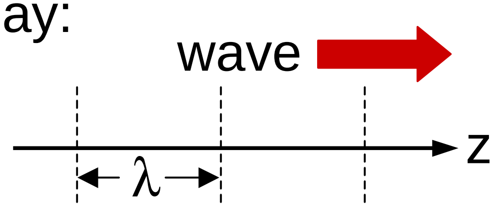
PLANE WAVES AT BOUNDARIES

Wave equation: $\left(\nabla^2 + \underbrace{\omega^2\mu\varepsilon}_{k^2}\right)\underline{\overline{E}} = 0$ in source-free region

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, $k = \dfrac{2\pi}{\lambda} = \dfrac{\omega}{c} = \omega\sqrt{\mu\varepsilon}$

z-Directional Wave:

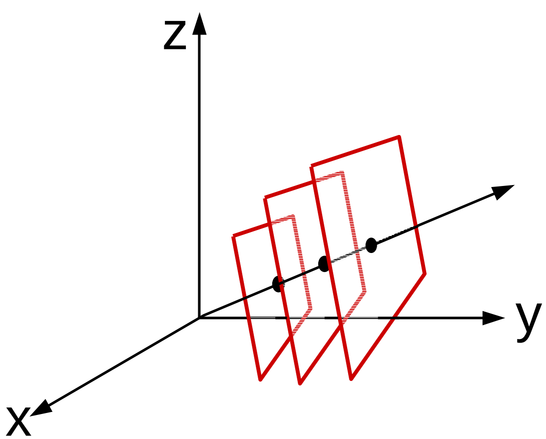
Say:



"Phase fronts"

$$\underline{\overline{E}} = \underline{\overline{E}}_o e^{-jkz}$$

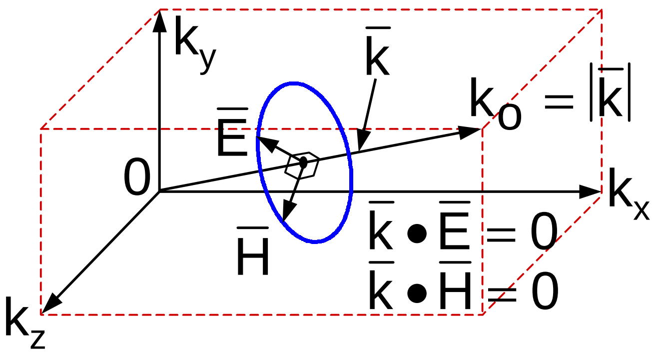
3-D Wave:

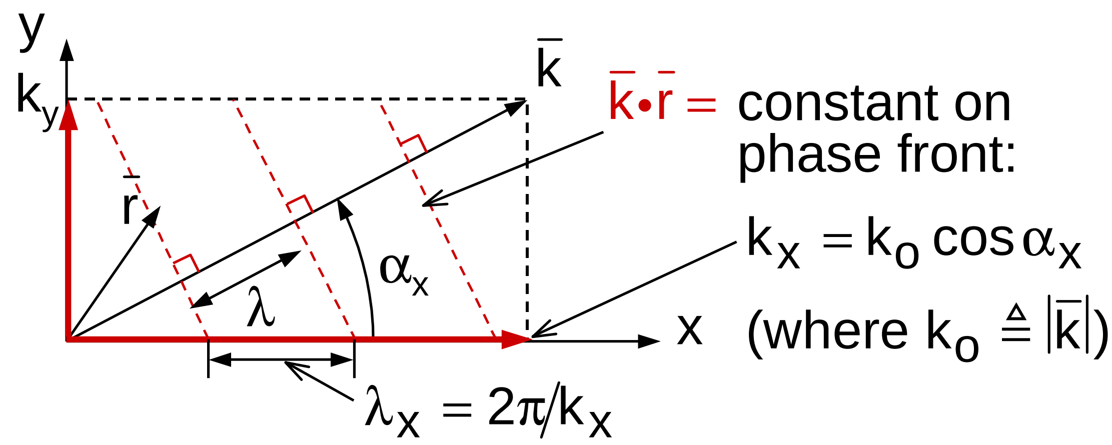


$$\underline{\overline{E}} = \underline{\overline{E}}_o e^{-jk_x x - jk_y y - jk_z z}$$

$$\underline{\overline{E}} = \underline{\overline{E}}_o e^{-j\overline{k}\cdot\overline{r}} \quad \text{where} \begin{cases} \overline{k} \triangleq \hat{x}k_x + \hat{y}k_y + \hat{z}k_z \\ \overline{r} \triangleq \hat{x}x + \hat{y}y + \hat{z}z \end{cases}$$

WAVES PROPAGATING IN THREE DIMENSIONS





Dispersion Relation:

Substitute $\underline{E}_o e^{-j\bar{k}\bullet\bar{r}}$ into wave equation $\left(\nabla^2 + k_o^2\right)\underline{E} = 0$ where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \Rightarrow \boxed{k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon \triangleq k_o^2} \Rightarrow$$

Wave Vector $\bar{k}$:

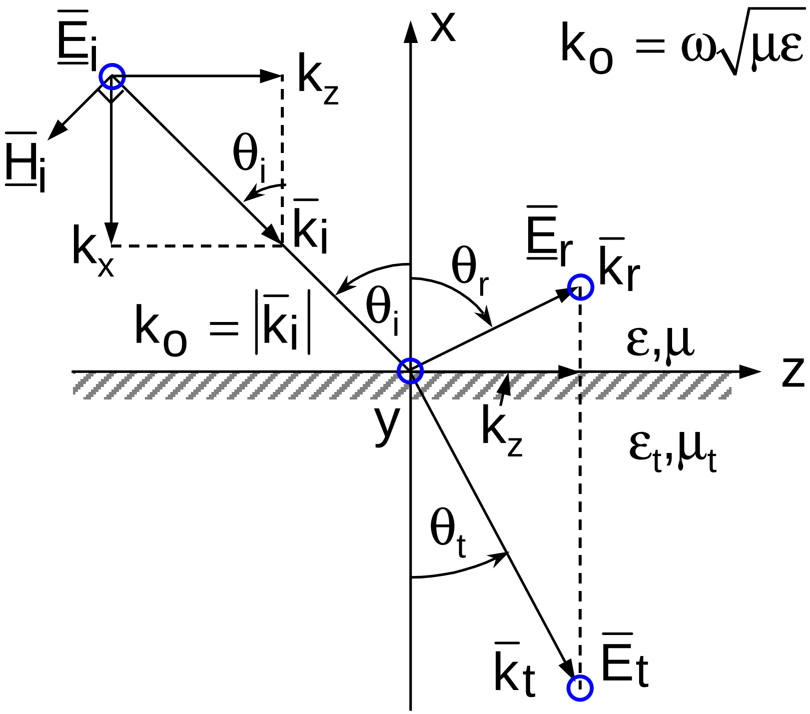
$$\nabla\bullet\underline{E} = 0 = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\bullet\left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right)e^{-j(k_x x + k_y y + k_z z)}$$

$$\Rightarrow -j\left(k_x\underline{E}_x + k_y\underline{E}_y + k_z\underline{E}_z\right) = -j\bar{k}\bullet\underline{E} = 0, \quad \therefore \quad \boxed{\bar{k}\perp\underline{E}}$$

$$\nabla\bullet\underline{H} = -j\bar{k}\bullet\underline{H} = 0, \quad \therefore \quad \boxed{\bar{k}\perp\underline{H}}$$

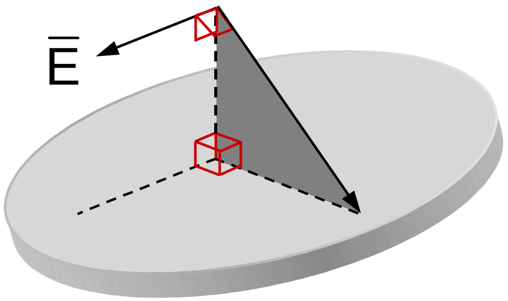
CONSIDER UPW AT PLANAR BOUNDARY

Case I: TE Wave



"Transverse Electric"

$\triangleq \bar{E} \perp$ Plane of incidence



Trial Solutions:

Incident: $\bar{\underline{E}}_i = \hat{y}E_o e^{+jk_x x - jk_z z} = \hat{y}E_o e^{+jk_o \cos\theta_i\, x - jk_o \sin\theta_i\, z}$

Reflected: $\bar{\underline{E}}_r = \hat{y}\underline{\Gamma}E_o e^{-jk_o \cos\theta_r\, x - jk_o \sin\theta_r\, z}$

Transmitted: $\bar{\underline{E}}_t = \hat{y}\underline{T}E_o e^{jk_t \cos\theta_t\, x - jk_t \sin\theta_t\, z}$

IMPOSE BOUNDARY CONDITIONS @ x = 0

$\underline{\overline{E}}_{//}$ is continuous

At x = 0:

$$\underline{E}_o e^{-j\overbrace{k_o \sin\theta_i}^{k_z} z} + \underline{\Gamma}\underline{E}_o e^{-jk_o \sin\theta_r z} = \underline{T}\underline{E}_o e^{-jk_t \sin\theta_t z} \quad \text{for all } z$$

Therefore $\underbrace{k_o \sin\theta_i}_{k_{iz}} = \underbrace{k_o \sin\theta_r}_{k_{rz}} = \underbrace{k_t \sin\theta_t}_{k_{tz}} = k_z$

and $\theta_r = \theta_i$ Angle of incidence equals angle of reflection

Therefore:

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_o}{k_t} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\sqrt{\mu_t\varepsilon_t}} = \frac{v_t}{v_i} = \frac{n_i}{n_t}$$ "Snell's Law"

where $n \triangleq c/v_{phase} = c\sqrt{\mu\varepsilon}$ "Refractive Index"

$n_{vacuum} = 1$

$n_{glass} \cong 1.5 - 1.66$

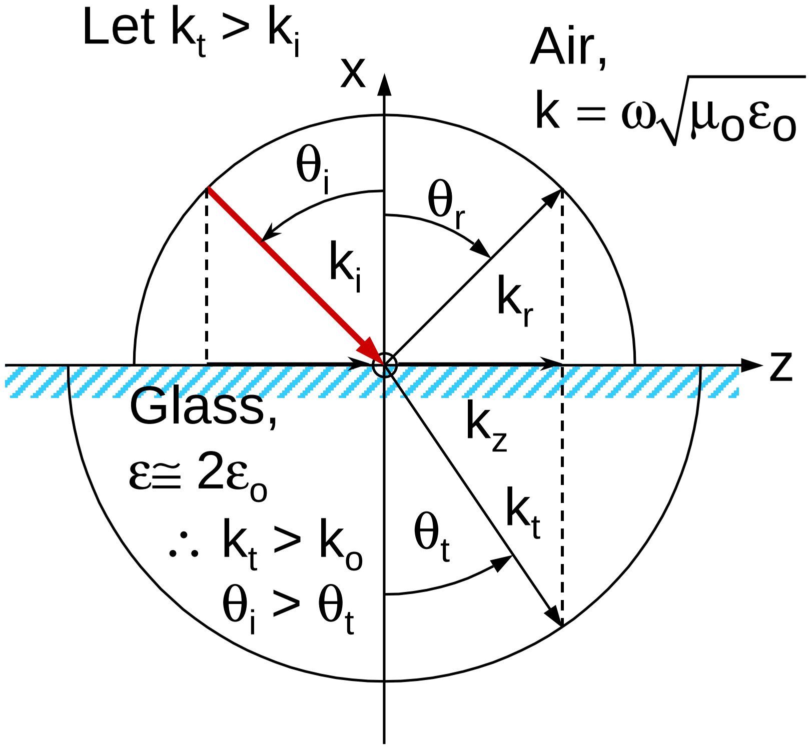
$n_{water} \cong 1.3$ at visible wavelengths

$\lesssim 9$ at audio-radio frequencies

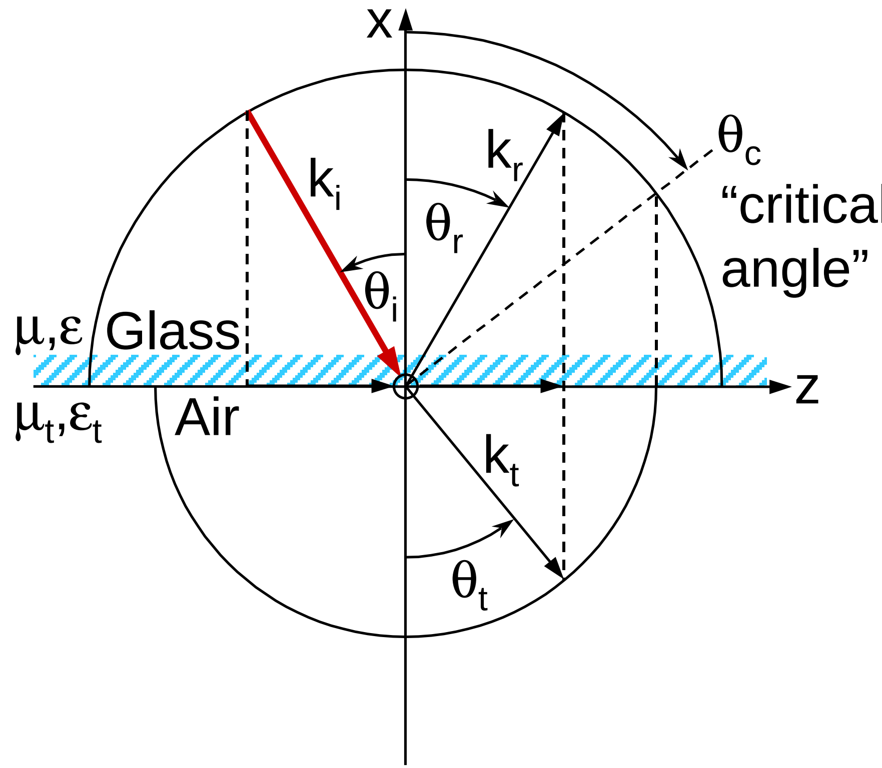
ONE WAY TO VISUALIZE SNELL'S LAW

Recall: $k_o \sin\theta_i = k_t \sin\theta_t$

Let $k_t > k_i$



Let $k_t < k_i$; then $\theta_t \geq \theta_i$



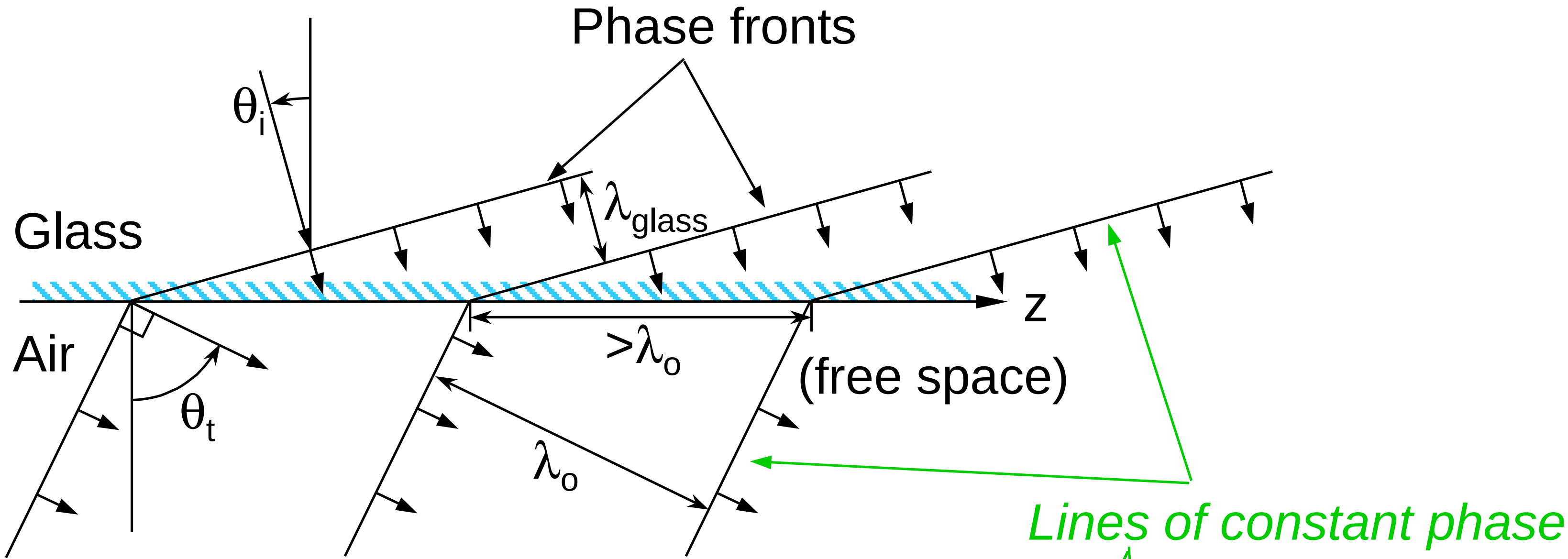
We know: $\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{n_i}{n_t}$ But when $\sin\theta_t = 1$, then $\theta_i = \theta_c$ where

$$\theta_c = \sin^{-1}(n_t/n_i) \quad \text{"critical angle"}$$

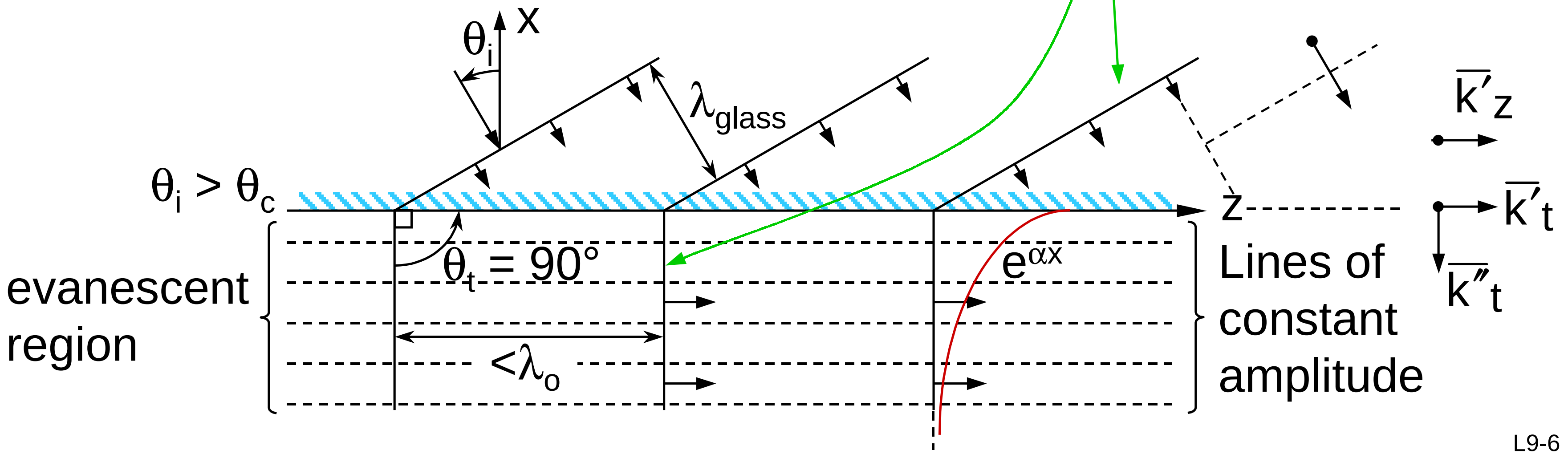
$$\left(\text{e.g. } [\varepsilon_i = 2\varepsilon_o, \mu = \mu_o] \Rightarrow \left[n_i = \sqrt{2}\right] \Rightarrow [\theta_c = 45°]\right)$$

NON-UNIFORM PLANE WAVES (NUPW)

Normal refraction: $\theta_i < \theta_c$



Beyond the critical angle, evanescence:

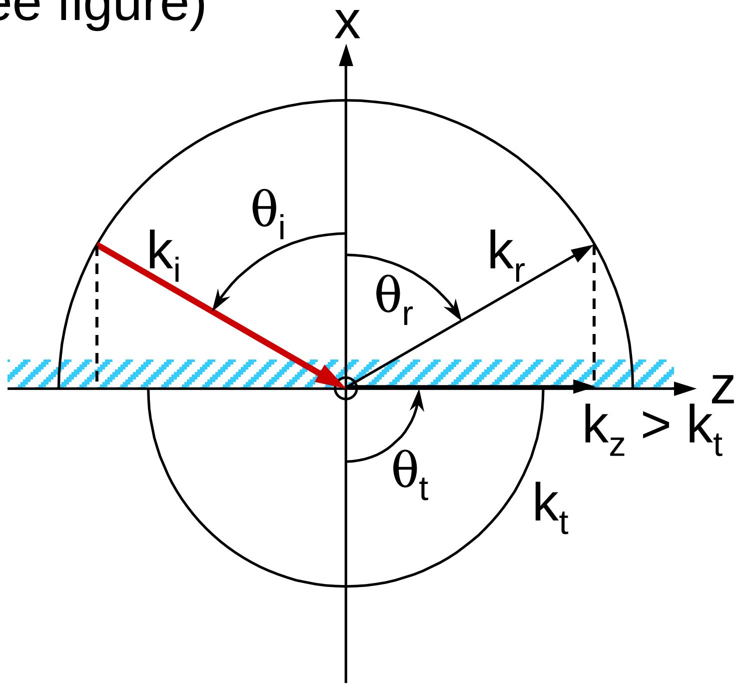


WHAT HAPPENS WHEN θ_i > THE CRITICAL ANGLE θ_c?

Since: $k_t^2 = \omega^2\mu_t\varepsilon_t = k_z^2 + k_{tx}^2$

Therefore: $k_{tx}^2 = k_t^2 - k_z^2$, < 0 for $\theta_i > \theta_c$! (see figure)

For $\theta_i > \theta_c$ $\quad \underline{k}_{tx} = \pm j\alpha = \pm j\sqrt{k_z^2 - k_t^2}$



where: $k_z^2 = \omega^2\mu_i\varepsilon_i \sin^2\theta_i$

$k_t^2 = \omega^2\mu_t\varepsilon_t$

and $\quad \underline{\overline{E}}_t = \hat{y}\underline{T}E_o e^{-jk_z z + jk_{tx}x} = \hat{y}\underline{T}E_o e^{-jk_z z + \alpha x}$

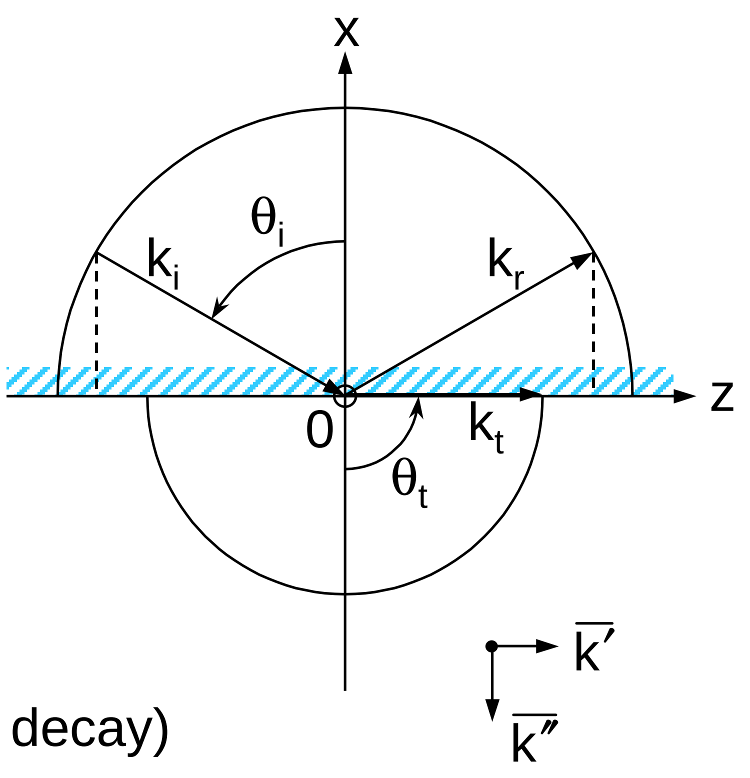
NON-UNIFORM PLANE WAVES (2)

Beyond the critical angle:

$$\overline{\underline{E}}_t = \hat{y}\underline{T}E_o e^{+\alpha x - jk_z z} \qquad (x < 0)$$

$$= \hat{y}\underline{T}E_o e^{-j\overline{\underline{k}}_t \bullet \overline{r}}$$

where: $\overline{\underline{k}}_t = k_z\hat{z} + j\alpha\hat{x} \triangleq \overline{k}' - j\overline{k}''$



Called:

"non-uniform plane wave"
"evanescent wave" (no power in direction of decay)
"surface wave"
"inhomogeneous plane wave"

Therefore, for the evanescent wave:

If lossless medium, $\overline{k}' \bullet \overline{k}'' = 0$ $\quad \overline{\underline{E}}, \overline{\underline{H}} \ \alpha \ e^{-j(\overline{k}' - j\overline{k}'')\bullet \overline{r}}$

(general expression)