6.013 Lecture 24:
Overview of 6.013; Perturbations of Resonators

A. Overview of 6.013, Electromagnetics and Applications

"Electromagnetics and Applications" presents in a one-semester subject those fundamentals of electromagnetism that underlie most systems encountered by electrical engineers and computer scientists. Although these systems are highly diverse, the simplicity of Maxwell's equations and the power of basic physical concepts make this objective practical. The applications addressed range from wireless communications to analog and digital circuits, motors and generators, MEMS devices, microwave and photonic devices and systems, and even acoustics.

The mathematical techniques employed included vector calculus, phasor representation, partial differential equations, difference equations, and their precursors. Several problem solving techniques were also used, such as perturbation methods, energy methods, duality, and methods for solving boundary value problems.

The broad nature of electromagnetics requires an understanding not only of Maxwell's equations, but also of mechanics, basic quantum phenomena, circuits, signals, and linear systems in general. All of these domains were exercised in 6.013 in the context of a wide range of practical applications and systems, making 6.013 a sort of capstone subject for much of the basic undergraduate electrical engineering curriculum, and a stepping stone to professional practice.

Follow-on electromagnetics subjects tend to focus on either the quasistatic limit, where device dimensions are very small compared to a wavelength so that wave phenomena become irrelevant, or on the wave limit where the dimensions are comparable to a wavelength or much larger. Both types of follow-on subject typically present more rigorous and complete solutions to Maxwell's equations, and address narrower sets of applications and physical issues in greater depth. 6.013 is merely an introduction to this broad technical area of wide application and impact.

B. Perturbations of LC Resonators

Lossless resonators are characterized by their resonant frequencies f_i, of which ideal distributed systems have an infinite number and simple ideal LC resonators have but one. It is often useful to shift these resonances dynamically or during manufacture, or to use an observed resonant frequency to characterize some perturbation of interest. For example, 1) a receiver might be tuned to various frequencies by perturbing a resonant filter, 2) a manufacturer could tune a single type of filter to serve diverse customers, 3) humidity can be sensed by its impact on the permittivity of air in a resonant cavity and therefore on the cavity's resonant frequency, and 4) the electromagnetic field distribution within a resonator can be measured by noting how much its resonant frequency is shifted by a small dielectric probe sphere as it moves to various positions within the resonator.

In this lecture two approaches to calculating these shifts in resonant frequency will be studied, one in the context of LC resonators, and one based on photons, forces, and energy. The second approach is simple and powerful, and can even be applied to acoustic resonators and the formation of vowels by the human vocal tract.

Consider the LC resonator illustrated in Figure 24-1a. Its resonant frequency $\omega_o = (LC)^{-0.5}$. If we perturb this resonator by bringing the capacitor plates a little closer together, C will increase slightly and ω_o will decrease slightly.

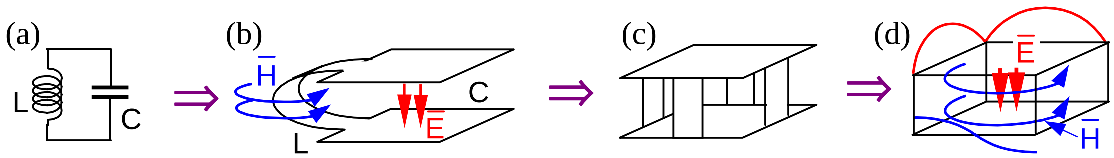


Figure 24-1. Transformation of an LC resonator into a cavity resonator

This LC resonator resembles a cavity resonator in its lowest frequency mode, and can be similarly perturbed. It is easy to see how an LC resonator might consist of two plates connected by a ribbon (Figure 24-1b), where most of the electric energy is stored between the plates, and most magnetic energy is stored next to the conducting ribbon. But we could have four ribbons (Figure 24-1c), and could even widen them to form a closed cavity resonator, where the electromagnetic fields are standing waves that match the boundary conditions (Figure 24-1d). These fields correspond to the TE_{10} mode in rectangular waveguide, where the waveguide is shorted at both ends, $\lambda_g/2$ apart; this is designated the TE_{101} resonance of the cavity resonator, where the letters and the first two integers indicate the waveguide mode, and the third integer corresponds to the number of half-waveguide-wavelengths in the longest dimension of the cavity.

Figure 24-1d shows the electric and magnetic fields in the cavity at some instant of time; both have the same sinusoidal spatial character (indicated by the truncated sinusoids in the figure) and oscillate sinusoidally in time. The f_{101} resonant frequency can be lowered slightly by pressing the top and bottom surfaces of the cavity resonator together near the middle of the top or bottom cavity walls where the electric field is strong and the magnetic field is weak; lowering 'C' in this way increases ω_o according to the formula. The resulting frequency shift can be estimated quantitatively using the energy method presented below.

C. Energy Method for Estimating Resonant Frequency Perturbations

This method is based upon the facts that: 1) electromagnetic waves can alternatively be considered as collections of photons at frequency f, each with energy hf [J], so that the total energy w_T in a resonator simply equals nhf, where n is the number of photons, and 2) the number n of photons in a closed container remains constant if the walls of the container move only slowly, i.e. they move much less than a wavelength in much less than one cycle. Therefore if wall motion causes the electromagnetic energy trapped in a lossless resonator to increase by Δw_T, then the resonant frequency must increase by Δf, where:

$$\Delta f = \Delta w_T / nh \tag{1}$$

Wall motion changes the stored electromagnetic energy because that motion does work on the fields if the motion is in a direction opposite to the electromagnetic force on the wall or, alternatively, extracts energy from the fields if the wall motion is in the same direction. Therefore to compute the frequency increase Δf as resonator walls are indented or protruded slightly we need only to compute the electromagnetic force density vector $\overline{F}$ [Nm^{-2}] and the resulting Δw_T before using (1).

We have seen earlier in Equations R13-6 and L13-15 that electric and magnetic force densities acting on perfectly conducting non-magnetic walls in vacuum are simply equal to the corresponding electric and magnetic energy densities adjacent to the walls, where the electric fields pull on the walls and the parallel magnetic fields push, as suggested in Figure 24-2a for the cavity resonator of Figure 24-1d. That is,

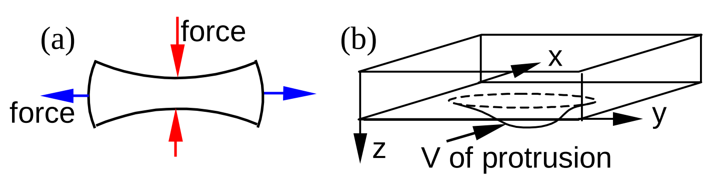
$$F_e = W_e = \varepsilon_o |\underline{E}|^2 / 4 \quad \text{(attractive force)} \quad \left[Nm^2\right]\left[Jm^{-3}\right] \tag{2}$$

$$F_m = W_m = \mu_o |\underline{H}|^2 / 4 \quad \text{(repulsive force)} \quad \left[Nm^2\right]\left[Jm^{-3}\right] \tag{3}$$

The net attractive force density F_{em} at any point on the wall is thus:

$$F_{em} = W_e - W_m \ [Nm^{-2}] \tag{4}$$

Figure 24-2. Cavity resonator shape perturbations



The increase in electromagnetic energy Δw_T equals the area integral of the net attractive electromagnetic force density F_{em} on the wall times the distance the wall protrudes outward as a result of the perturbation. That is:

$$\Delta w_T = \int dz \int F_{em}(x,y)dxdy = \int_V F_{em}(x,y)dv = \int_V (W_e - W_m)dv \ [J] \tag{5}$$

$$= \Delta(w_e - w_m) \ [\text{Joules in volume V added by the change in shape}] \tag{6}$$

where V is the volume enclosed by the combined original and deformed surface contours, as illustrated in Figure 24-2b. We also assume that V is sufficiently small that $\overline{E}$ and $\overline{H}$

are constant within it, and that the shape of the deformation is without sharp edges so that $\overline{E}$ and $\overline{H}$ are minimally changed by it and remain roughly perpendicular and parallel, respectively, to the new wall position.

We can now find the frequency perturbation by substituting (6) into (1) to obtain:

$$\Delta f/f = \Delta(w_e - w_m)/w_T \tag{7}$$

where w_T is the total energy in the resonator prior to the perturbation, and w_e and w_m are the average electric and magnetic energies stored within the small perturbation volume V, assuming the field strengths in V equal their unperturbed values at the wall.

D. Examples of Resonator Frequency Perturbations

An LC resonator provides a simple example of frequency perturbation. Suppose the plate separation of the capacitor C increases from d to d + Δd, then C will decrease and the resonant frequency will increase from f_o to $f_o + \Delta f$; what is Δf [Hz]?

If we calculate Δf directly using $f = (LC)^{-0.5}/2\pi$ and $C = \varepsilon A/d$ as $d \rightarrow d+\Delta d$, then we find $\Delta f \cong (\Delta d/2d)f_o$. This result can readily be compared to that for the energy perturbation method of Equation (7) by first finding $\Delta w_e/w_T$. First we evaluate Δw_e:

$$\Delta w_e = W_e V = (\varepsilon\,|\,\overline{\underline{E}}\,|^2/4)(A\Delta d)\ [J] \tag{8}$$

Since $\Delta w_m = 0$ in V, and since $w_T = 2w_e = (2\varepsilon|\,\overline{\underline{E}}\,|^2/4)Ad$, it follows that:

$$\Delta f/f_o \cong \Delta(w_e - w_m)/w_T \cong (\varepsilon\,|\,\overline{\underline{E}}\,|^2/4)(A\Delta d)/\,(2\varepsilon|\,\overline{\underline{E}}\,|^2/4)Ad \tag{9}$$

$$\cong \Delta d/2d$$

which is the same answer as determined above directly.

A second example of the use of the frequency perturbation equation is that of the metallic cavity resonator with flexible walls, illustrated in Figures 24-1 and 24-2. In this case we shall merely identify those wall areas where indentation increases or decreases resonant frequencies.

Consider the cavity resonator illustrated in Figure 24-3a; it is simply an elongated version of that in Figure 24-1. Referring to Equation (7) it is clear that if we indent the cavity where the electric energy density exceeds the magnetic energy density (and therefore where the attractive electric forces exceed the repulsive magnetic ones), then Δw_e and also $\Delta(w_e - w_m)$ within the perturbation volume V will be negative (negative because W_e in V is removed by indentation), and therefore Δf will be negative too. Qualitatively, the illustrated fields suggest w_e will exceed w_m only at the central portions of the top and bottom plates, as indicated by small ellipses. Elsewhere any indentation will press against a greater magnetic repulsive force, and will therefore increase the stored electromagnetic energy and the corresponding resonant frequency f_o so that $\Delta f > 0$.

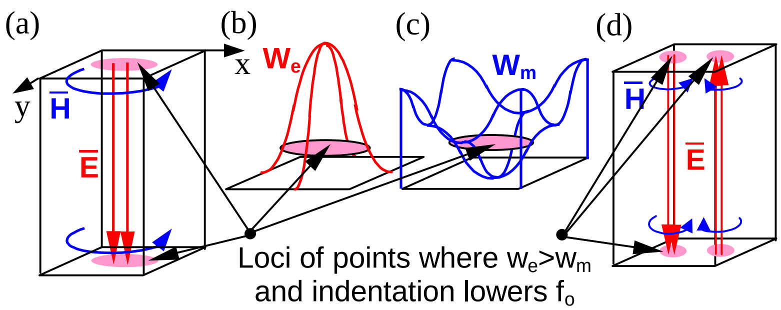
A quantitative answer requires examination of the expressions for the electric and magnetic fields so that the stored energy densities can be evaluated exactly. In this case we shall only examine the forms of those equations, which are:

$$W_e \propto \sin^2(\pi x/a)\,\sin^2(\pi y/a) \quad [\mathrm{J\ m^{-3}}] \tag{10}$$

$$W_m \propto \cos^2(\pi x/a) + \cos^2(\pi y/a) \quad [\mathrm{J\ m^{-3}}] \tag{11}$$

as sketched in Figure 24-3b and 24-3c, respectively.

Figure 24-3. Loci of points where indentation reduces the resonant frequency of a cavity resonator



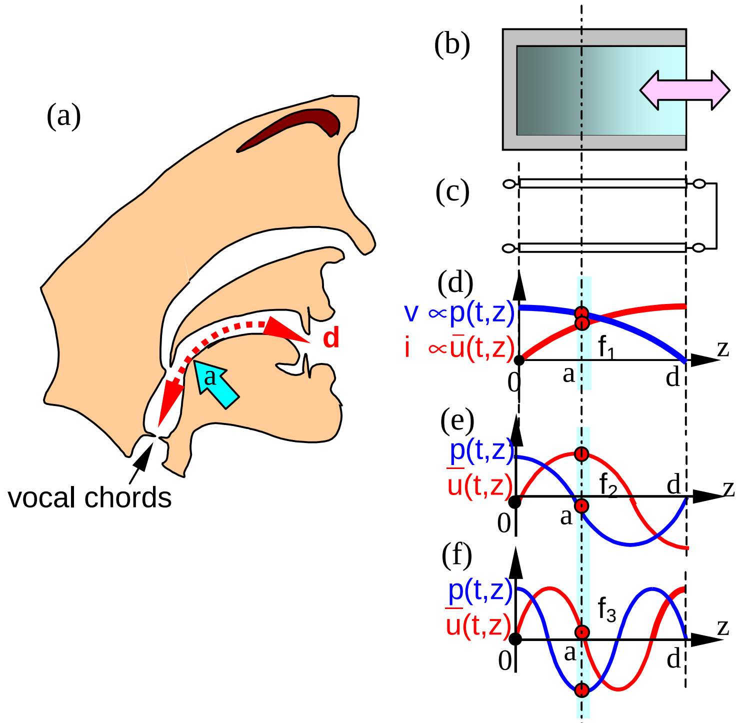
One application of this technique is determination of the mode responsible for a particular observed resonant frequency. Each resonant mode has a unique set of regions within which indentation reduces the resonant frequency, so an experiment that measures Δf for a series of indentations at critical points can determine the mode shape and the mode. For example, the f_{210} resonance pictured in Figure 24-3d can readily be distinguished from that of f_{110} and all other modes by its distinct double pattern of ellipses.

E. Human Acoustic Resonators

The human vocal tract is a remarkable organ that enables complex speech and communication. Speech consists of voiced and unvoiced sounds, where the voiced sounds are dominated by the natural resonances of the vocal tract. The unvoiced sounds are generally broadband and are not generated by the vocal chords. For example, the 'sss' sound is generated by acoustic turbulence between the teeth and the tongue.

During voiced sounds, typically vowels, the vocal cords vibrate rapidly, releasing periodic impulses of air from the lungs at a frequency called pitch. Each pulse is so brief that it has significant power up to 5-10 kHz. The broad acoustic spectrum of these impulses is then slowly modulated by the vocal tract, which acts like a transmission-line cavity resonator extending from the vocal chords to the teeth, as illustrated in Figure 24-4a. When the small passage to the nasal cavity is open, it can modulate the characteristics of the tract to produce nasal sounds. Not shown in the figure is the esophagus lying behind the trachea and connecting to the stomach; the trachea connects the vocal tract to the lungs.

Figure 24-4.
Vocal tract resonances



The vocal tract in Figure 24-4a has length d and is terminated by the vocal chords at the left end. The vocal chords act approximately like a rigid (but leaky) wall and reflect sound well. The other end of the vocal tract is open to the air, which has a lower impedance than the tract, and therefore this junction at the teeth also reflects well. The result is that each person has a natural fundamental (lowest) resonant frequency at birth that becomes still lower during childhood and adolescence and then becomes permanent. This frequency plus the natural frequency of vocal chord vibration, called pitch, together are responsible for the major distinctions between the speech of one person versus another. Pitch generally distinguishes young from old (and high musical notes from low ones), while the tract length d generally distinguishes men from women. The best voice-disguising electronics can alter both the pitch and d for the regenerated speech.

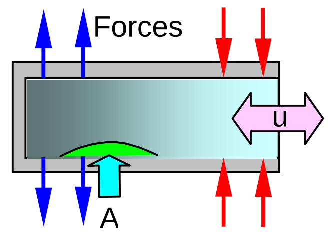
Figure 24-4b illustrates the behavior of the lowest resonant frequency for this quarter-wave cavity resonator; there is a velocity null at the vocal chords and a pressure null at the (open) teeth. The resonances corresponding to a closed mouth can generally be ignored. Figure 24-4c illustrates the equivalent TEM resonator, where pressure p is analogous to voltage, and velocity u is analogous to current. The fundamental acoustic resonant mode is illustrated in Figure 24-4d, and the first two harmonics are illustrated in Figure 24-4e-f. These natural frequencies are a function of d, and therefore are generally fixed and unchanging for each adult.

Vowels are formed by articulating the vocal tract so as to perturb its resonant frequencies. These resonant frequencies then shift up, down, or remain unchanged; different combinations of shifts correspond to different basic sounds or "phonemes".

These resonances can also modulate the unvoiced phonemes (consonants), but such acoustic effects are less obvious because most consonants are generated near the front of the mouth, not at the back of the resonator. Opening the passage to the nasal cavity has a strong effect on the resonances and is responsible for creating "nasal speech".

Using the energy method of Section D, the perturbations of the resonances can be readily related to the shape of the vocal tract as modulated by the position and shape of the tongue. The acoustic force on the walls of a resonator is attractive where the velocity is maximum and repulsive where the pressure is maximum. The attractive force is associated with the Bernoulli effect, in which rapid fluid flows parallel to a surface tend to pull on that surface more than at places where the fluid flow is slower.

Figure 24-5. Acoustic forces on resonator walls



For example, the net acoustic pressure at point A in Figures 24-4a and 24-5 is attractive if the oscillatory velocity u is relatively large there, and is repulsive if the oscillatory pressure p is relatively large, as suggested in Figure 24-5. If p is large at point A for a particular resonant mode and the force outward is positive, then constricting the cross-section of the vocal tract at point A would increase the total acoustic energy w_{Ta} stored in the acoustic resonator. But acoustic waves are quantized in units of phonons, just as electromagnetic waves are quantized in units of photons, and therefore:

$$w_{Ta} = nhf = w_p + w_k \qquad (12)$$

where n is the number of phonons in the resonator at the resonant frequency f, h is Planck's constant, w_p is the total potential energy associated with p, and w_k is the total kinetic energy associated with u. As before, n remains constant in a closed system if any shape changes are slow compared to the velocity of sound, and therefore any change in stored energy must result in a corresponding change in the resonant frequency f. Therefore vocal tract constrictions by the tongue at point A will increase the resonant frequency for a particular mode if the repulsive force at A due to p exceeds the attractive Bernoulli force due to u; equivalently, resonant frequencies are increased when $W_p > W_k$ at point A, where W corresponds to potential or kinetic energy density. By analogy with (7) and without derivation we find:

$$\Delta f/f = \Delta(w_p - w_u)/w_{Ta} \qquad (13)$$

where Δw is defined as positive for protrusions of resonator walls, and negative for indentations, corresponding to adding or removing volume from the resonator.

We can now understand how vowels are formed by considering an indentation by the tongue at point A along the vocal tract. Referring to Figure 24-4 we see that at point

A the relative values of W_p and W_u are approximately equal for the f_1 resonance (Figure 24-4d), and that W_p and W_u dominate the f_2 and f_3 resonances, respectively (Figures 24-4 c and d, respectively). Therefore Equation (13) predicts that indentation at point A will have little effect on f_1 and will decrease and increase the resonances f_2 and f_3, respectively, as suggested in Figure 24-6, where the unperturbed vocal tract is perturbed at time t_o. In this fashion we can form an enormous number of distinctive and recognizable phonemes, only a fraction of which are used in any given language. The ease with which we can predict the acoustic effects of indentations and protrusions on resonators clearly illustrates the power of energy methods.

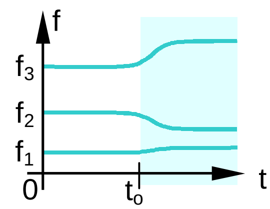


Figure 24-6. Perturbations of acoustic resonant frequencies by indentation at point A along the vocal tract at time t_o.

Many musical instruments are modeled after the vocal tract. They typically have a vibrating reed or lips at one end, analogous to vocal chords, and are connected to a tube open at the other end. Usually the tube has fixed length and holes in the sides are opened or closed in various combinations to perturb the resonances or to select the dominant one. In some cases the vibrational frequency of the reed may be influenced by which resonance is stronger, and so feedback can enhance one resonant frequency at the expense of others. For example, placing open holes at the pressure maxima of undesired resonances f_i would sap their strength in favor of one or more other desired musical notes. Lip and lung pressure can also raise or lower the favored frequency band. Furthermore, the lengths of some tubes can also be modulated, as in trombones, or parallel tubes of different lengths can be selected by valves, as in some horn instruments. Almost all such resonant waveguide instruments radiate principally from their open end, which is usually matched to open air by means of a bell, which is an exponentially widening section that functions, for example, like a series of quarter-wave impedance transformers.