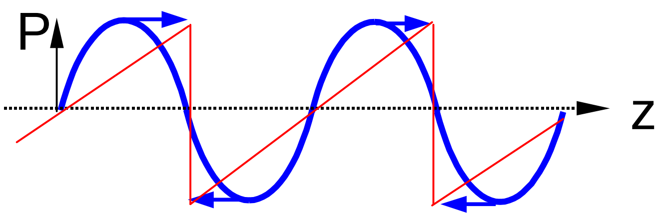
ACOUSTIC WAVES (GASES)

Basic Differences with EM Waves:

Electromagnetic Waves	Acoustic Waves
$\overline{E}, \overline{H}$ are vectors $\perp$ $\overline{S}$	$\overline{U}$(velocity) // $\overline{S}$, P(pressure) is scalar
Linear physics	Non-linear physics, use perturbations

Non-linearities:

1) Compression heats the gas; cooling by conduction and radiation (adiabatic assumption—no heat transfer)
2) Compression and advection introduce position shifts in wave
3) Wave velocity depends on pressure, varies along wave (loud sounds form shock waves)



Choice of Acoustic Variables:

Velocity: $\overline{U}\left[ms^{-1}\right] = \overline{U}_o + \overline{u} = \overline{u}$ (set $\overline{U}_o = 0$ here)

Pressure: $\mathbf{P}\left[Nm^{-2}\right] = P_o + p$

Density: $\rho\left[kg\ m^3\right] = \rho_o + \rho_1$

use perturbations

ACOUSTIC EQUATIONS

Mass Conservation Equation:

Recall: $\nabla \bullet \bar{J} = \nabla \bullet \rho_e \bar{u} = -\partial \rho_e / \partial t$ Conservation of charge

Acoustics: $\nabla \bullet \rho \bar{u} = -\partial \rho / \partial t$ Conservation of mass

Linearize: $\nabla \bullet (\rho_o + \rho_1)(\bar{U}_o + \bar{u}) = -\partial (\rho_o + \rho_1)/\partial t = -\partial \rho_1 / \partial t$ (with $\bar{U}_o \to 0$)

Drop 2nd Order term $(\rho_1 \bar{u})$,

Linearized Conservation of Mass:

$$\rho_o \nabla \bullet u \cong -\partial \rho_1 / \partial t$$

Linearized Force Equation (f = ma):

$$\nabla p = -\rho_o \partial \bar{u} / \partial t$$

Constitutive Equation:

Fractional changes in gas density and pressure are proportional, i.e.,

$d\rho/\rho = (dP/P)/\gamma \Rightarrow$ $\rho_1 = (\rho_o/\gamma P_o)p$

"adiabatic exponent" γ = 5/3 monotomic gas, ~1.4 air, 1-2 else

3 Equations, 3 Unknowns:

Reduce to 2 unknowns $(p, \bar{u})$

ACOUSTIC EQUATIONS

Acoustic Differential Equations:

Newton's Law (f = ma): $\nabla p = -\rho_o \partial \bar{u}/\partial t \quad [\text{Nm}^{-3}]\ [\text{kg m}^{-2}\text{s}^{-2}]$

Conservation of Mass: $\nabla \bullet \bar{u} = -(\gamma P_o)^{-1} \partial p/\partial t \quad [\text{s}^{-1}]$

Acoustic Wave Equation:

Combine the acoustic differential equations, eliminating ū:

$$\nabla \bullet \nabla p \Rightarrow \nabla^2 p - (\rho_o/\gamma P_o)\, \partial^2 p/\partial t^2 = 0 \text{ "Acoustic Wave Equation"}$$

2nd derivative in space = 2nd derivative in time

Solution to Wave Equation:

$$p(t,\bar{r}) = p(\omega t - \bar{k} \bullet \bar{r}) \quad [\text{Nm}^{-2}]$$

Example: $p(t,\bar{r}) = A\cos(\omega t - kz)$

$$\bar{u} = -\hat{z}\int \rho_o^{-1} \nabla p\ dt = \underbrace{(k/\rho_o\omega)}_{\eta_s^{-1}} A\cos(\omega t - kz)$$

Acoustic Impedance of Gas:

($\eta_s \cong 425$ Nsm^{-3} [$\neq \Omega$] for air 20°C)

$$\eta_s = \omega\rho_o/k = (\rho_o/\gamma P_o)^{0.5}$$

Substituting solution into wave equation

⇒ "Acoustic Dispersion Relation":

$$k = \omega(\rho_o/\gamma P_o)^{0.5}$$

ACOUSTIC PLANE WAVES

Acoustic Wave Example:

$$p(t,\bar{r}) = A\cos(\omega t - kz), \qquad k = \omega(\rho_o/\gamma P_o)^{0.5}$$

Velocity of Sound:

Phase velocity: $v_p = \omega/k = (\gamma P_o/\rho_o)^{0.5} = c_s$

Group velocity: $v_g = (\partial k/\partial \omega)^{-1} = (\gamma P_o/\rho_o)^{0.5} = c_s$

Example: Air 0°C, surface P_o

$(\Rightarrow \gamma = 1.4,\ \rho_o = 1.29\ \text{kg m}^{-3},\ P_o = 1.01 \times 10^5\ \text{Nm}^{-2})$

$\Rightarrow$ $c_s \cong 330\ \text{ms}^{-1}$

Velocity of Sound in Liquids and Solids:

$c_s = (K/\rho_o)^{0.5} \cong 1{,}500\ \text{ms}^{-1}$ in water, $\cong$ 1,500 – 13,000 in solids

"Bulk modulus"

ACOUSTIC POWER AND ENERGY

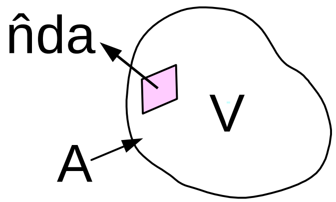
Poynting Theorem:

Recall: $\nabla p = -\rho_o \partial \bar{u}/\partial t \qquad \nabla \bullet \bar{u} = -(\gamma P_o)^{-1} \partial p/\partial t$

Note: Wave intensity $[\mathrm{Wm}^{-2}] = p\bar{u}(\mathrm{Nm}^{-2})(\mathrm{ms}^{-1})$

[Watts]

Derivation, try: $\nabla \bullet \bar{u}p = \bar{u} \bullet \nabla p + p \nabla \bullet \bar{u}$



$$= -\rho_o \bar{u} \bullet \partial \bar{u}/\partial t - (\gamma P_o)^{-1} p\, \partial p/\partial t$$

$$= -0.5\,\partial\left[\rho_o |\bar{u}|^2 - (\gamma P_o)^{-1} p^2\right]\Big/\partial t$$

Integral form: $$\int_A p\bar{u} \bullet \hat{n}da = -(\partial/\partial t)\int_V \left[\rho_o |\bar{u}|^2 \big/ 2 + p^2 \big/ 2\gamma P_o\right] dv$$

Kinetic and potential energy densities → $W_k[\mathrm{Jm}^{-3}]$ $\quad W_p$

Acoustic intensity $\quad I[\mathrm{Wm}^{-2}]$

ACOUSTIC POWER AND ENERGY (2)

Power P of Plane Wave = $p\bar{u} \bullet \hat{n} \left[Wm^{-2} \right]$:

$$p = p^2/\eta_s = \eta_s |\bar{u}|^2 \text{ instantaneously, } I = \langle P \rangle = p_o{}^2/2\eta_s = \eta_s |\bar{u}_o|^2/2$$

Where: $\eta_s = (\rho_o/\gamma P_o)^{0.5}$ {$\cong$ 425 Nsm^{-3} in surface air}

Example: $\langle I \rangle = 1\, Wm^{-2}$ at 1 kHz at the beach

$$\Rightarrow p_o = (2\eta_s I)^{0.5} = 850^{0.5} \cong 30\ Nm^{-2}$$

$$|\bar{u}_o| = p_o/\eta_s \cong 0.07\ ms^{-1};\ \Delta z \cong |\bar{u}|/\omega \cong 1\ \text{micron}$$

Threshold of Hearing:

$I_{thresh} \cong 0$ dB (acoustic scale) = 10^{-12} [Wm^{-2}]

$p_o = (2\eta_s I)^{0.5} = (850 \times 10^{-12})^{0.5} \cong 3 \times 10^{-5}$ [Nm^{-2}]

$u_o = p_o/\eta_s = 3 \times 10^{-5}/425 \cong 7 \times 10^{-8}$ [ms^{-2}]

$\Delta z \cong u_o T/\pi = u_o/\pi f \cong 2 \times 10^{-11}$ m = 0.2 Å

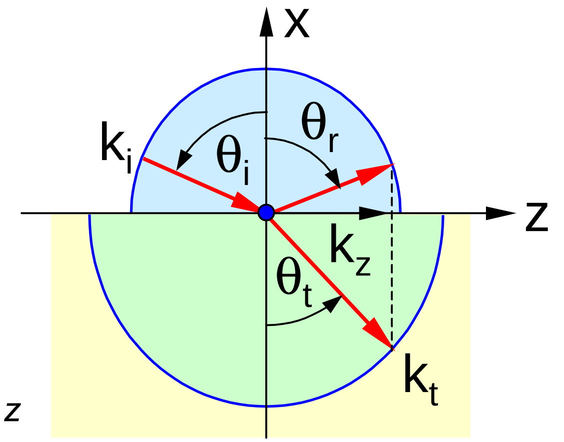
(less than an atomic diameter)

BOUNDARY CONDITIONS

At Interface Having ρ_o, γ_o, P_o (gases); ρ_o, K (solids, liquids):

Pressure: A discontinuity in p would accelerate zero-mass gas boundary with ∞ acceleration; therefore $\Delta p = 0$

Velocity: $\bar{u}_\perp$ must be continuous across boundary (to avoid ∞ mass density accumulation at boundary)



Rigid Boundaries:

Any p is accommodated by a rigid boundary

$\bar{u}_\perp = 0$ (because rigid body is motionless)

Reflection at Non-Rigid Boundaries:

Incident wave: $\underline{p}_i = p_o e^{-j\bar{k}\bullet\bar{r}} = p_o e^{+jk_o \cos\theta_i x - jk_o \sin\theta_i z}$

Reflected wave: $\underline{p}_r = \underline{p}_r e^{-j\bar{k}\bullet\bar{r}} = \underline{p}_{ro} e^{-jk_o \cos\theta_r x - jk_o \sin\theta_r z}$

Transmitted wave: $\underline{p}_t = \underline{p}_t e^{-j\bar{k}_t\bullet\bar{r}} = \underline{p}_{to} e^{-jk_t \cos\theta_t x - jk_t \sin\theta_t z}$

Velocities: Same, but $\underline{p}_k \rightarrow \underline{\bar{u}}_k$

Matching Phases: ⇒

Snell's Law: $\sin\theta_i / \sin\theta_t = k_t/k_o = (\rho_i\gamma_t/\rho_t\gamma_i)^{0.5}$

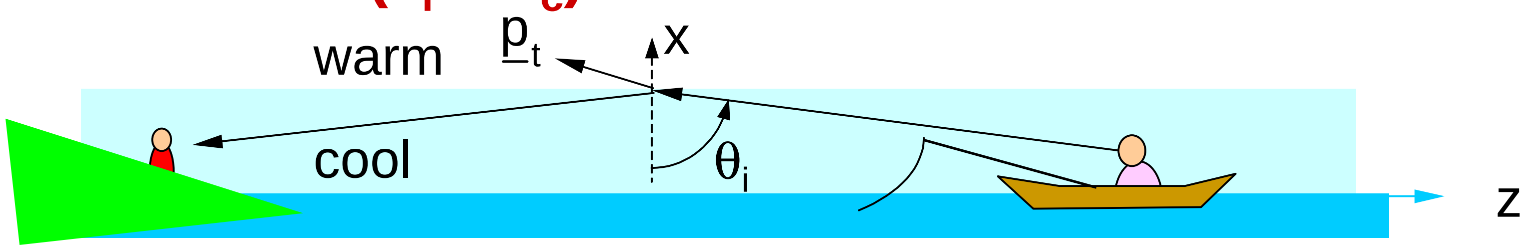
$(k = \omega(\rho_o/\gamma P_o)^{0.5})$

$\theta_r = \theta_I$

Critical angle: $\theta_c = \sin^{-1}(\rho_i\gamma_t/\rho_t\gamma_i)^{0.5}$

REFLECTIONS AT BOUNDARIES

Evanescent Waves ($\theta_i > \theta_c$):

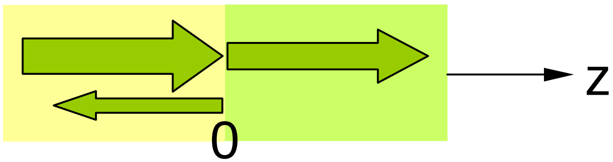


Recall: $\underline{p}_t = \underline{p}_{to} e^{-jk_t \cos\theta_t x - jk_t \sin\theta_t z} = \underline{p}_{to} e^{-\alpha x - jk_t \sin\theta_t z}$

Where: $\cos\theta_t = (1 - \sin^2\theta_t)^{0.5}$ and $\sin\theta_t = (\rho_i\gamma_t/\rho_t\gamma_i)^{0.5} \sin\theta_i > 1$

So: $\cos\theta_t = \pm j\alpha/k_t$

Normal Incidence, two gases:



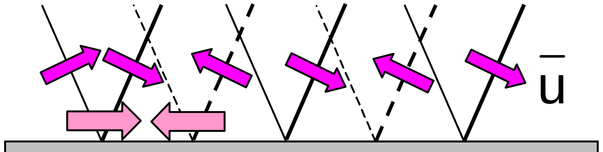
$\underline{p}: \quad p_o e^{-jk_o z} + p_{ro} e^{+jk_o z} = p_{to} e^{-jk_t z} \rightarrow 1 + \underline{\Gamma} = \underline{T}$ at $z = 0$ $(\Delta p = 0)$

Define $\underline{\Gamma} = \underline{p}_{ro}/p_o$, $\underline{T} = \underline{p}_{t0}/p_o$

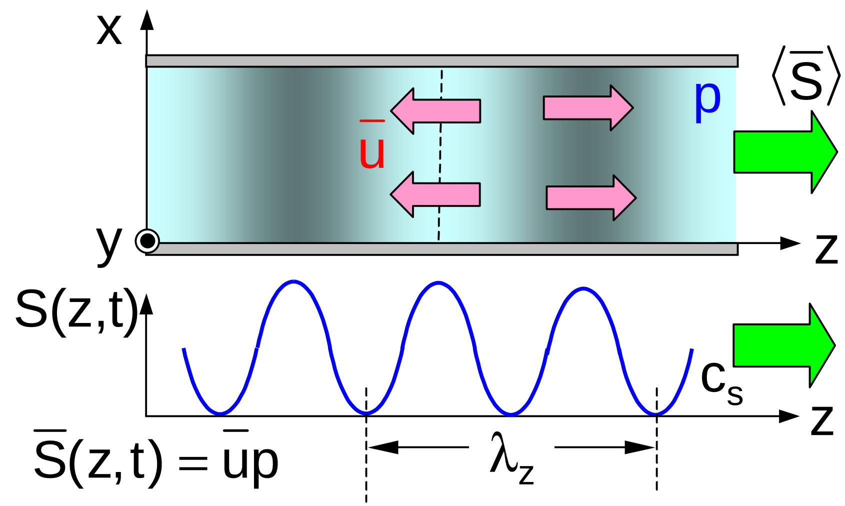
$u: \quad p_o/\eta_o - p_{ro}/\eta_o = p_{to}/\eta_t \rightarrow 1 - \underline{\Gamma} = \underline{T}\eta_o/\eta_t$ at $z = 0$ $(\Delta\bar{u} = 0)$

Solving: $\underline{T} = 2/(1 + \eta_o/\eta_t)$ where $\eta_o = \omega\rho_o/k = (\rho_o/\gamma P_o)^{0.5}$

Reflections from Solid Surface $(\hat{n} \bullet \bar{u} = 0)$:



ACOUSTIC WAVEGUIDES



A_{00} Mode ($\partial/\partial x = \partial/\partial y = 0$):

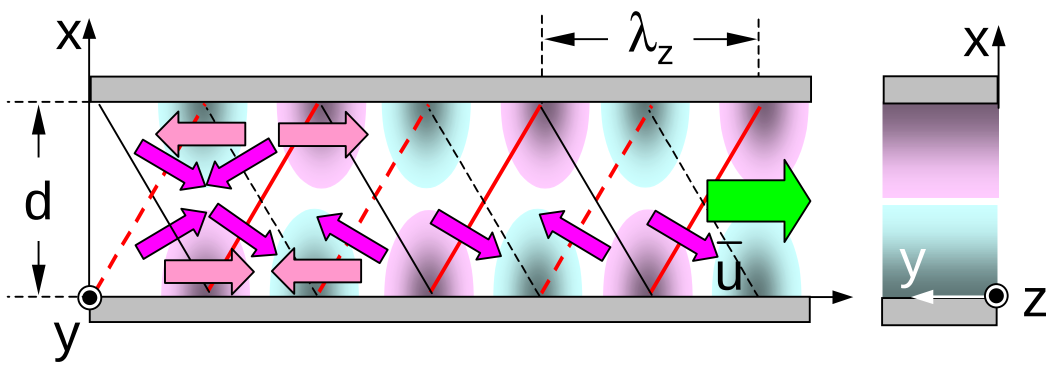
Satisfies boundary conditions
$(\overline{u}_\perp = 0)$

$p = A \cos(\omega t - kz)$

A_{m0} Mode ($\partial/\partial y = 0$):

$d = m\lambda_x/2$

A_{10} mode→

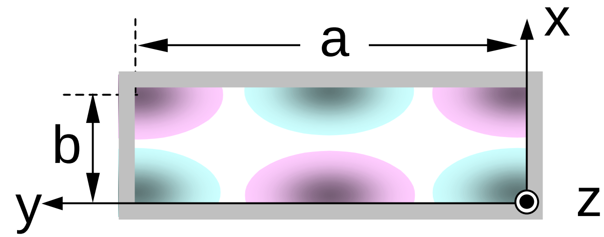


$p = A \cos(m\pi x/2d)$
$\cos(\omega t - kz)$

A_{mn} Mode:

$a = m\lambda_y/2$, $b = n\lambda_x/2$

A_{21} mode→



$p = A \cos(m\pi y/2a) \cos(n\pi x/2b) \cos(\omega t - kz)$