WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

Seasonal Inventories and the Use of Product-Flexible Manufacturing Technology

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WP #3011-89-MS May 1989

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ABSTRACT

As advancing technology makes Flexible Manufacturing Systems (FMS's) a viable option for an increasing number of firms and products, determining their economic value has become increasingly important. Despite considerable research in this area, the effect of the ability or inability to hold interperiod inventories on the value of FMS's has received relatively little attention.

This paper proposes a stochastic dynamic programming model for the production capacity investment decision that explicitly allows the firm to carry interperiod inventories for safety stock and seasonal stock motives, and it analyzes a simplified convex programming model that considers seasonal stock motives only. A variety of qualitative results are obtained for the case of periodic market conditions that are not known when the firm makes its investment decision. The complete solution is reported on for the case of deterministic periodically varying market conditions. Examples are given for both cases.

The analysis leads to some surprising results, including the fact that interperiod inventories and flexible capacity can be complements as well as substitutes. Hence, the analysis can be an important supplement to unguided intuition.

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1 Introduction and Overview

A variety of explanations have been offered for U.S. corporations' relatively low levels of investment in flexible manufacturing systems (FMS's). One explanation that has received a great deal of attention attributes this situation to their strict financial requirements for justifying investments. These requirements can stifle investment in FMS's because current cost/benefit accounting procedures fail to adequately assess the value of flexibility (Michael and Millen, 1984; Suresh and Meredith, 1984; Kaplan, 1986; Kelly, 1988; Port et. al., 1988). As a result, companies have relied instead on managerial judgement (Gerwin, 1981; Miles, 1988; Schiller, 1988) which, without the support of well-designed decision support tools, is often inadequate (Kaplan, 1986).

In response, researchers have tried to capture quantitatively the benefits of FMS's that conventional accounting procedures overlook. Early efforts to develop comprehensive models met with limited success because FMS's yield such a diverse set of benefits. Progress has been made, however, by developing models that focus on one type at a time. Models have been developed to analyze product life cycle effects on the value of FMS's (Hutchinson and Holland, 1982; Fine and Li, 1987), the benefits of FMS's that arise from their modularity in comparison with the indivisibility of transfer lines (Burstein, 1986), benefits from being able to produce with either of two sets of factor inputs (Kulatilaka and Marks, 1985; 1986; Kulatilaka, 1986; 1987), the ability to deter market entry by competitors and to credibly threaten entry into markets in which the firm does not currently compete (Fine and Pappu, 1988), market related benefits associated with flexibility (Gaimon, 1986; Roth et. al., 1986), and the ability to hedge against uncertainty about future demand (Fine and Freund, 1986; 1987; He and Pindyck, 1988).

One restrictive assumption of all these models is that the firm is not allowed to hold interperiod inventories. Since production and inventory are used together to meet demand, intuitively one would expect the ability to hold inventory to affect the value of flexible production capacity.

Ultimately one would like to know how a firm with the option of holding inventory and/or investing in flexible capacity would optimally respond to dynamic, non-stationary uncertainty, particularly unpredictable variability in market conditions. This problem is too complex to be addressed in full generality, but by decomposing the variability into its constituent parts, steps can be taken toward analyzing it.

Firms may hold inventories and/or invest in product-flexible capacity for at least three reasons: cycle stock motives, safety stock motives, and seasonal stock motives. Technology investment decisions and cycle stock considerations have been studied by Karmarkar and Kekre (1987) and Vander Veen and Jordan (1987), although they do not look explicitly at the economics of flexibility in their analyses. In addition, Graves (1988) has examined safety stock requirements when facilities have product mix flexibility. However, the interaction between flexible capacity and seasonal stocks has largely been ignored.

This paper extends the Fine and Freund (1987) model for analyzing the benefits of product flexibility to hedge against uncertainty in a multiperiod setting in which the firm can hold interperiod seasonal and safety inventories. Section 2 presents a stochastic dynamic programming formulation which allows for both safety and seasonal stocks. The rest of the paper analyzes a convex programming model which excludes safety stock considerations and focuses on seasonal stock issues. Section 3 presents six theorems that analyze the model when market conditions are seasonal but uncertain when the firm makes its technology investment decision.

Section 4 reports on a complete solution for the case of deterministic, periodically varying market conditions. Market conditions are rarely known with certainty, so the solution described in Section 4 is unlikely to be applied directly in practice, but we believe that studying this solution provides insights about the general problem. Examples are given in both Sections 3 and 4. Section 5 discusses results and conclusions.

2 The General Stochastic Demand Problem

2.1 Problem Description

The problem considered here is the following. At time zero, a firm must make an irreversible capacity investment decision in the face of uncertainty about future market conditions. The firm can invest in a mix of non-flexible production capacities each dedicated to one of its N products and/or more expensive flexible capacity that can be switched at zero cost among any of the N products. Each period's demand uncertainty is resolved before the firm makes that period's production and inventory decisions but after it is committed to its investment in production capacity. Because of the uncertainty, the firm would like to have flexible production facilities that can produce any of the N products it sells, but flexible production capacity requires a larger initial investment per unit than does capacity dedicated to a single product. The crux of the problem is to find the optimal trade-off between the benefits of flexible production capacity and its higher investment cost.

Fine and Freund (1986, 1987) address this problem for the case when all subsequent production and sales activity can be collapsed into a single period. This simplification yields useful results, but it completely suppresses the issues posed by interperiod inventories. This paper extends their model to an arbitrary number of production and sales periods and allows the firm to carry inventory from any period into the subsequent period.

2.2 Formulation

The formulation uses the following indices:

$i \in \{1, \ldots, N\}$ indexes the product,

$s \in \{1, \ldots, S\}$ indexes the state, and

$t \in \{1, \ldots, T\}$ indexes the period;

parameters:

α = the per period discount rate and

$\mathbf{r} = [r_1, r_2, \ldots, r_N, r_F]$ is the vector of capacity purchase costs, r_i is the purchase cost of one unit of capacity dedicated to product i, and r_F is the purchase cost of one unit of flexible capacity;

variables:

$\mathbf{K} = [K_1, K_2, \ldots, K_N, K_F]$ is the vector of production capacities, K_i is the amount of capacity dedicated to product i, and K_F is the amount of flexible capacity,

Y_{it} = amount of product i produced on dedicated capacity in period t,

Z_{it} = amount of product i produced on flexible capacity in period t,

I_{it} = amount of product i held in inventory from period t to period t+1, and

σ_t = the state in period t;

and functions:

$R^s_{it}(x)$ = revenues generated by selling x units of product i in state s,

$C^s_{it}(x)$ = cost of producing x units of product i in state s, and

$h^s_{it}(x)$ = cost of inventorying x units of product i from a period in which state s is realized to the next period.

It is assumed that the future state of the market depends only on the time and its current state and not on the firm's past decisions. So in the formulation, $P_t(\sigma_{t+1} = s' | \sigma_t = s)$ is the probability the market will be in state s' in period t+1 given that it is in state s in period t.

Boldface will be used for vectors. The dimension of vectors can be determined by noting which subscripts have been omitted. For example, $\mathbf{Y}^s_i = \left[Y^s_{i1}, Y^s_{i2}, \ldots, Y^s_{iT}\right] \in \Re^T$.

The decision variables are $\mathbf{K}$, the vector of capacities chosen at time 0, and for t = 1, ..., T, $p_t(s, \mathbf{I}_{t-1}, \mathbf{K})$ the period t production and inventory levels as a function of the current state, the current inventory vector, and the vector of capacities.

The General Stochastic Demand Problem (GSDP) has the following dynamic programming formulation:

$$\underset{\substack{\mathbf{K}, p_t(s, \mathbf{I}_t, \mathbf{K}) \\ t=1,\dots,T}}{\text{Max}} \quad V = -\sum_{i=1}^{N} r_i K_i - r_F K_F +$$

$$E_{\sigma_0, I_0}\left[\sum_{t=1}^{T} \alpha^t \sum_{i=1}^{N}\left[R_{it}^{\sigma_t}\left(Y_{it} + Z_{it} + I_{i(t-1)} - I_{it}\right) - C_{it}^{\sigma_t}\left(Y_{it} + Z_{it}\right) - h_{it}^{\sigma_t}\left(I_{it}\right)\right]\right]$$

subject to:

$$Y_{it} \le K_i \qquad i = 1,\dots,N;\ t = 1,\dots,T,$$

$$\sum_{i=1}^{N} Z_{it} \le K_F \qquad t = 1,\dots,T,$$

$$I_{it} \le Y_{it} + Z_{it} + I_{i(t-1)} \qquad i = 1,\dots,N;\ t = 1,\dots,T,$$

with all variables nonnegative.

Let $\psi(\mathbf{K}, t, s, \mathbf{I}_{t-1})$ be the maximum expected subsequent earnings if at time t the firm's capacity vector is $\mathbf{K}$, current inventory is $\mathbf{I}_{t-1}$, and the market is in state s. Then the problem has the following recursive relationship.

For $1 \le t < T$,

$$\psi(\mathbf{K}, t, s, \mathbf{I}_{t-1}) = \underset{\mathbf{Y}_t, \mathbf{Z}_t, \mathbf{I}_t}{\text{Max}}\left[\sum_{i=1}^{N}\left(R_{it}^{s}(Y_t + Z_t + I_{(t-1)} - I_t) - C_{it}^{s}(Y_t + Z_t) - h_{it}^{s}(I_t)\right) + \alpha \sum_{\tilde{s}=1}^{S} P\left(\sigma_{t+1} = \tilde{s} \mid \sigma_t = s\right) \psi(\mathbf{K}, t+1, \tilde{s}, \mathbf{I}_t)\right]$$

subject to:

$$Y_{it} \le K_i \qquad i = 1,\dots,N;\ t = 1,\dots,T,$$

$$\sum_{i=1}^{N} Z_{it} \le K_F \qquad t = 1,\dots,T,$$

$$I_{it} \le Y_{it} + Z_{it} + I_{i(t-1)} \qquad i = 1,\dots,N;\ t = 1,\dots,T,$$

with all variables nonnegative

and

$$\psi(\mathbf{K}, T, s, \mathbf{I}_{T-1}) = 0 \qquad \text{for all } \mathbf{K},\ s, \text{ and } \mathbf{I}_{T-1}.$$

In the sequel, Ξ will denote the vector of all production and capacity decision variables and the superscript * will denote optimal values.

Since at any given time, future demand is uncertain, this formulation clearly includes safety stock motives. Since the state of the market can be a function of time, it captures seasonal stock motives as well. However, this formulation is difficult to solve numerically because the solution space is large, and in general it is difficult to obtain analytic results for stochastic dynamic programs.

To our knowledge, the only work that has made progress on the interaction between product mix flexibility and safety stocks is that of Graves (1988). That work characterizes the difference in safety inventories required for a system with complete mix flexibility relative to one with no mix flexibility. Ultimately one would desire to analyze both safety and seasonal effects in one model. The work in the remainder of this paper moves toward this goal by analyzing the interaction between seasonally varying market conditions and flexible manufacturing capacity.

3 The Unknown Seasonal Demand Problem

3.1 Formulation

The General Stochastic Demand Problem (GSDP) considered in the previous chapter is too complex to yield to tractable analysis. However, by imposing more structure on the revenue and cost functions, it can be simplified into one that does. We accomplish this by making these functions periodic, thereby excluding safety stock issues and focusing on the seasonal stock problem. The result is a convex programming model that we label the Unknown Seasonal Demand Problem (USDP). This formulation requires the following assumptions.

A1: The revenue, production cost, and holding cost functions are periodic with period two. This can be modelled as:

$$P_t(\sigma_t = s \mid \sigma_{t-1} = r) = \begin{cases} 1 & \text{if } s \text{ is odd and } s = r - 1 \\ 1 & \text{if } s \text{ is even and } s = r + 1 \\ 0 & \text{otherwise.} \end{cases}$$

However, it is more natural to identify the market conditions for both odd and even periods with a single state and the time subscript t. Then

$$C^s_{it}(x) = C^s_{i1}(x), \quad R^s_{it}(x) = R^s_{i1}(x), \text{ and } h^s_{it}(x) = h^s_{i1}(x) \quad \text{for all odd } t, \text{ and}$$
$$C^s_{it}(x) = C^s_{i2}(x), \quad R^s_{it}(x) = R^s_{i2}(x), \text{ and } h^s_{it}(x) = h^s_{i2}(x) \quad \text{for all even } t.$$

A2: At time zero, when the technology investment is undertaken, the state s is unknown. However, the firm learns the state, and therefore both sets of functions, at the beginning of period one.

Assumptions A1 and A2 imply that all uncertainty is resolved before production and inventory decisions are made for any of the T periods. This is a strong assumption, but it captures the key ideas that (1) inventory policies can be used to respond to uncertainty in demand, (2) technology investments must be made before the resolution of this uncertainty, (3) after uncertainty is resolved, the production and inventory policies are constrained by the sunk investments, and (4) holding interperiod inventories can affect the utility of flexible capacity.

Periodicity in production, inventory, and sales conditions is common. For example, producing and transporting bulk commodities in the Great Lakes Region is more expensive in the winter than in the summer because the Inland Waterway is closed. Likewise, holding costs may be seasonal. For U.S. automobile manufacturers, holding cars from summer to winter is more expensive than holding them from winter to summer because many new models are released in the fall, reducing the value of cars held in inventory from summer to winter. Cost differences can arise from seasonal variations within the company itself, not just from weather. For example, holding costs for a toy manufacturer may be lower in spring than in fall because less of its warehouse space is devoted to stockpiling toys for the Christmas Season. Seasonal variations in demand are extremely common.

The subsequent analysis depends on the following assumptions about the capacity costs, revenue functions, and cost functions. Only the assumption about production costs is ever likely to be restrictive in practice.

A3: The revenue functions $R_{it}^{s}(x)$ are bounded, strictly concave, differentiable functions that are nondecreasing when their argument is zero.

A4: The production cost functions $C_{it}^{s}(x)$ are strictly increasing, convex, differentiable functions that are finite when their argument is zero. Production costs are a function only of the total quantity of each product produced in the period, not of which kind of capacity is used.

Strictly convex production costs are relatively rare because they imply diseconomies of scale, but linear production costs, which are reasonably common, are also encompassed in our formulation. The assumption that unit production costs are linear and roughly equal for flexible and dedicated capacity is reasonable if, for example, raw material costs are the dominant component of variable production costs.

A5: The holding cost functions $h_{it}^{s}(x)$ are strictly increasing, strictly convex, differentiable functions that are nonnegative and are finite when their argument is zero.

Note, if the holding cost functions are convex but not strictly convex, sales are uniquely determined, but not the production and inventory levels separately.

A6: Production in advance of demand and storage costs more than production in the subsequent period.

A7: $r_F > r_i > 0$ for all i, but $r_F < \sum_{i=1}^{N} r_i$.

If r_F were less than or equal to r_i for some i, then it would never be economical for the firm to purchase capacity dedicated to product i. If r_F were greater than or equal to the sum of the r_i's, the firm would have no incentive to purchase flexible capacity.

A8: $(R_{it}^{s}(x) - C_{it}^{s}(x))$ is nondecreasing at $x = 0$.

This is a technical condition which would generally be true and is used only in the proof of the uniqueness of the Karush-Kuhn-Tucker (KKT) multipliers.

With these eight assumptions the formulation reasonably models the seasonal stock issues, but three more assumptions will greatly simplify the analysis.

A9: The analysis is restricted to a two-product-family model; that is, $N = 2$.

This assumption greatly facilitates explication of the model and has applications in a number of settings. (See, e.g., Fine and Freund, 1987.) To distinguish the product and period indices, the two products are labelled 'A' and 'B' in the sequel. Also, the flexible capacity will be identified by the subscript 'AB' instead of the subscript 'F'.

A10: The planning horizon is infinite.

A11: The firm can carry inventory into the first period at a cost equal to the production cost in an even period plus the holding cost from an even period to an odd period.

Assumptions A10 and A11 eliminate the boundary effects associated with a terminal period and the complicating transient effects associated with building up to a desired level of inventory. They also create symmetry between even and odd periods that is exploited extensively. With an infinite time horizon and no boundary effects, the (ordered) set of even periods cannot be distinguished from the (ordered) set of odd periods. Or, to put it another way, the odd periods do not necessarily have to be thought of as coming "before" the even periods or "after" them. Thus, the optimal production and inventory levels are the same for all odd periods and for all even periods.

Letting $j \in \{1,2\}$ be the index identifying odd and even periods respectively, $p_s = P(\sigma_1 = s)$, and $\gamma \equiv \frac{1}{1-\alpha^2}$, the Unknown Seasonal Demand Problem (USDP) is:

$$\text{Max } V = -r_A K_A - r_B K_B - r_{AB} K_{AB} + \gamma \sum_{s=1}^{S} p_s \sum_{i=A}^{B} \sum_{j=1}^{2} \left(R_{ij}^s\left(Y_{ij}^s + Z_{ij}^s + I_{i\bar{j}}^s - I_{ij}^s \right) - C_{ij}^s\left(Y_{ij}^s + Z_{ij}^s \right) - h_{ij}^s\left(I_{ij}^s \right) \right)$$

subject to:

$$I_{ij}^{s} - Y_{ij}^{s} - Z_{ij}^{s} \leq 0 \qquad i = A,B;\ j = 1,2;\ s = 1..S \qquad (\delta_{ij}^{s})$$

$$Y_{ij}^{s} - K_i \leq 0 \qquad i = A,B;\ j = 1,2;\ s = 1..S \qquad (\alpha_{ij}^{s})$$

$$\sum_{i=A}^{B} Z_{ij}^{s} - K_{AB} \leq 0 \qquad j = 1,2;\ s = 1..S \qquad (\gamma_{j}^{s})$$

$$Y_{ij}^{s} \geq 0 \qquad i = A,B;\ j = 1,2;\ s = 1..S \qquad (s_{ij}^{s})$$

$$Z_{ij}^{s} \geq 0 \qquad i = A,B;\ j = 1,2;\ s = 1..S \qquad (u_{ij}^{s})$$

$$I_{ij}^{s} \geq 0 \qquad i = A,B;\ j = 1,2;\ s = 1..S \qquad (q_{ij}^{s})$$

$$K_i \geq 0 \qquad i = A,B, \text{ and } AB \qquad (m_i).$$

The variables in parentheses identify the Karush-Kuhn-Tucker multipliers which are used in the next section.

3.2 Analysis of General Case

With the additional structure of the USDP, it is possible to say much more about the optimal investment, production, and inventory decisions. Lemma 3.1 gives some simple properties of optimal solutions. Theorem 3.1 states that a unique optimal solution exists. Theorem 3.2 gives formulas for the optimal Karush-Kuhn-Tucker (KKT) multipliers and provides insight into their economic interpretation. Theorem 3.3 gives a necessary and sufficient condition for it to be optimal to purchase flexible capacity. Theorem 3.4 shows that the optimal value of the USDP is a convex, nonincreasing function of the capacity cost vector **r**, and its rate of decrease with respect to changes in a component of **r** is equal to the amount of the corresponding capacity that it is optimal to purchase. Theorems 3.5 and 3.6 in Section 3.3 prove additional properties of the optimal solution as a function of the capacity cost vector **r** for the special case of a USDP with quadratic revenue and holding cost functions and linear production costs.

Lemma 3.1:

(i) In any optimal solution Ξ, for all product/state pairs (i,s), $I^s_{i1} = 0$ or $I^s_{i2} = 0$ or both. That is, it is never optimal for the firm to carry inventory of any product in both odd and even periods.

(ii) In any optimal solution Ξ, $K_i + K_{AB} = \max_{s=1}^{S} \max_{j=1}^{2} \left\{ Y^s_{ij} + Z^s_{ij} \right\}$ and $K_A + K_B + K_{AB} = \max_{s=1}^{S} \max_{j=1}^{2} \left\{ Y^s_{Aj} + Y^s_{Bj} + Z^s_{Aj} + Z^s_{Bj} \right\}$. That is, for all products there is some state/period pair in which the firm uses all available capacity, both dedicated and flexible, to produce that product. And, in some state/period pair, production capacity will be fully utilized.

Proof: Straightforward. See Caulkins (1988). ■

Theorem 3.1: The USDP has a unique optimal solution.

Actually **Y** and **Z** are not necessarily uniquely determined because if there is surplus capacity the firm is indifferent between producing with dedicated or flexible capacity, but their sum **X** = **Y** + **Z** is uniquely determined, and, because of this indifference, knowledge of **K**, **X**, and **I** is essentially a complete solution.

Proof: Straightforward. See Caulkins (1988). ■

The next theorem gives formulas for the optimal Karush-Kuhn-Tucker (KKT) multipliers. It is possible to find a feasible solution for which all the constraints are satisfied as strict inequalities, so the USDP is strongly consistent; i.e., it satisfies Slater's condition. Thus the KKT conditions given by I - VII below are both necessary and sufficient for optimality. Primes denote derivatives. If j = 1 then $\bar{j}$ = 2 and vice versa.

Necessary and Sufficient Conditions for Optimality:

(I) $\gamma\, p_s\left(R_{ij}^{s\,'}\left(Y_{ij}^{s} + Z_{ij}^{s} + I_{i\bar{j}}^{s} - I_{ij}^{s}\right) - C_{ij}^{s\,'}\left(Y_{ij}^{s} + Z_{ij}^{s}\right)\right) = \alpha_{ij}^{s} - \delta_{ij}^{s} - s_{ij}^{s}$

i = A,B; j = 1,2; s = 1..S

(II) $\gamma\, p_s\left(R_{ij}^{s\,'}\left(Y_{ij}^{s} + Z_{ij}^{s} + I_{i\bar{j}}^{s} - I_{ij}^{s}\right) - C_{ij}^{s\,'}\left(Y_{ij}^{s} + Z_{ij}^{s}\right)\right) = \gamma_{j}^{s} - \delta_{ij}^{s} - u_{ij}^{s}$

i = A,B; j = 1,2; s = 1..S

(III) $\gamma\, p_s\left(R_{i\bar{j}}^{s\,'}\left(Y_{i\bar{j}}^{s} + Z_{i\bar{j}}^{s} + I_{ij}^{s} - I_{i\bar{j}}^{s}\right) - h_{ij}^{s\,'}\left(Y_{ij}^{s} + Z_{ij}^{s}\right) - R_{ij}^{s\,'}\left(Y_{ij}^{s} + Z_{ij}^{s} + I_{i\bar{j}}^{s} - I_{ij}^{s}\right)\right) = \delta_{ij}^{s} - q_{ij}^{s}$ i = A,B; j = 1,2; s = 1..S

(IV) $r_i = \sum_{s=1}^{S}\sum_{j=1}^{2} \alpha_{ij}^{s} + m_i$ i = A,B

$r_{AB} = \sum_{s=1}^{S}\sum_{j=1}^{2} \gamma_{j}^{s} + m_{AB}$

(V) $\alpha_{ij}^{s}\left(K_i - Y_{ij}^{s}\right) = 0$ i = A,B; j = 1,2; s = 1..S

$\gamma_{j}^{s}\left(K_{AB} - \sum_{i=A}^{B} Z_{ij}^{s}\right) = 0$ j = 1,2; s = 1..S

$\delta_{ij}^{s}\left(Y_{ij}^{s} + Z_{ij}^{s} - I_{ij}^{s}\right) = 0$ i = A,B; j = 1,2; s = 1..S

(VI) $s_{ij}^{s}\ Y_{ij}^{s} = 0$ i = A,B; j = 1,2; s = 1..S

$q_{ij}^{s}\ I_{ij}^{s} = 0$ i = A,B; j = 1,2; s = 1..S

$u_{ij}^{s}\ Z_{ij}^{s} = 0$ i = A,B; j = 1,2; s = 1..S

$m_i\ K_i = 0$ i = A, B, and AB

(VII) $\alpha_{ij}^{s}, \delta_{ij}^{s}, \gamma_{j}^{s}, q_{ij}^{s}, s_{ij}^{s}, u_{ij}^{s}$, and m_i are all nonnegative.

Theorem 3.2: When the derivatives of the revenue, production cost, and holding cost functions are evaluated at the optimal solution, the multipliers defined below are optimal KKT multipliers Furthermore, the multipliers are unique if **K** > 0.

$\bar{\delta}_{ij}^{s} = \gamma\, p_s \left[R_{i\bar{j}}^{s\,'} - h_{ij}^{s\,'} - R_{ij}^{s\,'}\right]^{+}$ i = A,B; j = 1,2; s = 1..S

$\bar{q}_{ij}^{s} = -\gamma\, p_s \left[R_{i\bar{j}}^{s\,'} - h_{ij}^{s\,'} - R_{ij}^{s\,'}\right]^{-}$ i = A,B; j = 1,2; s = 1..S

$\bar{\alpha}_{ij}^{s} = \gamma\, p_s \left[\mathrm{Max}\left\{ R_{i\bar{j}}^{s\,'} - h_{ij}^{s\,'},\ R_{ij}^{s\,'}\right\} - C_{ij}^{s\,'}\right]^{+}$ i = A,B; j = 1,2; s = 1..S

$$\overline{s}_{ij}^{s} = -\gamma\, p_s \left[\mathrm{Max}\left\{ R_{ij}^{s'} - h_{ij}^{s'},\ R_{ij}^{s'} \right\} - C_{ij}^{s'} \right]^{-} \qquad i = A,B;\ j = 1,2;\ s = 1..S$$

$$\overline{\gamma}_{j}^{s} = \mathop{\mathrm{Max}}_{i=A}^{B} \left\{ \overline{\alpha}_{ij}^{s} \right\} \qquad j = 1,2;\ s = 1..S$$

$$\overline{u}_{ij}^{s} = \overline{\gamma}_{j}^{s} - \left[\overline{\alpha}_{ij}^{s} - \overline{s}_{ij}^{s} \right] \qquad i = A,B;\ j = 1,2;\ s = 1..S$$

$$\overline{m}_{i} = r_i - \sum_{s=1}^{S} \sum_{j=1}^{2} \overline{\alpha}_{ij}^{s} \qquad i = A,B$$

$$\overline{m}_{AB} = r_{AB} - \sum_{s=1}^{S} \sum_{j=1}^{2} \overline{\gamma}_{j}^{s}$$

Proof: See Appendix.

These multipliers have straightforward economic interpretations. The multiplier α_{ij}^{s} is simply the positive part of the shadow value of having another unit of capacity dedicated to product i available in state/period pair (s,j). Multiplier s_{ij}^{s} is the absolute value of the corresponding negative part of the shadow value. Similarly δ_{ij}^{s} is the positive part of the shadow value of being able to inventory one more unit of product i from period j to period $\bar{j}$ in state s, whereas q_{ij}^{s} is the absolute value of the negative part. The multiplier γ_{j}^{s} is the shadow value of having an additional unit of flexible capacity in state/period pair (s,j). Finally, u_{ij}^{s} is the amount the firm should be willing to pay to 'upgrade' one unit of capacity dedicated to product i so that in state/period pair (s,j) it can be used as one unit of flexible capacity.

Suppose that at optimality the per unit purchase price of dedicated capacity for product A exceeds the sum of the shadow values of having an additional unit of capacity dedicated to product A in each period and each state. Then by IV, $m_A > 0$, and thus by VI, $K_A^* = 0$. In other words, m_A is the amount by which the price of dedicated capacity for product A must fall before it is optimal to purchase that technology. Similar results hold for m_B and m_{AB}, with respect to K_B and K_{AB}.

These interpretations of the multipliers are consistent with the left hand sides of conditions I - III. For example, the left hand side of condition I gives the marginal value of having an additional unit of capacity dedicated to product i in period j that is subject to the restriction that whatever it produces must be sold in period j. Similar statements can be made about the left hand sides of conditions II and III.

The next theorem gives a necessary and sufficient condition for it to be optimal to purchase flexible capacity. Its proof uses the concept of product subproblems. We consider these subproblems because the only linkage between products A and B is their sharing of flexible production capacity. If the firm does not purchase flexible capacity, the problem decomposes into two subproblems, one for each product. The subproblems are convex programs that can be solved independently. Subproblem i for i = A or B is:

$$\text{Max } V = -r_i K_i + \gamma \sum_{s=1}^{S} p_s \sum_{j=1}^{2} \left[R^s_{ij}\left(Y^s_{ij} + I^s_{ij^-} - I^s_{ij} \right) - C^s_{ij}\left(Y^s_{ij} \right) - h^s_{ij}\left(I^s_{ij} \right) \right]$$

subject to:

$$Y^s_{ij} - K_i \le 0 \qquad j = 1,2;\ s = 1..S,$$

$$I^s_{ij} - Y^s_{ij} - I^s_{ij^-} \le 0 \qquad j = 1,2;\ s = 1..S,$$

with all variables nonnegative.

The corresponding KKT conditions are:

$$\text{(I)} \quad \gamma p_s \left(R^{s\prime}_{ij}\left(Y^s_{ij} + I^s_{ij^-} - I^s_{ij} \right) - C^{s\prime}_{ij}\left(Y^s_{ij} \right) \right) = \alpha^s_{ij} - \delta^s_{ij} - s^s_{ij} \qquad j = 1,2;\ s = 1..S$$

$$\text{(II)} \quad \gamma p_s \left(R^{s\prime}_{ij^-}\left(Y^s_{ij^-} + I^s_{ij} - I^s_{ij^-} \right) - h^{s\prime}_{ij}\left(I^s_{ij} \right) - R^{s\prime}_{ij}\left(Y^s_{ij} + I^s_{ij^-} - I^s_{ij} \right) \right) = \delta^s_{ij} - \varsigma^s_{ij}$$

$$j = 1,2;\ s = 1..S$$

$$\text{(III)} \quad r_i = \sum_{s=1}^{S} \sum_{j=1}^{2} \alpha^s_{ij} + m_i$$

(IV) $\alpha_{ij}^{s}\left(K_i - Y_{ij}^{s}\right) = 0$ $\quad j = 1,2;\ s = 1..S$

$\delta_{ij}^{s}\left(Y_{ij}^{s} - I_{ij}^{s}\right) = 0$ $\quad j = 1,2;\ s = 1..S$

(V) $s_{ij}^{s}\, Y_{ij}^{s} = 0$ $\quad j = 1,2;\ s = 1..S$

$q_{ij}^{s}\, I_{ij}^{s} = 0$ $\quad j = 1,2;\ s = 1..S$

$m_i\, K_i = 0$

(VI) $\alpha_{ij}^{s}, \delta_{ij}^{s}, q_{ij}^{s}, s_{ij}^{s}$, and m_i are all nonnegative.

Theorems 3.1 and 3.2 can easily be specialized to each subproblem. Hence they have unique solutions, the formulas below give an optimal set of KKT multipliers, and these multipliers are unique if $K_i > 0$.

$$\bar{\delta}_{ij}^{s} = \gamma\, p_s \left[R_{i\bar{j}}^{s\prime} - h_{ij}^{s\prime} - R_{ij}^{s\prime}\right]^{+} \qquad j = 1,2;\ s = 1..S$$

$$\bar{q}_{ij}^{s} = -\gamma\, p_s \left[R_{i\bar{j}}^{s\prime} - h_{ij}^{s\prime} - R_{ij}^{s\prime}\right]^{-} \qquad j = 1,2;\ s = 1..S$$

$$\bar{\alpha}_{ij}^{s} = \gamma\, p_s \left[\mathrm{Max}\left\{R_{i\bar{j}}^{s\prime} - h_{ij}^{s\prime},\ R_{ij}^{s\prime}\right\} - C_{ij}^{s\prime}\right]^{+} \qquad j = 1,2;\ s = 1..S$$

$$\bar{s}_{ij}^{s} = -\gamma\, p_s \left[\mathrm{Max}\left\{R_{i\bar{j}}^{s\prime} - h_{ij}^{s\prime},\ R_{ij}^{s\prime}\right\} - C_{ij}^{s\prime}\right]^{-} \qquad j = 1,2;\ s = 1..S$$

$$\bar{m}_i = r_i - \sum_{s=1}^{S}\sum_{j=1}^{2} \bar{\alpha}_{ij}^{s}$$

Now that the subproblems have been introduced and described, it is possible to state and prove a necessary and sufficient condition for it to be optimal to purchase flexible capacity.

<u>Theorem 3.3:</u> $K_{AB} > 0$ is optimal if and only if,

$$r_{AB} < \sum_{s=1}^{S} p_s \left[\underset{i=A}{\overset{B}{\mathrm{Max}}}\left\{R_{i1}^{s\prime} - C_{i1}^{s\prime},\ R_{i2}^{s\prime} - h_{i1}^{s\prime} - C_{i1}^{s\prime},\ 0\right\} + \underset{i=A}{\overset{B}{\mathrm{Max}}}\left\{R_{i2}^{s\prime} - C_{i2}^{s\prime},\ R_{i1}^{s\prime} - h_{i2}^{s\prime} - C_{i2}^{s\prime},\ 0\right\}\right]$$

where these revenue and cost functions are evaluated at the optimal solutions to subproblems A and B.

The right hand side of this condition is the marginal benefit of a unit of flexible capacity, so this theorem simply asserts that the firm should invest in flexible capacity if and only if its marginal cost is less than the expected marginal benefit. It is useful because it shows how to measure the marginal benefit of investing in flexible capacity without solving the full problem. Only the subproblem solutions are needed, and generally solving both subproblems is much easier than solving the full problem.

Proof: Denote the right hand side of the inequality in the statement of Theorem 3.3 by S. Suppose $r_{AB} \geq S$. Then augment the solutions and multipliers for the subproblems with $\mathbf{Z} = \mathbf{0}$, $K_{AB} = 0$, $\bar{u}$, $\bar{\gamma}$ and $\bar{m}_{AB}$ as defined in Theorem 3.2. This yields a solution to the full problem and a set of KKT multipliers that satisfy all the KKT conditions. By Theorem 3.1 the solution is unique, so $r_{AB} \geq S$ implies $K_{AB}^* = 0$.

Now suppose $r_{AB} < S$. The formulas in Theorem 3.2 yield a set of optimal multipliers. If $K_{AB} = 0$, then the multipliers for the subproblems satisfy the formulas of Theorem 3.2. But since $r_{AB} < S$, this implies $\bar{m}_{AB} < 0$ which contradicts the nonnegativity of the KKT multipliers. Hence, $r_{AB} < S$ implies $K_{AB} = 0$ is not optimal. Since an optimal solution always exists, $r_{AB} < S$ implies $K_{AB}^* > 0$. ■

Let $\mathbf{r} = (r_A, r_B, r_{AB})$ and denote the optimal value of problem USDP as a function of $\mathbf{r}$ by $V^*(\mathbf{r}) = -\mathbf{r}^t \mathbf{K} + f(\Xi)$.

Theorem 3.4:

(i) $V^*(\mathbf{r})$ is convex in $\mathbf{r}$,

(ii) $V^*(\mathbf{r})$ is nonincreasing in $\mathbf{r}$, and

(iii) $\dfrac{\partial V^*(r)}{\partial r_i} = -K_i$ for i = A, B, and AB.

That $V^*(\mathbf{r})$ is nonincreasing in $\mathbf{r}$ is quite intuitive; decreasing costs of capacity acquisition increase the maximum profit the firm can earn. The convexity of $V^*(\mathbf{r})$ suggests that as the cost of acquiring advanced manufacturing technology continues to decline, firms will enjoy increasingly large profit improvements. Part (iii) tells exactly how rapidly profits improve as capacity acquisition costs decline.

Knowledge of $V^*(\mathbf{r})$ is useful for evaluating potential returns to reductions in acquisition costs and because there is often uncertainty about these costs, so sensitivity analysis with respect to the capacity costs can be informative. In general, obtaining $V^*(\mathbf{r})$ is difficult, but Theorem 3.4 describes its form and guarantees that it is a well-behaved function, so it is reasonable to draw inferences about $V^*(\mathbf{r})$ from points obtained numerically.

Proof: Proof of (i). Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two capacity cost vectors satisfying Assumption A7. For arbitrary $\lambda \in [0,1]$ let $\mathbf{r} = \lambda \mathbf{r}_1 + (1-\lambda) \mathbf{r}_2$. Let $(\mathbf{K},\Xi)$ be the unique optimal solution corresponding to $\mathbf{r}$. Since $(\mathbf{K},\Xi)$ is feasible for all cost vectors, $V^*(\mathbf{r}_1) \geq -\mathbf{r}_1^t \mathbf{K} + f(\Xi)$ and $V^*(\mathbf{r}_2) \geq -\mathbf{r}_2^t \mathbf{K} + f(\Xi)$. Thus $\lambda V^*(\mathbf{r}_1) + (1-\lambda)V^*(\mathbf{r}_2) \geq -(\lambda \mathbf{r}_1^t + (1-\lambda) \mathbf{r}_2^t)\mathbf{K} + f(\Xi) = V^*(\mathbf{r}) = V^*(\lambda \mathbf{r}_1 + (1-\lambda) \mathbf{r}_2)$, so $V^*(\mathbf{r})$ is convex in $\mathbf{r}$.

Proof of (ii). Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two capacity cost vectors satisfying Assumption A7 such that $\mathbf{r}_1 \geq \mathbf{r}_2$ and $\mathbf{r}_1 \neq \mathbf{r}_2$. Let $(\mathbf{K},\Xi)$ be the unique optimal solution corresponding to $\mathbf{r}_1$. Then $V^*(\mathbf{r}_1) = -\mathbf{r}_1^t \mathbf{K} + f(\Xi)$. Since $(\mathbf{K},\Xi)$ is feasible for $\mathbf{r}_2$, $V^*(\mathbf{r}_2) \geq -\mathbf{r}_2^t \mathbf{K} + f(\Xi)$. But then $V^*(\mathbf{r}_2) - V^*(\mathbf{r}_1) \geq (\mathbf{r}_1^t - \mathbf{r}_2^t)\mathbf{K} \geq 0$, so $V^*(\mathbf{r})$ is nonincreasing in $\mathbf{r}$.

Proof of (iii). This is a direct application of the envelope theorem (Varian, 1978) as extended to nondifferentiable functions by Fine and Freund (1987). ■

3.3 Analysis of Special Case of Quadratic Revenue and Holding Cost Functions and Linear Production Costs

As discussed above, knowledge of $V^*(\mathbf{r})$ is valuable for several reasons. For a general USDP, $V^*(\mathbf{r})$ is a well-behaved function of $\mathbf{r}$, but even more can be said when the USDP has quadratic revenue and holding cost functions and linear production costs. Theorem 3.6 gives an exact, second order Taylor expansion for $V^*(\mathbf{r} + \Delta\mathbf{r})$. Theorem 3.5 shows that the optimal values of the decision variables are continuous, piecewise linear functions of **r**. Hence for this special case of the USDP, the entire solution, not just the optimal value function, is a well-behaved function of **r**, and its dependence on **r** is of a particularly simple nature.

Theorem 3.5: For the USDP with quadratic revenue and holding cost functions and linear production costs, all decision variables' optimal values are continuous, piecewise linear functions of the capacity cost vector **r**, and the optimal value function is a continuous, piecewise quadratic function of **r**.

Proof: The second statement follows directly from the first and the fact that the objective function is a quadratic function. The first statement is a consequence of the theory of quadratic programming and Theorem 3.1 which guarantees the solution is unique. ■

Theorem 3.6: If **r** is in the interior of a region for which $V^*(\mathbf{r})$ is quadratic and **K** is the optimal capacity vector, then for Δr sufficiently small,

(1) $V^*(\mathbf{r} + \Delta\mathbf{r}) = V^*(\mathbf{r}) - r_A K_A - r_B K_B - r_{AB} K_{AB} + \frac{1}{2} \Delta\mathbf{r}^t M \Delta\mathbf{r}$, where

$$M = -\begin{bmatrix} \frac{\partial K_A}{\partial r_A} & \frac{\partial K_B}{\partial r_A} & \frac{\partial K_{AB}}{\partial r_A} \\ \frac{\partial K_A}{\partial r_B} & \frac{\partial K_B}{\partial r_B} & \frac{\partial K_{AB}}{\partial r_B} \\ \frac{\partial K_A}{\partial r_{AB}} & \frac{\partial K_B}{\partial r_{AB}} & \frac{\partial K_{AB}}{\partial r_{AB}} \end{bmatrix}$$

and M is positive semi-definite.

Proof: Formula (1) follows directly from Taylor's Theorem. M is positive semi-definite because $V^*(\mathbf{r})$ is convex. By Theorem 3.3, $\frac{\partial V^*(\mathbf{r})}{\partial r_i} = -K_i$ for i = A,B, and AB, where K_i is the optimal capacity.

Hence, $M_{r_i r_j} = \frac{\partial^2 V^*(\mathbf{r})}{\partial r_i \partial r_j} = \frac{\partial}{\partial r_i}\left(\frac{\partial V^*(\mathbf{r})}{\partial r_j}\right) = \frac{\partial}{\partial r_i}(-K_j) = -\frac{\partial K_j}{\partial r_i}$. ■

3.4 Example

This section gives an example of a USDP with linear cost functions and unitary inelastic demand. A firm facing unitary inelastic demand earns constant revenues per unit sold, up to some maximum volume. If per unit production and holding costs are constant, then production costs can be absorbed into the revenue and holding cost functions. Hence, without loss of generality, production costs will be assumed to be zero.

Revenue functions generated by unitary inelastic demand curves are concave but not strictly concave, so they violate Assumption A3. Similarly, linear holding cost functions are convex but not strictly convex, so they violate Assumption A5. As a result, uniqueness of the optimal solution is not guaranteed, but that is not a concern for this example.

The USDP with linear costs and unitary inelastic demand can be restated as follows. Let

$k \in \{1, 2\}$ be the index of the sales period,

D^s_{ik} be the maximum sales volume for product i in period k if state s is realized,

R^s_{ik} be the constant revenue per unit sold of product i in period k if state s is realized,

h^s_{ij} be the cost of carrying one unit of product i from period j to period $\bar{j}$ if state s is realized,

$\pi^s_{ijk} = R^s_{ik} - h^s_{ij}$ be the profit from producing one unit of product i in period j and selling it in period k if state s is realized,

Y^s_{ijk} be the amount of product i produced on dedicated capacity in period j and sold in period k if state s is realized, and

Z^s_{ijk} be the amount of product i produced on flexible capacity in period j and sold in period k if state s is realized.

So $Y^s_{ij} = \sum_{k=1}^{2} Y^s_{ijk}$, $Z^s_{ij} = \sum_{k=1}^{2} Z^s_{ijk}$, and $I^s_{ij} = Y^s_{ij\bar{j}} + Z^s_{ij\bar{j}}$.

The objective function can be written as:

$$V = -r^t \mathbf{K} + \sum_{s=1}^{S} p_s \left[\sum_{i=A}^{B} \sum_{j=1}^{2} \sum_{k=1}^{2} \pi_{ijk}^{s} \left(Y_{ijk}^{s} + Z_{ijk}^{s} \right) \right].$$

The maximum sales condition imposes the following constraint:

$$Y_{i1k}^{s} + Z_{i1k}^{s} + Y_{i2k}^{s} + Z_{i2k}^{s} \le D_{ik}^{s} \text{ for all i,k, and s}$$

in addition to the usual constraints.

This is a linear program, so it can be solved by standard techniques such as the simplex method. It can also be solved numerically because solving the second stage problem is trivial. One simply continues meeting, to the greatest extent possible, the remaining demand for the product/production-period/sales-period triplet yielding the greatest per unit profit until either (1) all production capacity has been allocated or (2) all demand has been met from product/production-period/sales-period combinations yielding a positive profit.

If one lists the π's in descending order of profitability, then a general formula for solving the second stage problem with unitary inelastic demand and linear cost functions is:

$$Y_{ijk} = \mathrm{Min}\{ D_{ik} - U(\pi_{i\bar{j}k}, \pi_{ijk})(Y_{i\bar{j}k} + Z_{i\bar{j}k}),\ K_i - U(\pi_{ij\bar{k}}, \pi_{ijk}) Y_{ij\bar{k}} \}$$

$$Z_{ijk} = \mathrm{Min}\{ D_{ik} - U(\pi_{i\bar{j}k}, \pi_{ijk})(Y_{i\bar{j}k} + Z_{i\bar{j}k}) - Y_{ijk},$$
$$K_{AB} - U(\pi_{ij\bar{k}}, \pi_{ijk}) Z_{ij\bar{k}} - U(\pi_{\bar{i}jk}, \pi_{ijk}) Z_{\bar{i}jk} - U(\pi_{\bar{i}j\bar{k}}, \pi_{ijk}) Z_{\bar{i}j\bar{k}} \}$$

where $U(\pi_x, \pi_y)$ is a modified unit step function. $U(\pi_x, \pi_y) = 1$ if $\pi_x > \pi_y$ or $\pi_x = \pi_y$ and π_x precedes π_y in the list of π_{ijk}'s. Otherwise, $U(\pi_x, \pi_y) = 0$.

Example The example (parameterized by ξ) considers how the nature of the uncertainty about market conditions affects the relationship between inventories and flexible capacity. Specifically, it considers two extreme situations: (1) when the firm knows with certainty what total demand for each product will be, but it does not know how seasonal the demand will be (ξ = 1) and (2) when the firm knows with certainty what total demand for each period will be, but it does not know how it will be divided between the two products (ξ = 0).

The underlying demand is assumed to be for one unit of each product in each period, and total demand is always four units. However, the actual demand in product/period pair (A,1) is a random variable with the following distribution:

$$P(D_{A1}=x) = \begin{cases} P_{A1} & x=0 \\ 1-2P_{A1} & x=1 \\ P_{A1} & x=2 \end{cases}$$

Similarly,

$$P(D_{B2}=x) = \begin{cases} P_{B2} & x=0 \\ 1-2P_{B2} & x=1 \\ P_{B2} & x=2 \end{cases}$$

Assume for the moment that these two random variables are independent.

Furthermore, let $0 \le \xi \le 1$ be the fraction of the change in demand for product/period pairs (A,1) and (B,2) that comes from the same product but the other period, and $1 - \xi$ be the fraction that comes from the same period but the other product. Thus if $\xi = 0$, all the variability is between products, and if $\xi = 1$, all the variability is between periods. Table 3.1 summarizes the demand distribution in each of the nine possible states of the world.

Table 3.1
Description of Demand States

State s	P(State = s)	D^s_{A1}	D^s_{A2}	D^s_{B1}	D^s_{B2}
1	$(1-2P_{A1})(1-2P_{B2})$	1	1	1	1
2	$(1-2P_{A1})P_{B2}$	1	ξ	$1-\xi$	2
3	$(1-2P_{A1})P_{B2}$	1	$2-\xi$	$1+\xi$	0
4	$P_{A1}(1-2P_{B2})$	0	$1+\xi$	$2-\xi$	1
5	$P_{A1}P_{B2}$	0	2ξ	$2(1-\xi)$	2
6	$P_{A1}P_{B2}$	0	2	2	0
7	$P_{A1}(1-2P_{B2})$	2	$1-\xi$	ξ	1
8	$P_{A1}P_{B2}$	2	0	0	2
9	$P_{A1}P_{B2}$	2	$2(1-\xi)$	2ξ	0

First consider what happens when $\xi = 1$ so all variability is between periods.

Let the per unit holding costs h be equal for all product/period pairs. Similarly, let revenues per unit sold be equal for all product/period pairs and be high enough to ensure the firm always meets demand. With these assumptions revenues are the same for all states and all solutions, so they can be omitted from the objective function. Demand over both periods and both products is four in every state, so the firm can always meet demand if $K_{AB} \geq (1/2)(4 - 2(K_A + K_B))$. Finally, assume capital investment costs are large relative to holding costs (Specifically, $r_{AB} > [1 - (1 - 2P_{A1})(1 - 2P_{B2})]h$) so the firm does not idle any purchased capacity; i.e., $K_{AB} = (1/2)(4 - 2(K_A + K_B))$. With these assumptions it is easy to solve the second stage problems to determine how much inventory will be held under each state s.

Ignoring the constant revenues and the part of holding costs that do not depend on the decision variables, the objective function reduces to:

$$\text{Max } V = (r_{AB} - r_A - 2\, P_{A1}P_{B2}\, h)K_A + (r_{AB} - r_B - 2\, P_{A1}P_{B2}\, h)K_B.$$

So, if $r_{AB} < r_A + 2\, P_{A1}P_{B2}\, h$, then $K_A = 0$ is optimal. Otherwise, $K_A = 1$ is optimal. Likewise, if $r_{AB} < r_B + 2\, P_{A1}P_{B2}\, h$, then $K_B = 0$ is optimal; otherwise, $K_B = 1$ is.

Hence, for this problem and these parameter values, increasing holding costs favors investment in flexible capacity, and decreasing the cost of flexible capacity reduces the expected amount of inventory. So flexible capacity and inventory are substitutes. Note, however, that even if only flexible capacity is purchased, in six of the nine states inventory is still used. This is simply because no amount of flexible capacity can shift a unit from one period to the next; only inventory can. Also, increasing variability, i.e. increasing P_{A1} or P_{B2}, favors investment in flexible capacity. (However, if $P_{A1} \leq \frac{r_{AB} - r_A}{2h}$, then $K_{AB} = 0$ is optimal no matter how large P_{B2} is, and vice versa.)

Intuitively one would expect that if demand for the two products were positively correlated the firm would be less likely to purchase flexible capacity and, conversely, if their demand were negatively correlated, the firm would be more likely to invest in flexible capacity. This intuition is correct.

Suppose $P(s = 6) = P(s = 8) = P_{A1}P_{B2} - \varepsilon$ and $P(s = 5) = P(s = 9) = P_aP_b + \varepsilon$. Then the correlation coefficient between demand for A and demand for B in the same period is $\rho = \frac{2\varepsilon}{\sqrt{P_{A1}P_{B2}}}$. The objective function is now

$$\text{Max } V = (r_{AB} - r_A - 2(P_{A1}P_{B2} - \varepsilon)h)K_A + (r_{AB} - r_B - 2(P_{A1}P_{B2} - \varepsilon)h)K_B.$$

So, positively correlated demand favors larger inventories, and negatively correlated demand favors investment in flexible capacity.

Now consider the other extreme, $\xi = 0$, so all variability is between products. The firm knows for certain what demand in each season will be, but it does not know how demand will be divided between products A and B. If revenues are sufficiently high, or equivalently, stockout costs are sufficiently severe, the firm will meet demand under all states. Then as long as P_{A1} and P_{B2} are both nonzero, the firm must have $K_{AB} \geq 2 - \text{Min}\{K_A, K_B\}$ because it is possible there will be four units of demand for one product. Since $r_{AB} < r_A + r_B$, this implies that it is always optimal to have $K_A = 0$, $K_B = 0$, and $K_{AB} = 2$, and it is not optimal to carry inventory even if holding costs are zero!

Clearly in this case inventory is not a substitute for flexible capacity. The only way to meet uncertainty about which product will be demanded is to purchase flexible capacity. No amount of inventory of product A can create even a single unit of product B.

$K_{AB} \geq 2 - \text{Min}\{K_A, K_B\}$ is not necessarily optimal, however, if stockout costs are not arbitrarily large. In a sense, stocking out converts one product into another. Shortage of one product is converted into dollar losses which can be recovered by reduced initial outlays or increased sales of another product.

This example suggests that insight can be gained by decomposing total demand variability into variability between products and variability between periods. Flexible capacity is best suited for meeting variability between products and inventory is best suited for meeting variability between periods. However, since the objective function is measured in dollars, which can be generated by either product in either period, stocking out provides a means by which flexible capacity can compensate for variability between periods and inventory can compensate for variability between products.

In this example, both increasing variability in general and negatively correlated demand for the two products favored investment in flexible capacity. This matches intuition and suggests how safety stock considerations would modify the results obtained in this paper for seasonal stock considerations.

4 The Known Seasonal Demand Problem

This section reports results for the Known Seasonal Demand Problem (KSDP), a special case of the USDP in which market conditions vary periodically, but are deterministic. The formulation of the KSDP is identical to that of the USDP except that there is just one state.

Market conditions are rarely deterministic, so it is unlikely this model will be applied in practice, but it gives useful insights nonetheless. For example, we observe that (1) holding inventory and investing in flexible capacity can be complements, (2) it can be optimal to purchase less capacity dedicated to a product than is sold in any period, and (3) it may be optimal to use flexible capacity to produce a product even if its demand is known and constant.

Section 4.1 gives properties of optimal solutions to the KSDP and shows by example that some 'properties' one might expect to be true do not, in fact, always hold. It is possible to obtain closed form expressions for the optimal values of the decision variables when revenue functions are quadratic and costs are linear. This special case is called the Quadratic Known Seasonal Demand Problem or Q-KSDP. Section 4.2 gives properties of optimal solutions to the Q-KSDP. Section 4.3 describes how the solution is derived. Section 4.4 gives sensitivity analysis results.

The derivations of these results are quite involved, and if they were compressed sufficiently to present here, they would no longer be informative. They are reported in their entirety in Caulkins (1988). Interested readers are invited to request a copy.

4.1 Properties of Optimal Solutions

The following statements about production and inventory hold for all optimal solutions to the KSDP.

1) If it is optimal to purchase flexible capacity, it will all be used to produce one product in one period and the other product in the other period.

2) For at least one product, the dedicated capacity that is acquired is fully utilized in both periods.

3) All capacity is fully utilized in at least one of the two periods.

4) It is never optimal to carry inventory out of both product/period pairs in which flexible capacity is used.

5) In any product/period pair in which flexible capacity is used, all dedicated capacity for that product will also be employed.

It is also possible to make statements relating production and sales.

6) There is never slack capacity for a product in a period in which sales for that product are at least as great as they are in the other period.

7) If sales for a product are constant, capacity dedicated to that product will be fully utilized in both periods.

8) Sales for at least one of the product/period pairs in which flexible capacity is used must be greater than sales for that product in the other period.

9) If the optimal solution has sales of both products equal in each period, then no inventory or flexible capacity will be used.

Combining these properties gives the following useful result.

Theorem 4.1: If it is optimal to purchase flexible capacity for the KSDP, then in the optimal solution either:

$Z_{A1} = Z_{B2} = K_{AB}$, $Z_{A2} = Z_{B1} = 0$, $Y_{A1} = K_A$, $Y_{B2} = K_B$,
($Y_{A2} = K_A$ or $Y_{B1} = K_B$ or both),
($I_{A1} = 0$ or $I_{B2} = 0$ or both), and
($S_{A1} > S_{A2}$ or $S_{B1} < S_{B2}$ or both).

or

$Z_{A1} = Z_{B2} = 0$, $Z_{A2} = Z_{B1} = K_{AB}$, $Y_{A2} = K_A$, $Y_{B1} = K_B$,
($Y_{A1} = K_A$ or $Y_{B2} = K_B$ or both),
($I_{A2} = 0$ or $I_{B1} = 0$ or both), and
($S_{A1} < S_{A2}$ or $S_{B1} > S_{B2}$ or both).

And, if it is optimal to purchase no flexible capacity, then $Z_{A1} = Z_{A2} = Z_{B1} = Z_{B2} = 0$.

As a result, all optimal solutions belong to one of three categories, one with $K_{AB} = 0$ and two with $K_{AB} > 0$. The two categories with $K_{AB} > 0$ are symmetric (See Figure 4.1.) because either product could be called product A and, by Assumptions A10 and A11, either the odd or the even periods could be labelled with a '1'.

Figure 4.1
Optimal Patterns of Allocation of Flexible Capacity

	$Z_{A1} = Z_{B2} = K_{AB} > 0$ Period 1	Period 2		$Z_{B1} = Z_{A2} = K_{AB} > 0$ Period 1	Period 2
Product A	X		Product A		X
Product B		X	Product B	X	

X denotes a product/period pair in which flexible capacity is used.

The following example of a KSDP with unitary inelastic demand and linear costs demonstrates that several 'properties' one might intuitively expect to be true do not, in fact, always hold. In particular, it shows that the following intuitively appealing heuristic can lead to suboptimal solutions. "Use dedicated capacity to meet 'base demand', i.e. demand present in every period and state, and use flexible capacity and/or dedicated capacity coupled with inventories to meet 'swing demand', the demand in excess of base demand." The heuristic does not specify how to meet swing demand. It just tells the firm to purchase at least as much dedicated capacity as the minimum that will be sold in any state and period. In symbols,

$$K_i^* \geq \underset{j,s}{\text{Min}} \left\{ S_{ij}^s \right\} \quad \text{for } i = A,B.$$

This heuristic simplifies the problem by 'subtracting out' base demand so the analysis can focus on the more difficult problem of meeting swing demand.

<u>Example</u>

$r_A = r_B = 1.0$ $\qquad$ $r_{AB} = 1.1$

$D_{A1} = 5$ $\quad$ $D_{A2} = 5$ $\quad$ $D_{B1} = 0$ $\quad$ $D_{B2} = 10$

$h_{A1} = h_{A2} = 0.25$ $\qquad$ $h_{B1} = h_{B2} = \infty$

$R_{A1} = R_{A2} = R_{B1} = R_{B2} = \infty.$

By $h_{B1} = h_{B2} = \infty$ it is meant that holding costs for product B are so high the firm would never inventory product B. Similarly, $R_{A1} = R_{A2} = R_{B1} = R_{B2} = \infty$ means that stockout costs are high enough to ensure demand is always met. Using the properties of the KSDP, the problem reduces to the following linear program:

$$\begin{aligned}
\text{Max} \quad & -K_A - K_B - 1.1\,K_{AB} - 0.1\,I_{A1} \\
\text{s.t.} \quad & K_B + K_{AB} \geq 10 \\
& K_A + K_{AB} - I_{A1} \geq 5 \\
& K_A + I_{A1} \geq 5
\end{aligned}$$

with K_A, K_B, K_{AB}, and I_{A1} nonnegative.

The heuristic $K_i^* \geq \underset{j,s}{\text{Min}}\left\{ S_{ij}^s \right\}$ implies that it is safe to assume $K_A \geq 5$ and $K_B \geq 0$. The optimal solution constrained by $K_A \geq 5$ and $K_B \geq 0$ is: $K_A = 5$, $K_B = 10$, $I_{A1} = 0$, and $K_{AB} = 0$. Its cost is 15. The true optimal solution: $K_A = 0$, $K_B = 0$, $I_{A1} = 5$, and $K_{AB} = 10$ with a cost of 12.25, is completely different. So, for the given parameter values, the heuristic leads to a suboptimal solution. It gives the optimal solution if $r_{AB} > 1.375$ or if $h_{A1} = h_{A2} > 0.8$.

This example leads to some striking conclusions.

1) It is not always optimal to purchase as much dedicated capacity as the minimum of sales over all state/period pairs.

2) It may be optimal to use flexible capacity to produce a product even if its demand is known and constant. In fact, it may be optimal to purchase no capacity dedicated to that product.

3) Flexible capacity and inventory can be complements. If r_{AB} > 1.375 no inventory or flexible capacity is used, but if r_{AB} < 1.375, it is optimal to inventory 5 units of product A from each odd period to each even period and to acquire 10 units of flexible capacity. Thus, decreasing the cost of flexible capacity can increase optimal inventory levels as well as investment in flexible capacity. Similarly, with r_{AB} = 1.1, if holding costs for product A increase above 0.8 per unit, it is no longer optimal to purchase flexible capacity. So, increasing holding costs can lead to less investment in flexible capacity as well as smaller inventories.

4.2 Properties of Optimal Solutions to the Q-KSDP

The KSDP with quadratic revenue functions and linear holding and production costs is called the Quadratic-Known Seasonal Demand Problem (Q-KSDP). Since the holding costs for the Q-KSDP are linear, they are convex but not strictly convex, and so violate Assumption A5. The only consequence of this is that uniqueness of the optimal solution cannot be guaranteed, but if the probability distribution for the model parameters is any continuous distribution over any open set, then nonuniqueness occurs with probability 0.

For the Q-KSDP many additional properties of optimal solutions hold unless the model parameters exactly satisfy a certain "knife-edge" condition. (There are different conditions for each property.) If the parameters are randomly distributed according to any continuous probability distribution over any open set, these conditions are satisfied with probability 0. Furthermore, even if the condition is satisfied, it is not always the case that an optimal solution will violate the property, and there is always an optimal solution that does not violate the property. Hence, although these properties could in principle be violated by an optimal solution, it can safely be assumed that they hold. See Caulkins (1988) for details.

With probability 1, in any optimal solution using flexible capacity:

(1) The firm will not carry inventory into both product/period pairs in which flexible capacity is used.
(2) If one or both types of dedicated capacity are used then inventory of both products will not be carried into the same period.
(3) If capacity is not fully utilized in some product/period pair then no inventory of either product will be carried into that period.
(4) If both types of dedicated capacity are used, inventory will be carried into at most one of the four product/period pairs.

4.3 Solution of the Q-KSDP

It is possible to completely solve the Q-KSDP, and this solution leads to some useful insights. The derivation of the solution, which is very long, is only briefly described here but can be found in its entirety in Caulkins (1988).

There are 15 decision variables: I_{ij}, Y_{ij}, and Z_{ij} for i = A,B and j = 1,2 and r_i for i = A, B, and AB. A strategy is defined as a specification of which variables are zero, which are positive but not at their upper bound, and which are at their upper bound. By definition, variables that are positive but not at their upper bound cannot be on the boundary of the region corresponding to a strategy, so setting the derivative of the objective function with respect to one of these free variables equal to zero always gives a necessary condition for optimality. Solving the system of equations comprised of the first order conditions of all the free variables yields closed form expressions for the decision variables' optimal values when that strategy is optimal.

In principle there are over four million strategies, but using the properties in Section 4.1, one can show that only 137 can ever be optimal. It is possible to derive necessary and sufficient conditions for each of the strategies in the category with $K_{AB} = 0$ to be optimal within that category. It is also possible to find necessary and sufficient conditions for a strategy in a category with $K_{AB} > 0$ to be optimal within its category for a fixed value of K_{AB}. Then using the expressions for the decision variables derived from the first order conditions, one can compute the optimal value within the category as a function of K_{AB}. Comparing the optimal solutions within each of the three categories for all possible values of K_{AB} yields the overall optimal solution.

The necessary and sufficient conditions for a strategy to be optimal within its category are of three types: conditions comparing the cost of dedicated capacity to the demand and holding cost parameters, conditions on the relative strength of demand in each period, and conditions on K_{AB}. The conditions all have straightforward economic interpretations, as do the expressions for the optimal values of the decision variables and objective function. No one of these economic interpretations by itself is particularly significant, but being able to give the interpretations is the key to obtaining the following sensitivity analysis results and developing intuition for the problem in general.

4.4 Sensitivity Analysis for the Q-KSDP

Since the solution gives closed form expressions for the decision variables for all strategies in terms of the problem parameters and K_{AB}, the sensitivity of any function Y of the decision variables with respect to changes in a parameter x can be computed by the chain rule:

$$\frac{dY^*}{dx} = \frac{\partial Y^*}{\partial x} + \frac{\partial Y^*}{\partial K^*_{AB}} \frac{\partial K^*_{AB}}{\partial x} . \qquad (4.1)$$

The partial derivatives $\frac{\partial Y^*}{\partial x}$ and $\frac{\partial Y^*}{\partial K^*_{AB}}$ can be computed directly. If the optimal solution Ξ is in the category of solutions with $K^*_{AB} = 0$, and there is no alternate solution with $K^*_{AB} > 0$, then $\frac{\partial K^*_{AB}}{\partial x} = 0$. For solutions with $K^*_{AB} > 0$

$$\frac{\partial K^*_{AB}}{\partial x} = \frac{-V^*_{K_{AB},x}}{V^*_{K_{AB},K_{AB}}} , \qquad (4.2)$$

where V_x denotes the partial derivative of V with respect to x, V_{xx} denotes the second partial derivative, etc.... This formula can be obtained by differentiating the first order condition

$$\frac{\partial V^*(K_{AB}(x), x)}{\partial K_{AB}} = 0.$$

$$\frac{\partial}{\partial x}\left[\frac{\partial V^*(K_{AB}(x), x)}{\partial K_{AB}}\right] = \frac{\partial^2 V^*(K_{AB}(x), x)}{\partial K_{AB}^2}\frac{\partial K_{AB}(x)}{\partial x} + \frac{\partial^2 V^*(K_{AB}(x), x)}{\partial K_{AB}\partial x} = 0.$$

Rearranging terms gives (4.2).

By direct computation, $V_{K_{AB}K_{AB}}$ is strictly negative for all strategies that can be optimal (with nonzero probability), so (4.2) does not entail division by zero. Furthermore, $\operatorname{sgn}(\frac{\partial K_{AB}^*}{\partial x}) = \operatorname{sgn}(\frac{\partial^2 V^*}{\partial K_{AB}\partial x})$.

With (4.1) and (4.2), obtaining the sensitivity of any decision variable to any of the reduced model's parameters is a straightforward calculation. The values of the derivatives were computed but are not reported here because they depend on the particular strategy used, and hence do not yield general insights. The signs of many of the derivatives, however, are the same for all combinations of strategies and have interesting economic interpretations, so they will be described. (Since most, if not all, of the derivatives can be zero for certain combinations of strategies, the term 'increases' (increasing, greater than, more, etc..) will be used to describe quantities that increase or remain the same. Similarly for the term 'decreases'.)

The most important of these results is that holding inventory and using flexible capacity are not always substitutes. The optimal amount of flexible capacity is increasing in the cost of carrying inventory into a product/period pair in which flexible capacity is used, and decreasing in the cost of carrying inventory out of those product/period pairs. Hence, using flexible capacity in a product/period pair is a substitute for carrying inventory into that product/period pair and a complement to carrying inventory out of that product/period pair. Since it is not known a priori to which product/period pairs flexible capacity will be allocated, general statements cannot be made about the complementarity or substitutability of inventory and flexible capacity.

The sensitivity analysis reveals how decision variables associated with one product are affected by changes in parameters associated with the other product. For any function Y of the decision variables associated with one product, $\frac{\partial Y}{\partial x} = 0$ for all parameters x associated with the other product, so the sensitivity is determined entirely by the second term of (4.1), $\frac{\partial Y^*}{\partial K_{AB}^*} \frac{\partial K_{AB}^*}{\partial x}$. Hence, if changing a parameter associated with product i does not influence the optimal value of K_{AB}, then it will not influence the optimal value of $K_{\bar{i}}$, the amount of capacity dedicated to the other product. Furthermore, since $-1 \leq \frac{\partial K_{\bar{i}}}{\partial K_{AB}} \leq 0$, if changing a parameter associated with product i changes the optimal value of K_{AB}, it will change the optimal value of $K_{\bar{i}}$ in the opposite direction and by an amount no greater in absolute value. Hence $\mathrm{sgn}\left(\frac{d(K_{\bar{i}} + K_{AB})}{d\_}\right) = \mathrm{sgn}\left(\frac{dK_{AB}}{d\_}\right)$. Thus changes in product i's parameters affect the optimal amount of capacity dedicated to the other product only through their effect on the optimal amount of flexible capacity.

The sensitivity analysis also shows how changes in demand and holding costs affect the optimal solution. If demand for a product increases in a product/period pair in which flexible capacity is used, then it is optimal to purchase more flexible capacity and thus less capacity dedicated to the other product. Production of the other product may increase or decrease, but overall production will increase. As one would expect, the firm will inventory less of the first product out of that period, but it will also inventory less of the other product out of that period. Increased demand in the other period generally has the opposite effect on inventories. It also leads to increased investment in capacity dedicated to the first product, but the direction of the effect on investment in flexible capacity and in capacity dedicated to the other product depends on which strategy is in use.

If it costs more to carry inventory out of a product/period pair in which flexible capacity is used, production of that product will decrease. Production of the other product may increase, and it is possible that its increase will more than offset the decrease in production of the first product, giving rise to an increase in overall production. Of course it is certainly possible that both production of the second product and overall production will decrease. Not surprisingly, the firm will inventory less out of that product/period pair. On the other hand, it will inventory more of the other product out of that period, but that increase will be smaller in magnitude than the decrease in the inventory of the first product. Inventory out of the other period is not affected for either product, so total inventory levels decrease.

If it costs more to carry inventory into a product/period pair in which flexible capacity is used, the firm will purchase more flexible capacity and less capacity dedicated to the other product. The direction of the effects on capacity dedicated to the first product, overall capacity, and production levels depend on the parameter values. Not surprisingly, if it costs more to carry inventory into a product/period pair in which flexible capacity is used, the firm will carry less inventory into such a pair. It will, however, increase its inventory of the other product between the same two periods.

If the price of one of the dedicated capacities increases the firm will purchase less of both kinds of dedicated capacity and more flexible capacity. Production of the product whose dedicated capacity became more expensive will decrease; production of the other product may increase or decrease.

Finally, the derivatives of the optimal objective function value are all consistent with one's intuition about the model. When demand increases, profits increase. When costs go up, profits go down.

5 Conclusions and Discussion

The advent of flexible manufacturing technology has made assessing the economic value of flexible manufacturing systems (FMS's) an active research topic. Holding inventory and investing in flexible capacity are both ways firms can respond to varying market conditions so it is useful to explore both in one model. This paper extends the Fine and Freund (1986, 1987) model for assessing the value of product-flexible FMS's to the case when the firm can hold inventory, reports on the complete solution for a special case, and gives some intuition building insights.

The FMS investment decision problem was first formulated as a T+1 period investment and production dynamic programming problem called the General Stochastic Demand Problem (GSDP) that explicitly allows the firm to hold inventory. Few analytic results can be obtained for the GSDP unless some structure is imposed on the uncertainty about market conditions.

Section 3 considers the GSDP when the variability in market conditions is periodic and known when production and inventory decisions are made but unknown at the time the firm makes its investment decision. With this structure it is possible to show a unique optimal solution exists; give explicit formulas for the optimal Karush-Kuhn-Tucker multipliers; state necessary and sufficient conditions for it to be optimal to purchase flexible capacity; show the optimal value is a well-behaved function of the capacity costs; and characterize the optimal solution as a function of the capacity costs for the special case of quadratic revenue and holding cost functions and linear production costs.

Section 4 reports on the special case when the variability in market conditions is periodic and known at the time the firm makes its investment decision. Many properties of optimal solutions are stated for general revenue and cost functions as are conclusions drawn from the complete solution for the case of quadratic revenue functions and linear production and holding costs.

Two overall conclusions from Section 4 are that flexible capacity can be useful even in a world of certainty and that determining the value of a FMS investment when inventories are allowed is difficult. The model in Section 4 only considers the benefits of product flexibility, and more importantly, it is deterministic. Nevertheless, the derivation of the solution is long, the solution itself is complex, and the example showed that unguided intuition can be misleading. Hence it seems unlikely that closed form expressions for the value of flexible capacity when inventories are allowed will be obtained for more realistic models. In practice, intuition and numerical solutions will probably be the rule. The ability to draw inferences from numerical solutions depends on results, such as those in Section 3, that guarantee the solution is a well-behaved function of the parameters. Likewise, it is hoped that the analysis of this special case and the examples in this paper will contribute to the intuition of modellers and practitioners. Some results from the examples and Section 4 are a consequence of the special structure of those problems, but the following can be expected to hold more generally.

1) Although holding inventory and investing in flexible capacity are both ways firms respond to market variability, the two are not always substitutes. They may be complements, substitutes, or neither. Hence it is not possible to make general statements about how changes in the cost of holding inventory affect the value of flexible capacity or how changes in the cost of flexible capacity affect optimal inventory levels.

2) It may be optimal to purchase flexible capacity even if demand is known with certainty. In fact, it can be optimal to produce a product exclusively with flexible capacity even if all demand is known with certainty and demand for that product does not vary over time.

3) Sometimes it is useful to decompose uncertainty about demand into two components: uncertainty between products and uncertainty between periods. Flexible capacity is most useful for the first kind of uncertainty; inventory is most useful for the second.

4) It is not always optimal to use dedicated capacity to meet 'base demand', demand the firm knows for certain will exist in every state and period. Thus heuristics that 'subtract out' base demand, to be met with dedicated capacity, leaving a smaller and hopefully simpler problem can lead to suboptimal solutions.

The ultimate objective of research in the economic evaluation of FMS's is to develop tools practitioners can use to assist technology investment decision making. Developing practical tools from the models in the literature will require a great deal of work, including testing them on real world problems. This will be a long and difficult process, so it is important that it begin now, but there is also more to be done at the level of academic research.

This paper was based on the Fine and Freund (1986, 1987) model for quantifying the value of product-flexible FMS's. There are a number of models in the literature that focus on benefits of FMS's other than product flexibility. It is likely that extending them to consider inventory effects would yield useful insights.

There are at least three reasons why firms hold inventory and/or invest in flexible capacity: varying market conditions, product life cycle considerations, and economies of scale. These factors can have quite different and even opposing effects on the relationship between inventories and flexible capacity. Hence comprehensive models that include all three are unlikely to yield simple, qualitative statements about the overall nature of the relationship. This paper focuses exclusively on the effects of varying market conditions, but the other two deserve study.

Further work could center on these related problems, but there is also certainly more to be learned about how holding inventory affects the value of product-flexible FMS's when market conditions vary. The analysis in this paper rests on many assumptions and simplifications. It would be useful to know what results presented here continue to hold when these assumptions and simplifications are relaxed.

Direct generalizations of the model presented here, however, may not be the most productive avenue for further research. The problem of determining how holding inventory affects the value of flexibility with no particular structure to the variability in market conditions is too general to analyze. The analysis in this thesis was possible because it considered periodically varying market conditions. Periodic conditions are complex enough to bring out the dynamics of the interaction between inventory and flexible capacity, but simple enough to facilitate analysis. It would be useful to explore other types of market variability that are complex enough to be interesting but simple enough to analyze and compare the results obtained with those presented here. The key to this may be the idea mentioned above, that overall variability can be decomposed into simpler constituents, such as variability between periods and variability between products.

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Appendix: Proof of Theorem 3.2

Proof: Let $\bar{\chi}$ denote the collection of multipliers defined in the theorem. By Theorem 3.1, Ξ is uniquely determined so the multipliers above are uniquely specified. By direct substitution they satisfy KKT conditions I - IV, and it is obvious that all variables except perhaps $\bar{m}_i$ for i = A, B, and AB are nonnegative. So, the proof will be complete if it is shown that these three multipliers are nonnegative and $\bar{\chi}$ satisfies V and VI. Let $\tilde{\chi}$ be any set of optimal KKT multipliers.

First it will be shown that $\bar{\chi}$ satisfies V. $\bar{\delta}^s_{i1} - \bar{q}^s_{i1} = \gamma p_s\left(R^{s'}_{i2} - h^{s'}_{i1} - R^{s'}_{i1}\right)$. So since the optimal solution is unique, by III, $\bar{\delta}^s_{i1} - \bar{q}^s_{i1} = \tilde{\delta}^s_{i1} - \tilde{q}^s_{i1}$. Likewise, $\bar{\delta}^s_{i2} - \bar{q}^s_{i2} = \tilde{\delta}^s_{i2} - \tilde{q}^s_{i2}$. Since $\bar{\delta}^s_{ij}\,\bar{q}^s_{ij} = 0$, $\tilde{\delta}^s_{ij} \geq \bar{\delta}^s_{ij}$ and $\tilde{q}^s_{ij} \geq \bar{q}^s_{ij}$ for all i,j, and s. Similarly, the formulas in the Theorem and KKT condition I imply $\bar{\alpha}^s_{ij} - \bar{\delta}^s_{ij} - \bar{s}^s_{ij} = \tilde{\alpha}^s_{ij} - \tilde{\delta}^s_{ij} - \tilde{s}^s_{ij}$. Transposing terms, $\tilde{\alpha}^s_{ij} = \bar{\alpha}^s_{ij} + \left(\tilde{\delta}^s_{ij} - \bar{\delta}^s_{ij}\right) - \bar{s}^s_{ij} + \tilde{s}^s_{ij}$. Since $\bar{\alpha}^s_{ij}\,\bar{s}^s_{ij} = 0$, $\tilde{\alpha}^s_{ij} \geq \bar{\alpha}^s_{ij}$. Similarly, $\tilde{\gamma}^s_j \geq \bar{\gamma}^s_j$. Because $\tilde{\chi}$ is an optimal set of KKT multipliers, it satisfies V. Hence, since $\tilde{\alpha} \geq \bar{\alpha}$, $\tilde{\gamma} \geq \bar{\gamma}$, and $\tilde{\mathbf{q}} \geq \bar{\mathbf{q}}$, $\bar{\chi}$ also satisfies V.

Next it will be shown $\bar{\chi}$ satisfies VI. Since $\tilde{\chi}$ satisfies VI, $\bar{s}^s_{ij} Y^s_{ij} = 0$ if $\bar{s}^s_{ij} \leq \tilde{s}^s_{ij}$. So $\bar{s}^s_{ij} Y^s_{ij} = 0$ if $\bar{s}^s_{ij} > \tilde{s}^s_{ij}$ implies $Y^s_{ij} = 0$. Suppose $\bar{s}^s_{ij} > \tilde{s}^s_{ij}$. Then since $\bar{\alpha}^s_{ij}\,\bar{s}^s_{ij} = 0$, $\bar{\alpha}^s_{ij} = 0$. Also, from the formulas in the Theorem and KKT condition I, $\bar{s}^s_{ij} = \tilde{s}^s_{ij} + \left(\tilde{\delta}^s_{ij} - \bar{\delta}^s_{ij}\right) - \left(\tilde{\alpha}^s_{ij} - \bar{\alpha}^s_{ij}\right)$. So if $\bar{s}^s_{ij} > \tilde{s}^s_{ij}$, it must be that $\tilde{\delta}^s_{ij} > \bar{\delta}^s_{ij}$. Since $\tilde{\chi}$ is an optimal set of multipliers, by V $\tilde{\delta}^s_{ij} > 0$ implies that $I^s_{ij} = Y^s_{ij} + Z^s_{ij}$. Furthermore, since $\bar{\delta}^s_{ij} - \bar{q}^s_{ij} = \tilde{\delta}^s_{ij} - \tilde{q}^s_{ij}$, $\tilde{\delta}^s_{ij} > \bar{\delta}^s_{ij}$ implies that $\tilde{q}^s_{ij} > \bar{q}^s_{ij} \geq 0$. By VI this implies $I^s_{ij} = 0$ and thus $Y^s_{ij} = 0$. Hence $\bar{s}^s_{ij} > \tilde{s}^s_{ij}$ implies $Y^s_{ij} = 0$. So since $\tilde{s}^s_{ij} Y^s_{ij} = 0$, $\bar{s}^s_{ij} Y^s_{ij} = 0$.

Next it will be shown that $\bar{u}_{ij}^{s} Z_{ij}^{s} = 0$. Suppose $\bar{u}_{ij}^{s} > \tilde{u}_{ij}^{s}$. Then since $\bar{u}_{ij}^{s} = \tilde{u}_{ij}^{s} + \left(\tilde{\delta}_{ij}^{s} - \bar{\delta}_{ij}^{s}\right) - \left(\tilde{\gamma}_{j}^{s} - \bar{\gamma}_{j}^{s}\right)$ and $\tilde{\gamma}_{j}^{s} \geq \bar{\gamma}_{j}^{s}$, $\tilde{\delta}_{ij}^{s} > \bar{\delta}_{ij}^{s}$. Since $\bar{\delta}_{ij}^{s} - \bar{q}_{ij}^{s} = \tilde{\delta}_{ij}^{s} - \tilde{q}_{ij}^{s}$ this implies that $\tilde{q}_{ij}^{s} > \bar{q}_{ij}^{s} \geq 0$. Since $\tilde{\chi}$ is an optimal set of multipliers, by VI, $\tilde{q}_{ij}^{s} > \bar{q}_{ij}^{s} \geq 0$ implies $I_{ij}^{s} = 0$. Likewise, $\tilde{\delta}_{ij}^{s} > 0$ implies that $I_{ij}^{s} = Y_{ij}^{s} + Z_{ij}^{s}$, so $Z_{ij}^{s} = 0$. Hence $\bar{u}_{ij}^{s} > \tilde{u}_{ij}^{s}$ implies $Z_{ij}^{s} = 0$, so $\bar{u}_{ij}^{s} Z_{ij}^{s} = 0$.

Next it will be shown that $\bar{m}_A K_A = 0$. If $\bar{m}_A K_A > 0$, then $\bar{m}_A > 0$ and $K_A > 0$. $K_A > 0$ implies $\tilde{m}_A = 0$. So by IV, $r_A = \sum_{s=1}^{S} \sum_{j=1}^{2} \tilde{\alpha}_{Aj}^{s}$, and thus

$$\bar{m}_A = \sum_{s=1}^{S} \sum_{j=1}^{2} \left(\tilde{\alpha}_{Aj}^{s} - \bar{\alpha}_{Aj}^{s}\right) > 0.$$

This implies there exists some state/period pair (s,j) such that $\tilde{\alpha}_{ij}^{s} > \bar{\alpha}_{ij}^{s}$. Then by V, $Y_{Aj}^{s} = K_A > 0$. Also $\tilde{\alpha}_{ij}^{s} > \bar{\alpha}_{ij}^{s}$ implies $\tilde{\delta}_{ij}^{s} > \bar{\delta}_{ij}^{s}$, which by V implies $I_{Aj}^{s} = Y_{Aj}^{s} + Z_{Aj}^{s} > 0$. But $\tilde{\alpha}_{ij}^{s} > \bar{\alpha}_{ij}^{s}$ also implies $\tilde{q}_{ij}^{s} > \bar{q}_{ij}^{s}$, which by VI implies $I_{Aj}^{s} = 0$. Contradiction. So $\bar{m}_A K_A = 0$. Similarly $\bar{m}_B K_B = 0$. Several more steps are required to show $\bar{m}_{AB} K_{AB} = 0$, but the reasoning is similar.

Also, since $\tilde{q}_{ij}^{s} \geq \bar{q}_{ij}^{s}$ and $\tilde{q}_{ij}^{s} I_{ij}^{s} = 0$, $\bar{q}_{ij}^{s} I_{ij}^{s} = 0$. So $\bar{\chi}$ satisfies VI.

Finally, it will be shown that $\bar{m}_A$, $\bar{m}_B$, and $\bar{m}_{AB}$ are nonnegative. Since $\tilde{\alpha}_{ij}^{s} \geq \bar{\alpha}_{ij}^{s}$, $\bar{m}_A = r_A - \sum_{s=1}^{S} \sum_{j=1}^{2} \bar{\alpha}_{Aj}^{s} \geq r_A - \sum_{s=1}^{S} \sum_{j=1}^{2} \tilde{\alpha}_{Aj}^{s} = \tilde{m}_A \geq 0$. Similarly, $\bar{m}_B \geq 0$ and $\bar{m}_{AB} \geq 0$. So $\bar{\chi}$ is a set of optimal KKT multipliers.

Furthermore, $\bar{\chi}$ is the unique set of optimal KKT multipliers if **K** > 0. Suppose $\bar{\chi}$ and $\tilde{\chi}$ are two distinct vectors of optimal KKT multipliers. **K** > 0 implies $\bar{m}_A = \bar{m}_B = \bar{m}_{AB} = 0 = \tilde{m}_A = \tilde{m}_B = \tilde{m}_{AB}$. Hence by IV and the formulas in the Theorem, $\sum_{s=1}^{S} \sum_{j=1}^{2} \bar{\alpha}_{Aj}^{s} = r_A = \sum_{s=1}^{S} \sum_{j=1}^{2} \tilde{\alpha}_{Aj}^{s}$. So since $\tilde{\alpha}_{ij}^{s} \geq \bar{\alpha}_{ij}^{s}$, $\tilde{\alpha}_{ij}^{s} = \bar{\alpha}_{ij}^{s}$ for all i,j, and s. Similarly, $\tilde{\gamma} = \bar{\gamma}$.

Then by I, $\bar{\delta}_{ij}^{s} + \bar{s}_{ij}^{s} = \tilde{\delta}_{ij}^{s} + \tilde{s}_{ij}^{s}$. Since $\tilde{\delta}_{ij}^{s} \geq \bar{\delta}_{ij}^{s}$, $\bar{\delta}_{ij}^{s} + \bar{s}_{ij}^{s} = \tilde{\delta}_{ij}^{s} + \tilde{s}_{ij}^{s}$ unless $\tilde{\delta}_{ij}^{s} > \bar{\delta}_{ij}^{s} \geq 0$ and $\bar{s}_{ij}^{s} > \tilde{s}_{ij}^{s} \geq 0$. But since $\tilde{\chi}$ satisfies VI, $\bar{s}_{ij}^{s} > \tilde{s}_{ij}^{s}$

implies $Y_{ij}^{s} = 0$. Similarly, since $\tilde{\gamma} = \bar{\gamma}$, $\bar{\delta}_{ij}^{s} + \bar{u}_{ij}^{s} = \tilde{\delta}_{ij}^{s} + \tilde{u}_{ij}^{s}$, so $\tilde{\delta}_{ij}^{s} > \bar{\delta}_{ij}^{s}$ implies $\bar{u}_{ij}^{s} > \tilde{u}_{ij}^{s} \geq 0$ which implies $Z_{ij}^{s} = 0$. $Y_{ij}^{s} = Z_{ij}^{s} = 0$ implies $I_{ij}^{s} = 0$. By Assumption A6, $Y_{ij}^{s} = 0 < K_i$ implies $I_{ij}^{s} = 0$. Then by I, $\gamma p_s R_{ij}^{s\prime}(0) - C_{ij}^{s\prime}(0) = \bar{\alpha}_{i1}^{s} - \bar{\delta}_{i1}^{s} - \bar{s}_{i1}^{s}$. But $\bar{s}_{ij}^{s} > 0$ implies $\bar{\alpha}_{ij}^{s} = 0$, so $R_{ij}^{s\prime}(0) - C_{ij}^{s\prime}(0) < 0$ which contradicts Assumption A8. Thus $\bar{\delta} = \tilde{\delta}$, $\bar{s} = \tilde{s}$, and $\bar{\mathbf{u}} = \tilde{\mathbf{u}}$. And since $\bar{\delta}_{ij}^{s} + \bar{s}_{ij}^{s} = \tilde{\delta}_{ij}^{s} + \tilde{s}_{ij}^{s}$, $\bar{\mathbf{q}} = \tilde{\mathbf{q}}$. Thus $\bar{\chi} = \tilde{\chi}$, contradicting the hypothesis that they are distinct. So if $\mathbf{K} > 0$, the optimal multipliers given by the formulas above are unique. ■

Date Due

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