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Abstract

A study is made of the limits within which diffusion phenomena control the breakdown of a high-frequency discharge. The discussion is based on proper variables for dimensional analysis, using the parameters $p\Lambda$, $p\lambda$ and $E\Lambda$, where p is the pressure, Λ the characteristic diffusion length, λ the wavelength of the excitation, and E the breakdown electric field. The limits of applicability of the diffusion theory are found to be a uniform field limit, a mean free path limit, and an oscillation amplitude limit. Within these limits, a single function for the effective breakdown voltage, $E\Lambda$, and the energy per mean free path, E/p, correlated the breakdown voltages for all published data tested, covering a wavelength range from 10 cm to 17,000 cm.

LIMITS FOR THE DIFFUSION THEORY OF HIGH-FREQUENCY GAS DISCHARGE BREAKDOWN

The breakdown mechanism for a high-frequency a-c gas discharge is much simpler than that for the corresponding d-c breakdown. It is for this reason that relatively simple exact theories may be written for breakdown at the high frequencies. It is important to determine over how wide a range in frequency and pressure these discharge theories are applicable, and such a study for the case of hydrogen is the subject of this paper.

I. Dimensional Analysis

If a gas contained in a vessel is placed in an alternating electric field, for a certain value of the electric field, the gas will break down into an electrical discharge. This breakdown field can be expressed as

$$E_b = E(u_i, \Lambda, \lambda, p)$$

where E is the electric field intensity, u_i is the ionization potential, Λ is a parameter describing the vessel, λ is the wavelength of the exciting field, and p is the pressure. The field E is measured in volts per cm, and u_i is measured in volts. The term Λ has the units of length, and its appearance in explicit calculations also involves various known dimensionless ratios to describe the shape of the vessel. It is customary to measure pressure in millimeters of mercury, and the mean free path, which is inversely proportional to pressure, is measured in centimeters. A relation between pressure and mean free path is obtained by introducing a quantity, P_c, which is the number of collisions per mean free path at 1 mm Hg pressure. Thus, P_c, which is called the probability of collision, may be regarded as having the units of reciprocal length, even though this is not its true dimension.

Treating the breakdown problem dimensionally, there are five variables with but two fundamental dimensions, volts and centimeters. This leads to three independent dimensionless variables between which there is a functional relation (1). It is often convenient in physical problems to introduce variables which are not dimensionless but are nevertheless proper variables for dimensional analysis because the completely dimensionless variables contain one or more physically invariant quantities which need not be carried along in a practical discussion. There are a number of sets of such proper variables in a gas discharge problem which may be transformed into one another and their relative convenience depends on the purpose for which they are to be used.

One very useful set of proper variables is

$$E\Lambda, \ p\Lambda, \ p\lambda \quad . \qquad (1)$$

The advantage of these variables lies in the fact that p, Λ, and λ are the experimentally independent parameters which determine the dependent variable E, the observed breakdown field. These are the same variables which give the Paschen law in d-c breakdown, since in d-c, $p\lambda$ has no meaning; $E\Lambda$ is analogous to voltage, and $p\Lambda$ to the parallel plate pd.

Another set of proper variables which we shall use is obtained by dividing the first variable in (1) by the second and so obtaining

$$E\Lambda, \ E/p, \ p\lambda \quad . \tag{2}$$

This set has the particular advantage, in a discussion of breakdown phenomena, that we may define (2) an ionization coefficient $\zeta = 1/E^2\Lambda^2$ which is a function of E/p and $p\lambda$. In many ways, ζ is equivalent to the Townsend first coefficient η (volts^{-1}) which is a function of E/p alone.

II. Diffusion Theory

The simplest breakdown condition to calculate is for the high-frequency case in which the ionization rate is balanced by the loss of electrons by diffusion. The simplicity lies in the absence of complicating secondary phenomena. This breakdown criterion may be derived from the solution of the equation

$$\nabla^2\psi + \frac{\nu}{D}\psi = 0 \tag{3}$$

where $\psi = Dn$. D is the diffusion coefficient for electrons and n is the electron density, and the product must be equal to zero on the boundary. The quantity ν is the net production rate per electron. For the case of infinite parallel plates with a uniform field, the solution to Eq. (3) is

$$\psi = A \sin \frac{z}{\Lambda} \tag{4}$$

where z is the distance from one plate to an arbitrary point in the cavity, and A is a constant. The length parameter Λ used in the above dimensional analysis is called the characteristic diffusion length, and can be calculated exactly for a few common shapes of discharge tube. For example, with a cylinder of radius R and length L,

$$\frac{1}{\Lambda} = \sqrt{\left(\frac{\pi}{L}\right)^2 + \left(\frac{2.405}{R}\right)^2} \quad . \tag{5}$$

When the sideways diffusion to the walls is negligible, the parallel plate case results, namely $\Lambda = L/\pi$, while only the last term of Eq. (5) is important for very long cylindrical tubes.

Certain basic assumptions are made in the calculations of breakdown as a balance between the ionization rate and the loss of electrons by diffusion. We examine here the limits which the assumptions place on the application of the theory to various experimental conditions. These can be discussed in terms of the proper independent variables of Eq. (1), $p\Lambda$ and $p\lambda$. One can plot on the $p\Lambda$-$p\lambda$ plane the conditions for all breakdown data for a single gas and derive limits in these variables which will define the applicability of the diffusion theory.

III. Uniform Field Limit

The solution of Eq. (3) in the form given in Eq. (4) assumes a uniform field between infinite parallel plates. At low frequencies, the experimental measurements of breakdown are always taken in vessels whose dimensions are small compared to a wavelength of the exciting power. For this case, the uniform field assumption may be very good. At very high frequencies, there exists a limit to the size of the discharge tube consistent with the uniform field assumption of the diffusion theory. This can be written in terms of the size of vessel allowable to sustain a single loop of a standing wave of the electric field. The relation between the parallel plate separation, the wavelength, and the diffusion length given in Eq. (5) may be written as

$$\frac{\lambda}{2} = \pi\Lambda \quad . \tag{6}$$

For parallel plates where the separation is small, so that sideways diffusion can be neglected, $\lambda/2 = L$. In general, however, we must use both terms in Eq. (5). Thus one arrives at a limit which may be written in terms of $p\lambda$ and $p\Lambda$ as

$$p\lambda = 2\pi(p\Lambda) \quad . \tag{7}$$

This limiting line is plotted in Fig. 1 and designated as the uniform field limit.

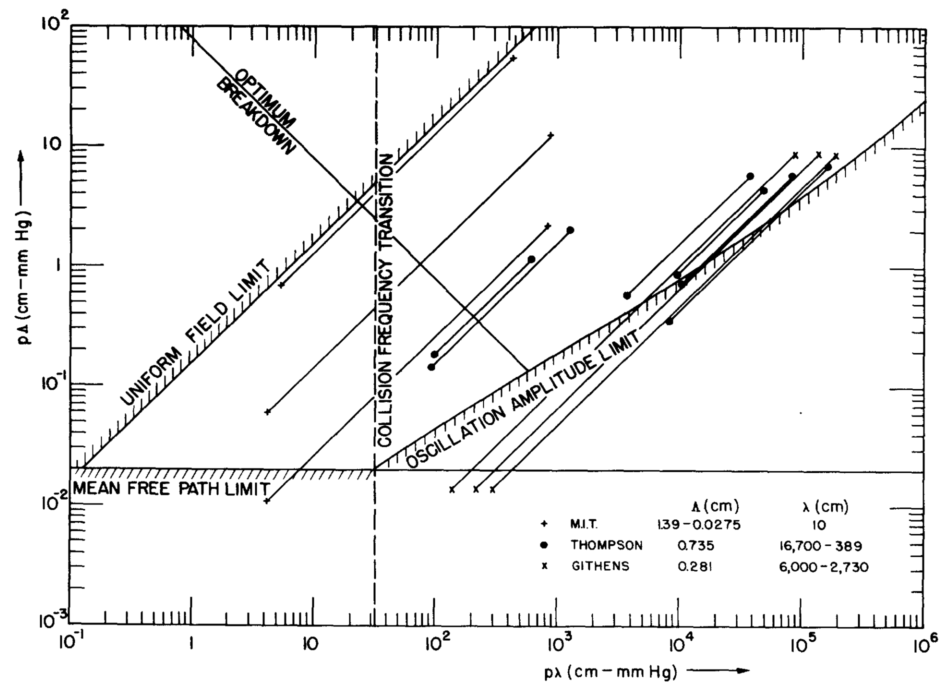

Fig. 1

A plot of the limits of the diffusion theory for breakdown at high frequencies in terms of variables derived from dimensional analysis. The extent of experimental data is represented by 45° lines.

IV. Mean Free Path Limit

The diffusion theory will not apply where the electron mean free path becomes comparable to the tube size. In the limiting case, this can be expressed as the mean free path, ℓ, being equal to Λ. The probability of collision, P_c, is equal to $1/p\ell$. To plot this condition in Fig. 1, we may write

$$p\Lambda = 1/P_c \quad . \tag{8}$$

The value of P_c is not a constant, but depends upon the electron energy. Assuming that the average electron has an energy equal to one-third of the ionization potential, the average electron energy would be 5.1 volts for hydrogen. Using Brode's (3) measured value for the probability of collision in hydrogen for the average electron, $P_c = 49$ $(\text{cm} - \text{mm Hg})^{-1}$. With this value, we obtain the horizontal line in Fig. 1 marked mean free path limit.

V. Collision Frequency Transition

Within the limits of experimental conditions in which diffusion theory adequately explains the breakdown behavior, several different phenomena may occur. One of the phenomenological changes which is important is the transition from many collisions per oscillation of the electron to many oscillations per collision. This can be written as the condition that $\nu_c = \omega$ where ν_c, the collision frequency, is the ratio of the average velocity v to the mean free path, and ω is the radian frequency of the applied field. From Brode's data, we can obtain a relation, $\nu_c = 5.93 \times 10^9$ p. Putting this in terms of the proper variables, we obtain

$$p\lambda = 32 \quad . \tag{9}$$

This relation is plotted in Fig. 1 as the dotted line marked as the collision frequency transition.

On the low pressure side of this transition, the oscillatory velocity of the electron lags the applied field by nearly 90°, and little energy is transferred from the applied field to the electron. As the pressure increases, the increasing frequency of interruption of electron oscillation decreases the lag, and the resulting in-phase component of the velocity represents an increasing absorption of energy. This may be taken into account by introducing an effective field (4) for energy transfer, defined by the relation

$$E_e^2 = E^2 \frac{\nu_c^2}{\nu_c^2 + \omega^2} \quad , \tag{10}$$

where E is the rms value of the applied alternating field and ω is the radian frequency. E_e is therefore the effective field which would produce the same energy transfer as a steady field.

VI. Optimum Breakdown

The most striking characteristic of the effects of changing pressure on the breakdown field intensity is the fact that at high pressure the field decreases with decreasing pressure, and at low pressure the field increases with decreasing pressure. In discussing this effect let us start with high pressure. Here the power which goes into the electron from the electric field is dissipated in elastic collisions between the electrons and the gas molecules. The region corresponds to the lowest values of E/p measured experimentally. The data on either the Townsend first ionization coefficient or the a-c ionization coefficient (5) show that here E/p is a constant equal in hydrogen to 10 rms volts/cm/mm Hg. Thus for a discharge in which nearly all the loss goes into non-ionizing collisions, the field and pressure are related by the equation

$$E = 10\,p \quad . \tag{11}$$

In the low pressure region, the electrons make many oscillations per collision and the breakdown field may be determined by equating the number of collisions to ionize to the number of collisions to diffuse out of the tube. The change in energy of an electron on collision is $\Delta u = (m/2e)(\overline{\Delta v^2})$ where $\overline{\Delta v^2}$ is the average value of the square of the change in velocity on collision. Since

$$\overline{\Delta v^2} = \left(\frac{E_e e}{m}\right)^2 \overline{t_c^2}$$

where E_e is the effective value of the field, and the mean squared collision time $\overline{t_c^2}$ is given by $2\ell^2/v^2$ from kinetic theory, we may write for the change in energy on collision

$$u = \frac{E_e^2 e}{m} \frac{\ell^2}{v^2} = \frac{E_e^2 e}{m} \frac{1}{\nu_c^2} \quad .$$

An electron can reach ionization energy in n collisions when $n\Delta u = u_i$. The number of collisions to diffuse out of a tube of size Λ is calculated from kinetic theory (6) to be $(3/2)(\Lambda^2/\ell^2)$ and hence

$$\frac{u_i \nu_c^2}{E_e^2/m} = \frac{3}{2} \frac{\Lambda^2}{\ell^2} \quad .$$

Substituting in this equation the value of the effective field from Eq. (10) and rearranging terms, the resulting expression for E becomes

$$E = \frac{2\pi c}{\Lambda \lambda P_c p} \sqrt{\frac{2}{3} \frac{u_i}{e/m}} \quad . \tag{12}$$

For hydrogen, this equation reduces to

$$E = \frac{785}{p\Lambda\lambda} \quad . \tag{13}$$

Equation (11) gives the relation between field and pressure at high pressure and Eq. (13) the relation at low pressure. Eliminating the field between these two equations will allow us to calculate the pressure at which breakdown will occur most easily. In terms of the variables of Fig. 1, this leads to the equation

$$p\lambda = \frac{78.5}{p\Lambda} \quad . \tag{14}$$

This relation is plotted in Fig. 1 as the line marked as the optimum breakdown.

VII. Oscillation Amplitude Limit

When the amplitude of the electron oscillation in the electric field is sufficiently high, the electrons can travel completely across the tube and collide with the walls on every half cycle. The presence of the gas complicates the problem somewhat, but Gill and Donaldson (7) showed that this phenomenon accounts for abrupt changes in the breakdown behavior.

The electric field may be written in the form $E = E_p \sin \omega t$ where E_p is the peak value. Under the action of the field, an electron is accelerated an amount Ee/m for the time between collisions t_c to attain a velocity given by

$$v = at_c = \frac{eE}{m} \frac{1}{\nu_c} \tag{15}$$

where ν_c is the collision frequency. Putting in the sinusoidal variation of the field with time

$$v = \frac{eE_p}{m} \frac{1}{\nu_c} \sin \omega t$$

$$x = \frac{eE_p}{m\omega \nu_c} \cos \omega t \quad .$$

The limiting case on the diffusion mechanism in which all of the electrons will hit the walls would be calculated by setting the oscillation amplitude equal to one half of the electrode separation. In order to take account of the greater density of electrons at the center due to the sinusoidal spatial distribution, the distance the electrons must travel is multiplied by the average value of the sine function. Thus, the oscillation amplitude becomes equal to

$$\frac{eE_p}{m\omega \nu_c} = \frac{L}{\pi} \quad . \tag{16}$$

Substituting λ in terms of ω, v/ℓ in place of ν_c, and $1/pP_c$ for ℓ, we obtain

$$p\lambda = \left(\frac{2mc}{e} vP_c\right) \frac{pL}{(E_p/p)} \quad . \tag{17}$$

Examination of Brode's data for the probability of elastic collision of an electron with hydrogen shows that over most of the energy range, the hyperbolic form of the curve gives $vP_c = 5.96 \times 10^9$ (sec – mm Hg)$^{-1}$. Putting this numerical value in Eq. (17), and using the parallel plate relation that $L = \pi\Lambda$, Eq. (17) reduces to

$$p\lambda = 2\pi \times 10^5 \left(\frac{p\Lambda}{E_p/p}\right) \quad . \tag{18}$$

This equation can be solved numerically where experimental values of the breakdown field are available. Experiments of this sort have been carried out for parallel plate geometry using hydrogen gas, and numerical values for Eq. (18) can therefore be determined. This calculation yields the oscillation amplitude limit of Fig. 1.

VIII. Correlation of Data of Various Workers

Many experimenters have studied the breakdown of a gas discharge at various frequencies and in various different geometrical arrangements. Most workers have obtained breakdown data for hydrogen and several have included in their reports sufficient detail to enable the parameters p, λ, and Λ to be determined. Since the data were determined by measuring breakdown voltage in a given size container at a given applied frequency at various pressures, a single run plots as a line at 45° in Fig. 1. In this figure, the range of data in $p\lambda$ and $p\Lambda$ taken by several observers is indicated by such lines. Many of the breakdown curves which have been obtained have complicated shapes and it is difficult to visualize a correlation in a two-dimensional plot between different workers. On the other hand, if we make a three-dimensional plot in which the horizontal axes are the $p\lambda$, $p\Lambda$ parameters of Fig. 1 and the vertical axis the effective breakdown voltage $E\Lambda$, all available data may be represented on a single surface. This is done in Fig. 2. Breakdown voltage data between parallel plates for hydrogen for the experimental conditions of $p\lambda$ and $p\Lambda$ (shown in Fig. 1) were used to construct this surface. The data covers a wavelength range from 17,000 cm to 10 cm and a range in Λ from 1.4 cm to 0.0275 cm and the surface extends only as far as experimental data exist from which to construct it. One can see that in the diffusion theory regions, the curves are smooth and well behaved, as one would expect if a single theory were applicable. In crossing over the oscillation amplitude limit, the breakdown curves appear to be much more complicated due to the effects of secondary electrons produced by the collision of the electrons with the walls or electrodes of the discharge vessel.

Sufficient data are available to check the generality of the breakdown theory at high pressure where the electrons make many collisions per oscillation. In the region where the diffusion theory is valid, one should observe the same value of the effective breakdown voltage $E\Lambda$ at a given value of the energy per mean free path E/p, independent of the frequency at which the measurements are made. Figure 3 shows this to be the case since a single curve is determined, within the limits of experimental accuracy, for breakdown in hydrogen using data taken by various observers over a wide range of wavelength and discharge tube size.

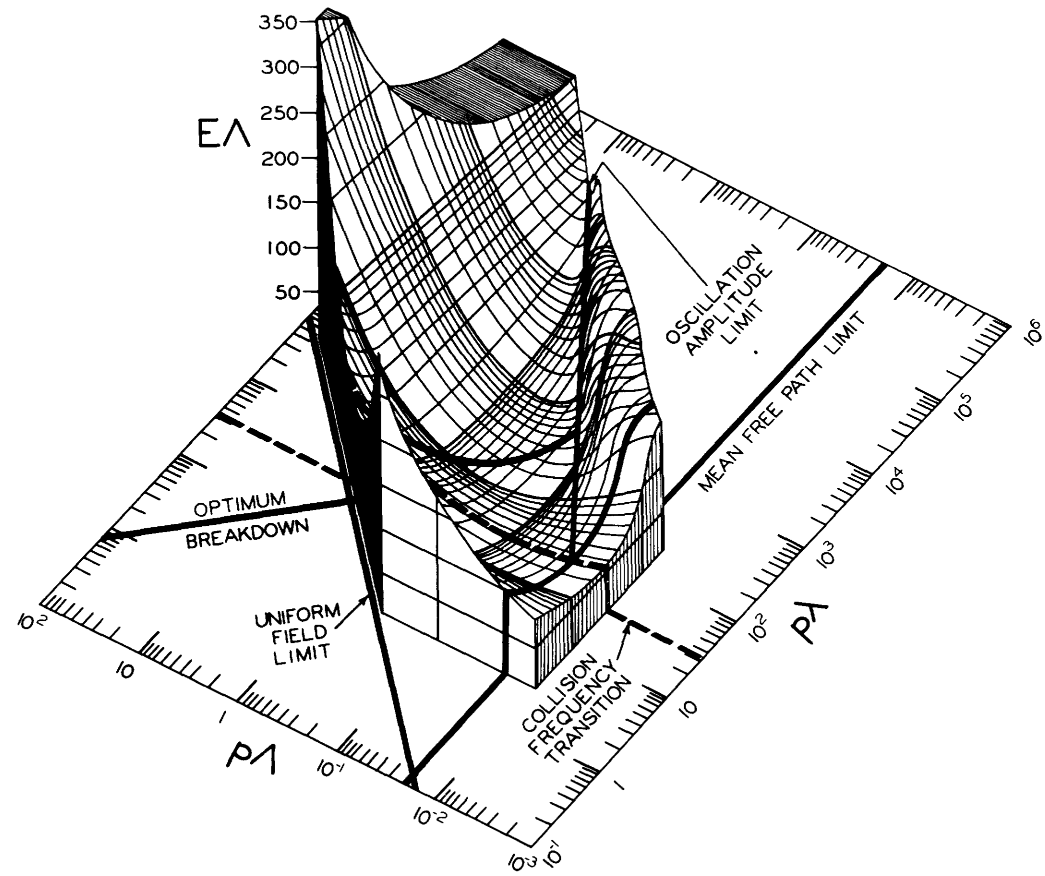

Fig. 2 A surface constructed from the experimental determinations indicated in Fig. 1, correlating the breakdown voltage measurements in terms of variables derived from dimensional analysis.

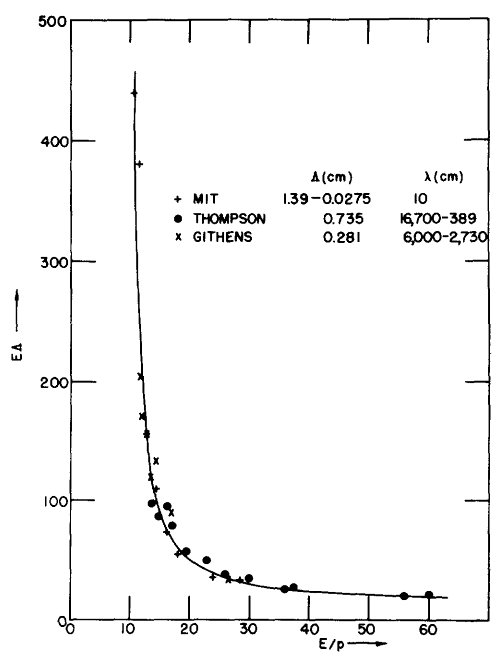

Fig. 3 Effective breakdown voltages as a function of E/p calculated from breakdown measurements of numerous workers (5)(8)(10).

IX. Discussion

The general agreement of the shape of the experimental curves with the limits shown in Fig. 1 is illustrated by the position of these limits on the three-dimensional projection of Fig. 2. To illustrate this agreement in a more quantitative fashion, sets of curves by different workers are shown in Figs. 4 and 5. Figure 4 shows data taken with a given electrode spacing, hence fixed Λ, for different wavelengths (8). The calculated positions of the oscillation amplitude limit for the given frequencies are shown by the appropriately marked arrows. The portions of the curves to the right of these arrows lie in the region where electron diffusion controls, and it is from these parts of the curves that the data in Fig. 3 were taken. The complicated shape of the curves to the left of the oscillation amplitude limit involves the production of secondary electrons at the electrodes or walls of the container. Such curves have been reported by numerous observers (8)(9)(10). Well-defined changes in the data can be recognized where the data cross the mean free path limit.

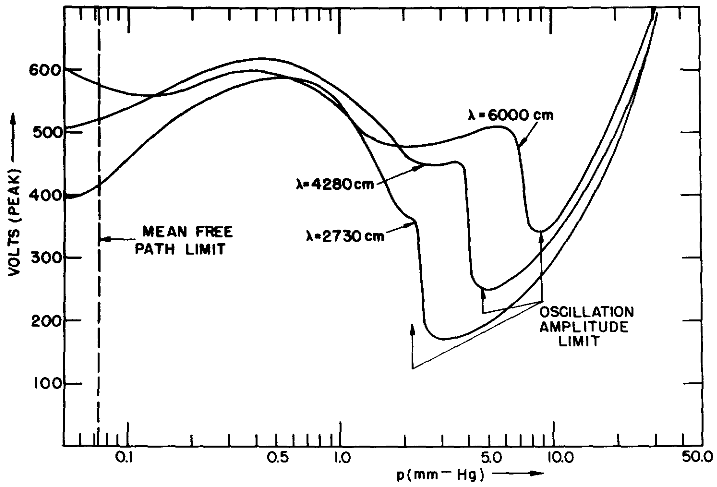

Fig. 4

Breakdown measurements in hydrogen by Githens (8) showing calculated and observed transition points.

Experiments carried out at microwave frequencies (5) allow one to explore the transition through the optimum breakdown condition. Figure 5 shows a series of runs for breakdown field as a function of pressure taken at a constant frequency for different values of Λ. When Λ is high, corresponding to large plate separation, the curves cross only one controlling transition, that from many collisions per oscillation to many oscillations per collision where the shape of the curve is given by expression (1). When the value of Λ is very small, on decreasing pressure, the data first cross the optimum breakdown condition and then the mean

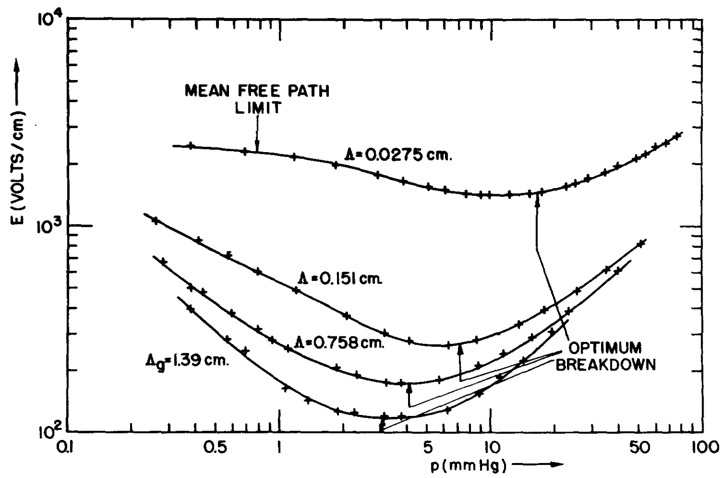

Fig. 5

Breakdown measurements in hydrogen (5) at 10-cm wavelength showing calculated and observed transition points.

free path limit. The values of Λ shown in Fig. 5 are the geometrical values calculated from Eq. (5), since it is these values which are used in computing $p\Lambda$-$p\lambda$ values for inclusion in Fig. 1. In calculating the $E\Lambda$ values in Figs. 2 and 3, a correction must be applied for those cases where the height is so great that the cavity cannot be considered a parallel plate system. The two largest cavities of Fig. 5 required this correction which was carried out in accordance with the non-uniform field theory given in a recent paper by Herlin and Brown (11).

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