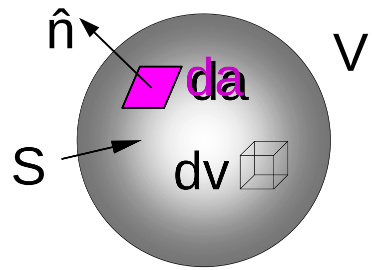
ELECTROMAGNETIC FIELDS AT BOUNDARIES

Differential Form of Maxwell's Equations:

$$\nabla\times\overline{E} = -\partial\overline{B}/\partial t, \qquad \nabla\times\overline{H} = \overline{J} + \partial\overline{D}/\partial t \qquad \nabla\bullet\overline{D} = \rho, \qquad \nabla\bullet\overline{B} = 0$$

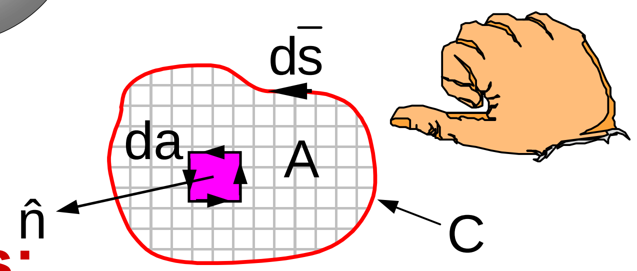
Gauss's Divergence Theorem:

$$\int_V (\nabla\bullet\overline{D})\,dv = \int_S \overline{D}\bullet\hat{n}\,da$$



Stoke's Theorem:

$$\int_A (\nabla\times\overline{G})\bullet\hat{n}\,da = \oint_C \overline{G}\bullet d\overline{s}$$



Integral Form of Maxwell's Equations:

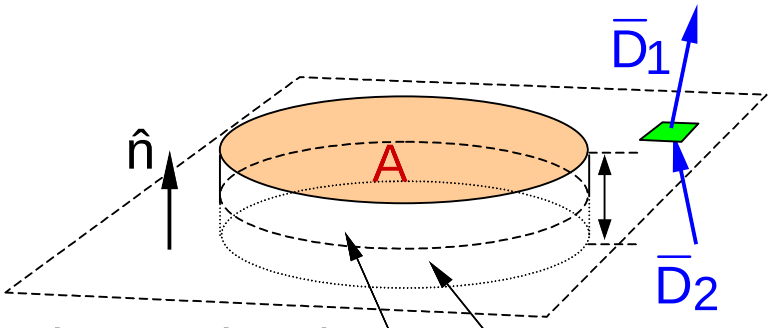
$\int_S \overline{D}\bullet\hat{n}\,da = \int_V \rho\,dv, \quad \int_S \overline{B}\bullet\hat{n}\,da = 0$	Gauss's Laws
$\oint_C \overline{E}\bullet d\overline{s} = -\frac{d}{dt}\int_A \overline{B}\bullet\hat{n}\,da$	Faraday's Law
$\oint_C \overline{H}\bullet d\overline{s} = \int_A \overline{J}\bullet\hat{n}\,da + \frac{d}{dt}\int_A \overline{D}\bullet\hat{n}\,da$	Ampere's Law

FIELDS PERPENDICULAR TO BOUNDARIES

Using Gauss's Law $\left(\int_S \overline{D} \bullet \hat{n}\, da = \int_V \rho\, dv\right)$:

$$\int_S \overline{D} \bullet \hat{n}\, da = (D_{1n} - D_{2n})A \qquad (\lim \delta \to 0)$$

$$= \int_V \rho\, dv = \sigma_S A$$



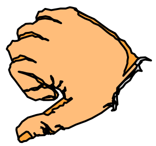
Therefore:

$$D_{1n} - D_{2n} = \sigma_S \quad \text{yields:} \qquad \hat{n} \bullet (\overline{D}_1 - \overline{D}_2) = \sigma_S$$

$$\int_S \overline{B} \bullet \hat{n}\, da = (B_{1n} - B_{2n})A = 0 \quad \text{yields:} \qquad \hat{n} \bullet (\overline{B}_1 - \overline{B}_2) = 0$$

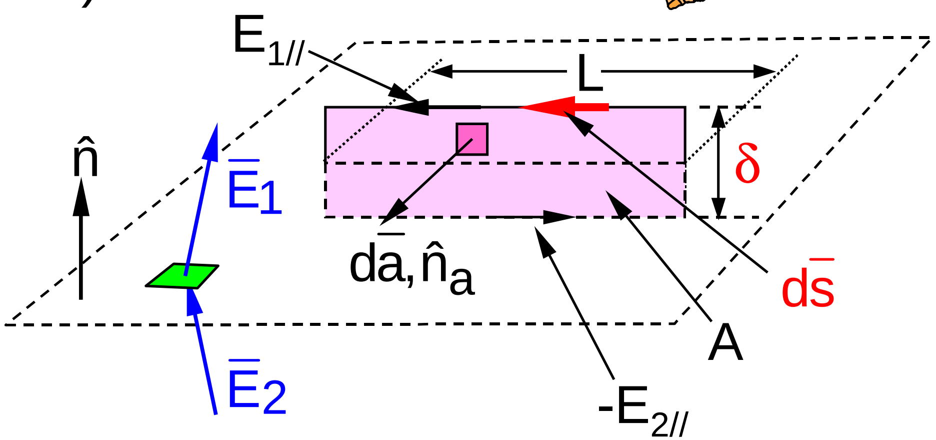
BOUNDARY CONDITIONS FOR PARALLEL FIELDS

Using Faraday's Law: $\oint_C \overline{E} \bullet d\overline{s} = -\frac{d}{dt}\int_A \overline{B} \bullet \hat{n} da$:



$$\oint_C \overline{E} \bullet d\overline{s} = (E_{1//} - E_{2//})L \qquad (\lim \delta \to 0)$$

$$= -\frac{d}{dt}\int_A \overline{B} \bullet d\overline{a} \to 0$$



Therefore $E_{1//} = E_{2//}$ and

$$\hat{n} \times (\overline{E}_1 - \overline{E}_2) = 0$$

Using Ampere's Law: $\oint_c \overline{H} \bullet d\overline{s} = \int_A \overline{J} \bullet d\overline{a} + \frac{d}{dt}\int_A \overline{D} \bullet d\overline{a}$

$$\oint_c \overline{H} \bullet d\overline{s} = (H_{1//} - H_{2//})L \qquad (\lim \delta \to 0)$$

$$= \int_A \overline{J} \bullet d\overline{a} + \frac{d}{dt}\int_A \overline{D} \bullet d\overline{a} \to (\overline{J}_s \bullet \hat{n}_a)L$$

Therefore: $\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s$

FIELDS INSIDE PERFECT CONDUCTORS

Electric Fields Inside Perfect Conductors:

If $\sigma \to \infty$ and $E \neq 0$: Then $\bar{J} = \sigma\bar{E} \to \infty$

But if $\bar{J} \to \infty$: Then $\bar{H} \to \infty$ since $\nabla \times \bar{H} = \bar{J} + \partial\bar{D}/\partial t$

But if $\bar{H} \to \infty$: Then $W_m = \mu H^2/2 \to \infty$ and $w_m \to \infty$

But w_m cannot $\to \infty$: Therefore **$\bar{E} = 0$ inside** perfect conductors

Since $\bar{E} = 0$ inside: Therefore **$\rho = 0$ inside** since $\nabla \bullet \varepsilon\bar{E} = \rho$

Magnetic Fields Inside Perfect Conductors:

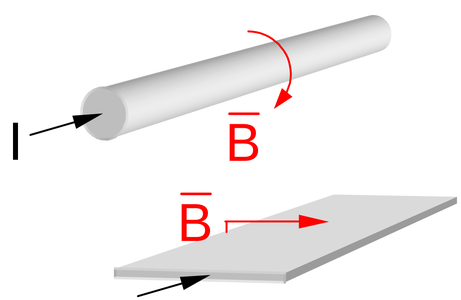
Since $\bar{E} = 0$ and $\nabla \times \bar{E} = -\partial\bar{B}/\partial t$, therefore $\partial\bar{B}/\partial t = 0$

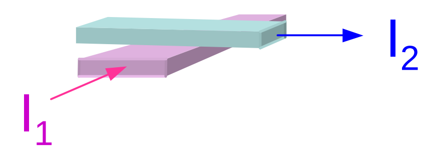
And therefore: **$\bar{B} = 0$ inside** perfect conductors if $\sigma = 0$ first

Superconductors:

$\bar{B} \cong 0$ inside

Superconductivity fails above a critical $\bar{B}_{crit} = f(T_{emperature})$ outside[1]





Thus currents in superconducting wires are limited (use ribbons) Cryotrons (Prof. Dudley Buck ~'58)

I_1 turns off I_2 (switches and logic)

BOUNDARY CONDITIONS, PERFECT CONDUCTORS

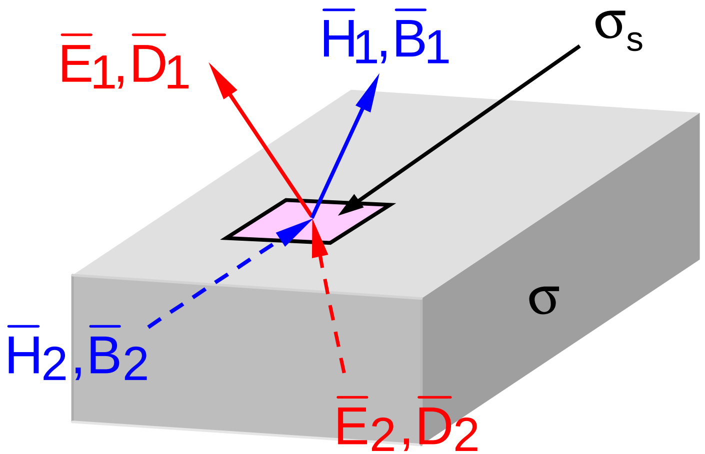
General Boundary Conditions:

$$\hat{n} \bullet (\overline{D}_1 - \overline{D}_2) = \sigma_s$$

$$\hat{n} \bullet (\overline{B}_1 - \overline{B}_2) = 0$$

$$\hat{n} \times (\overline{E}_1 - \overline{E}_2) = 0$$

$$\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s$$



$\overline{D}_2 = \overline{B}_2 = \overline{E}_2 = 0$ inside $\sigma = \infty$:

$$\hat{n} \bullet \overline{D}_1 = \sigma_s$$

$$\hat{n} \bullet \overline{B}_1 = 0$$

$$\hat{n} \times \overline{E}_1 = 0$$

$$\hat{n} \times \overline{H}_1 = \overline{J}_s$$

$\Rightarrow \overline{B}$ is parallel to perfect conductors

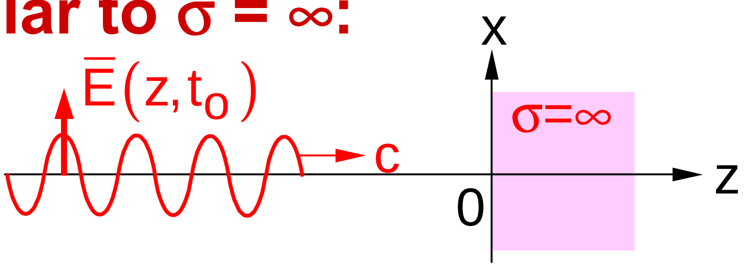
$\Rightarrow \overline{E}$ is perpendicular to perfect conductors

REFLECTIONS FROM PERFECT CONDUCTORS

Solution Method for Boundary Value Problems:

1) Write fields in terms of unknown coefficients (no boundaries); typically a sum of terms
2) Write equations for fields that satisfy boundary conditions
3) Solve for unknowns and check answer with Maxwell Equations

Example—Plane Wave Perpendicular to σ = ∞:



1) Incident: $\underline{\overline{E}}_i = \hat{y}E_o e^{-jkz}$

Reflected: $\underline{\overline{E}}_r = \hat{y}\underline{\overline{E}}_r e^{+jkz}$

Transmitted: $\underline{\overline{E}}_t = \hat{y}\underline{\overline{E}}_t e^{-jkz} = 0$ here

2) Match B.B.: $\underline{\overline{E}}_{//}(0) = 0$:

$$\underline{\overline{E}}_i(z=0) + \underline{\overline{E}}_r(z=0) = \underline{\overline{E}}_t(z=0) = 0$$

3) Solve: $\hat{y}\left[E_o e^{-jk0} + \underline{E}_r e^{+jk0}\right] = 0 \Rightarrow \underline{E}_r = -E_o$

"Standing wave"

PURE STANDING WAVES

Waves Reflected by Perfect Conductor:

Incident: $\underline{\overline{E}}_i = \hat{y}E_o e^{-jkz}$ Reflected: $\underline{\overline{E}}_r = -\hat{y}E_o e^{+jkz}$

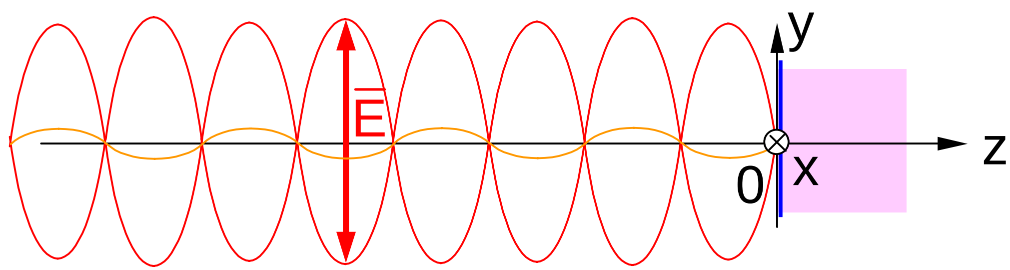
$\underline{\overline{H}}_i = -\hat{x}(E_o/\eta_o)e^{-jkz}$ $\underline{\overline{H}}_r = \hat{x}(E_o/\eta_o)e^{-jkz}$

Total: $\underline{\overline{E}} = \hat{y}E_o\left(e^{-jkz} - e^{+jkz}\right) = -2j\hat{y}E_o \sin kz$

$\underline{\overline{H}} = \hat{x}(E_o/\eta_o)\left(e^{-jkz} + e^{+jkz}\right) = 2\hat{x}(E_o/\eta_o)\cos kz$

Time Domain: $\overline{E}(t,z) = R_e\left\{\underline{\overline{E}}e^{j\omega t}\right\} = 2\hat{y}E_o(\sin kz)\sin\omega t$

$\overline{H}(t,z) = R_e\left\{\underline{\overline{H}}e^{j\omega t}\right\} = 2\hat{x}(E_o/\eta_o)(\cos kz)\cos\omega t$



Surface Charge: $\sigma_s = \hat{n} \bullet \overline{D}_n = 0 \quad \left[Cm^{-2}\right]$

Surface Current: $\overline{J}_s = \hat{n} \times \overline{H}_1 = 2\hat{y}(E_o/\eta_o)\cos\omega t \quad \left[Am^{-1}\right]$

POWER AND ENERGY IN STANDING WAVES

Waves Reflected by a Perfect Conductor:

$$\underline{\overline{E}} = -2j\hat{y}E_o \sin kz$$
$$\underline{\overline{H}} = 2\hat{x}(E_o/\eta_o)\cos kz$$

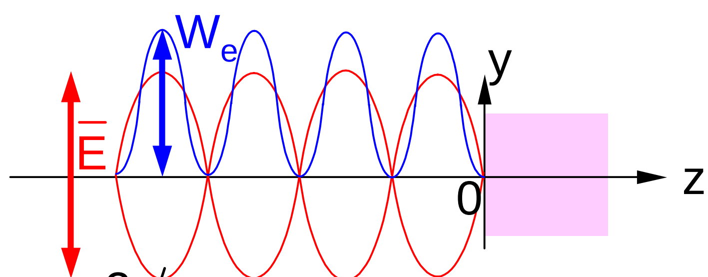
Since: $\overline{E}(t,z) = R_e\{\underline{\overline{E}}e^{j\omega t}\} = 2\hat{y}E_o(\sin kz)\sin\omega t$

$\overline{H}(t,z) = R_e\{\underline{\overline{H}}e^{j\omega t}\} = 2\hat{x}(E_o/\eta_o)(\cos kz)\cos\omega t$

Then: $\overline{S}(t,z) = \overline{E}\times\overline{H} = \hat{z}(E_o^2/\eta_o)(\sin 2kz)\sin 2\omega t$

$\underline{\overline{S}}(z) = \underline{\overline{E}}\times\underline{\overline{H}}^* = -2j\hat{z}(E_o^2/\eta_o)\sin 2kz$

And: $W_e(t,z) = \varepsilon|\overline{E}(t,z)|^2/2 = 2\varepsilon E_o^2(\sin^2 kz)\sin^2\omega t \quad [Jm^{-3}]$



$W_m(t,z) = \mu|\overline{H}(t,z)|^2/2 = 2\mu(E_o/\eta_o)^2(\cos^2 kz)\cos^2\omega t \quad [Jm^{-3}]$