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RESEARCH LABORATORY OF ELECTRONICS
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Abstract

The diffusion equation for electrons in a nonuniform field is solved and the breakdown condition derived. The breakdown condition is expressed in such a manner that an effective characteristic diffusion length Λ_e is determined; the meaning of Λ_e expresses the equivalent characteristic diffusion length for uniform electric fields. From the experimental breakdown fields, Λ_e is determined and used to predict theoretical breakdown curves which are compared with experiment.

ELECTRON DIFFUSION IN A SPHERICAL CAVITY

Electrical breakdown of a gas at microwave frequencies has been considered theoretically and experimentally for the case in which the electric field is uniform (1)(2)(3). It is sometimes important to consider cases in which the electric field is not uniform, so that the shape of the volume from which diffusion takes place may be varied. The case of a cylindrical cavity of arbitrary length has been solved by Herlin and Brown (4), and the procedure of this paper is similar to theirs. The diffusion equation for electrons in a nonuniform field is solved and the breakdown condition derived. The breakdown condition is expressed in such a manner that an effective characteristic diffusion length Λ_e is determined; the meaning of Λ_e expresses the equivalent characteristic diffusion length for uniform electric fields. From the experimental breakdown fields, Λ_e is determined and used in the theories developed in References 2 and 3 to predict theoretical breakdown curves which are compared with experiment.

I. Diffusion Theory

The differential equation to be solved for the characteristic diffusion length of electrons in a gas is

$$\nabla^2\psi + \zeta E^2\psi = 0 \qquad (1)$$

where $\psi = Dn$, the electron diffusion current density potential; D is the electron diffusion coefficient; n is the electron concentration; E is the rms value of the electric field; and ζ is defined by the equation

$$\zeta = \nu/DE^2$$

where ν is the ionization rate per electron.

The boundary condition on Eq. (1) is that the electron concentration and thus ψ go to zero within a mean free path of the walls of the metal cavity.

ζ may be expressed as a function of E/p and pλ where p is the pressure and λ is the free space wavelength of the applied electric field. Integration of Eq. (1) is simplified by the use of the approximation employed by Herlin and Brown (4)(5).

$$\zeta = \zeta_o\left(\frac{E}{E_o}\right)^{\beta-2} = \left(\frac{k}{E_o}\right)^2\left(\frac{E}{E_o}\right)^{\beta-2} \qquad (2)$$

where ζ_o is the value of the ionization coefficient at the maximum field point; k is introduced for mathematical convenience and has the units of reciprocal length. The quantity β-2 is obtained as the slope of the ζ vs. E/p plot on a logarithmic scale. The slope required is that of ζ for which pλ is kept constant. The approximation of Eq. (2) is very good where the ionization is high. It is inaccurate only where there is little ionization and since the regions in which there is little ionization contribute only slightly to the determination of breakdown fields, the procedure obtained leads to correct results.

The electric field in the lowest electric mode in a spherical cavity may be given by (6)

$$E_r = E_o \frac{a}{2.75r} \cos\theta \left[j_1(2.75\ r/a) \right] \tag{3}$$

$$E_\theta = E_o \frac{a}{2.75r} \sin\theta \frac{d}{dr} \left[(2.75\ r/a)\, j_1(2.75\ r/a) \right] \tag{4}$$

$$E_\phi = 0$$

where r, θ and φ are the spherical coordinates, a is the radius of the sphere and j_1 is the first-order spherical Bessel function.

It is seen that the electric field depends on both r and θ, the introduction of which makes Eq. (1) inseparable. Since, in breakdown, we are interested only in the energy transfer from the field to the electrons, we need take into account only the magnitude of the field at a given point. Near the center of the cavity, where the field (and therefore the ionization) is high, the magnitude of the electric field is approximately spherically symmetric. This fact is illustrated in Fig. 1 which shows the maximum variation of fields with θ, as a function of r. If we assume that the electric field may be expressed as the average of these values over the whole of the cavity, we may write

$$E = E_o \left[1 - (r/a)^2 \right] \quad . \tag{5}$$

Equation (5) is also plotted in Fig. 1 where it is seen to be a good approximation to the average electric field except near the boundaries where it does not matter.

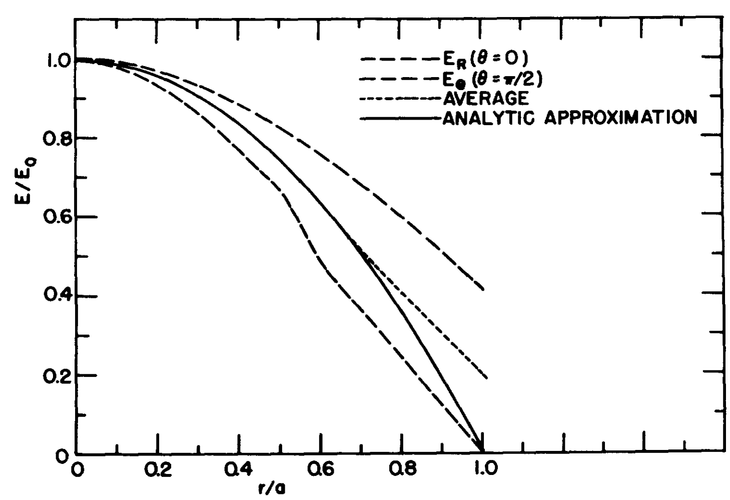

Fig. 1 Electric field in a spherical cavity showing the maximum variation and the analytic approximation to the average field as a function of radius.

The assumption that the electric field is independent of θ and φ leads one to the independence of ψ on these variables. Therefore, we may write Eq. (1) with the assumption of Eq. (2)

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) + k^2 \left(\frac{E}{E_o} \right)^\beta \psi = 0$$

and introducing the value of E from Eq. (5), we have

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + k^2\left[1 - (r/a)^2\right]^{\beta}\psi = 0 \quad . \tag{6}$$

We expand the term in r/a by the binomial theorem and drop powers of (r/a) greater than 2. This makes an appreciable error only near the boundaries, where again the accuracy of the method is unimportant. Then

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + k^2(1 - \mu^2 r^2)\psi = 0 \tag{7}$$

where $\mu^2 = \beta/a^2$. $1/\mu$ is the radius at which the ionization goes to zero under the above assumptions. Beyond $1/\mu$, these assumptions lead to a negative ζ which is not physically correct so we set $\zeta = 0$ for $r > 1/\mu$.

For mathematical convenience we transform to a dimensionless independent variable and let $k\mu r^2 = x$; Eq. (7) becomes

$$\frac{d^2\psi}{dx^2} + \frac{3}{2x}\frac{d\psi}{dx} + \frac{k^2}{x}\left(\frac{1}{4k\mu} - \frac{x}{4k^2}\right)\psi = 0 \qquad x < \frac{k}{\mu} \quad . \tag{8}$$

We transform the dependent variable by letting $\psi = e^{-x/2}g$ and then

$$\frac{d^2g}{dx^2} + \frac{dg}{dx}\left(\frac{3}{2x} - 1\right) - \frac{\alpha}{x}g = 0 \tag{9}$$

where $\alpha = 3/4 - k/4\mu$.

Equation (9) is the equation for the confluent hypergeometric equation in parameters 3/2 and α (7). The second solution is not allowed by the boundary condition and therefore

$$\psi_1 = e^{-x/2}\,M(\alpha\,;\tfrac{3}{2};\,x) \tag{10}$$

where we designate by ψ_1 that part of ψ for which r is less than $1/\mu$ or $x < k/\mu$. When $x > k/\mu$, ζ is zero and the differential equation (1) becomes

$$\frac{d}{dx}\left(x^{3/2}\frac{d\psi}{dx}\right) = 0$$

whose solution is

$$\psi_2 = C\left[1 - \left(\frac{x_o}{x}\right)^{1/2}\right] \tag{11}$$

where $x_o = k\mu a^2$ and is determined by the condition that ψ_2 be zero on the boundary; C is an arbitrary constant. We must match the solutions of Eqs. (10) and (11) at the point where $r = 1/\mu$ and therefore

$$\frac{\psi_1'}{\psi_1} = \frac{\psi_2'}{\psi_2}$$

which gives us

$$\frac{2/3\,\alpha M(\alpha + 1;\ 5/2;\ x_1)}{M(\alpha;\ 3/2;\ x_1)} - \frac{1}{2} = \frac{1}{2x_1\left[\left(\frac{x_1}{x_o}\right)^{1/2} - 1\right]} \tag{12}$$

where $x_1 = k/\mu = ka/\beta^{1/2}$.

Equation (12) relates a, the radius of the cavity, β determined from the slope of the ζ curve and k which is inversely proportional to the characteristic diffusion length, and may be written

$$\left[\frac{2/3\,\alpha M(\alpha + 1;\ 5/2;\ y)}{M(\alpha;\ 3/2;\ y)} - \frac{1}{2}\right] 2y = \frac{x}{1 - x} \tag{13}$$

where $\alpha = 1/4(3 - y)$, $y = ka/\beta^{1/2}$ and $x = \beta^{1/2}$. Equation (13) is an equation in which the left-hand side is a function of $ka/\beta^{1/2}$ only and the right-hand side is a function of $\beta^{1/2}$ only. Therefore, it is a simple matter to find ka as a function of β. For the case

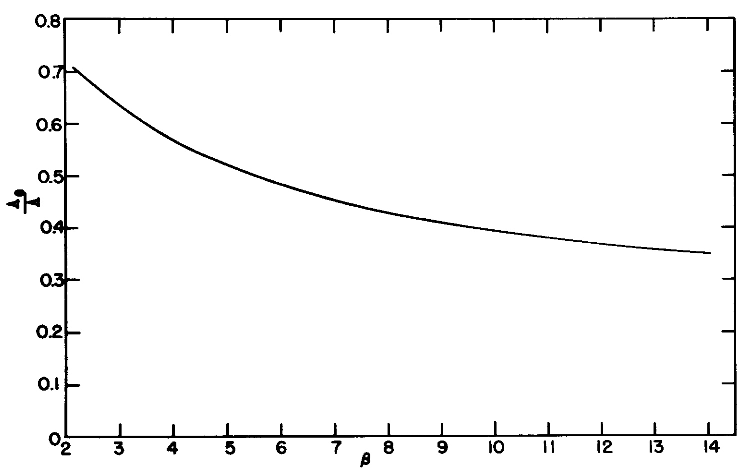

Fig. 2 Ratio of effective diffusion length to characteristic diffusion length.

of a uniform field, the characteristic diffusion length in a sphere is a/π and inspection of Eq. (1) indicates that for a uniform field, $k = \pi/a$. k now may be considered as a measure of the effective radius of the discharge for diffusion and ka/π then is Λ/Λ_e where Λ is the characteristic diffusion length as determined by the geometry of the container and Λ_e is the effective characteristic diffusion length. Equation (13) is solved numerically and plotted in Fig. 2.

II. Experiment

Breakdown fields have been measured for hydrogen in a spherical cavity operating in the lowest electric mode. The details of experimental method are similar to those previously reported (2). The microwave apparatus is shown in Fig. 3. Microwave power with a free space wavelength of approximately 10 cm generated by a c-w magnetron is coupled to a microwave resonant cavity through coaxial transmission lines. A known fraction of the power delivered is measured by a bolometer. The power absorbed by the cavity is combined with the cavity Q and the known field configuration to determine the electric field by standard methods (8)(9). The cavities in which breakdown takes place

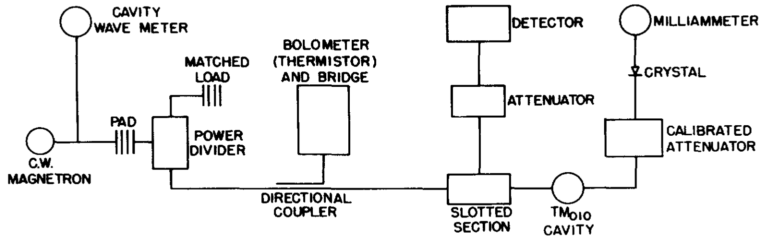

Fig. 3 Block diagram of experimental microwave apparatus.

are made of oxygen free high conductivity copper and connected through Kovar to an all-glass vacuum system. The vacuum system would hold at a pressure of better than 10^{-7} mm of Hg for a period of about two hours with the pumps turned off. A single series of breakdown measurements takes about this time.

III. Calculation of Effective Diffusion Length

Figure 4 is a ζ vs. E/p plot for hydrogen. Constant $p\lambda$ lines are plotted, the data being combined from breakdown measurements of cavities with different Λ's. This plot illustrates the use of the theory developed in this paper for computing an effective Λ. Both the corrected and uncorrected data are presented.

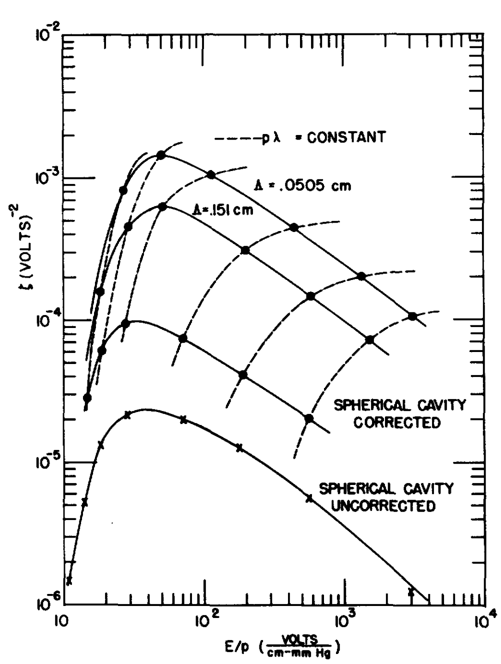

Fig. 4 Plot illustrating the actual calculation of effective Λ's. Both the corrected and uncorrected ζ curves for the spherical cavities are shown.

The procedure used in finding Λ_e begins with a plot of ζ vs. E/p, using the geometrical value which is a first approximation. If the geometrical characteristic diffusion length is equal to the actual diffusion length, we shall find Λ_e/Λ from Fig. 2 to be equal to 1. This is not generally true, so we take the value of β for the slope of a constant $p\lambda$ curve for the pressure and wavelength in which we are interested, find Λ_e/Λ and recalculate ζ to obtain a second approximation. The process is convergent and is continued until successive calculations agree.

The sphere used in the experiment had a radius of 4.69 cm and a geometrical Λ of 1.49 cm. The effective Λ determined from the theory varies from 0.43 cm for a low E/p to 0.61 cm for a high E/p. (The effective radius of the discharge depends on the ionization coefficient.)

In Fig. 5 are plotted the theoretical values of E as a function of p, using the calculated values of Λ_e. These are calculated from the theory developed in two previous papers (2)(3). Experimental breakdown fields are compared with theory on the same figure. The good agreement illustrates the validity of the method.

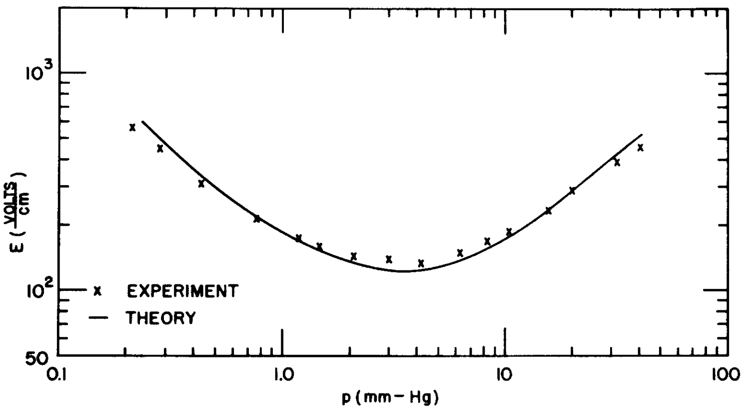

Fig. 5 Comparison of experimental electric field and values predicted for theory.

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