6.013 Lecture 13:
Reluctance and Permanent Magnet Motors; Photon Forces

A. Overview

Reluctance motors generally incorporate soft-iron rotors that are pulled toward magnetized poles. By switching the excitation of the poles synchronously with rotor motion, the motion can continue. The forces acting on the rotor can be found by differentiating the total system energy with respect to rotor position. This calculation can be simplified if the inductance $L(\theta)$ of the system is known as a function of rotor angle θ where the magnetic energy stored in L is: $w_m = LI^2/2$. The simple expression for flux linkage $\Lambda = N\int_A \overline{B} \bullet d\overline{a} = LI$ can help relate the magnetic field $\overline{B}$ to L, where N is the number of turns in the excitation coils circling the area A of the stator; the flux linked in the stator approximately equals that at the rotor poles.

Permanent magnets cling to high permeability surfaces with a force density of $\sim B_{gap}^2/2\mu_o$ [Nm^{-2}], where B_{gap} is the magnetic flux density in the thin gap separating the two bodies. They can also provide the fields needed to exert force on currents flowing in permanent-magnet motors or generators.

Electromagnetic waves exert pressure on objects that partially or completely absorb or deflect them. These forces can be computed using the Lorentz force law or by computing changes in photon momentum for the wave. Photon momentum is $\overline{p} = \hat{z}\, hf/c$ [Nms^{-1}], where $\hat{z}$ is a unit vector in the direction of wave propagation. The force on an object $\overline{f} = -n(d\overline{p}/dt)$, where n is the number of incident photons per second and $d\overline{p}/dt$ is the momentum change per photon induced by the object.

B. Reluctance Motors

To find the torque on the rotor for the *reluctance motor* illustrated in Figure 13-1, the magnetic fields must first be found. The high permeability of the *stator* confines the magnetic fields produced by the N-turn coil and guides them to the pole faces where the small gaps b between stator and rotor offer the path of "least resistance" for the field lines to close on themselves around the loop. $\overline{B}$ must be continuous across the two gaps because $\nabla \bullet \overline{B} = 0$, so $\overline{B}_{stator} \cong \overline{B}_{gap} \cong \overline{B}_{rotor}$, and $\overline{H}_{stator} \cong (\mu_o/\mu)\ \overline{H}_{gap} << \overline{H}_{gap}$.

Next we must relate NI to the fields. The integral form of $\nabla \times \overline{H} = \overline{J}$ becomes

$$\int_A \overline{J} \bullet d\overline{a} = \int_c \overline{H} \bullet d\overline{s}, \text{ so that} \tag{1}$$

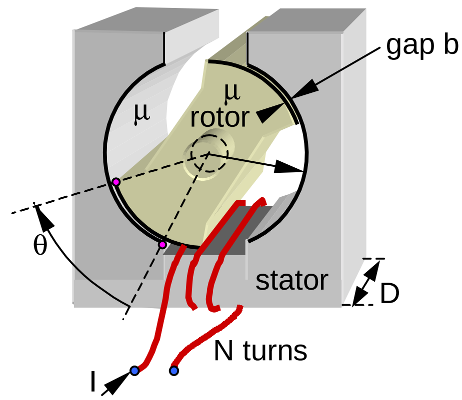
$$NI = \int_c (\overline{H}_{gap} + \overline{H}_{stator}) \bullet d\overline{s} \cong 2bH_{gap} \tag{2}$$

It follows that

$$H_{gap} \cong NI/2b \quad (3)$$

independent of the cross-sectional area of the gaps (provided they don't approach zero so that the non-gap field leakage becomes a significant fraction of the total magnetic flux).

Figure 13-1. Reluctance motor



We might be tempted to compute the torque on the rotor by differentiating the magnetic energy in the gap with respect to rotor position. The total energy in the two gaps equals the volumes of the two gaps times the magnetic energy density stored there, or:

$$w_{gap} \cong 2bR\theta D\ \mu_o|H_{gap}|^2/2\ \ [\mathrm{Jm^{-3}}] \quad (4)$$

where the gap area is $R\theta D$, R is the rotor radius, θ is the angular overlap (radians) between rotor and stator, and D is the depth of the structure. However, if we compute torque T using the usual formula:

$$T = -\partial w_{gap}/\partial\theta = -bRD\ \mu_o|H_{gap}|^2 \quad (5)$$

we obtain the wrong sign! The torque actually acts to pull the rotor between the stator poles, so its true sign is positive, not negative as in (5). The error here is that the total system energy expression did not include the power supply driving the current I through the windings.

Although we might include the change in power supply energy ($\int Vi(t)dt$) as the rotor turns in order to obtain the correct value for force, it is easier here to change the problem definition slightly by short-circuiting the input coil so that I continues to flow. We note that Faraday's law $\nabla \times \overline{E} = -\partial\ \overline{B}/\partial t$ becomes:

$$\oint_{c\ coil} \overline{E} \bullet d\overline{s} = -N\frac{d}{dt}\int_A \overline{B} \bullet d\overline{a} = -d\Lambda/dt = 0 \quad (6)$$

where the contour integral is zero when the coil is short-circuited. The magnetic flux linkage Λ is therefore constant:

$$\Lambda = N\int_A \overline{B} \bullet d\,\overline{a} = NB_{gap}A_{gap} = N\mu_o H_{gap}A_{gap} = N^2\mu_o IA_{gap}/2b \quad (7)$$

where we used the fact that $H_{gap} = NI/2b$ (see (3)). It follows that:

$$L = \Lambda/I = N^2\mu_o A_{gap}/2b \tag{8}$$

To find the torque we differentiate the system energy with respect to θ using an expression that contains only θ and constants such as Λ. Expressions containing L and I are problematic because they are not known to be constant.

We know $I = \Lambda/L$ so, using (8):

$$w_m = LI^2/2 = \Lambda^2/2L = \Lambda^2 2b/\ N^2\mu_o A_{gap} = \Lambda^2 b/\ N^2\mu_o RD\theta \tag{9}$$

The torque T [Nm] is:

$$T = -\partial w_m/\partial\theta = -(\Lambda^2 b/N^2\mu_o RD)\partial(\theta^{-1})/\partial\theta = \Lambda^2 b/N^2\mu_o RD\theta^2 \tag{10}$$

Using (7) to replace Λ we find:

$$T = N^2\mu_o I^2 RD/4b\ \ [\mathrm{Nm}] = (\mu_o H_{gap}{}^2/2)(2bRD)\ \ \neq f(\theta) \tag{11}$$

This can be expressed more simply and physically as:

$$T = W_{mgap}(dV_{olume}/d\theta)\ \ [\mathrm{Nm}] \tag{12}$$

Thus (12) expresses an important result, which is essentially the same result we found for electric motors—the torque is limited by the maximum energy density in the electromagnetic fields, and by the rate at which the volume of the energized gap changes per radian of rotation.

Equation (11) suggests that to maximize torque we should maximize NI and RD, and minimize the gap b. In practice, the ratio μ/μ_o is sufficiently large that usually the gap width b, and therefore the torque, is limited in part by manufacturing, bearing, and life stress tolerances.

The power supply for a motor such as that shown in Figure 13-1 normally provides current I starting when the rotor angle θ is such that the gap area is minimum, and it stops when that area becomes maximum. The rotor then coasts with I = 0 until the area is again minimum, when the cycle repeats.

There are ways to increase this duty cycle so drive currents operate continuously. Figure 13-2 illustrates an example with three stator poles and four rotor poles. The stator fields will pull the rotor poles to close the gaps. Here, if windings A and B are excited, then rotor pole 1 will be pulled clockwise into stator pole B. The gap area for stator pole A is temporarily constant and contributes no additional torque. After the rotor moves π/3 radians, the currents are switched to poles B and C so as to pull rotor pole 2 into stator pole C, while rotor pole 1 contributes nothing to the torque. Next C and A are excited, and the cycle is repeated twice per revolution. Counter-clockwise torque is obtained by reversing the excitation sequence. Many pole combinations are possible, with the higher number of poles yielding higher torques because the derivative in (12) is proportional to the number of active poles.

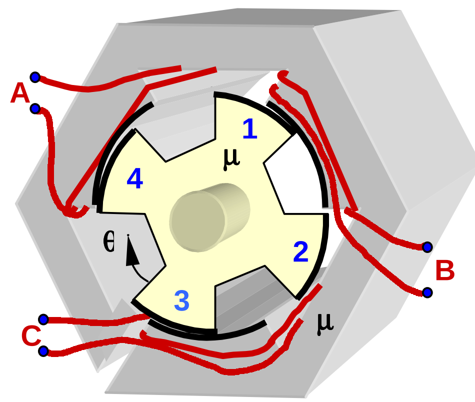


Figure 13-2. Reluctance motor with 3 rotor and 4 stator poles

C. Permanent Magnets

The force f [N] attracting a *permanent magnet* to a high-permeability material can be found using:

$$f = dw_m/db \tag{13}$$

where b[m] is the separation between the two, as illustrated in Figure 13-3, and w_m is the total energy in the magnetic fields [J]. The changing magnetic energy in the high-permeability material is negligible compared to that in air because: 1) W_m [Jm^{-3}] $\alpha\ \mu|\overline{H}|^2$ where $\mu >> \mu_o$, and 2) the ratio of $\overline{H}$ in the two media is $H_\mu/H_{\mu o} = \mu_o/\mu << 1$ because boundary conditions require that $\overline{B}_\perp = \mu\ \overline{H}_\perp$ be continuous across the boundary between the two media; thus the energy density in air is greater by $\mu/\mu_o >> 1$.

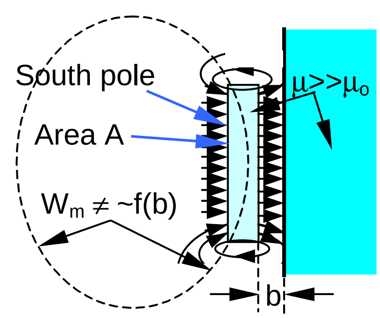


Figure 13-3. Magnet clinging to metal

The variable magnetic energy is dominated by the energy w_m in the gap, which is the energy density, $W_m = \mu_o H_{gap}^2/2 = B_{gap}^2/2\mu_o$, times the volume of the gap Ab, where A is the area of the magnet face. Thus:

$$w_m \cong \mu_o H_{gap}^2 bA/2 \ \ [J] \tag{14}$$

Differentiating w_m with respect to b yields the force $f \cong \mu_o H_{gap}^2 A/2$ [N], and the force density:

$$F \cong \mu_o H_{gap}^2/2 \ [Nm^{-2}] = W_{gap}[Jm^{-3}] \quad (15)$$

This can be expressed in terms of B: $F = B_{gap}^2/2\mu_o$ [Nm^{-2}].

Most permanent magnets have magnetic flux densities B less than one Tesla (10^4 gauss). A magnet this powerful with an area A = 10 cm^2 (~the size of a silver dollar) would therefore exert an attractive force of $AF = 0.001\times1^2/2\times4\pi10^{-7} \cong 400N$ (~100 pound force). A more typical permanent magnet the same size might exert only a 20-pound force.

Permanent magnets fail above their *Curie point*, which is the critical temperature above which the magnetic domains become scrambled. Cooling them in a strong external magnetic field can generally restore them. Some types of permanent magnets can also fail at very low temperatures, and should not be used where that is a risk.

D. Forces Arising from Electromagnetic Waves

Electromagnetic wave forces on media can be computed using the Lorentz force law or Newton's law for photons. First consider the Lorentz forces exerted by a uniform z-directed plane wave normally incident upon a perfect conductor, as illustrated in Figure 13-4. At the surface of the conductor the electric and magnetic fields are:

$$\overline{E}_x(z = 0) = 0 = E_+\cos(\omega t - kz) + E_-\cos(\omega t + kz) \quad (16)$$

$$\overline{H}(z = 0) = \hat{y}\,(2E_+/\eta_o)\cos \omega t \quad (17)$$

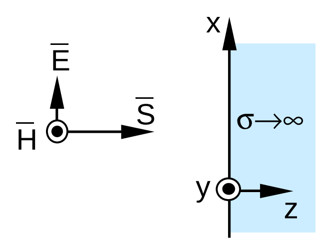


Figure 13-4. Uniform plane wave impacting a perfect conductor

The boundary condition for a perfect conductor is that the surface current density $\overline{J}_s$ [Am^{-1}] is:

$$\overline{J}_s = \hat{n}\times \overline{H} = \hat{x}\,(2E_+/\eta_o)\cos \omega t \quad (18)$$

The Lorentz force law

$$\overline{f} = q(\overline{E} + \overline{v}\times\mu_o \overline{H}) \quad (19)$$

yields the *electromagnetic pressure* P [Nm^{-2}], which is found as follows. First we calculate the force density:

$$F[Nm^{-3}] = nqv\mu_o H = J\mu_o H \quad (20)$$

where n is the number of moving electrons per cubic meter, q is their charge, and v is their velocity. But n is a function of depth into the conductor, so we must integrate the force density over depth z to obtain the pressure P:

$$P = \int_0^\infty [\,\overline{J}(z) \times \mu_o\, \overline{H}(z)]\, dz \quad (21)$$

Inside the conductor $\overline{J}(z) = \nabla \times \overline{H}(z)$ for $\sigma \to \infty$, and so:

$$\overline{J}(z) = \nabla \times \overline{H} = -\hat{x}\,\partial Hy/\partial z \quad (22)$$

Therefore (21) becomes:

$$P(t) = -\hat{z}\,\mu_o \int_0^\infty (\partial H_y/\partial z) H_y\, dz = -\hat{z}\,\mu_o \int_{H_y(z=0)=H_{y0}}^{H_y=0} H_y dH_y = \hat{z}\,\mu_o H_{y0}^2(t)/2 \quad (23)$$

which is the same answer as before—the *magnetic pressure* equals the magnetic energy density when the magnetic energy density in the adjacent medium is negligible in comparison. In the sinusoidal steady state the time average pressure is half the value given by (23), and can also be expressed in terms of the time-average Poynting vector $<\overline{S}(t)>$:

$$<P(t)> = \hat{z}\,\mu_o <H_{y0}^2(t)>/4 = 2<S(t)>/c\ [Nm^{-2}] \quad (24)$$

where $H_{y0} = 2H_+$ and $<S(t)> = \eta_o H_+^2/2 = (\mu_o/c)H_+^2/2$. By expressing the pressure in terms of $<S(t)>$ it is easy to relate it to the photon momentum flux, which also yields pressure.

We recall that photon energy is hf [J]. If the photon had mass "m", its kinetic energy K = hf would be "m"c^2 and its momentum M would be "m"c or K/c. Therefore:

$$\textit{photon momentum}\ M = hf/c\ [Nms^{-1}] \quad (25)$$

The momentum transferred to a mirror upon perfect reflection backwards of a single photon is therefore 2hf/c. We recall from mechanics that the force f required to change momentum mv is:

$$f = d(mv)/dt\ [N] \quad (26)$$

so that the total *radiation pressure* on a mirror reflecting directly backwards n photons $s^{-1}m^{-2}$ is:

$$<P> = n2hf/c = 2<S(t)>/c \ [Nm^{-2}] \quad (27)$$

The factor of two arises because the photon momentum was not just zeroed, but was reversed. If these photons were absorbed rather than reflected, the rate of momentum transfer to the absorber would be half. In general if the incident and reflected power densities are $<S_1>$ and $<S_2>$, respectively, then the average radiation pressure on the mirror is:

$$<P> = <S_1 + S_2>/c \ [Nm^{-2}] \quad (28)$$

Consider the simple example of a *solar sail* blown by radiation pressure across the solar system, sailing from planet to planet. At earth the solar radiation intensity is ~1.4 kW/m^2, so (28) yields a total force on a sail intercepting one square kilometer of radiation of $<P> = 10^6 \times 2800/3\times10^8 = 9$ N. A sail this size one micron thick and having the density of water would have a mass of 1000 kg. Since the sail velocity v = at = (f/m)t where a is acceleration, it follows that after one year the accumulated velocity of a sail facing such pressure would be $(9/1000)3\times10^7 \cong 3\times10^5\ ms^{-1} = c/1000$. Since gravity also acts on such sails, orbital mechanics must also be considered in order to obtain accurate results.