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TURNPIKE THEOREMS FOR
INTEGER PROGRAMMING PROBLEMS

by

Jeremy F. Shapiro

October 10, 1968

Working Paper 350-68

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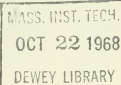
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ABSTRACT

The structure of integer programming problems is investigated by means of a dynamic programming formulation. A group theoretic result of Gomory is then given the following interpretation. Consider the class of integer programming problems with a variable right hand side resource vector b but with fixed activities and cost coefficients. There exists an integer m -vector b^* such that for all integer m -vectors b satisfying $B^{-1} b \geq B^{-1} b^*$, where B is an optimal linear programming basis, the optimal immediate decision when given resources b is to use any one of the optimal basic activities once, and proceed optimally thereafter. Integer programming problems which satisfy this property are called steady-state. The result is extended to all integer programming problems. In effect, the extension identifies the steady-state and transient components of a given integer programming problem. Finally, there is a discussion of the relationship of these ideas to the concept of shadow prices in integer programming.

INTRODUCTION

The purpose of this paper is the development of theorems which expose certain asymptotic properties of the integer programming (IP) problem. The problem is written in canonical form as

$$\begin{aligned} & \text{minimize} && cw && (1) \\ & \text{subject to} && Aw = b \\ & && w \text{ non negative integer} \end{aligned}$$

where A is an $m \times (m+n)$ matrix of integers with columns a_j , c is an $m+n$ row vector of integers with coefficients c_j , and b is an m dimensional column vector of integers. It is assumed that there is a $m \times m$ identity matrix among the columns a_j .

The IP problem can also be formulated as a dynamic programming problem with the shortest route recursion

$$\begin{aligned} F(\ell) = \text{minimum} \{c_j + F(\ell - a_j)\} & \text{ if } \ell \neq 0, \\ j = 1, \dots, m+n & \end{aligned} \quad (2)$$

$$F(0) = 0,$$

where ℓ is an integer m -vector. The cost of an optimal solution in (1) is found by finding $F(b)$ and the optimal solution with this value can in principle be found by the usual backtracking procedure of dynamic programming. Although it is very inefficient to solve problems of any size with the recursion (2), we use it here to gain insight into the structure of integer programming problems.

As a preliminary step, solve (1) as a linear programming (LP) problem (assume an optimal solution exists). Let B be any optimal LP basis and suppose $A = (R, B)$ and $c = (c_R, c_B)$. Problem (1) becomes

$$\begin{aligned} \min \quad & z + \bar{c}x \\ \text{s.t.} \quad & y = \bar{b} - \bar{R}x \\ & x, y \text{ non-negative integer} \end{aligned}$$

where $z = c_B \bar{b}$, $\bar{c} = c_R - c_B B^{-1} R \geq 0$, $\bar{b} = B^{-1} b \geq 0$, and $\bar{R} = B^{-1} R^*$. Let π denote the optimal shadow prices with respect to B ; that is, $\pi = c_B B^{-1}$. In the discussion below, we will often consider (3) with a parametric integer requirements vector b rather than a unique given requirements vector. A non-negative integer vector x is called a correction in problem (3). Let

$$\begin{aligned} C(b) = \text{minimum} \{ & \bar{c}x : x \text{ non-negative integer} \\ & \text{and } \bar{b} - \bar{R}x \geq 0 \} \end{aligned} \quad (4).$$

Notice that $F(b) = \pi b + C(b)$.

In [1], Gomory transformed (3) into an optimization problem over an abelian group G of order D where $D = |\det B|$. The problem is: Find $G(\mathfrak{B})$ where

$$\begin{aligned} G(\mathfrak{B}) = \text{minimum} \quad & \sum_{j=1}^m c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^m \alpha_j x_j = \mathfrak{B} \\ & x_j \text{ non-negative integer, } j = 1, \dots, m, \end{aligned} \quad (5)$$

where the α_j are elements of the group corresponding to the activities a_j , and \mathfrak{B} is the group element corresponding to b . Let Λ_k , $k = 0, 1, \dots, D-1$,

* In the discussion below, the symbol " $\bar{\quad}$ " over a vector or matrix will denote pre-multiplication by B^{-1} .

denote the elements of the group, and let $x^*(\Lambda_k)$, $k = 0, 1, \dots, D - 1$, denote optimal solutions to (5) when \mathfrak{B} is replaced by Λ_k .

The following is a restatement of theorem 1 of [1].

LEMMA 1: (Gomory) There exist non-negative integers u_i^* such that for all b satisfying $\bar{b}_i \geq u_i^*$, $i = 1, \dots, m$,

$$G(\mathfrak{B}) = C(b).$$

In words, if b satisfies the conditions of lemma 1, then (5) solves (1) in the sense that there is an optimal solution to (5) which is optimal in (1).

The group has a variety of isomorphic representations. For our purposes, the most convenient representation of G is as the factor group $M(I)/M(B)$ where $M(I)$ is the group of all lattice points in m -space and $M(B)$ is the subgroup of $M(I)$ consisting of all integer combinations of the columns of B . The group operation for $M(I)$ is ordinary addition. Let g be the mapping (canonical homomorphism) from $M(I)$ to $M(I)/M(B)$; that is, g maps integer vectors in m -space into their corresponding group elements.

Each element Λ in $M(I)/M(B)$ can be regarded as an equivalence class of elements in $M(I)$ where the integer vectors b and b^1 are in the same class Λ if there exists an integer vector w such that $b = b^1 + Bw$. If we transform the original space of the a_j with respect to B^{-1} , then \bar{b} and \bar{b}^1 are in the same class Λ if $\bar{b} = \bar{b}^1 + w$ for some integer vector w . We shall restrict our attention to $b \in K_B$ where

$$K_B = \{b: b \in M(I) \text{ and } \bar{b} \geq 0\} \quad (6)$$

Since the entire feasible region of m -space can be partitioned into sets of the form (6), our results obtain for all requirements vectors. Define*

* Let $[a]$ denote the integer part of a .

$$\lambda = b - B[B^{-1}b] \quad \text{for any } b \in \Lambda; \quad (7)$$

notice that λ is the minimal element in the set $\Lambda \cap K_B$ in the sense that $\bar{\lambda} \leq \bar{b}$ for any $b \in \Lambda \cap K_B$. Thus, for each $b \in \Lambda \cap K_B$, we have

$$\bar{b} = \bar{\lambda} + \sum_{i=1}^m [\bar{b}_i] e_i \quad (8)$$

where e_i is the i th unit vector in m -space.

TURNPIKE THEOREMS

The basis activities a_{n+i} , $i = 1, \dots, m$, are called optimal turnpike activities for the set K_B . The reason for this terminology is made clear by the following theorem which can be inferred directly from lemma 1.

TURNPIKE THEOREM 1: There exist non-negative integers t_i^* such that for all b satisfying $\bar{b}_i \geq t_i^*$, $i = 1, \dots, m$,

$$(i) \quad F(b) = c_{n+i} + F(b - a_{n+i}), \quad i = 1, \dots, m;$$

That is, an optimal immediate decision when the requirements are sufficiently deep in K_B is to choose any one of the optimal turnpike activities and proceed optimally thereafter.

Upper bounds on the t_i^* are (see [1] or [4])

$$t_i^* \leq (D - 1) \cdot \max \{0, \max_{j=1, \dots, n} [\bar{r}_{ij}]\} + 1.$$

The turnpike terminology can now be explained. Suppose the right hand side b in (1) is anywhere in the set $\{b: b \in K_B \text{ and } \bar{b}_i \geq t_i^*, i = 1, \dots, m\}$. The theorem states, in effect, that a minimal cost solution to (1) is a shortest route path connecting b to the origin with the property that an optimal first step is to choose any one of the optimal turnpike activities at zero relative cost (relative to the optimal LP solution). This myopic rule remains optimal through a sequence of decisions until b is reduced to b' such that $\bar{b} \geq \bar{b}'$ and either (i) b' is no longer in K_B and therefore some of the activities in the basis B are no longer LP optimal, or (ii) \bar{b}' is close to the zero vector and therefore unit amounts of all the turnpike activities cannot be economically chosen. Of course, (i) and (ii) can occur together. Figure 1 illustrates these ideas. The basic activities are shown with solid lines and the non-basics with dotted lines.

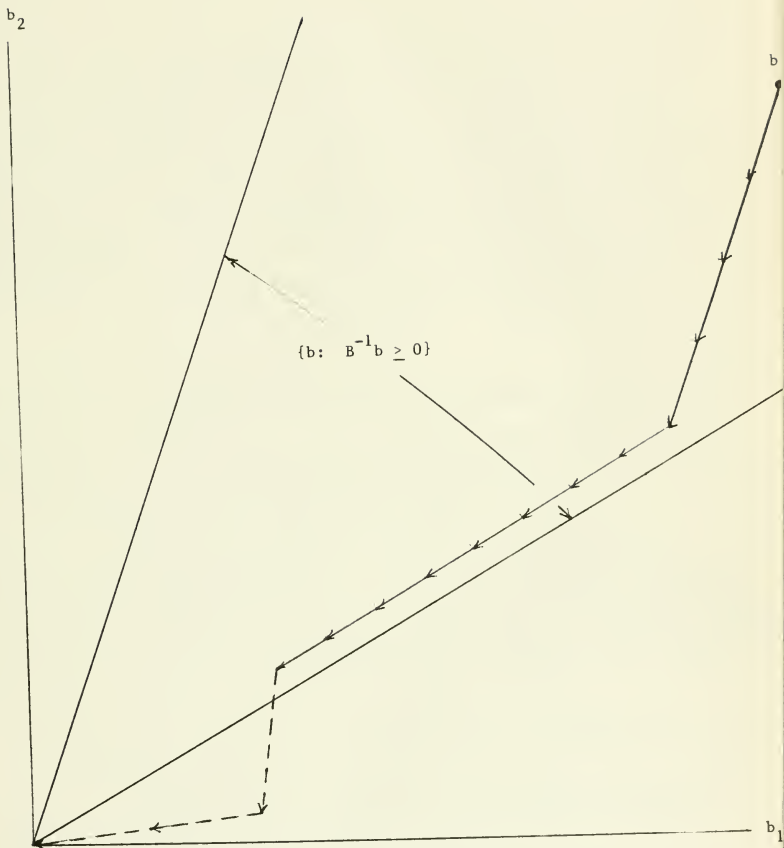


FIGURE 1

Thus, there is some justification for describing IP problems with b satisfying theorem 1 as steady-state. If b does not satisfy the condition of theorem 1, it may be that b is only partially transient in the sense of theorem 2 below. This theorem is the main result of this paper. Turnpike theorems for other discrete optimization problems can be found in [2] and [3].

TURNPIKE THEOREM 2: Given any set $I \subset \{1, \dots, m\}$, there exist non-negative integers t_i^{**} , $i \in I$, such that for all $b \in K_B$ satisfying $\bar{b}_i \geq t_i^{**}$, $i \in I$,

$$F(b) = c_{n+i} + F(b - a_{n+i}), \quad i \in I;$$

That is, an optimal immediate decision when the requirements are sufficiently deep in K_B with respect to a_{n+i} , $i \in I$, is to choose anyone of the turnpike activities a_{n+i} for $i \in I$, and proceed optimally thereafter.

To prove this theorem, we need some preliminary results.

LEMMA 2: If $b^1, b^2 \in \Lambda \cap K_B$ for some Λ and $\bar{b}^2 \geq \bar{b}^1$, then $C(b^2) \leq C(b^1)$

Proof: The proof follows directly from the fact that a feasible correction for (3) with \bar{b}^2 . To see that this is so, note that $\bar{b}^2 = \bar{b}^1 + u$, u non-negative integer. Thus, if x is non-negative integer and $\bar{b}^1 - \bar{R}x \geq 0$, it follows that $\bar{b}^2 - \bar{R}x \geq \bar{b}^1 - \bar{R}x \geq 0$.

A sequence $\mu = \{b^v\}_{v=1}^{\infty} \in \Lambda \cap K_B$ with the property that $\bar{b}^{v+1} \geq \bar{b}^v$,

$v = 1, 2, \dots$, is called a chain. The following lemma exposes a fundamental asymptotic property of integer programming problems.

LEMMA 3: Given a chain $\mu = \{b^v\}_{v=1}^{\infty} \in \Lambda \cap K_B$, there exists a v^* such that for all $v \geq v^*$

$$C(b^v) = C(b^{v^*}).$$

Proof: We consider the non-trivial case when $C(b^v) < \infty$ for all but a finite number of b^v . Then for any b^v such that $C(b^v) < \infty$, let $x^*(b^v)$ be an optimal correction. We have

$$C(b^v) = \bar{c}x(b^v) = (c_R - c_B B^{-1}R)x^*(b^v) = q(b^v)/D$$

where $q(b^v)$ is some non-negative integer because c_R , c_B , R and $x^*(b^v)$ are integers, and B^{-1} is a matrix of numbers of the form q/D . Since by lemma 2, the $q(b^v)$, $v = 1, 2, \dots$, are monotonically decreasing, and constitute a closed set bounded below by zero, we have $\lim_{v \rightarrow \infty} C(b^v)$ is attained; namely

$$\lim_{v \rightarrow \infty} C(b^v) = \frac{1}{D} \cdot \lim_{v \rightarrow \infty} q(b^v) = \frac{q(b^{v^*})}{D} \quad \text{for all } v \geq v^*.$$

Although theorem 2 could now be proven directly, we prove it by defining and using a new problem similar to (5). The problem is: Find

$$\begin{aligned} G(\Lambda; t_i, i \in I^C) &= \min \sum_{j=1}^n \bar{c}_j x_j \\ \text{s.t.} \quad \sum_{j=1}^n \alpha_{ij} x_j &= \Lambda \\ \sum_{j=1}^n \bar{r}_{ij} x_j &\leq (\bar{\lambda}_i + t_i), i \in I^C \end{aligned} \quad (9)$$

$$x_j = 0, 1, 2, \dots; j = 1, \dots, n,$$

where Λ , $I \subset \{1, \dots, m\}$, and the non-negative integers t_i , $i \in I^C$, are given.

It is important to recognize that for $b \in \Lambda$ with $I = \phi$ (the empty set) and $t_i = \lfloor \bar{b}_i \rfloor$, problem (9) is equivalent to (1). Thus, for $\mathcal{B} = g(b)$, $C(b) \geq G(\mathcal{B}; \lfloor \bar{b}_i \rfloor, i \in I^C) \geq G(\mathcal{B})$. Corollary 1 below is required for the proof of lemma 4.

COROLLARY 1: Given a chain $u = \{b^v\}_{v=1}^{\infty} \in \Lambda \cap K_B$, where

$$b^v = \bar{\lambda} + \sum_{i \in I^c} t_i' e_i + v \sum_{i \in I} e_i,$$

for given non-negative integers t_i' , $i \in I$, there exists a v^+ such that for all $v \geq v^+$

$$G(\Lambda; t_i', i \in I^c) = C(b^v).$$

Proof: This corollary can be proven either directly or by using lemma 3.

We give the direct proof. Solve (9) for $G(\Lambda; t_i', i \in I^c)$ thereby obtaining the optimal correction x^* . Then choose v^+ large enough so that the constraints $\sum_{j=1}^n \bar{v}_{ij} x_j^* \leq \lambda_i + v$, $i \in I$, hold for all $v \geq v^+$.

LEMMA 4: Given any set $I \subset \{1, \dots, m\}$, there exist non-negative integers u_i^{**} , $i \in I$, such that for all $b \in K_B$ satisfying $\bar{b}_i \geq u_i^{**}$, $i \in I$,

$$G(\mathcal{B}; [\bar{b}_i]), i \in I^c) = C(b)$$

where $\mathcal{B} = g(b)$ and $\bar{\lambda} = \bar{\beta} \equiv \bar{b} - [\bar{b}]$ in (9). In other words, if b satisfies the above inequalities, then there is an optimal non-negative integer correction from (9) that is optimal in (3).

Proof: The proof is by induction on $|I^c|$. The induction hypothesis is: The lemma holds for all sets I such that $|I^c| = p$, $p = 0, 1, 2, \dots, m-1$. The induction hypothesis is true when $p = 0$ because that is the case of lemma 1.

Suppose the induction hypothesis is true for $p = \ell$. It clearly suffices to show that the result holds for $p = \ell + 1$ and $b \in \Lambda \cap K_B$ because the collection of distinct Λ is finite.

Let J_0, J_1, \dots, J_S be the family of subsets of $\{1, \dots, m\}$ containing I and suppose $J_0 = \{1, \dots, m\}$ and $J_S = I$. For $s = 0, 1, \dots, S-1$, $|J_s^c| < |I^c|$ and thus the induction hypothesis holds for I replaced by J_s in the lemma.

Let $\{u_i^{**}(J_s) : i \in J_s\}$ denote a set of u_i^{**} satisfying the theorem for $s = 0, \dots, S-1$. Define

$$u_i^+ = \begin{cases} \max & u_i^{**}(J_s), i \in I \\ & s = 0, 1, \dots, S-1 \\ \\ \max & u_i^{**}(J_s), i \in I^c \\ & s = 0, 1, \dots, S-1 \\ & \text{and} \\ & i \in J_s \end{cases}$$

where $u_i^{**}(J_0) = u_i^*$, $i = 1, \dots, m$, from lemma 1. We partition $\{b : b \in \Lambda \cap K_B\}$ as follows:

$$\{b : b \in \Lambda \cap K_B\} = \bigcup_{s=0}^S Q_s$$

where

$$Q_s = \{b : b \in \Lambda \cap K_B \text{ and } \bar{b}_i \geq u_i^+, i \in I^c \cap J_s, \text{ and } \bar{b}_i < u_i^+, i \in I^c \cap J_s^c$$

$$s = 0, 1, 2, \dots, S.$$

Again, it clearly suffices to exhibit u_i^{**} , $i \in I$, for which the lemma holds for b in each of the sets Q_s in the finite family.

For Q_0 , take $u_i^{**} = u_i^*$, $i \in I$. To see that this choice is valid, note that for $b \in Q_0$ such that $\bar{b}_i \geq u_i^+$, $i \in I^c$, and $\bar{b}_i \geq u_i^*$, $i \in I$, we have by construction that $\bar{b}_i \geq u_i^*$, $i = 1, \dots, m$. Thus by lemma 1.

$$G(\Lambda) = C(b)$$

and the lemma holds.

For Q_s , $s = 1, \dots, S-1$, take $u_i^{**} = u_i^*(Q_s)$, $i \in I$. Thus for $b \in Q_s$ such $\bar{b}_i \geq u_i^*(Q_s)$, $i \in I$, we have by construction that $\bar{b}_i \geq u_i^+ \geq u_i^*(Q_s)$, $i \in I^c \cap Q_s$, and by the induction hypothesis,

$$G(\Lambda; \mathbf{[b_i]}, i \in I^c) = C(b).$$

Finally, consider the set

$$Q_S = \{b: b \in \Lambda \cap K_B \text{ and } \bar{b}_i < u_i^+, i \in I^c\}$$

As we have mentioned in the introduction, any $b \in Q_S$ when mapped into \bar{b} is of the form $\bar{\lambda} + \sum_{i=1}^m t_i e_i$ for any non-negative t_i . For the moment, consider

$b \in Q$ such that $\bar{b} = \bar{\lambda} + \sum_{i \in I^c} t_i e_i$. Since there can only be a finite number of the t_i , $i \in I^c$, which satisfy the conditions defining Q_S , there are only a finite number of \bar{b} of this form.

Pick an arbitrary one, say $\lambda + \sum_{i \in I^c} t_i e_i$ and construct the chain

$$\lambda + \sum_{i \in I^c} t_i e_i + v \sum_{i \in I} e_i, v = 0, 1, 2, \dots \quad (10)$$

By corollary 1, there is a v^+ such that for all $v \geq v^+$,

$$\begin{aligned} & C(\lambda + \sum_{i \in I^c} t_i e_i + v \sum_{i \in I} e_i) \\ &= G(\Lambda; t_i, i \in I^c). \end{aligned}$$

For each of the finite number of $\lambda + \sum_{i \in I^c} t_i e_i$, let $u_i^{***} = v^+$, and let u_i^{**} be the maximum of these.

Thus, if $b \in Q_S$ and $t_i \geq u_i^{**}$, $i \in I$, then by construction b is far enough

out on one of the chains (10) so that

$$C(b) = G(\Lambda; t_i^1, i \in I^C).$$

where $t_i = [\bar{b}_i]$, $i \in I^C$. This completes the proof.

COROLLARY 2: If $I_1 \subset I_2$ and $t_i^2 \geq t_i^1$, $i \in I_2^C$, then $G(\Lambda; t_i, i \in I_2^C) \leq G(\Lambda; t_i, i \in I_1^C)$.

Proof of Theorem 2: Consider any $b' \in K_B$ such that

$$\bar{b}' = \beta + \sum_{i \in I^C} [\bar{b}'_i] e_i + \sum_{i \in I} u_i^{**} e_i.$$

By lemma 4, there is an optimal correction x^* in (9) which is also optimal in

(3). The corresponding basic value y_i , $i \in I$, are

$$y_i = \bar{b}'_i - \sum_{j=1}^n \bar{r}_{ij} x_j^*.$$

Consider any other $b'' \in K_B$ such that

$$\bar{b}'' = \beta + \sum_{i \in I^C} [\bar{b}''_i] e_i + \sum_{i \in I} (u_i^{**} + t_i) e_i$$

where the t_i are non-negative integers greater than zero. It is clear that

x^* remains optimal in (9) derived from b'' because (9) is unaffected by the

change. By lemma 4, x^* remains optimal in (3). The corresponding basic values

y_i , $i \in I$, are

$$y_i' = \bar{b}''_i - \sum_{j=1}^n \bar{r}_{ij} x_j^* = y_i + t_i \geq y_i + 1 \geq 1.$$

Therefore, take $t_i^{**} = u_i^* + 1$, $i \in I$. Then if $b \in K_B$ such that $\bar{b}_i \geq t_i^{**}$, $i \in I$,

by the above argument, there is an optimal correction x^* in (3) such that for

$i \in I$,

$$y_i = \bar{b}_i - \sum_{j=1}^n \bar{r}_{ij} x_j^* \geq 1.$$

This proves the theorem.

SHADOW PRICES FOR INTEGER PROGRAMMING**

Our results can be used to investigate the concept of shadow or dual prices for integer programming problems. We consider the simplest case first. When all of the requirements vectors b , $b - e_i$, $b + e_i$ in (1) satisfy the conditions of theorem 1, it is clear that

$$\Delta F^+ \equiv F(b + e_i) - F(b) = \pi_i + G(\mathcal{B} + \varepsilon_i) - G(\mathcal{B}),$$

while

$$\Delta F^- \equiv F(b) - F(b - e_i) = \pi_i + G(\mathcal{B}) - G(\mathcal{B} - \varepsilon_i)$$

where $\varepsilon_i = g(e_i)$. The terms $G(\mathcal{B} + \varepsilon_i) - G(\mathcal{B})$ and $G(\mathcal{B}) - G(\mathcal{B} - \varepsilon_i)$ may be independently positive or negative. Their signs as well as their magnitudes depend on the coefficients c_j and a_{ij} of the given problem*. Thus, we may conclude that the change in objective function that results from a unit change in one requirement is π_i plus a positive or negative correction which depends on the combinatorial structure of the given problem. Moreover, the change in objective function is not symmetric for corresponding positive and negative changes in requirements.

Suppose now that we consider an IP problem (1) such that b , $b - e_i$ and $b + e_i$ do not all satisfy the conditions of theorem 1. Suppose, however, that there is a set I such that the conditions of lemma 4 hold for these vectors. In this case,

* It can be shown, however that $G(\Lambda) \leq (D - 1) \cdot \max_{i=1, \dots, m} \pi_i$ for any Λ (see theorem 2 of [1]).

** The author is indebted to Laurence Wolsey for valuable discussions about the ideas in this section.

$$\Delta F = \pi_i + G(\mathcal{B} + \epsilon_i, [\bar{b}_i + \bar{e}_i], i \in I^C) \\ - G(\mathcal{B}; [\bar{b}_i], i \in I^C),$$

while

$$\Delta F^- = \pi_i + G(\mathcal{B}; [\bar{b}_i], i \in I^C) \\ - G(\mathcal{B} - \epsilon_i; [\bar{b}_i - \bar{e}_i], i \in I^C).$$

By corollary 2, the $G(\cdot; \cdot)$ terms tend to be smaller as $|I|$ and $[\bar{b}_i], i \in I^C$, get larger. Thus, in this qualitative sense, the π_i become better estimates of the change in objective function as an IP problem becomes more steady-state.

Other sensitivity or post-optimality tests can be devised. For example, suppose the new activity a_{m+n+1} with cost coefficient c_{m+n+1} is to be considered after (1) has been solved. Suppose, in the simple case, that (5) solved (1). Then a sufficient condition that the new activity cannot reduce the cost of an optimal solution to (1) is

$$\bar{c}_{m+n+1} \geq G(\alpha_{m+n+1}) \quad (11)$$

where $\bar{c}_{m+n+1} = c_{m+n+1} - \pi a_{m+n+1}$, and $\alpha_{m+n+1} = g(a_{m+n+1})$. However, even if $\bar{c}_{m+n+1} < G(\alpha_{m+n+1})$, and (5) is reoptimized, it may be that the optimal solution is unchanged.

If (5) did not solve (1), a sufficient condition can be given for ignoring the new activity.

LEMMA 5: The new activity a_{m+n+1} with cost coefficient c_{m+n+1} cannot be used to reduce the cost $C(b)$ of an optimal correction in (3) if

$$(1) \quad \bar{c}_{m+n+1} \geq 0,$$

and

$$(ii) \quad k \bar{c}_{m+n+1} + G(\mathcal{B} - k\alpha_{m+n+1}) \geq C(b)$$

for $k = 1, 2, \dots, p$,

where p is the order of α_{m+n+1} in G .

Proof: Consider the consequences of using the new activity k' times for $k' \leq p$. The effect is to change b in (1) to $b' = b - k'a_{m+n+1}$. The value $G(\mathcal{B} - k'\alpha_{m+n+1})$ is a lower bound on the residual problem (1) with requirements vector b' . Therefore $k' \bar{c}_{m+n+1} + G(\mathcal{B} - k'\alpha_{m+n+1})$ is a lower bound on the value of using a_{m+n+1} k' times and proceeding optimally thereafter.

If

$$k' \bar{c}_{m+n+1} + G(\mathcal{B} - k'\alpha_{m+n+1}) \geq C(b),$$

Then for any non-negative integer i ,

$$(k' + ip) \bar{c}_{m+n+1} + G(\mathcal{B} - k\alpha_{m+n+1} - ip\alpha_{m+n+1})$$

$$\geq k' \bar{c}_{m+n+1} + G(\mathcal{B} - k'\alpha_{m+n+1}) \geq C(b).$$

The first inequality follows because $\bar{c}_{m+n+1} \geq 0$ and $ip\alpha_{m+n+1}$ equals the group identity element. Thus, an improvement cannot be achieved by using a_{m+n+1} $k + ip$ times, for $i = 0, 1, 2, \dots$. A simple induction argument gives the desired result.

As a final remark, we note that these last considerations indicate how a column generating procedure for integer programming might be devised. In the simple case when (5) solve (1), it is sufficient to generate an

optimal LP basis plus those non-basics required for a true optimization of (5) for Λ_k , $k = 0, 1, \dots, D - 1$. The test for the basics is the usual one of non-negative relative cost factor. The test for the useful non-basics is (11). Condition (ii) of lemma 5 is relevant when (5) does not solve (1). The implementation of these ideas to decomposable integer programming problems is an area of current research.

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