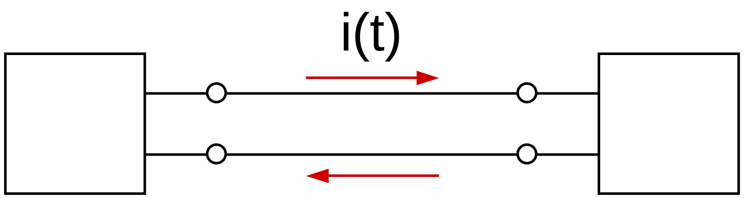
INDUCTORS

Inductance is ubiquitous:



Ampere's Law: $\nabla \times \overline{H} = \overline{J} + \partial \overline{D}/\partial t$

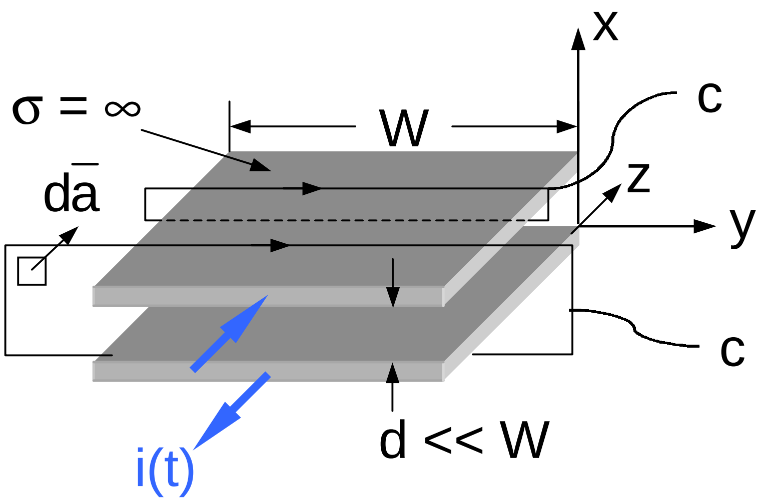
Let $\partial/\partial t \cong 0$

Example—printed circuit:

$$\oint_c \overline{H} \bullet d\overline{s} = \int_A \overline{J} \bullet d\overline{a}$$

= 0 around both wires

= i(t) around the top wire



Magnetic Field H:

Outside: $\overline{H} \cong 0$

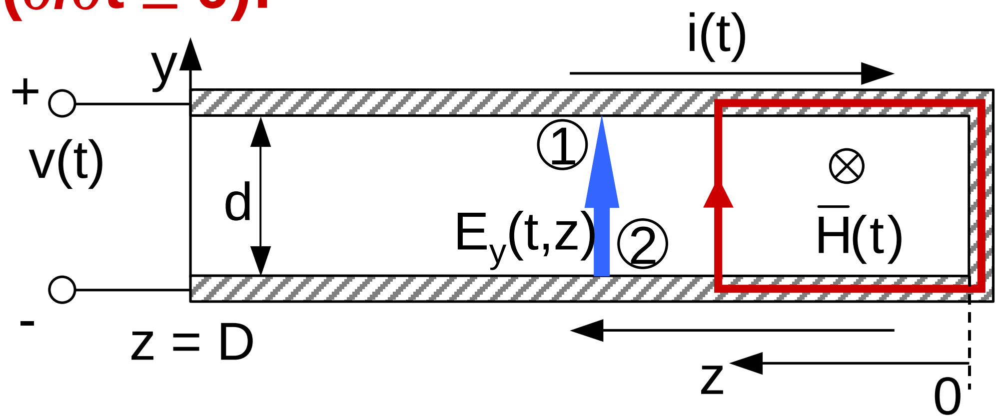
Inside: $\overline{H} \cong -\hat{y} i/W$

(since d << W we ignore fringing fields)

INDUCTANCE

Quasistatic Behavior ($\partial/\partial t \cong 0$):

Side view:



Maxwell's Equations:

Faraday's Law: $\nabla \times \overline{E} = -\partial \overline{B}/\partial t$

$$\Rightarrow \int_c \overline{E} \bullet d\overline{s} = -\frac{d}{dt}\int_A \mu \overline{H} \bullet d\overline{a}$$

$$E_y(t,z)d = -(\mu z d/W)\,di(t)/dt \quad \text{[recall } H = i/W\text{]}$$

Therefore when z = D: $v(t) = \int_1^2 \overline{E} \bullet d\overline{s} = -E_y d = (\mu D d/W)(di(t)/dt)$

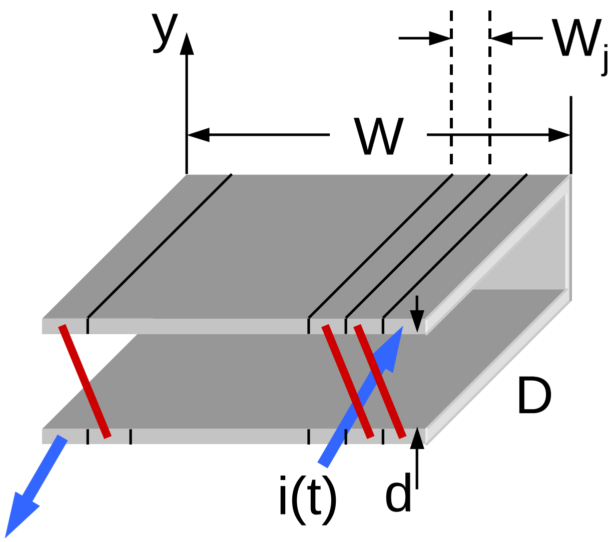
Note: Dd = cross-sectional area A

v(t) = L di(t)/dt where L = μA/W Henries

Note: Kirchoff's voltage law not obeyed here; $E_y = f(z)$

SOLENOIDAL INDUCTORS

N-Turn Solenoidal Inductor:



$$v(t) = \int_1^2 \overline{E} \bullet d\overline{s} = -E_y d = (\mu Dd/W)(di(t)/dt)$$

$$v_j(t) = (\mu Dd/W_j)(di(t)/dt) \text{ for } j^{th} \text{ turn}$$

But: $W = NW_j$

$$v(t) = Nv_j(t)$$

Therefore: $v(t) = (N^2 \mu Dd/W)(di(t)/dt)$; $Dd = A$

Inductance of N-Turn Solenoid:

$L = N^2 \mu A/W$ Henries

Magnetic Energy Storage:

$$W_m = \mu |\overline{H}(t)|^2 / 2 \quad [\text{J m}^{-3}]$$

$$w_m = \mu DdW |\overline{H}(t)|^2 / 2 = \mu AW (Ni/W)^2 / 2 \quad [\text{J}] \quad [\text{recall } H = Ni/W]$$

Therefore: $w_m = Li^2(t)/2 \quad [\text{J}]$

RESISTIVE INDUCTORS

Single-Turn Inductor:

Conductance of slab of cross-sectional area S: σS [Siemens m]

Resistance of slab of length D: $D/\sigma S$ [ohms]

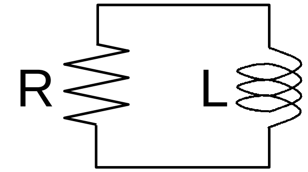
Resistance of a single-turn inductor: $2(D + d)/\sigma S$ [ohms] ($S = \delta W$)

Resistance of an N-turn inductor: $2N(D + d)/\sigma(S/N) = 2N^2(D + d)/\sigma S$

L/R Time Constant of Solenoidal Inductor:

$\tau = L/R$ seconds

$\left(\text{e.g. } i(t) = i_o e^{-\tau/\tau}\right)$

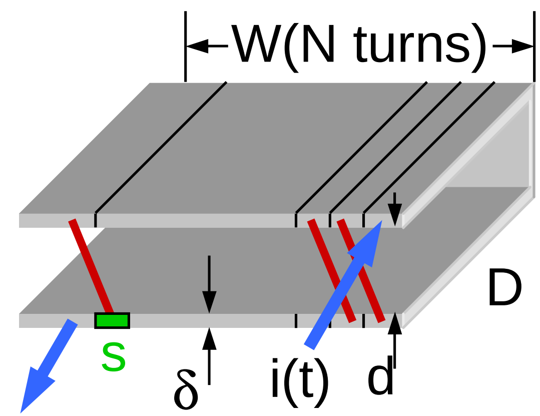


$$= \left(N^2 \mu A / W\right) \Big/ \left(2N^2 (D + d)/\sigma S\right) \cong \mu d \delta \sigma / 2N$$

where $D >> d$, $S = \delta W$, $A = Dd$

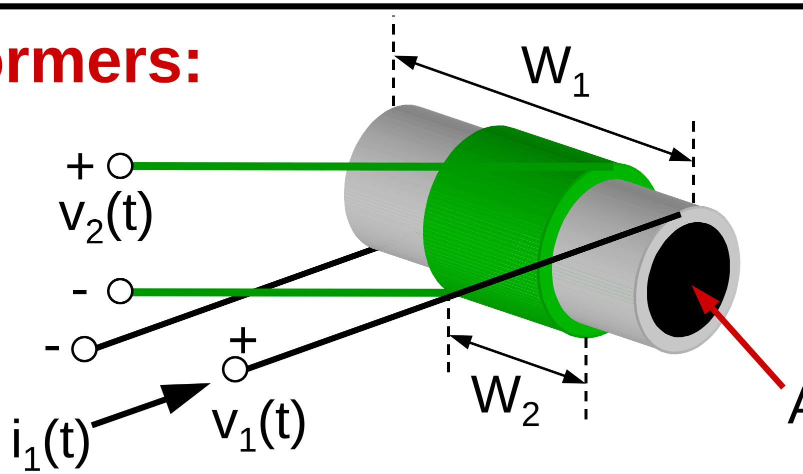
For finite size and mass, τ is limited

Want $d \rightarrow D$, $\delta \gtrsim d/3$, $N \rightarrow 1$, $d \rightarrow W$



TRANSFORMERS

Air-Wound Solenoidal Transformers:



$$\oint_C \overline{E} \bullet d\overline{s} = -\frac{d}{dt}\int_A \mu \overline{H} \bullet d\overline{a}$$

Say $A_1 = A_2$, $W_1 = W_2$ here
N_i = number of turns in coil i

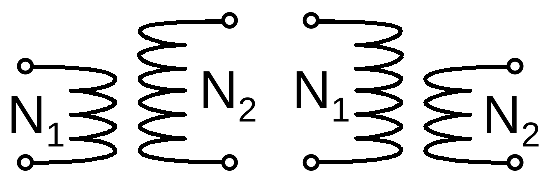
Therefore:

The voltage induced in one turn of coil 2 is the same as induced in one turn of coil 1,

And the total voltage induced in coil 2 is N_2/N_1 times the total voltage induced in coil 1, regardless of whether it is generated by i_1 or i_2.

N_2/N_1 is called the transformer turns ratio

Step-up and Step-down Transformers:



Step-up or step-down the output voltage, correspondingly.
The flux coupling between the two coils may be imperfect and the output voltage is correspondingly reduced

[flux $\Lambda = \mu HA$, and linked flux = $N\mu HA$].

IRON-CORE TRANSFORMERS (1)

Boundary Conditions:

$H_{//}$ and $\overline{B}_{\perp}$ are continuous across the boundary

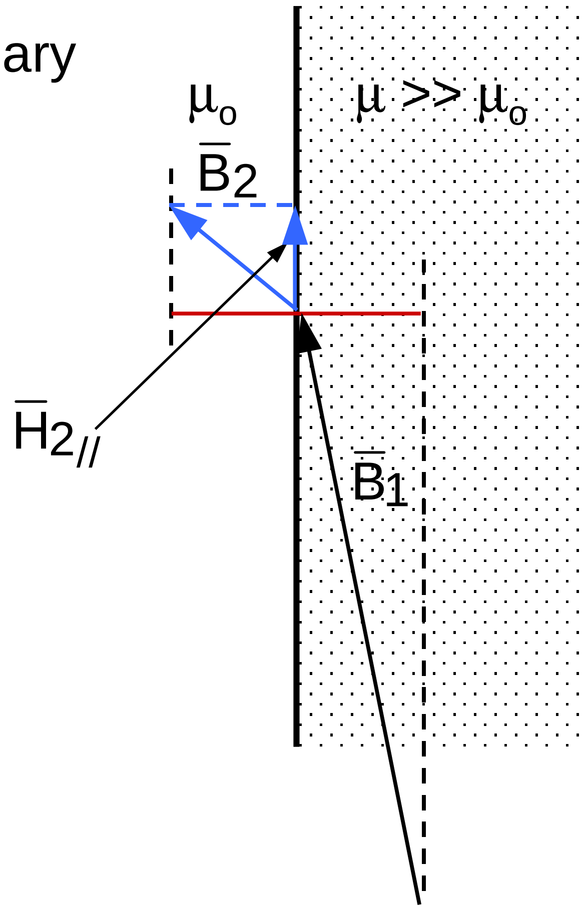
$$\left(\nabla \times \overline{H} = \overline{J} + \partial \overline{D} / \partial t;\ \nabla \bullet \overline{B} = 0\right)$$

$\overline{H} // \overline{B}$ and $\overline{B} = \mu \overline{H}$

μ / μ_o can be as large as 10^6.

Since $\mu >> \mu_o$, $\overline{B}_1$ is essentially parallel to the interface, and trapped within the high permeability medium.

The magnetic flux $\overline{B}$ is "trapped" inside.

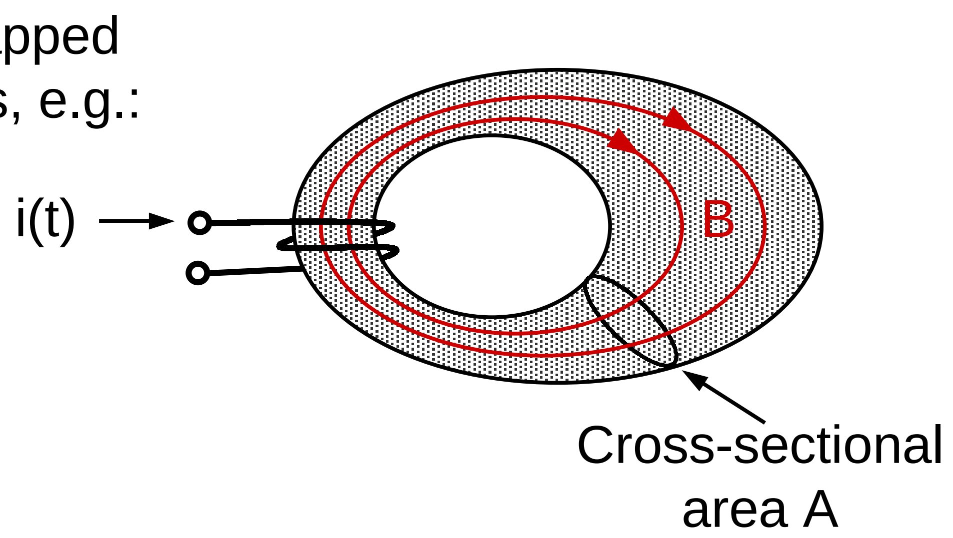


IRON-CORE TRANSFORMERS (2)

Flux trapping inside high permeability materials:

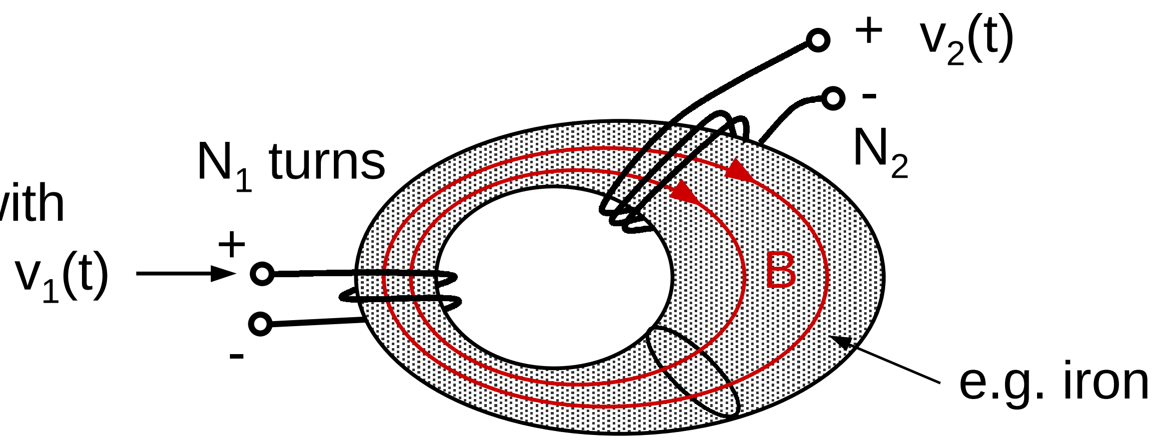
The magnetic flux density $\bar{B}$ is trapped inside high-permeability materials, e.g.:

$$\text{Flux } \Lambda = \int_A \bar{B} \bullet d\bar{a} \cong \text{constant}$$



Transformer Output:

$v_2(t) = (N_2/N_1)v_1(t)$
The flux is highly linked with little leakage

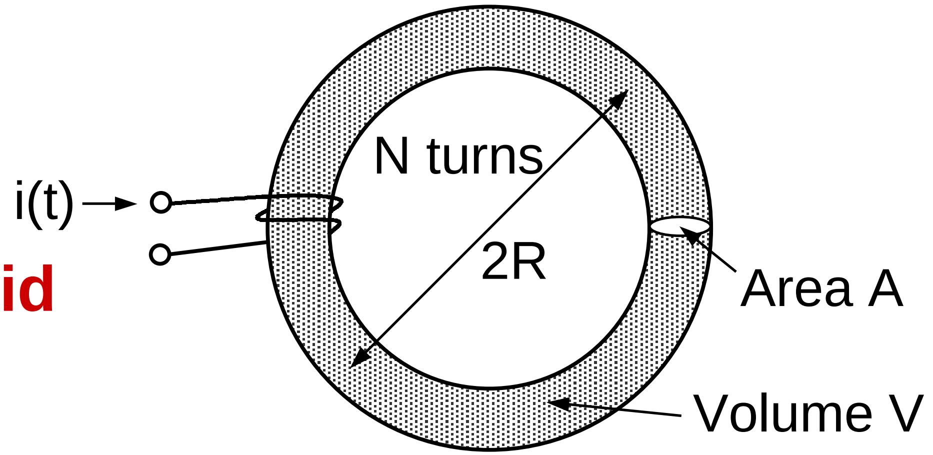


INDUCTANCE OF IRON TOROID (1)

Inductance L of N turns around toroid ($N_2 = 0$):

Recall: $w_m = Li^2(t)/2 = \int_V W_m dv \;\; [J] = \int_V \left(\mu|\overline{H}|^2/2\right)dv$

where $W_m = \mu|\overline{H}(t)|^2/2 \quad \left[Jm^{-3}\right]$



Example: Constant Area Toroid

Since:

$$L = \left(\mu\int_V|\overline{H}(t)|^2 dv\right)/i^2(t)$$

$$\nabla\times\overline{H} = \overline{J} + \partial\overline{D}/\partial t \Rightarrow \oint_c \overline{H}\bullet d\overline{s} \cong Ni(t)$$

($\partial\overline{D}/\partial t$) $\cong 0$

Therefore:

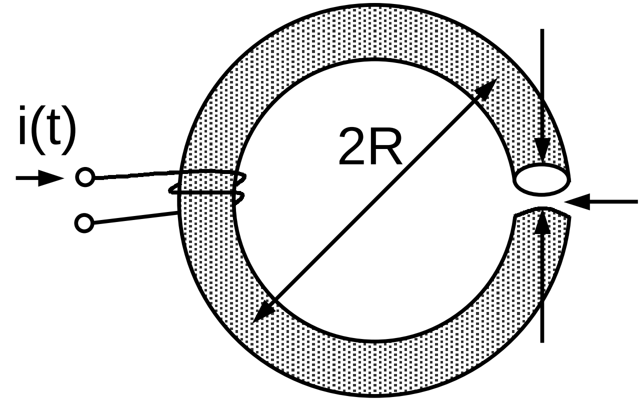
$$2\pi R|\overline{H}| \cong Ni \quad \text{(R varies slightly over A)}$$

$$L \cong \mu\int_V (N/2\pi R)^2 dv \cong \mu(N/2\pi R)^2 V \cong \mu N^2 A/2\pi R \;\; [\text{Henries}]$$

where $V \cong 2\pi RA\left[m^3\right]$

INDUCTANCE OF IRON TOROID (2)

Inductance L of a toroid with a gap:



Recall: $w_m = Li^2(t)/2 = \int_V W_m dv$ [J]

where $W_m = \mu |\overline{H}(t)|^2/2$ $[Jm^{-3}]$

Therefore: $L = \left(\mu \int_V |\overline{H}(t)|^2 dv\right)/i^2(t)$, as before

Finding $|\overline{H}(t)|$:

Since: $\oint_c \overline{H} \bullet d\overline{s} \cong Ni(t)$

Therefore: $|\overline{H}_\mu|(2\pi R - d) + |\overline{H}_{\mu_o}| d \cong Ni(t)$

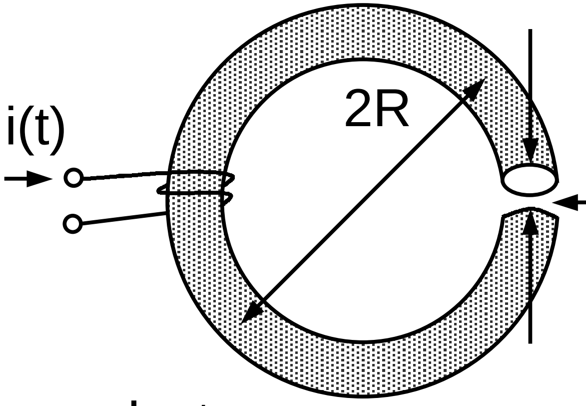
But $\nabla \bullet \overline{B} = 0$, so $\mu_o H_o = \mu H$ where we assume $\mu >> \mu_o$

Therefore: Magnetic energy density in gap is $(\mu/\mu_o)^2$ greater than inside the torus, and dominates unless $(\mu/\mu_o)^2 << 2\pi R/d$

$\left(\text{we require } d \lesssim 2\pi R (\mu_o/\mu)^2\right)$

INDUCTANCE OF IRON TOROID (3)

Inductance L of a toroid with a gap:



Recall: $L = \left(\mu(t)\int_V |\overline{H}|^2 dv\right) / i^2$

$$|\overline{H}_\mu|(2\pi R - d) + |\overline{H}_{\mu_o}| d \cong Ni(t)$$

Where: $|\overline{H}_{\mu_o}| d \cong Ni(t)$ for a small gap with little fringing ($d << A^{0.5}$) and we neglect the energy storage inside the torus

Therefore: $L \cong \mu A d (N/d)^2$ or, for a small-gap torus:

$$L \cong \mu_o A N^2 / d \quad \text{Henries}$$

provided that $d \gtrsim 2\pi R(\mu_o/\mu)^2$