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TIME-SHARING CHARACTERISTICS AND USER UTILITY *

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Introduction

As more and more economic activity shifts to service industries, meaningful evaluation of their "product" is becoming a topic of major concern. The service organization is constantly being evaluated by its customers and it is important for management to understand the bases for this evaluation. It is only in this way that management can prop e rly evaluate this customer feedback in order to improve the service offered.

This paper will outline a general procedure for formal evaluations utilitzing subjective data supplied by the facilities' customers. In particular, a systems analysis employing a utility theoretic approach is proposed for incorporating subjective data into the management decision making process. As an example of this methodology, the provision of computer time-sharing service is analyzed and sample customer evaluations are taken. The emphasis in this presentation is upon the use of such a procedure to improve future management decision making. It should be noted, however, that more economical procedures must be found for the assessment of personal preference data before the proposed methodology will find general practical application.

The Decision Theory Model

In this section we will describe the steps in a systems analysis approach and particularize it to the case of service facility evaluation. We will not attempt to formulate a new model of the management decision-making process but will instead modify one often professed by decision theorists:

1. Abstract and stucture the problem. Determine the courses of action available, the relevant attributes necessary to describe them, and the possible consequences which may result.

2. Evaluate the joint judgmental probability destribution for the consequences of each course of action.

3. Determine the best course of action from information obtained in the preceding steps. (9)

We can expand this process to include steps which are particularly pertinent to a manager dealing with provision of a service (such as a manager of a computing facility). In particular, we are interested in the preferences of the users of the service and of the manager's relative "preference" for each of them:

3. a. Determine the various types of users of the system affected by the action.

   b. Assess each user's relative preference for the various possible consequences.

   c. Determine the relative importance of each user.

   d. Combine the results of 3b and 3c into management's view of the community's relative preferences for the various possible consequences.

If we particularize our problem to those of the computer center manager charged with the provision of a time-sharing service, we can now describe our model for decision making in this area:

1. a. Determine the various courses of action available to improve service. For instance, hiring programmers, purchasing a disk drive, producing a system manual, refurbishing the computer room, etc. (This determination should take account of the various constraints on resources such as budget or the job market.)

   b. Determine the relevant attributes necessary to describe the consequences of these actions. For instance, system availability, reliability, response time, cost, or usability. (These should be broken down to a level where adequate measures of each characteristic can be determined. For example, system availability might be measured by the probability of a successful login, but this in itself might have component measures relating to system reliability and system capacity.)

   c. Determine possible consequences for each course of action. For instance, increased reliability by duplicating equipment, decreased response time by hiring a programmer (who improves the software), improved usability by writing a new manual.

2. Evaluate the manager's joint judgmental probability distribution for the magnitude of the consequences of each course of action. How likely is it that the additional programmer will improve the software and by how much?

3. a. Determine the various categories of users affected by a particular decision. For example, problem-oriented users, students, sub-system designers, administrative production users, etc.

   b. Access each user group's preference ordering for the various consequences of each action. (We assume that user groups can be formed, each with a characteristic utility function.)

c. Determine the importance of each user group (as seen by the manager) in the particular case. For example, a user group dependent upon a particular feature may be said to have a more important viewpoint in regard to decisions affecting that feature. Another example would be the relatively high importance of student opinion in decisions affecting computer-aided instruction or other educational use.

d. Combine the results of 3b and 3c to arrive at a management preference ordering of consequences of each of the actions.

4. Determine the best course of action from the information obtained in the preceding steps.

These steps form the basis for the formal evaluation procedure. In practice, management of the service facility should be able to determine, in general, relevant attributes and characteristics of the system necessary to evaluate the consequences of any proposed action (including the status quo) as in step 1b. In the next section we show the results of carrying out this step for the case of the time-sharing service facility. Steps 1c, 2 and 3a must be carried out in the framework of making a particular decision. Step 3b, however, can be performed in some general sense by determining user preferences for different levels of the characteristics determined in step 1b. That is to say, if we know that a user group has a low preference for a decrease in a particular characteristic, then by implication their preference for any action resulting in this effect will be low. The following sections of this paper describe procedures that can be used to carry out this analysis. A section describing the relevant concepts in decision theory precedes the procedural description.

Determination of Time-Sharing Characteristics

Review of the literature shows a number of studies concerned with measurement of the usefulness of time-sharing systems (4, 5, 8, 14, 15, 16, 18, 19, 20). Most simply chose system characteristics for study on an empirical basis and did not attempt to structure or relate them to show the analytical basis for the selection of any particular set. Miller (13), while not specifically concerned with computing systems, does establish a system for hierarchically relating the various system characteristics used in his study. He, however, goes to great pains to ensure mutual utility independence among characteristics included in the set (See Chapter II of Reference 13) - something not required in this study.

The following list of time-sharing characteristics is a combination of those presented in the references indicated:

Ability to let user feel "in control"
Ability to manipulate large data bases
Accounting
Availability of terminals

Backup
Batch Facility
Bulk I/O

Capacity
Cost

Debugging and editing systems
Documentation

File protection, storage, and maintenance
Flexibility

Human interface

Languages
Library routines

Machine independence of user programs

Output rate

Reliability
Response time

System availability
System down time

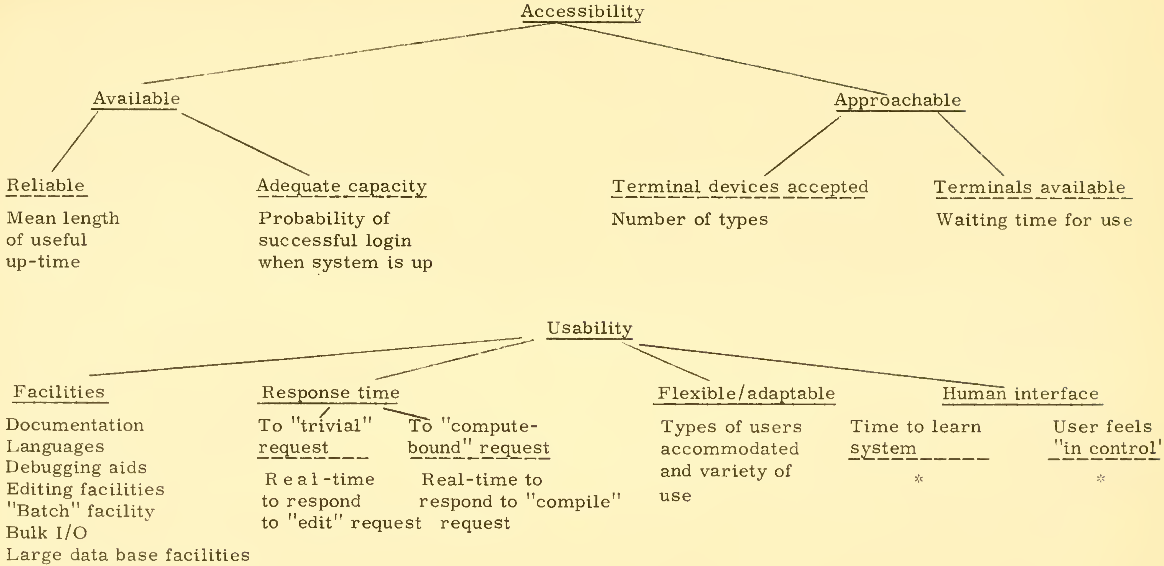
Time to learn the system

(A number of these characteristics appeared in several studies in slightly different forms.)

Drawing upon work in a previous paper by the author (8), we will structure these characteristics in a hierarchically related manner. On the "top" level, the general categories of accessibility, usability, and manageability are established. As shown in Table 1 all the characteristics (or variants) fall either one or two levels below these general categories.

The hierarchical structure of system characteristics was determined on the basis of relatedness, importance, and inclusiveness of the characteristics. The major divisions (accessibility, usability, manageability) indicated the nature of the general goals of providers of computer service. It is possible to classify their actions in managing as influencing one or more of these goals. The placement of system characteristics under one of the three major headings was aided by breaking each goal into a number of subsidiary goals (for instance,

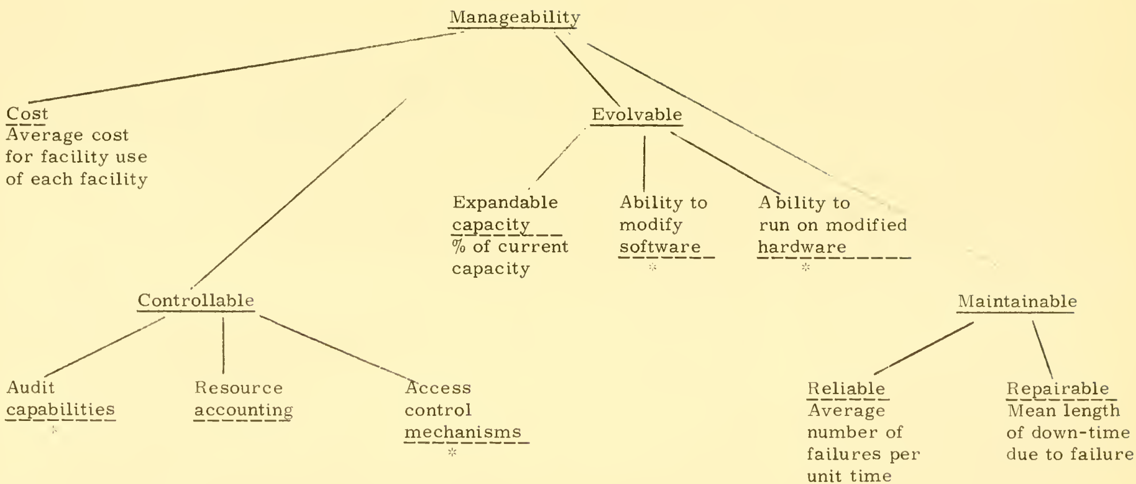
Table I. Criteria for Time-Sharing System Assessment



Notes under dashed line indicate possible measure

*Subjective estimate by expert, impartial observer

Table 1. (Continued)



*Subjective estimate of "quality" of facility obtained from expert, impartial observers.

accessibility into availability and approachability) which comprise it. The choices of subsidiary goals and their placement (see Table 1) are probably not uniquely determined but represent one possible set which accommodates a structured listing of system characteristics. These characteristics were analyzed for their major component in terms of contribution towards a goal and placed in the structure accordingly.

Establishing measures for these various characteristics is largely a matter of determining how each system treats them. For example, availability of terminals could be measured by the percent of time free or by the waiting time to use them. Reliability could be measured by the number of service interruptions (or perhaps by the mean time between failures) and the recovery rate of work performed (or by the mean amount of work lost per failure). Data does exist for a number of systems on parameters such as these and user preferences for different values can be found (see below). Such other characteristics as languages or debugging and editing facilities, however, require both a numerical (objective) and a qualitative (subjective) measure. Subjective measurement of these characteristics requires user interviews (with considerable attention to bias elimination caused by the user's preference for the particular value present at the time). The subject of subjective measurement is treated in a number of papers including references 12, 13, 18, and 22.

In order to demonstrate the methods of utility function determination and analysis, we will work with only two sub-goals. While it is theoretically possible to begin an analysis of the entire problem (that is, of all characteristics listed in Table 1), it should be noted that only those sets of characteristics which are expected to be affected by a particular decision (action) need enter into the discussion. In essence, we are calculating utility functions with certain parameter values held fixed. As will be seen below, this technique is perfectly valid if the subject has been properly introduced to the situation.

The two sub-goals in our analysis, availability and response time, subsume four system characteristics:

Reliability
Capacity
Response time to trivial requests
Response time to compute-bound requests

Further, reliability has been found to be such a dominating characteristic (if it is at a low value very little else matters to most users) that we will assume it at a uniformly high level for the remainder of our analysis. Thus, from Table 1, we see that we will be determining utility functions with the following three parameters:

Probability of successful login when the system is up

Real time to respond to "edit" requests

Real time to respond to "compile" request.

Table 1 indicates possible measures for a number of other characteristics which could be used if the utility function approach were to be expanded to many dimensions. It is felt that the characteristics chosen and their respective measures will serve to illustrate a number of pertinent points in the analysis. The next section deals with the theory and history of multi-dimensional utility function assessment and will lead us into a discussion of the actual experiments.

Utility Theory

An early paper by Yntema and Klem (24) deals with the problem of machine evaluation of alternatives. They devise an experiment to assess a formulation for a person's utility function involving three attributes. The formulation was as follows:

$$U(x, y, z) = A + Bx + Cy + Dz + Exy + Fxz + Gyz + Hxyz \qquad (5)$$

This initial formulation was chosen empirically to "allow the three attributes to interact, although only in certain restricted fashions".

The subjects in this experiment were pilots evaluating the safety of various landing situations. The attributes under investigation were ceiling, visibility, and the amount of fuel left on landing. They were given a set of triples containing a particular value for each of the attributes and told to arrange them in a preference ordering according

to the safety that the situation represented. They were told to place chips down on a scale such that the ratio of the distances between them would indicate the relative safety of the situations indicated on them. By providing the pilots with chips representing the eight possible "corner" situations (that is, the combinations of highest and lowest values of visibilities, ceiling and fuel) the experimenters were able to evaluate the eight constants in Equation 5. They noted that "this formula is equivalent to a linear three-way interpolation of the kind commonly done in using a table of a function of three variables". There was, at this time, no justification for this functional form but a statistical analysis of machine predictions for randomly selected triples compared very favorably with the actual pilots' assessed safety of the situation.

Keeney, in his Master's thesis (see 9),formalized the method of "interpolation between the corners" illustrated above. Keeney spoke of the fact that previous discussions of multi-attributed utility functions took the approach of assessing them as separable or additive. For two attributes, this could be written as follows:

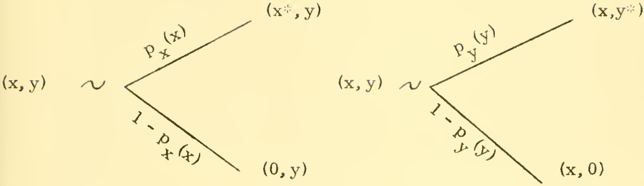
$$U(x,y) = U(x,0) + U(0,y) \qquad (6)$$

This functional form places constraints on the utility function that are not often true in practice. Keeney suggests the quasi-separable utility function as an alternative approach:

$$U(x,y) = U(x,0) + U(0,y) + kU(x,0)U(0,y) \qquad (7)$$

He then brings in the "quasi-separability assumptions" in terms of preferences for lotteries:

$0 < x < x^*$ $\qquad$ $0 < y < y^*$



The assumptions are that there is some probability, p_x, such that (x, y) for certain is indifferent to (x*, y) with probability p_x and (0, y) with probability $(1-p_x)$. An analogous situation is shown in the second lottery.

It is now possible to write down simple utility equations from these two loterries to evaluate the constant k in Equation 7.

$$U(x, y) = p_x(x)U(x^*, y) + [1 - p_x(x)]U(0, y) \tag{8}$$

$$U(x, y) = p_y(y)U(x, y^*) + [1 - p_y(y)U(x, 0) \tag{9}$$

If we set:

$$U(0, 0) = 0 \qquad\qquad U(x^*, y^*) = 1 \tag{10}$$

We can then combine Equations 4 and 5 to solve for "k":

$$U(x, y) = U(x, 0) + U(0, y) + \frac{1 - [U(x^*, 0) + U(0, y^*)]}{U(x^*, 0)U(0, y^*)} U(x, 0)U(0, y) \tag{11}$$

In words, the quasi-separability assumptions state that "the decision-maker's relative preferences for different amounts of one factor are not affected by common amounts of the other factor".

With the extension of the quasi-separability assumptions to utility functions of three attributes, it can be shown that the formulation of Yntema and Klem in Equation 5 is, in fact, a quasi-separable utility function.

Later work by Keeney in his doctoral dissertation (10) formulates the conditions for "utility independence" and shows the quasi-separability conditions to be those of mutual utility independence. For example, an attribute x is said to be utility independent of an attribute y if the relative utility for values of x is independent of the value of y:

$$U(x, y) = C_1(y) + C_2(y)U(x, y_o), \text{ all } y \tag{12}$$

It should be noted that "x utility independent of y" does not necessarily imply "y utility independent of x". However, if both of these conditions hold, then the utility function is separable as in Equation 7 above and x and y are "mutually utility independent". (The conditions of utility independence correspond to Raiffa's formulation of "strong" conditional utility independence (17).)

In dealing with utility functions of more than two attributes, it is possible that attribute x is utility independent of y but not of z, or that there be any other combination of utility independent or non-independent

variables. Keeney shows that it is possible to make use of any information on utility independence that is available to simplify the utility functions. For instance, if we have a situation where x is utility independent of (y, z) and (y, z) is utility independent of x, then we can formulate a utility function of vector attributes as follows:

$$A = (x) \qquad B = (y, z) \tag{13}$$

We see immediately that our three attribute utility function, under the above conditions, reduces to a two attribute quasi-separable utility function with vector arguments A and B.

Keeney's work goes on to show minimal combinations of preference curves, point utilities, or marginal utilities that can be combined with the various conditions of utility independence in order to evaluate the utility function over the entire space. For example, if we have a utility function of three attributes, $U(x, y, z)$, such that attribute x is utility independent of (y, z) and attribute y is utility independent of (x, z), then the complete utility function is specified by assessing six one attribute conditional utility functions: $U(x, y_o, z_o)$, $U(x_o, y, z_o)$, $U(x_o, y_o, z)$, $U(x_o, y_1, z)$, $U(x_1, y_o, z)$, and $U(x_1, y_1, z)$ for arbitrary values x_o, x_1, y_o, y_1, and z_o, subject to consistent scaling on the six. The proof of this statement follows directly from successive application of Equations 12 and 13 for the different variables. The total utility function of x, y, and z will then be a weighted sum and product form of the six conditional utility functions (see Reference 10, Chapter 3 for a more complete

discussion).

Further statements by Keeney indicate the usefulness of the above-mentioned techniques even under conditions of only approximate utility independence. He shows that when Equation 12 is solved, the resulting equation is a utility function with five degrees of freedom:

$$U(x, y) = U(x_0, y)[1 - U(x, y_0)] + U(x_1, y)U(x, y_0)$$

$$U(x_0, y_0) = 0; U(x_1, y_0) = 1 \qquad (14)$$

Using the techniques shown in Equation 13, we can then use this utility function as an approximation to one with greater than two scalar attributes (this is viewed as being of major importance in many operational situations, see below).

Winkler (22) and Keeney (10) both spend significant amounts of time discussing the procedures by which a subject should be interrogated about his probability and utility assessments. Both stress the importance of explaining the axioms of rational behavior and coherence. Subjects should be made aware of any violations of these rules and given a chance to review their assessments and remove inconsistencies.

In another work by Winkler (23), the topic of discussion is combining the subjective probability distributions of many assessors. Various theoretical approaches such as weighting of judgments according to the experimenter's subjective feelings about the assessor, or according to the subject's own feelings about his expertness in this situation,

are discussed. Various behavioral approaches to the problem, such as feedback and reassessment by each subject, or group reassessment of probabilities are also discussed. Winkler states that"the discussion indicates that different methods may well produce varying results. Of course, there is no correct weighting function, so it is impossible to select a method because it is more nearly "correct" than other methods".

If we assume that the ultimate aim of the assessment procedure is to make a decision about a multi-attributed situation, then the importance of Winkler's work is in allowing us to test the sensitivity of the decision to different weightings of subjective judgments. In the final analysis, this work is meaningful only so far as "mechanical rules simplify the decision-maker's problem and at the same time seem reasonable to him."

Experimental Determination of a Multi-Dimensional Utility Function

The management decision-making procedures outlined in the beginning of this paper hinge upon successful determination of user utility functions. An experiment was performed to show the feasibility of this determination for one group of users of a general purpose time-sharing system. The experiment consisted of a series of interviews with users to determine the minimum amount of information necessary to formulate their utility function. The determining factor as to what information was necessary was the utility independence relationships that existed among the system characteristics (see the

discussion in the previous section). The data was then combined according to the mathematical formulation of the utility function and tested for general correctness (no statistical tests of accuracy were made as the usefulness of these procedures does not rely upon extreme precision of utility values).

The interview procedure consisted of several parts:

1. General determination of users' usage patterns of the time-sharing service.
2. Introduction to basic theory of utility assessment.
3. Focusing of attention on the three attributes to be discussed: availability, response time to trivial requests, and response time to compute-bound requests.
4. Determination of utility dependence relations.
5. Determination of conditional utility functions and point utilities necessary for the calculation of the total (three-dimensional) utility function. (Which conditional and point utilities were needed was determined by application of mathematical utility theory.

All assessment procedures are based on the assumption that the subject can abstract his feelings regarding the comparison of situations which are not occurring at the present time. It is fairly easy to show that while such assessments for situations which the subject has experienced in the past require a good deal of care, determining

preference data for situations which he has not experienced becomes increasingly difficult to validate and reproduce as the hypothetical situation becomes more removed from the subject's experiences. For this reason, general conversations about the types of work that the subject performed on the time-sharing system were held prior to any assessment procedures. If, for example, the subject had never experienced extreme values of the variables to be assessed, then efforts were made to delay further analysis until these experiences occurred. Another approach to this problem would have been to simulate certain conditions during test periods when the user was active on the time-sharing system.

As mentioned earlier, another factor that is felt to be important in performing judgmental assessment procedures is the introduction of the subject to the theory and methodology. For this reason some simple everyday situations were analyzed with the various subjects before discussion of their utilities for the characteristics of the time-sharing system. In these discussions, points were made relating to "rationality", particularly as it pertained to the meaning of independence of variables and the transitivity of preference statements. Throughout the procedures, any inconsistancies in assessments found were discussed with the subject to determine whether they could be removed upon reassessment.

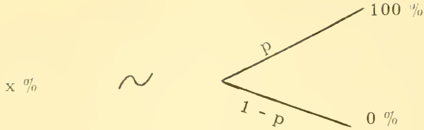
Focus was then directed to the three attributes to be considered by presenting a scenario which established their presence as the major variables. For instance, the question of reliability was removed from the picture by insuring that on the "experimental" time-sharing system it was at a uniformly high level as stated above, (as long as this situation was not too far removed from the user's experiences it was felt that he could and would accept the assumption and thus ignore reliability as influencing his preferences). Another example is that the features necessary for him to adequately perform his task were assumed to be fully operable (again, it was necessary that this condition not be too far removed from actuality). In essence, we were, therefore, computing utility functions conditional on high levels of all characteristics that were not explicitly considered.

Independence relations were determined by a series of questions regarding the shape of various conditional utility curves. For example, if the user asserts that his conditional utility function for availability has one shape when response time is at a favorable level and a fairly different shape when response time is at an unfavorable level, then we can say that availability is not utility independent of response time. If, on the other hand, the user asserts that the shape of his conditional utility curve for one variable is independent of the value of the second variable, then we can state that a utility independence relation exists. The utility independence

relations were then used to determine a mathematical form for a complete utility function (as shown in the section on theory) involving a relatively small number of conditional utility functions and point utilities.

The final step in the procedure is the actual determination of conditional utility functions and point utilities as indicated necessary for the calculation of the entire utility space. If a sufficient number of independence relations exist, the conditional utility functions to be determined are in fact functions of only one variable. A person's relative preferences for values of a single variable can be assessed in several ways including the presentation of "lottery" situations and by "fractile assessment". In the lottery assessment procedure a typical question might appear as follows:

> At the beginning of each month, you are given the opportunity to be assured successful logins x percent of the time, or to participate in a lottery, which gives you a certain probability of assuring all successful logins for the month and a certain probability of refusing you service for the month. That is to say, we are offering guaranteed success of login of x percent per month or the opportunity to try for a chance at 100 percent successful logins during the month. If you try for the chance at 100 percent successful you are risking the receipt of 0 percent. The lottery is a fair game and we will tell you in advance what the probability is that you will "win" (receive 100 percent successful logins during the month). For what value of this probability will you be indifferent between participating in the lottery or receiving immediate assurance of x percent of successful login?



This situation is presented, with discussion, for a number of different values of x in order to determine relative preferences for the values of the variable "probability of successful login." It was found, however, that a number of subjects had difficulty in placing themselves in the "lottery" experience and for them a method of fractile assessment was used. A typical presentation would then be as follows:

> Time-sharing service is contracted for in terms of an assures "probability of successful login". This can range from 0 percent (no service during the month) to 100 percent (guaranteed service at any time). What percentage of successful logins is worth the most to you? (This may be 100 percent or less than 100 percent if the person does not require guaranteed service.) What is the maximum percentage for which the service is useless? (This may be above 0 percent for people who require a minimum grade of service.) Now try and find a percentage for which the service would be worth half as much as it would at the maximum value. This is the .5 fractile. For what percentage would service be worth three quarters as much, one quarter as much, etc.?

Since the meaning of the word "worth" is not entirely clear, many subjects immediately fall into thinking about a monetary equivalent. Since it is established that the maximum amount of

money ever to be paid will be no more than their current expenditure, it is assumed that money is a reasonable proxy for worth and that its use does not introduce any additional non-linearities in the assessment procedures. (In general, this assumption cannot be made and tests must be performed to determine the user's utility for different amounts of money. What we are assuming is that these utilities are linear over the particular range.)

In order to insure that the one-dimensional utility functions assessed as above are in fact conditional utility functions for the appropriate specific values of the other variables, the experimentor first presented and discussed with each subject the environment (that is, values of all other variables) in which he was being asked to assess a one-dimensional utility function. For example, if he were asked to assess utilities for probabilities of login (as indicated in the sample questions above) then he would be asked to assume a response to trivial requests and a response to compute-bound requests as suggested by the experimentor. It can be seen, therefore, that the user must be asked to assess a particular utility function (say for probability of login) under a number of different sets of environmental conditions (different values of the other variables). The total number of conditional functions that must be assessed is determined by the types of independence relationships that exist between the variables. (The next section of the paper will show an example of this assessment and will re-emphasize these points.)

Finally, the user was asked to assess his utility for the eight possible "end points" generated by the combination of the highest and lowest values of the three variables. The method used for this assessment was very similar to that of Yntema and Klem (24) mentioned above. The eight possible end points were calculated and each was written on a separate paper marker. The subject was given simple scale on which to lay the markers according to their relative value to him. He was told to place a marker so that the distance from the base line represented his assessment of the "value" of the service quality specified by the indicated values of the variables. He was told that if the distance from the base line for one marker was twice the distance from the base line for another marker, then this should indicate that the quality of the situation represented by the first marker was twice the quality of the situation represented by the second marker. This procedure produced information for the determination of the total utility function by providing scaling points and fixing origins.

It should be noted that while all of these steps could be carried out sequentially, the experimental procedure was very iterative in nature: for instance, if determination of a conditional utility function showed a possibly incorrect utility independence relation, than this part of the procedure was redone. A number of interviews were held with selected users to ensure the validity of these results. Much general discussion was interspersed with the assessment procedure to give the

users a chance to "live with" their statements, understand the implications of other assessments, etc. Although time consuming, this is an important part of the assessment procedure. It is in this way that the experimentor attempts to ensure that all extraneous factors are removed from the assessment. In the next section we give an example which shows specific utility independence relations and the calculation of a total utility function.

An Example

The users interviewed were computer system application programmers whose main tasks were the input and editing of programs (trivial requests) coupled with compilation and testing thereof (compute-bound requests). The ratio of editing sessions to compile and testing sessions was fairly high (about five to one) indicating that a lot of "desk debugging" was being done (somewhat irrelevant to the problem, but interesting nonetheless). Initial testing of the subjects indicated the following utility independence relationships between two variables when the third variable was in the relevant range:

1. RT ui (utility independent) A - Relative preferences for trivial request response time was independent of availability.

2. RT ui RC - Relative preferences for response to trivial requests were independent of response to compute-bound requests.

(These both seem to make sense since response time to editing requests was the major determinant of how much work could be accomplished. It is suspected

that these may not hold for other classes of users - especially when there is a more even balance between trivial and compute bound requests.)

3. RC ui A - Relative preferences for compute bound response time was independent of availability.

(This also seems reasonable, as the subjects, in general, tended to separate preferences for the various values of system work conditions from anything else. This may also not hold for a less demanding user type.)

This, of course, leaves the following three pairs to be not utility independent:

A. A not ui RT,

B. A not ui RC, and

C. RC not ui RT.

(Again these seem to make logical sense, as indicated in the following statements. If either response has an unfavorable value, then the relative preferences for A may very well change as the programmer will spend most of his time having to contend with the unfavorable response and may not wish to log in as badly. Also, in any particular session, the programmer may set his relative

expectations for RC in terms of the RT he is experiencing - a not uncommon practice of changing sights in view of concurrent conditions.)

It was now possible, using similar methods to those developed by Keeney (10), to determine exactly which conditional utility functions and point utilities had to be determined to completely specify this class of users' complete utility function. The following was proven:

Given the above conditions of utility independence of the three attributes A, RT, RC, the following four utility points and seven one-attribute conditional utility functions are sufficient to completely specify U(A, RT, RC):

Point utility for (A_0, RT_1, RC_0) and (A_1, RT_0, RC_0), (A_1, RT_0, RC_1), (A_1, RT_1, RC_0), as well as scaling points for minimum and maximum utility.

Conditional utility functions: $U(A, RT_0, RC_0)$, $U(A, RT_0, RC_1)$, $U(A, RT_1, RC_0)$, $U(A, RT_1, RC_1)$, $U(A_0, RT, RC_0)$, $U(A_1, RT_0, RC)$, and $U(A_1, RT_1, RC)$.

The utility function thus obtained is as follows:

$$U(A, RT, RC) = \left\{ U(A, RT_0, RC_0) + [U(A, RT_0, RC_1) - U(A, RT_0, RC_0)] \cdot \left[\frac{U(A_1, RT_0, RC) - U(A_1, RT_0, RC_0)}{U(A_1, RT_0, RC_1) - U(A_1, RT_0, RC_0)} \right] \right\}$$

$$\cdot \left\{ 1 - \frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)} \right\}$$

Cont.

$$+\left\{U(A, RT_1, RC_1) + [U(A, RT_1, RC_0) - U(A, RT_1, RC_1)] \cdot \left[\frac{U(A_1, RT_1, RC) - 1}{U(A_1, RT_1, RC_0) - 1}\right]\right\}$$

$$\cdot\left\{\frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)}\right\} \quad (15)$$

The complete proof of this theorem is provided as an Appendix to this paper. The points and conditional utility functions indicated are shown graphically in Figure 1.

The above equation is not a unique determinant of the utility function as is shown below. The theory allows us to derive several equations, each of which is an equivalent statement of the utility function. For example, a slightly different derivation (see the Appendix) yields a function that can be specified by the following point and conditional utilities (see Figure 2):

Points: (A_0, RT_0, RC_1), (A_0, RT_1, RC_0), and (A_1, RT_1, RC_0) and scaling points.

Conditional utility functions: $U(A, RT_0, RC_0)$, $U(A, RT_0, RC_1)$, $U(A, RT_1, RC_0)$, $U(A, RT_1, RC_1)$, $U(A_0, RT, RC_0)$, $U(A_0, RT_0, RC)$, and $U(A_1, RT_1, RC)$

which yield a utility equation as follows:

$$U(A, RT, RC) =$$

$$\left\{U(A, RT_0, RC_0)\left[1 - \frac{U(A_0, RT_0, RC)}{U(A_0, RT_0, RC_1)}\right] + U(A, RT_0, RC_1)\left[\frac{U(A_0, RT_0, RC)}{U(A_0, RT_0, RC_1)}\right]\right\}$$

Cont.

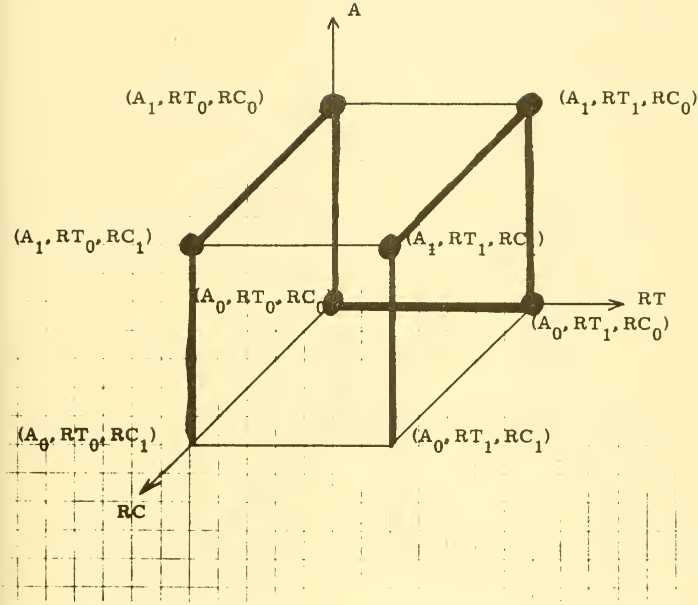


Figure 1

Points and one-dimensional utility functions necessary to evaluate utility values within the cube under following utility independence relations: RT ui A

RT ui RC

RC ui A

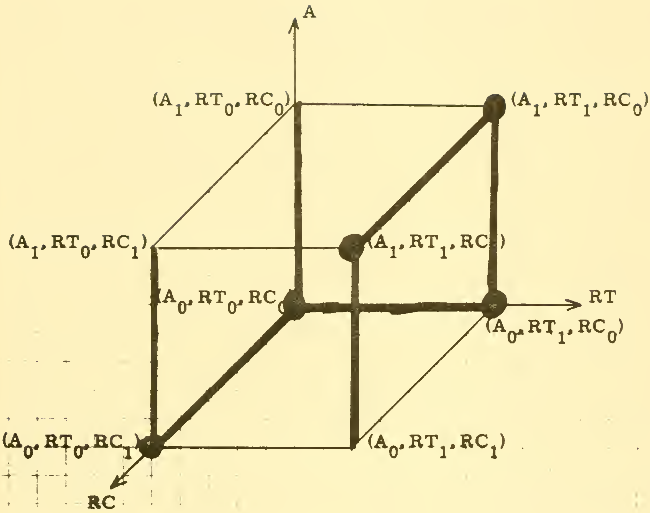


Figure 2

Possible alternative requirements for calculation of utility space having independence relations: RT ui A

RT ui RC

RC ui A

$$\cdot \left\{ 1 - \frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)} \right\}$$

$$+ \left\{ U(A, RT_1, RC_1) \left[1 - \frac{U(A_1, RT_1, RC) - 1}{U(A_1, RT_1, RC_0) - 1} \right] + U(A, RT_1, RC_0) \left[\frac{U(A_1, RT_1, RC) - 1}{U(A_1, RT_1, RC_0) - 1} \right] \right.$$

$$\left. \cdot \left(\frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)} \right) \right\} \tag{16}$$

We can now see that the determination of a three-dimensional utility function for this class of users has been reduced to the determination of several (conditional) one-dimensional utility functions and point utilities - a fairly simple assessment process for both experimenter and subject.

By using a similar procedure for each class of users (and remember, we predict that this will be a numerically small number), we can effectively determine utility functions for the entire user population.

A Utility Function

Figures 3 through 9 show computer generated plots of conditional utility functions derived from Equation 15. A time-sharing user whose main job was programming was interviewed (as described above) for the purposes of assessing actual point and conditional utility functions and/or determining the appropriateness of the utility independence relationships specified. After appropriate scaling (to bring utilities in the range 0 to 1000), the utility space was generated. The utility value is shown on the z coordinate and the two variable characteristics on the x and y. The third characteristic is held fixed as indicated.

The first plot (Figure 3) shows the user's utility for various values of response to trivial requests (RT) and response to compute bound requests (RC) when availability is at its maximum value (100% probability of login). As can be seen, the relative preferences for RT (that is, a conditional utility curve for RT) is independent (has the same shape) for all values of RC. On the other hand, it can also be seen quite clearly that the shape of conditional utility curves for RC vary as RT goes from its maximum to its minimum value. By noting the relative magnitude in change of utility between one set of values for RT and RC and another set of values, the decision-maker can determine where the greatest marginal improvement will occur. He might use this information to help him select the alternative which best assured him of the appropriate changes in the values of those characteristics.

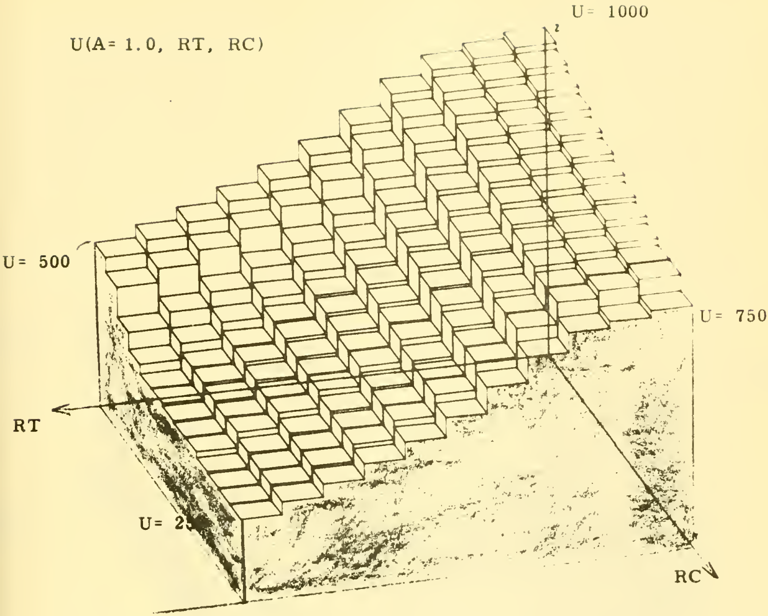


Figure 3

Utility space calculated using data from point and conditional utility functions in Equation 15 (three dimensional view shown is for highest value of availability).

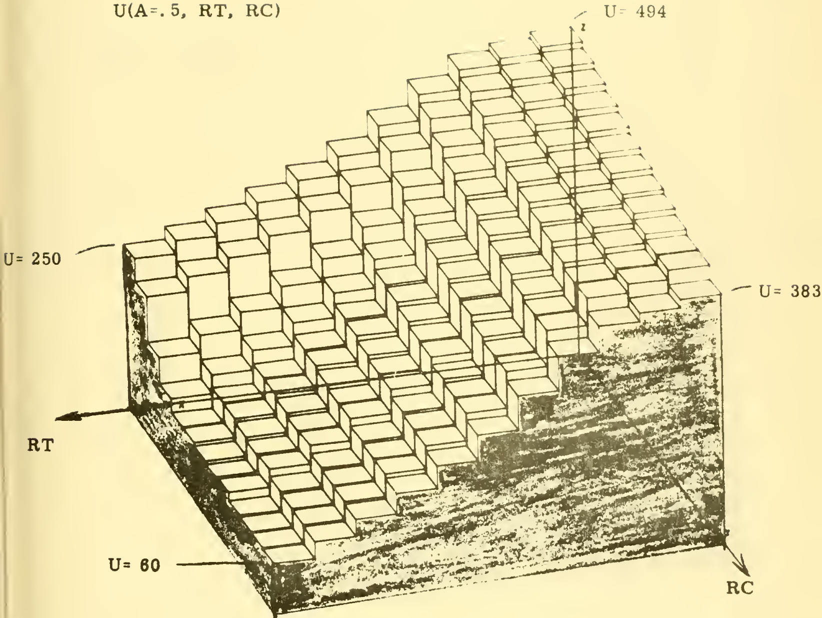


Figure 4

Conditional utility space with availability equal 50%.

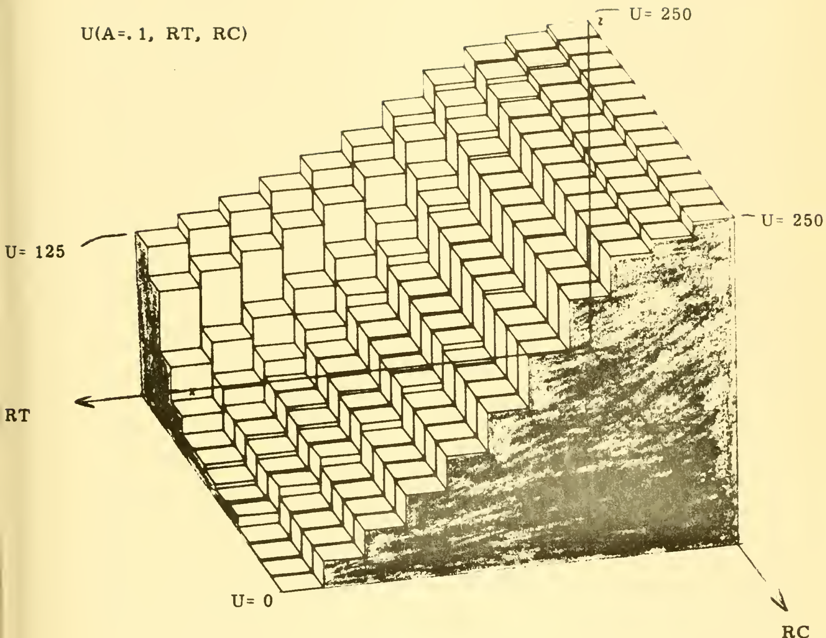


Figure 5

Conditional utility space at minimum value of availability.

Figures 4 and 5 show plots similar to Figure 3 except that the value of availability is 50% in Figure 4 and 10% in Figure 5. We note that the minimum and maximum values of utility have changed, but the approximate shapes of the conditional utility functions have not (indicating the fact that both RT and RC are utility independent of A).

Figures 6, 7, and 8 show utility space for A and RC at different values of RT. Again, we can look at the stairsteps in any particular figure to see that RC is utility independent of A but not vice versa. By comparing the curve that these stairsteps trace in the different diagrams, we can also see that neither A nor RC is utility independent of RT (the curves traced by any particular conditional utility function vary in shape in the three figures, that is, for different values of RT).

Finally, Figure 9 shows the utility space for A and RT with RC fixed at a low to medium level. We note here the relatively flat preference for good values of RT which trail off fairly rapidly. The utility independence relations between A and RT (A not utility independent of RT but RT utility independent of A) can again be seen by comparing the relative shape of the conditional utility curves at different values of the other variables.

The determination of utility spaces as was done above also serves the useful purpose of directing a decision-maker's search for alternatives. Simple mathematical analysis can show where the "steepest gradients" are relative to changes in the values of the characteristics. This can be

combined with the manager's knowledge of the probable consequences of various actions to arrive at a set of possible policies which would be most advantageous to users of the computation facility. Further exploratory work needs to be done in this area as well as others to be described below.

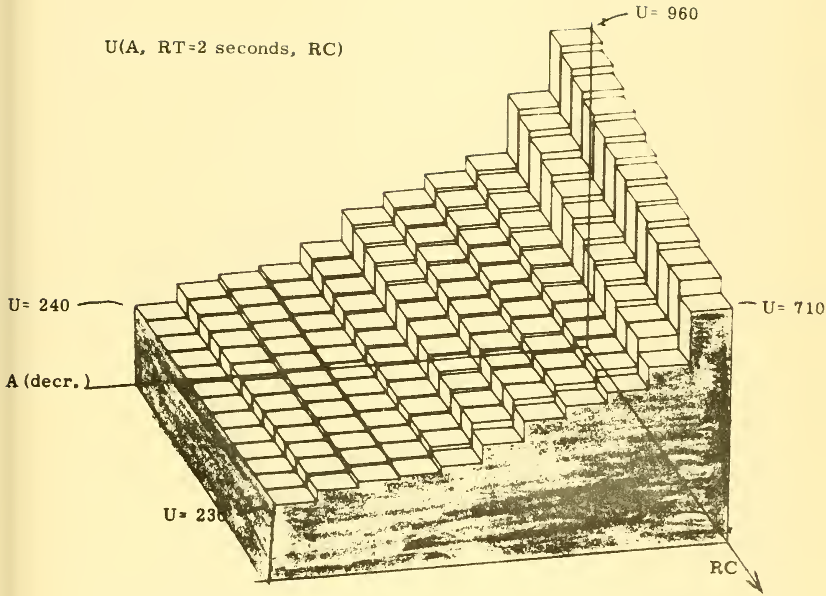


Figure 6

Conditional utility space with response time to trivial requests at most favorable value.

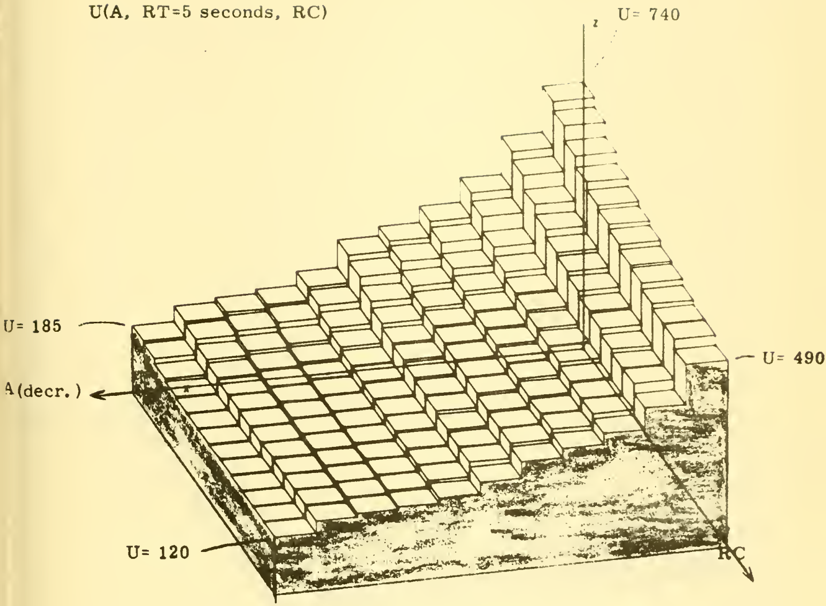


Figure 7

Conditional utility space with response time to trivial requests at intermediate value.

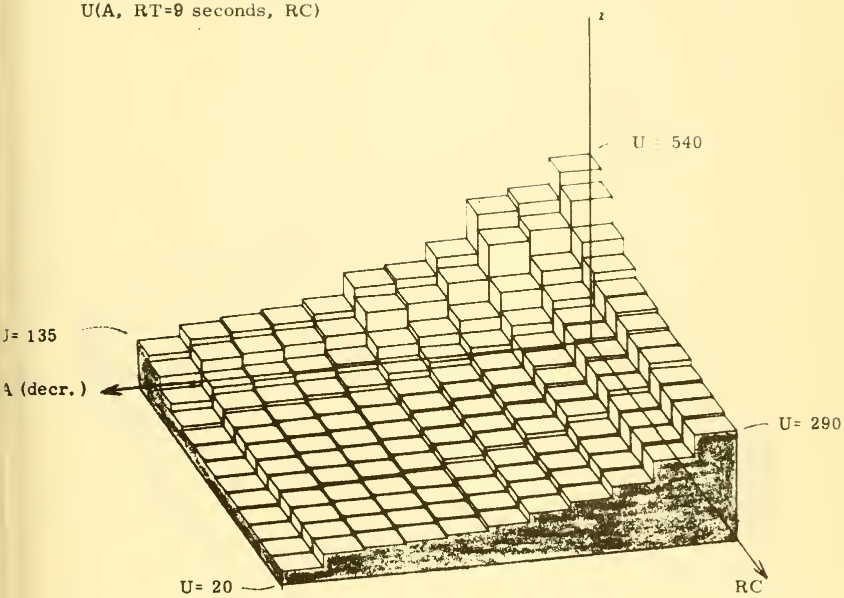


Figure 8

Conditional utility space with response time to trivial requests at least favorable value.

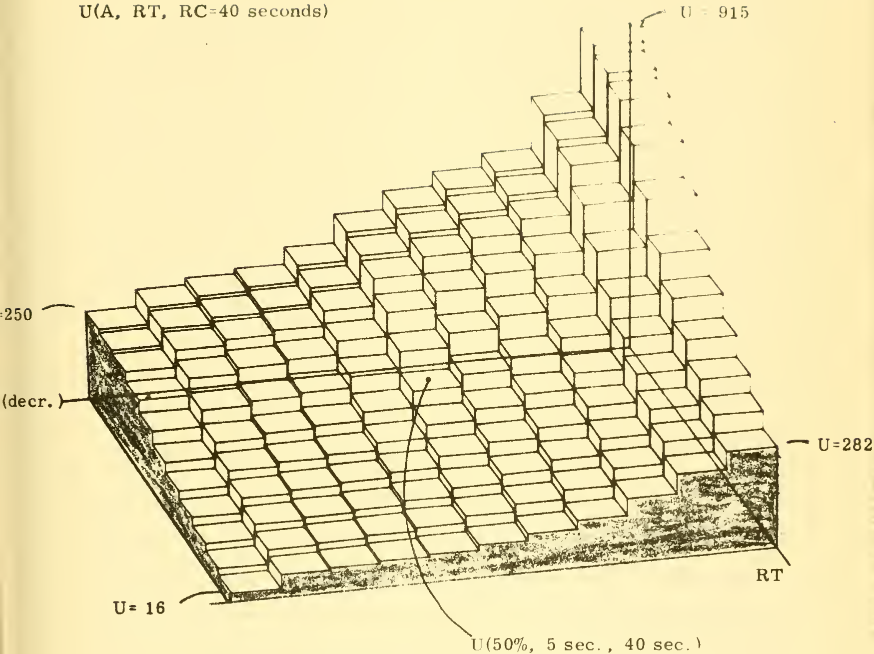


Figure 9

Conditional utility space with response time to compute-bound requests at intermediate value.

Observations and Conclusions

The experimental determination of a utility function of three parameters was successfully accomplished as outlined above. This procedure, however, as complex and time consuming as it was, is only part of the general management decision-making process described in the beginning of this paper. It is still necessary for the manager to determine various possibilities for action and the presumed affect of these actions on the characteristics of the service.

The calculation of a user's utility function is simply a tool for prediction of what the reaction of the community will be to the manager's decision. Since the manager is concerned with a number of user groups and with other factors as well (such as top management requirements, etc.), a single utility function is only a small part of the data needed to make a decision. If a user's utility function could only be used in this way, then we would question whether the effort was justified. However, it is often argued that perhaps the largest part of any manager's time and intellectual activity is spent in the process of "problem finding" (26). If the utility function could be used in this area as well, it would gain additional value. Similarly, Simon (27) has indicated that the problem solving activities of "intelligence" and "design" occupy a proportionately larger share of the total problem solving time and effort than does the final "choice" for a large class of problems. We will show in the next few paragraphs how the user's

utility functions can be used in searching for problems and for alternatives as well as in making a final decision. We assume below that the utility function depicted in the proceding diagrams represents the only user group that our hypothetical manager has to deal with.

In the first case we wish to see if there is any way that existing resources can be used to provide better services. For example, there is a general relationship between the number of users of a time-sharing system and the response time. If we currently set the maximum number of users at the value n, we can then measure the response time to trivial requests, r. Generally, by setting a lower value for the maximum number of users we will achieve a decreased response time (basically, because a smaller number of users will be demanding a smaller amount of the total system's resources). The question is whether that is a desirable thing to do in terms of increasing user utility, and if so, at what point should we stop. If we look, for instance, at Figure 9 and assume some current operating point of A = 50% and RT = 5 seconds, we can then determine that moving to a point (40%, 4 seconds) has approximately the same utility value where (40%, 3 seconds) has a significantly higher utility value. Similarly, moving to a point (60%, 6 seconds) has a lower utility value and (70%, 6 seconds) has a higher utility value (although not as high

as (40%, 3 seconds)). Thus, if our present resources can be arranged to take us to one or the other of the better operating points (or some other operating point that has a higher utility value), then we have improved our overall service at no additional cost. Similar analyses can be done for other variables as well.

If we now take the situation where an additional amount of money is to be spent and the determination of which resource to spend it on is in question, then we can again use the utility function to help us in our decision. If we look at our operating point in the overall utility surface, it should be possible to mathematically determine the direction of "steepest gradient" (fastest increase in utility value). We can determine "feasible" directions according to the characteristics which can be modified by the application of further resources (we may be able to change availability to some degree, response time to trivial requests to another degree, and response time to compute-bound requests not at all--therefore ruling out certain directions of moving). Finally, we can then compute that point in the feasible region which provides the greatest utility increase (this procedure is very similar to some used in the solution of non-linear programming problems). With the aid of the computer the mathematical manipulations may be fairly easily performed.

As utility assessment procedures become more widely used and accepted, we expect even further investigation of the use of utility functions as major policy reassessment tools. We have indicated in this papar several ways in which the use of user determined utility functions can enter into all stages of the management problem finding-decision making process. If further work in this area makes it easier to extrapolate the fairly simple example discussed above to problems involving numerous characteristics and several groups of users (and hence, several utility functions), we see general applicability of these techniques. The management of service facilities is a sufficiently important problem so as to almost guarantee the interest of researchers and practitioners alike in these areas. We look forward to the extension of this work in the near future.

Further Research

The major deterrent to the use of multi-dimensional utility calculations for management decision-making is in establishing the reliability of the utility assessment and in the relative cost of the assessment procedure. Management problems are characterized by a

1. large number of relevant variables
2. small number of utility independence relations
3. large number of conditional utility curves with thresholds, "knees", or other irregularities

all of which serve to increase the number or complexity of the utility points or conditional functions which must be assessed in order to specify the complete utility function. Assessment procedures currently require laborious and painstaking work to ensure the validity of the assessed values - work that, in general, must be repeated for each service facility user being interviewed.

Research areas which would foster the acceptance of the utility approach to complex managerial decision-making currently include:

1. "Assessment by questionnaire." If adequate questions could be formulated to cut down significantly on interviewer time, utility determination could be integrated into existing management information gathering procedures.

2. Machine-aided assessment. The use of conversational graphics systems should be experimented with to give the subject a pictorial "feel" for the meaning of his assessments. This should help increase reliability of assessed utilities.

3. Utility function approximation and sensitivity analysis. The issue is establishing procedures for easily determining the dominant variables and relationships in any problem. A side effect of this could be a decrease in number and dimension of conditional utility functions that must be assessed.

4. User utility equivalence determination. Ad hoc procedures were used in this paper to determine user "classes" on the basis of similar characteristics. More work needs to be done to make this determination more accurate. Correlation of membership in a class (at a given time) with the actual work performed on the service facility should be attempted.

5. Time variance of utility assessment. How often must a manager reassess utilities in his user community to ensure that his data truly reflects their preferences? Research could be devoted to controlled experiments over time periods relevent to the service facility in question (six months or more in the case of computer systems - about the length of time necessary for major

changes).

6. Utility functions of utility functions. In order to better understand the utility function of managers, we need to investigate the properties of utility functions whose attributes are other utility functions.

Research in these and other areas should be yielding further insights into the use of utility functions in *practice* in the near future.

Summary

We are concerned with the basic problem of providing a mechanism for decision-making regarding the provision of some service to a population of users. Problem analysis begins by defining a hierarchy of system characteristics and measures of effectiveness for them. By using the methodology of utility function assessment, it is possible to determine a multi-dimensional utility function for each group of users.

As an example, system characteristics of a general purpose time-sharing system were determined and several user utility functions were assessed. The decision-maker (the Computer Center Manager) must then go about the task of evaluating probable changes in the various attributes due to any action he might possibly take and determining the relative importance (to him) of each user group's utility for the action. The results of these analyses can

then be combined to evaluate a "most desirable" action.

As stated earlier, it is felt that the procedures illustrated in this paper hold considerable promise for the future. As management must deal with more and more complex problems with more variables, more "interested parties," and more possible actions, use of an integrated set of procedures for evaluation of alternatives will become more prevalent. Incorporation of individual preference data can only increase the depth and breadth of management understanding.

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APPENDIX

The following is the proof of the result stated in Equation 15 in the text. We will show that, given the utility independence relations shown below, the utility function U(A, RT, RC) can be evaluated by the expression in Equation 21 using only the indicated conditional and point utilities.

Utility independence relations:

$$RT \text{ ui } A \quad (1)$$

$$RT \text{ ui } RC \quad (2)$$

$$RC \text{ ui } A \quad (3)$$

Given the definitions of utility independence and mutual utility independence given by Keeney (10), we can write the following equations:

Using (1) and (2) above:

$$U(A, RT, RC) = C_1(A, RC) + C_2(A, RC)U(A_0, RT, RC_0) \quad (4)$$

for some value of $RC = RC_0$ and $A = A_0$

Using (3) above, and setting $RT = RT_0$:

$$U(A, RT_0, RC) = D_1(A, RT_0) + D_2(A, RT_0)U(A_1, RT_0, RC) \quad (5)$$

for some value of $A = A_1$

Using (3) above, and setting $RT = RT_1$:

$$U(A, RT_1, RC) = E_1(A, RT_1) + E_2(A, RT_1)U(A_1, RT_1, RC) \quad (6)$$

for some value of $A = A_1$

We can now set the scale by assigning

$$U(A_0, RT_0, RC_0) = 0$$
$$U(A_1, RT_1, RC_1) = 1 \quad (7)$$

By evaluating Equation 4 with $RT = RT_0$ we get

$$U(A, RT_0, RC) = C_1(A, RC) \quad (8)$$

By evaluating Equation 5 with $RC = RC_0$ we get

$$U(A, RT_0, RC_0) = D_1(A, RT_0) + D_2(A, RT_0)U(A_1, RT_0, RC_0) \quad (9)$$

and by evaluating Equation 6 with $RC = RC_1$ we get

$$U(A, RT_1, RC_1) = E_1(A, RT_1) + E_2(A, RT_1) \quad (10)$$

Substituting these results into our first equations, we arrive at

$$U(A, RT, RC) = U(A, RT_0, RC) + C_2(A, RC)U(A_0, RT, RC_0) \quad (11)$$

$$U(A, RT_0, RC) = U(A, RT_0, RC_0) + D_2(A, RT_0)\left[U(A_1, RT_0, RC) - U(A_1, RT_0, RC_0)\right] \quad (12)$$

$$U(A, RT_1, RC) = U(A, RT_1, RC_1) + E_2(A, RT_1)\left[U(A_1, RT_1, RC) - 1\right] \quad (13)$$

By setting $RT = RT_1$ in Equation11, we get an expression for C_2:

$$C_2(A, RC) = \frac{U(A, RT_1, RC) - U(A, RT_0, RC)}{U(A_0, RT_1, RC_0)} \quad (14)$$

Similarly, we set $RC = RC_1$ in Equation 12 to get an expression for D_2:

$$D_2(A, RT_0) = \frac{U(A, RT_0, RC_1) - U(A, RT_0, RC_0)}{U(A_1, RT_0, RC_1) - U(A_1, RT_0, RC_0)} \quad (15)$$

Similarly, we set $RC = RC_0$ in Equation 13 to get an expression for E_2:

$$E_2(A, RT_1) = \frac{U(A, RT_1, RC_0) - U(A, RT_1, RC_1)}{U(A_1, RT_1, RC_0) - 1} \quad (16)$$

Substituting these results into our last set of equations, we get

$$U(A, RT, RC) = U(A, RT_0, RC) + \left[U(A, RT_1, RC) - U(A, RT_0, RC)\right] \cdot \frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)} \qquad (17)$$

$$U(A, RT_0, RC) = U(A, RT_0, RC_0) + \frac{U(A, RT_0, RC_1) - U(A, RT_0, RC_0)}{U(A_1, RT_0, RC_1) - U(A_1, RT_0, RC_0)} \left[U(A_1, RT_0, RC) - U(A_1, RT_0, RC_0)\right] \qquad (18)$$

$$U(A, RT_1, RC) = U(A, RT_1, RC_1) + \frac{U(A, RT_1, RC_0) - U(A, RT_1, RC_1)}{U(A_1, RT_1, RC_0) - 1} \cdot \left[U(A_1, RT_1, RC) - 1\right] \qquad (19)$$

We rearrange Equation 17 for convenience

$$U(A, RT, RC) = U(A, RT_0, RC)\left[1 - \frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)}\right] + U(A, RT_1, RC) \cdot \frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)} \qquad (20)$$

Now, by simply substituting Equation 18 for $U(A, RT_0, RC)$ where it appears in Equation 20, and Equation 19 for $U(A, RT_1, RC)$ where it appears, we get our final result

$$U(A, RT, RC) = \Bigg\{U(A, RT_0, RC_0) + \left[U(A, RT_0, RC_1) - U(A, RT_0, RC_0)\right] \cdot \left[\frac{U(A_1, RT_0, RC) - U(A_1, RT_0, RC_0)}{U(A_1, RT_0, RC_1) - U(A_1, RT_0, RC_0)}\right]\Bigg\}$$

CON

$$\cdot \left\{1 - \frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)}\right\}$$

$$+ \left\{U(A, RT_1, RC_1) + \left[U(A, RT_1, RC_0) - U(A, RT_1, RC_1)\right] \cdot \left[\frac{U(A_1, RT_1, RC) - 1}{U(A_1, RT_1, RC_0) - 1}\right]\right\}$$

$$\cdot \left\{\frac{U(A_0, RT, RC_0)}{U(A_0, RT_1, RC_0)}\right\} \qquad (21)$$

It can now be seen that the following point and conditional utility relationships are sufficient to compute the utility function U(A, RT, RC) given the independence relations in Equations 1, 2 and 3:

Point utilities:

$U(A_0, RT_1, RC_0)$
$U(A_1, RT_0, RC_0)$
$U(A_1, RT_0, RC_1)$
$U(A_1, RT_1, RC_0)$

Conditional utility functions:

$U(A, RT_0, RC_0)$
$U(A, RT_0, RC_1)$
$U(A, RT_1, RC_0)$
$U(A, RT_1, RC_1)$
$U(A_0, RT, RC_0)$
$U(A_1, RT_0, RC)$
$U(A_1, RT_1, RC)$

This set of utilities is shown pictorially in Figure 1.

In order to arrive at text Equation 16, we simply allow $A = A_0$ instead of A_1 in Appendix Equation 5 and continue the analysis from there. The point and conditional utilities necessary to calculate the utility space from this equation are depicted in Figure 2.