, it amounts to less than 1/10 percent change in a length of 55 inches, and is accordingly negligible.

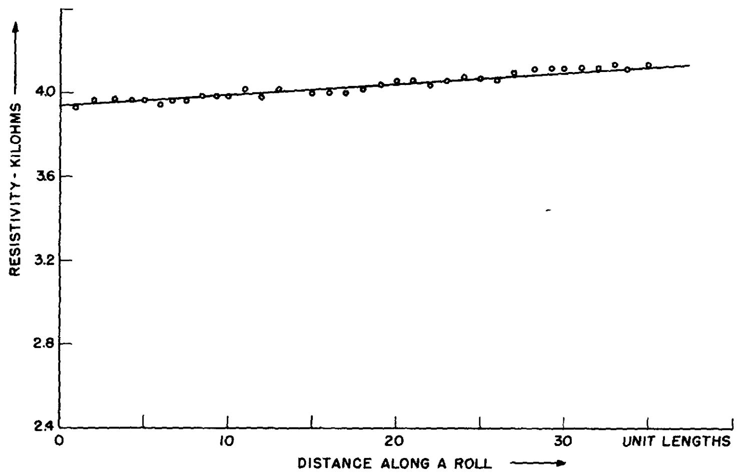


Fig. 27 Uniformity measurements.

The second investigation involved a comparison of calculated and experimentally determined equipotential lines. Small nonuniformities in the paper show up as deviations in the equipotential lines. In the experiment, a uniform current distribution was obtained in a sheet of Teledeltos paper 7.55 inches wide. The calculated equipotential lines were simply straight lines across the paper, and any deviations from them could be easily determined. The voltage variations along these lines were determined with respect to the voltage at one end of the line. These voltage variations are plotted in Fig. 28. They show that the maximum random deviation is 0.2 volt. The voltage gradient in the experiment was 5.4 volts per inch, and so the maximum deviation of the equipotential lines was 0.2/5.4 or 0.037 inch.

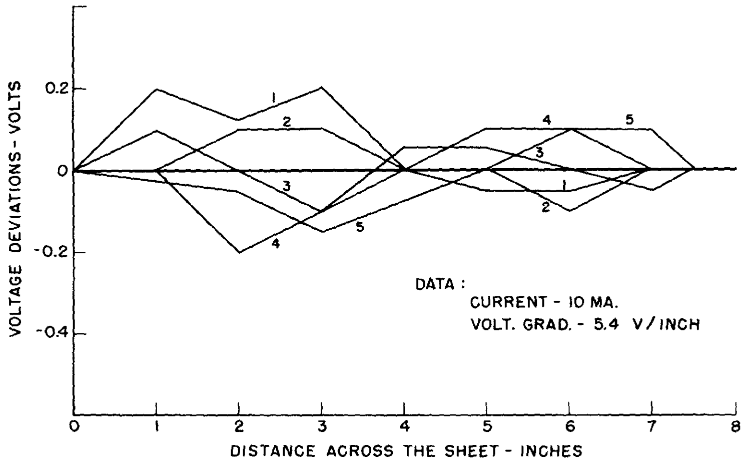


Fig. 28 Variations from equipotentials.

The Errors Introduced by Nonuniformities

In the logarithmic form of the analog, the output voltage is measured along the center line of a long rectangular strip of the conducting paper. The mirror image scheme is used. For magnitude measurements there is no current flowing at right angles to the axis of the measurements, and for phase measurements there is no current flowing along the axis of measurements.

Magnitude errors can arise only from a shift of the equipotential lines along the frequency axis. The maximum shift of these lines because of nonuniformities is 0.037 inch. This represents a constant error of 0.85 percent in the frequency at which a given potential occurs.

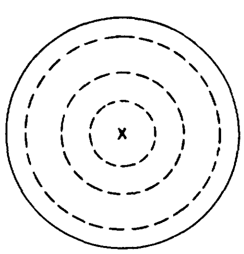
The error in the phase derivative at any point can be calculated from the spacing of the probes and the amount by which the voltages on the probes can be in error. Since the phase derivative is simply the difference in the voltages on the two probes, the error in the difference might appear to be the sum of the possible errors in the voltages at the two probes. From Fig. 28 it can be seen that the voltage error changes rather gradually. Indeed, the maximum rate at which the errors change is about 0.2 volt per inch. Converted into changes in equipotentials this is 0.037 inch change per inch of spacing. In the analog, the probes are 1/10 inch apart and accordingly the maximum effective shift between them is 0.0037 inch.

If the voltage gradient is E, the voltage between the probes is 1/10 E. The error in the voltage might be 0.0037 E, and the percentage error 0.0037 E/0.1 E = 3.7 percent.

Thus the maximum error which the nonuniformities are likely to cause in the phase derivative measurements is 3.7 percent. The error in the phase will in general be less than this because of the smoothing effect of integration.

The Errors Due to the Finite Size of the Plane

The error in the potential field of an individual source or sink occurs because of the distortion introduced into the equipotential lines by the zero potential boundary. At the edge of the analog, the equipotential lines are constrained to take up the shape of the boundary. If the boundary is a circle, distortion will occur for all poles and zeros except those at the exact center of the analog. The effect is shown qualitatively in Fig. 29.



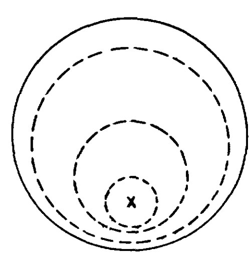


Fig. 29 The effect of the finite boundary upon the equipotential lines.

Source at Origin — Source not at Origin

In the logarithmic plane no errors are introduced by the finite width of the paper. The errors introduced by the finite length can be computed most conveniently for the equivalent radius in the original plane. The analysis procedes from a solution of Laplace's equation in polar coordinates to an exact solution for the error introduced by a conducting rim at a finite radius.

The solution of Laplace's equation in polar coordinates may be written in the form

$$\phi(r, \theta) = (K - J) \ln r - \sum_{n=1}^{\infty} A_n \frac{\cos n\theta}{r^n} \tag{45}$$

where

$$A_n = \frac{1}{n}\left[\sum_K r_k^n \cos n\theta_k - \sum_J r_j^n \cos n\theta_j\right] \tag{46}$$

K = number of zeros, J = number of poles, $r_k e^{j\theta_k}$ = location of a zero, $r_j e^{j\theta_j}$ = location of a pole, n = an arbitrary integer.

In this equation the first term represents the potential which would be present if all the poles and zeros were grouped at the origin. This term gives a circular equipotential at any radius. The remaining terms represent the deviation of the actual potential at any radius from an equipotential, and accordingly they represent the error produced at the boundary by the introduction of the conducting rim. This error may be extrapolated back into the plane by solving Laplace's equation for this potential on the boundary, and with no sources or sinks within the region. If this procedure is followed the error at any point in the plane for a finite radius R is found to be

$$\epsilon(r\,\theta) = -\sum_{n=1}^{\infty} \frac{r^n}{R^{2n}} A_n \cos(n\theta) \quad . \tag{47}$$

Along the imaginary axis the error is

$$= -\sum_{n\,\text{even}} \frac{r^n}{R^{2n}} A_n$$

or

$$= \frac{1}{2}\frac{r^2}{R^2}\frac{r_k^2}{R^2}\cos 2\theta_k + \frac{1}{4}\frac{r^4 r_k^4}{R^6}\cos 4\theta_k \; . \tag{48}$$

$$+ \ldots \quad .$$

Since $r_k \ll R$, the first term is the most important. An approximate formula for the error is

$$\delta = \frac{1}{2}\frac{r^2}{R^2}\frac{r_k^2}{R^2}\cos 2\theta_k \; . \tag{49}$$

For a single zero on the jω axis the error is

$$\delta_{j\omega} = -\frac{1}{2}\frac{r^2}{R^2}\frac{r_k^2}{R^2} \quad . \tag{50}$$

For a single zero on the real axis the error is

$$\delta_{real} = +\frac{1}{2}\frac{r^2}{R^2}\frac{r_k^2}{R^2} \quad . \tag{51}$$

For a single zero on the 135° axis the error is

$$\delta_{135} = 0 \text{ to the first approximation.} \tag{52}$$

As an experimental check on the above formulas, the voltages along the imaginary axis were measured for zeros on the real axis, on the 135° line, and on the imaginary axis. For convenience, the measurements were made on the logarithmic plane.

The voltages for the three cases are plotted on Fig. 30. The horizontal scale is the distance $U = \log r$ ($= \log f$), measured from an arbitrary zero at $U = \log r_k$. The vertical scale is the voltage ϕ measured with respect to the origin.

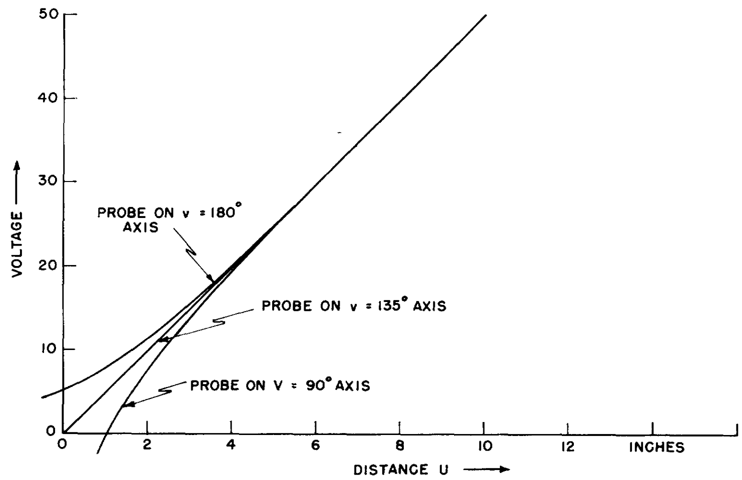


Fig. 30 The voltage gradient for single probes in the W plane.
Width of paper, 7.55 inches; 1 decade, 5.6 inches; probe diameter, 0.025 inch.

Since the measurements of $U = \log r$ are made from $r_k = 1$ ($\log r_k = 0$), the voltages should be

$$\phi(j\omega) = \ln r - \frac{1}{2}\frac{r_k^2}{R^2} \tag{53}$$

$$\phi(135°) = \ln r \text{ to first approximation} \tag{54}$$

$$\theta(\text{real}) = \ln r + \frac{1}{2}\frac{r_k^2}{R^2} \quad . \tag{55}$$

These expressions are verified on the graph to within the accuracy of the measurements.

In practice, the errors introduced by placing an equipotential at one decade are reduced because the measurements are not made out to this radius. The graph shows the error which would occur at a radius of R if an equipotential is placed at that radius.

From the above analysis it can be concluded that if the radius of the tank is 7 times the radius of the largest pole or zero, the errors due to the finite size of the plane will never be greater than 1 percent. In the present model, this ratio has been increased to 10 as an added precaution. The same situation occurs at the zero end of the strip, and it is likewise met by leaving a decade of the plane unused at the zero end.

The Errors Introduced by the Current Probes

The current probes can introduce errors in a number of ways. First and foremost the effect of the conformal mapping on the equivalent s plane shape of the current probes must be investigated.

Conformal Mapping and the Current Probes

If the current probes were of infinitesimal size they would not be changed in any way by the conformal mapping. Because of their finite size, they are changed and it is necessary to know how much. The situation is illustrated in Fig. 31.

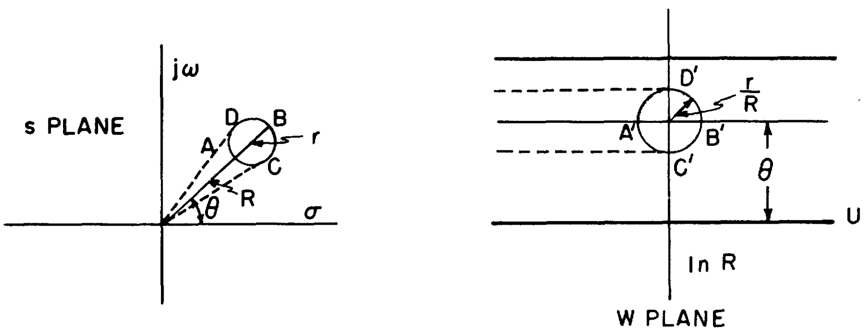


Fig. 31 A probe in the s plane and the W plane.

In the W plane, the center of the circle transforms to the point $V = \theta$, $U = \ln R$. The point A in the s plane transforms to the point $V = \theta$.

$$U = \ln\ R - r \doteq \ln R + \ln\left(1 - \frac{r}{R}\right)$$
$$= \ln R - \frac{r}{R} \text{ if } R \gg r,$$

and the point B to $V = \theta$, $U = \ln R + r/R$. The point C transforms to $V = \theta - r/R$, $U = \ln R$, and the point D transforms to $V = \theta + r/R$, $U = \ln R$, provided only that $R \gg r$.

Thus a circle in the s plane transforms to a circle in the W plane with its radius reduced by a factor R, providing only that the circle is away from the origin far enough that $R/r > 10$. Conversely a circle of constant size in the W plane transforms to a circle in the s plane which decreases in radius in proportion to the distance from the origin.

More directly perhaps, it can be seen that the angle subtended by the probes in the W plane is equal to that distance in V direction over which their diameters extend. This angle remains constant, no matter where the probe is situated in the plane, and hence the effective size of the probe must decrease directly with its distance from the origin. The angle subtended at the origin by a probe is

$$\frac{0.025''}{13.6} \times \pi = 0.00577 \text{ radian or } 0.331 \text{ degree.}$$

This result shows that the probe diminishes in size as the origin is approached, and hence there is no loss in accuracy when working at small values of R in the W plane. Conversely, of course, there is no gain when working at large values. This is in agreement with the general property of a constant percentage error which is often possessed by logarithmic devices.

The Errors Introduced by the Finite Size of the Probes

The effect of the finite probe size is, speaking generally, to distort the field in the immediate vicinity of the probe. For convenience the principle of superposition can be used and the field can be divided into two components. The first is the field which is produced by the probe itself, and the second is the field produced at the probe by all the other probes in the plane. The effect of the finite probe size on these two components will be considered separately.

Distortion in the Field Produced by the Probe Itself

The finite probe size has no effect on the field component which is produced by the probe itself. The equipotential lines about any source or sink will be circular in the neighborhood of the discontinuity. In the s plane, when the boundaries are infinite, the equipotentials are circles regardless of the radius. The transformation into the W plane has been shown to leave circles invariant in form if the distance of the circle from the origin is at least ten times as great as the radius of the circle. This is true for the region of the W plane which is used in the analog. It has also been shown that the finite boundary has no effect in the region used in the analog. Accordingly the equipotentials in the W plane must also be circles. This being so, it does not matter at what radius the current is introduced, provided that it is introduced along an equipotential, and that it has a constant magnitude.

Thus the finite size of a probe introduces no error into the field component produced by the probe itself.

Distortion in the Field Produced by the Other Probes

An error is introduced into the field produced by the other probes. According to the principle of superposition, a current probe is represented by an open circuit for the calculation of the field components produced by the other probes. The physical probe, however, remains present in the plane, and distorts the conductivity of the conducting layer in the region. This in turn distorts the field which is produced at this point by the currents from all the other probes in the plane. In order to obtain the order of magnitude of the effect, the field produced by the other probes will be assumed to be constant in the vicinity of the probe. The situation is shown in Fig. 32.

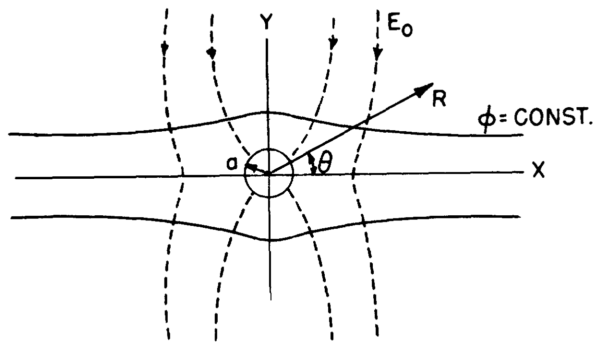


Fig. 32 A conducting probe in a uniform field.

In Fig. 32 the conducting probe is represented by the circular conducting area of radius a. The electric field at a large distance from this region is assumed to have a constant value of E_0, and to be directed downwards for the coordinates which are shown. The solution for the potential near the probe is

$$\theta = E_0 \, r \sin\theta \left(1 - \frac{a^2}{r^2}\right) \quad . \tag{56}$$

The field E is

$$-\text{grad}\,\phi = -\left(i_r \frac{\partial \phi}{\partial r} + i_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta}\right)$$

$$= i_r \, E_0 \sin\theta \left(1 + \frac{a^2}{r^2}\right) + i_\theta \frac{E_0}{r} \left(1 - \frac{a^2}{r^2}\right) r \cos\theta .$$

Along the line $\theta = \pi/2$ the field is

$$E_0 \left(1 + \frac{a^2}{r^2}\right) \quad .$$

The maximum value of the field is $2E_0$ at $r = a$ but it drops off rapidly in the r direction and when $r = 10a$, the distortion is only 1 percent.

Along the line $\theta = 0$ the field is

$$E_0 \left(1 - \frac{a^2}{r^2}\right) \quad . \tag{57}$$

The minimum value of the field is 0 at r = a, but it increases rapidly in the r direction and when r = 10a, the distortion is again only 1 percent. Thus the constant field will be distorted in the immediate vicinity of the probe and to a distance of about 10a, or 0.125 inch, from it.

The final step in the argument is to combine the constant field with the field produced by the probe itself, and to examine the distortion of the resultant field. The field produced by the probe itself is

$$E = -\frac{d\theta}{dr} = \frac{\rho I}{2\pi}\frac{1}{r}$$

where r is the distance from the probe.

At the probe the value is $80\ \rho\ I/2\pi$ since the probe radius is 0.025/2 inch. At ten times this radius $E = 8\ \rho\ I/2\pi$ volts/inch. The field at the probe produced by other probes may be taken approximately as the asympototic value which is approached when all the current flows uniformly towards infinity. For a single probe, this uniform value is obtained when the current density is I/W, where W is the width of the plane. Thus $J = I/13.6$ ma per inch, and $E = \rho\ I/13.6$ volts per inch. The maximum error in the field occurs right at the probe where the uniform field is reduced to zero. This corresponds to a percentage error of

$$\frac{\rho\ I/13.6}{80\ \rho\ I/2\pi} = 0.58 \text{ percent} \quad .$$

At a distance of 10a from the probe, the error is only 1 percent of the constant field. The probe field at this point is $8\ \rho\ I/2\pi$, and the percentage error is

$$\frac{\rho\ I/1360}{\rho\ I/13.6 + 8\ \rho\ I/2\pi} = 0.09 \text{ percent} \quad .$$

These errors are so small that they will not be significant even when the constant field is much stronger than the single probe field indicated here. The only place in which this error becomes of importance is when a probe is near the axis where the voltage is to be measured. In this case, the granularity errors and the errors produced because of the double row of probes are much greater than the errors discussed in this section.

The Effect of Holes Left in the Plane by the Current Probes

The effect of holes left in the plane by the current probes as they are moved to another spot is similar to the effect of the finite probe size. The same method may be used to calculate the errors and they are found to be of identical magnitudes.

The Finite Series Resistance

Ideally the current sources would have infinite internal impedance and no difficulty would be encountered because of the finite resistance of the paper. In practice, the current sources have only 300,000 ohms resistance, and it is necessary to know the resistance between a probe and the conducting rim.

This resistance was calculated theoretically by using bipolar coordinates, and then checked experimentally in the logarithmic plane. The results are compared in the graph of Fig. 33 where it can be seen that the change in resistance per decade is about 3000 ohms for each probe in the region. Thus a one-percent change in the current as a

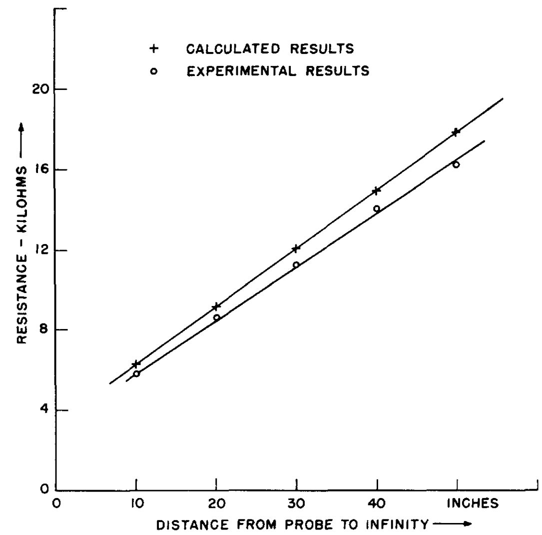


Fig. 33. W plane probe to infinity resistance.

single probe is moved through one decade is obtained with a series resistance of 300,000 ohms. In practice, if there is a large excess of poles or zeros in a region, the currents taken by each probe must be measured and equalized to obtain the maximum in accuracy. Accuracy of this order is, fortunately, seldom necessary. In any case, the criterion of one-percent error per decade of spread, per excess pole or zero in the region, may be used to estimate the error, and to determine whether or not a higher degree of accuracy is required.

The Errors Introduced by the Voltage Probes

The voltage probes can introduce errors in a number of ways. Because of their finite size they distort the field in their immediate vicinity in the same fashion as the current probes. The effect is more serious now, however, because the voltage probes are producing no field of their own. The symmetrical distribution of the probes alleviates the situation somewhat and the errors from this source will seldom rise above one percent. This is negligible in relation to the errors produced by inaccuracies in the probe positions, and by the finite spacings between the measuring probes.

The Errors Introduced by Inaccuracies in the Probe Positions

The spacing of the probes is 0.1 inch and the spacing between the rows of probes is also 0.1 inch. During the construction, the setting of a probe was controlled to within about 0.002 inch. Thus the maximum possible error which can occur in the probe spacing is 0.004 inch.

For magnitude measurements, this corresponds to a shift of 0.004 inch in an equipotential. It was shown above that an equipotential shift of 0.037 inch corresponded to an error of 0.85 percent in the frequency at which the voltage appeared in the graph. The shift here is only a fraction of this value, and hence the effect is negligible.

For measurements of the phase derivative, errors in the probe spacing are more serious. The voltage gradient may be considered to be approximately constant over the region in which the two probes are situated. This being so, the voltage between them will be directly proportional to the distance between them and the percentage error in this voltage will be the same as the percentage error in the distance, which in this case may be as high as 4 percent. In practice these errors occur in a random fashion and are smoothed out somewhat by the integration process. Thus, except under extreme conditions, the errors in the phase because of errors in the physical positions of the measuring probes will be less than 4 percent. In regions where the phase approaches zero, the percentage error may, of course, be larger than this. However, in general, the phase is of very little interest in these regions.

The Errors Introduced by the Two-Row Method of Measuring the Magnitude

The voltage along the imaginary axis is theoretically equal to the logarithm of the magnitude of the impedance function. In the analog plane there are no probes directly on the axis, but rather two rows at 0.05 inch on either side of the axis. The average is taken of the voltages measured along the two rows, and this result is assumed to be very nearly the same as the actual voltage along the axis. The use of the mirror image method reduces the errors involved in this approximation, but they are still not negligible. These errors will be calculated for two current probes set up in mirror-image fashion at a distance of a from the measurement axis, as shown in Fig. 34.

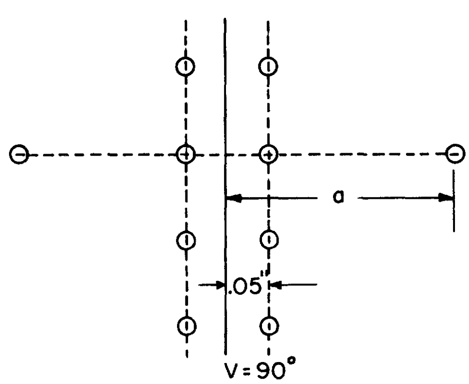


Fig. 34 Two mirror-image probes at a distance a from the 90° axis.

The potential at the nearest point on the axis from the two current probes is

$$\phi = 2\left(\frac{\rho I}{2\pi} \ln \frac{a}{0.025}\right) . \tag{58}$$

This is the voltage value which it is desired to measure. The actual value measured is the value at the voltage probes, which is

$$\phi = \frac{\rho I}{2\pi}\left(\ln\frac{(a-0.05)}{0.025} + \ln\frac{(a+0.05)}{0.025}\right) .$$

The error in the voltage is

$$\Delta\phi = \frac{\rho I}{2\pi}\left(-\ln\frac{(a-0.05)}{0.025} - \ln\frac{(a+0.05)}{0.025} + 2\ln\frac{a}{0.025}\right)$$

$$= \frac{\rho I}{2\pi}\left(2\ln a - \ln(a-0.05) - \ln(a+0.05)\right)$$

$$= \frac{\rho I}{2\pi}\left(\ln(1-\frac{0.05}{a}) + \ln(1+\frac{0.05}{a})\right)$$

$$= \frac{\rho I}{2\pi}\left(\frac{0.05^2}{a^2} + \dots\right) \qquad (59)$$

The percentage error in the voltage is

$$\frac{\frac{(0.05)^2}{a^2}}{2\ln\frac{a}{0.025}} \times 100 . \qquad (60)$$

For a one-percent error, a solution by cut and try gives a value of a = 0.24 inch. This value corresponds to an angle of $0.24/13.7 \times \pi = 0.055$ radians, or 3.16 degrees. Probes should be kept approximately 0.25 inch from the measurement axis, if errors greater than 1 percent are not to arise because of the off-center voltage measurements.

The maximum values of the Q of the poles and zeros which can be handled is $1/2\times 0.055$ radians = 9. When Q's higher than this are to be used, the paper must be shifted over until the imaginary axis coincides with one of the rows of probes. Then the voltage on this row must be used as the logarithm of the magnitude of the impedance function.

The Granularity Error Caused by the Sampling Process

The granularity error caused by the use of a finite number of voltage measuring probes is greatest when the current probes are near the imaginary axis. Thus the effect of granularity will again be a limitation on the Q's which can be handled, with a given percentage error in the result. The magnitude of the error is not changed in any way by the use of the mirror-image method of solving the problem, and so it will be investigated for the case of single pole at a distance of a from the measuring probes, as in Fig. 35.

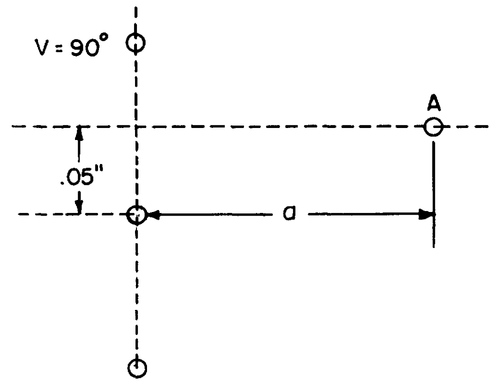


Fig. 35 A single pole at a distance a from the measuring probes.

The true voltage at B is

$$\phi = \frac{\rho I}{2\pi} \ln \frac{a}{0.025} \; . \tag{61}$$

The measured voltage is the voltage at the voltage probe or

$$\phi = \frac{\rho I}{2\pi} \ln \sqrt{\frac{0.05^2 + a^2}{0.025}}$$

The error is

$$\frac{\rho I}{2\pi} \left(\ln \frac{a}{0.025} - \ln \sqrt{\frac{0.05^2 + a^2}{0.025}} \right)$$

$$= \frac{\rho I}{2\pi} \left(\ln \frac{a}{0.025} - \frac{1}{2} \ln (a^2 + 0.05^2) \right.$$

$$= - \frac{\rho I}{2\pi} \left(\frac{1}{2} \ln \left(1 + \frac{0.05^2}{a^2} \right) \right)$$

$$= - \frac{\rho I}{2\pi} \, \frac{1}{2} \, \frac{0.05^2}{a^2} \; . \tag{62}$$

The percentage error in the voltage is

$$\frac{\frac{1}{2} \, \frac{0.05^2}{a^2} \times 100}{\ln \frac{a}{0.025}} \tag{63}$$

This is an identical expression with the one for the percentage error in the previous section, Eq. 60. Hence, again, for a one-percent error the value of a must be at least 0.24 inch.

From this we may conclude that if the granularity error is not to exceed 1 percent, current probes must not be placed nearer to the measuring axis than 0.05 inch. This corresponds to a Q of 9 if the probes are on the axis, and a Q of $9 \times 0.24/0.29 = 7.5$ if the measuring probes are 0.05 inch off the axis. It is only fair to add that the granularity errors are not as serious a problem as might be indicated by the analysis given here. Higher Q's may be used provided that the step curve is extrapolated with some intelligence to give the smooth curve which represents the desired function.

Miscellaneous Errors

In addition to the errors which have been discussed in the previous sections, there are others which arise in the actual machine. Hum pick-up has been kept to a low level by means of careful design and shielding. The effect of hum in the output is readily observable and is not likely to affect the results seriously. Nonlinearities in the amplifiers and integrator have been held to less than one percent. Dirt in the commutator can cause trouble; it makes the output trace on the cathode ray tube jittery and is therefore easily detected.

If higher accuracy is required than is available on the machine, it may be obtained by a process of iteration. This procedure is not recommended, though, because the computations involved subtract a great deal from the convenience of using the machine.

Conclusion

The analog device which is described in this report is the most advanced design that has been built. It is the first device of this kind which provides a continuous display of both gain and phase functions. It is the first device which possesses the necessary accuracy and convenience to be more than a laboratory curiosity. It provides a simple method of solving the approximation problem of network synthesis. The two most novel features in its construction are the use of Teledeltos paper as the conducting medium, and the use of a logarithmic transformation to overcome the errors due to the finite size of the plane. This transformation results in output curves of the so called "log-db" type which are very convenient for a great many network and servomechanisms problems.

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