SPHEROIDAL SOLUTION FOR UNBOUNDED ORBITS ABOUT AN OBLATE PLANET

by

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ABSTRACT

This paper derives the equations of motion for a satellite in an unbounded orbit about an oblate planet. The only force assumed present is the one due to Vinti's spheroidal gravitational potential, which can be fitted to the Earth so as to account exactly for the even zonal harmonics through the second and for most of the fourth. The equations of motion in integral form are obtained as a result of the separability of the Hamilton-Jacobi equation. Two sets of orbital elements are used for the solution. The first set can be obtained directly from a set of initial conditions. The second set, which allows analytical evaluation of the integrals, can be obtained from the first through numerical factoring of a quartic polynomial. The final solution, which is summarized in Chapter 10, is given in terms of this second set of orbital elements and certain uniformizing variables, whose periodic terms are correct to the second order in J_2.

The solution is valid for all inclinations, has no troublesome poles and reduces to Keplerian hyperbolic motion for a perfectly spherical planet. In order that certain series converge, however, we find it necessary to restrict out attention to trajectories whose extensions do not pass through a small spheroid in the center of the planet. A computer program is also presented which calculates the trajectory about both an oblate planet and a perfectly spherical planet for the same initial conditions.

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CHAPTER I

INTRODUCTION

If we let $\underline{r}$ be the position vector of a satellite relative to the center of mass of an oblate planet and if forces other than the gravitational forces between the two bodies can be neglected, then the motion of the satellite is determined by the equation

$$\ddot{\underline{r}} = -\nabla V \tag{1.1}$$

The gravitational potential V of the planet can be expressed in terms of spherical harmonics as

$$V = -\frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} \left(\frac{r_e}{r}\right)^n J_n P_n(\sin\theta)\right] + \text{tesseral harmonics} \tag{1.2}$$

where μ is the product of the gravitational constant and the mass of the planet, r is the magnitude of the position vector, r_e is the equatorial radius, θ is the declination, P_n is the nth Legendre polynomial, and the J_n are constants which characterize the planet's potential. For the Earth $J_2 \approx 1.08 \times 10^{-3}$ and all other J_n's are of the order 10^{-6} or smaller.

Most methods of solving the equations of motion involve determining a reference trajectory using the potential $V_o = -\frac{\mu}{r}$ and then determining the pertubations of this reference trajectory caused by the higher harmonics of the true potential.

Vinti [1], however, derived a form for the potential of an oblate planet as the solution of Laplace's equation in oblate spheroidal coordinates which leads to separability of the Hamilton-Jacobi equation. As used in this paper Vinti's spheroidal potential accounts for the second zonal harmonic J_2 and most of the fourth. Using his spheroidal potential Vinti [2,3] then solved the resulting equations of motion for the case of a bounded orbit, that is for total energies $\alpha_1 < 0$. The solution has an exact secular part and a periodic part correct to the second order in J_2.

The main object of the present paper is to provide a solution using Vinti's spheroidal potential for the case of an unbounded trajectory, that is for total energy $\alpha_1 > 0$. The approach will be to introduce certain uniformizing variables and then to evaluate the integrals of the equations of motion in terms of these variables in a manner analagous to that of Vinti for the bounded case. The equations of motion are then inverted so as to obtain the position and velocity vectors as functions of time. In order that the similarities between the bounded case and the unbounded case to be derived may be apparent, Vinti's notation has been preserved as far as possible in the present treatment. In addition we shall show that the solution for the unbounded case reduces to simple Keplerian hyperbolic motion for a perfectly spherical planet and that the solutions for bounded and unbounded spheroidal trajectories reduce to the same "parabolic" trajectory for $\alpha_1 = 0$.

A computer program is also developed which calculates the trajectory about both an oblate planet and a spherical planet for the same initial conditions. A numerical comparison then enables us to determine the effect of the oblateness and how it varies with the energy, inclination, and perigee distance of the trajectory.

CHAPTER 2

THE KINETIC EQUATIONS

2.1 The Oblate Spheroidal Coordinates

Of the eleven Staeckel coordinate systems, the oblate spheroidal system has the most appropriate symmetry for an oblate planet such as the Earth. If X, Y, Z are the rectangular coordinates and r, θ, ϕ are the spherical coordinates, then the oblate spheroidal coordinates, ρ, η, ϕ, are defined by

$$X + iY = r\cos\theta \exp i\phi = \left[(\rho^2+c^2)(1-\eta^2)\right]^{\frac{1}{2}} \exp i\phi \qquad (2.1)$$

$$Z = r\sin\theta = \rho\eta \qquad (-1 \leq \eta \leq 1) \qquad (\rho \geq 0) \qquad (2.2)$$

Here r and θ are the geocentric distance and declination, and ϕ is the right ascension. For the spheroidal coordinates, the surfaces of constant ρ are oblate spheroids, the surfaces of constant η are hyperboloids of one sheet, and the surfaces of constant ϕ are planes through the polar axis. The constant, c, is the radius of the focal circle in the equatorial plane. The portion of the equatorial plane inside the focal circle is the surface ρ = 0, while the portion outside is the surface η = 0. Note that for large r, $\rho \approx r$ and $\eta \approx \sin\theta$.

2.2 The Gravitational Potential

According to Vinti [1] the potential field of an oblate body may be closely represented by

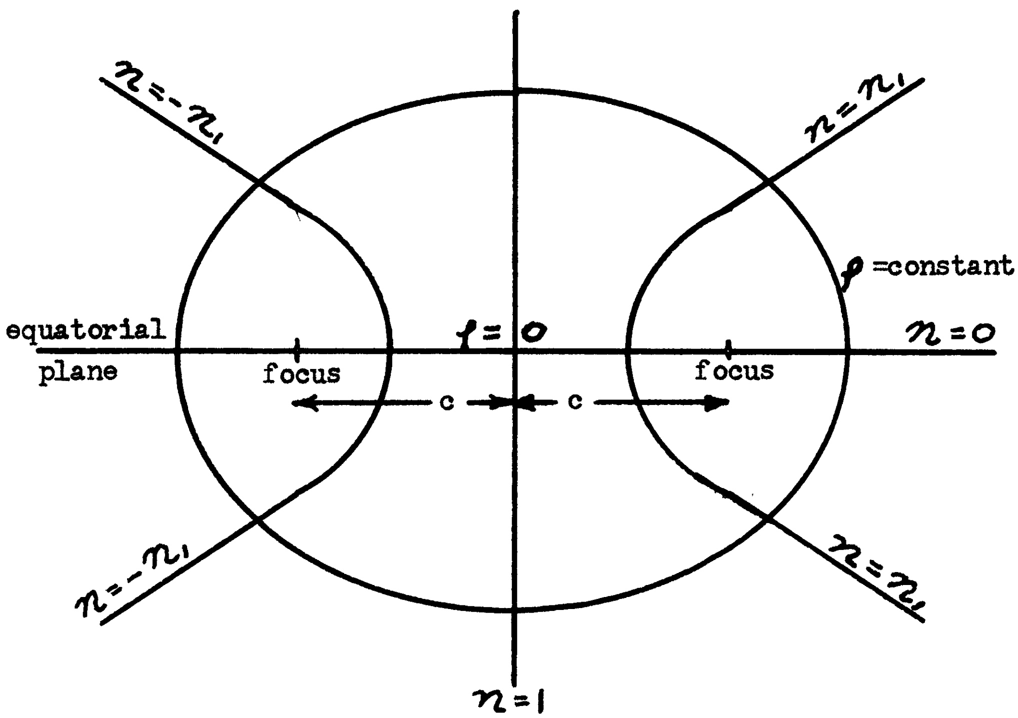

FIGURE 2.1 The Oblate Spheroidal Coordinates

$$V = -\mu\rho\ (\rho^2+c^2\eta^2)^{-1} \tag{2.3}$$

where μ is the product of the gravitational constant and the planetary mass, and

$$c^2 = r_e^2\ J_2 \tag{2.4}$$

where r_e is the planet's equatorial radius and J_2 is the coefficient of the second zonal harmonic in the expansion of the potential in spherical harmonics. For the Earth J_2 is approximately 1.082×10^{-3} and $c \approx 210$ km. In the equation for the potential, (2.3), we have taken the center of mass at the origin of the coordinate system.

The above potential leads to separability of the Hamilton-Jacobi equation in spheroidal coordinates. From Vinti [1], the resulting kinetic equations with the separation constants α_1, $\tilde{k}$, α_3 and the Hamilton-Jacobi function $S(\rho,\eta,\phi)$ are

$$\frac{\partial S}{\partial \alpha_1} = t + \beta_1 = \int_{\rho_1}^{\rho} \pm\ \rho^2\ F^{-\frac{1}{2}}(\rho)d\rho + c^2 \int_{\eta_1}^{\eta} \pm\ \eta^2\ G^{-\frac{1}{2}}(\eta)d\eta$$

$$\frac{\partial S}{\partial \tilde{k}} = \beta_2' = \frac{1}{2}\int_{\hat{\rho}_1}^{\rho} \pm\ F^{-\frac{1}{2}}(\rho)d\rho + \frac{1}{2}\int_{\eta_1}^{\eta} \mp\ G^{-\frac{1}{2}}(\eta)d\eta \tag{2.5}$$

$$\frac{\partial S}{\partial \alpha_3} = \beta_3 = \phi + c^2\alpha_3 \int_{\rho_1}^{\rho} \pm\ (\rho^2+c^2)\ F^{-\frac{1}{2}}(\rho)d\rho + \alpha_3 \int_{\eta_1}^{\eta} \mp\ (1-\eta^2)^{-1}\ G^{-\frac{1}{2}}(\eta)d\eta$$

where

$$F(\rho) = c^2\alpha_3^2 + (\tilde{k}+2\mu\rho+2\alpha_1\rho^2)(\rho^2+c^2) \tag{2.6}$$

and

$$G(\eta) = -\alpha_3^2 + (1-\eta^2)(-\tilde{k}+2\alpha^1 c^2\eta^2) \tag{2.7}$$

and the betas are Jacobi constants. For bounded motion, α_1 is negative. For such motion Vinti [1] has shown that the constant $\tilde{k}$ is negative, a fact which leads to considerable simplification of the integrals.

2.3 The Separation Constant $\tilde{k}$

We should now like to investigate the separation constant $\tilde{k}$, for unbounded motion $\alpha_1 > 0$. According to Vinti [1] the final separation of the Hamilton-Jacobi equation becomes (for the CM at the origin)

$$(\xi^2+1)\left(\frac{\partial S}{\partial \xi}\right)^2 - (\xi^2+1)^{-1}\alpha_3^2 - 2\mu c\xi - 2\alpha_1 c^2\xi^2 =$$

$$-(1-\eta^2)\left(\frac{\partial S}{\partial \eta}\right)^2 - (1-\eta^2)^{-1}\alpha_3^2 + 2\alpha_1 c^2\eta^2 = \tilde{k} \qquad (2.8)$$

where $\rho = c\xi$.

Since we have taken the center of mass of the earth to be at the origin of the coordinate system we can assume that the trajectory (or its extension) will pass through the equatorial plane. Furthermore, we will require this passage to occur on or outside the focal circle which is shown in Figure 2.2. Thus $\eta = 0$ will be reached by the trajectory and according to (2.8)

$$\tilde{k} + \alpha_3^2 = -\left(\frac{\partial S}{\partial \eta}\right)^2 \leq 0 \qquad (2.9)$$

For physically realizable motion the Kinetic equations (2.5) require that $F(\rho)$ and $G(\eta)$ be positive and our coordinate system requires that $\eta^2 \leq 1$ and $\rho \geq 0$. Rewriting $F(\rho)$ in the form

$$F(\rho) = c^2\alpha_3^2 + 2\alpha_1(\rho^2+c^2)\left[\left(\rho + \frac{\mu}{2\alpha_1}\right) + \frac{\tilde{k}}{2\alpha_1} - \frac{\mu^2}{4\alpha_1^2}\right] \qquad (2.10)$$

and setting $\alpha_3 = 0$, we find for the two real zeroes of $F(\rho)$

$$\rho = -\frac{\mu}{2\alpha_1} \pm \left[\frac{\mu^2}{4\alpha_1^2} - \frac{\tilde{k}}{2\alpha_1}\right]^{\frac{1}{2}} \qquad (2.11)$$

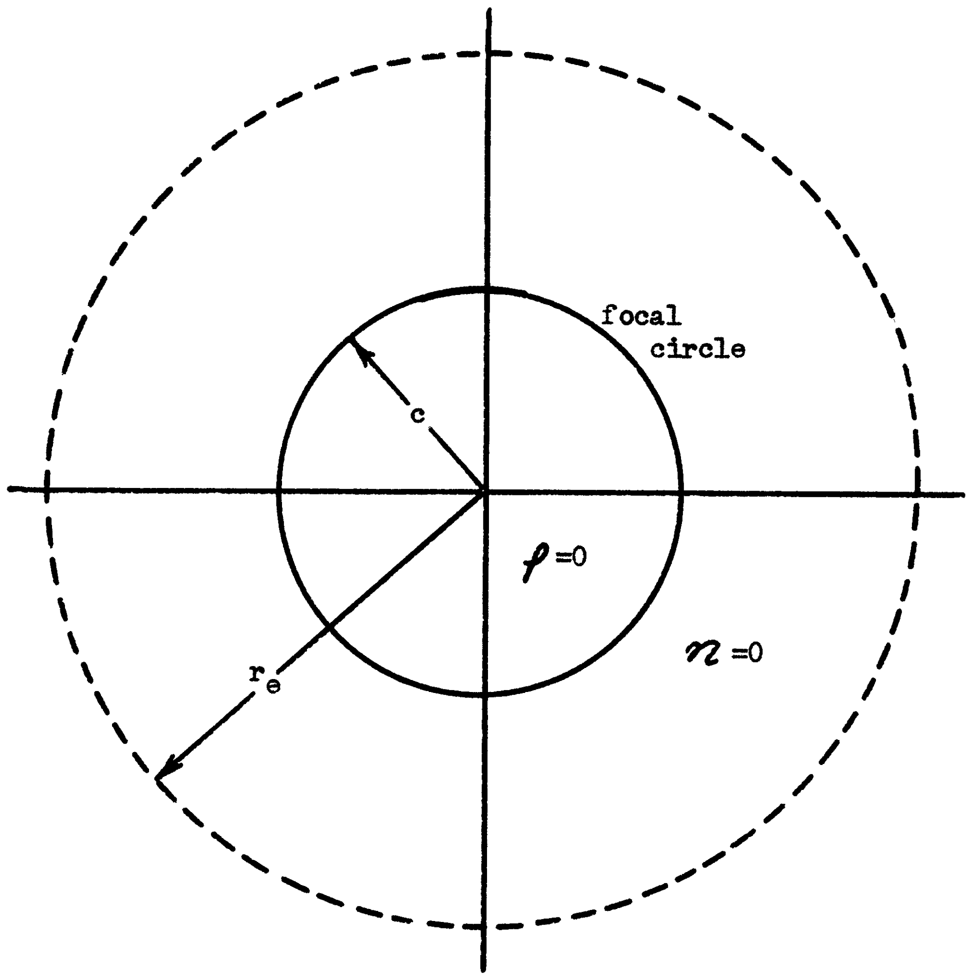

FIGURE 2.2 The Equatorial Plane

The more positive of the two roots will be labeled ρ_1, the other ρ_2. Equation (2.9) can be rewritten as

$$\tilde{k} \le -a_3^2 \tag{2.12}$$

A consideration of the derivatives of $F(\rho)$ and the values of $F(\rho)$ for $\rho = \pm\infty$ enables us to sketch $F(\rho)$ for $\alpha_3 = 0$ as in Figure 2.3. Realizable motion then takes place in the shaded region of the sketch from $\rho = \rho_1$ to $\rho = \infty$. Clearly ρ_1 is the ρ-perigee of the trajectory.

For the case $\alpha_3 \neq 0$ we put $F(\rho)$ in the form

$$F(\rho) = 2\alpha_1\rho^4 + 2\mu\rho^3 + (\tilde{k}+2\alpha_1 c^2)\rho^2 + 2\mu c^2\rho + c^2(\tilde{k}+\alpha_3^2) \tag{2.13}$$

At $\rho = 0$ we then have

$$F(0) = c^2(\tilde{k}+\alpha_3^2) \tag{2.14}$$

$F(0)$ will then be non-positive if (2.12) is imposed. For convergence of certain series we will later find it necessary to restrict our attention to trajectories with

$$2\alpha_1 c^2 \ll -\tilde{k} \tag{2.15}$$

Thus the third and fifth terms of (2.13) are negative for $\rho > 0$ so that, by Descarte's rule of sign, there are no more than three positive real zeroes of $F(\rho)$. A similar consideration for $\rho < 0$ reveals that $F(\rho)$ has no more than one negative real zero. $F(\rho)$ might then appear as in Figure 2.4 for $\alpha_3 \neq 0$. Again realizable motion occurs only for ρ and $F(\rho)$ non-negative. We shall later see that case (b) of Figure 2.4 occurs only for equatorial or very nearly equatorial orbits.

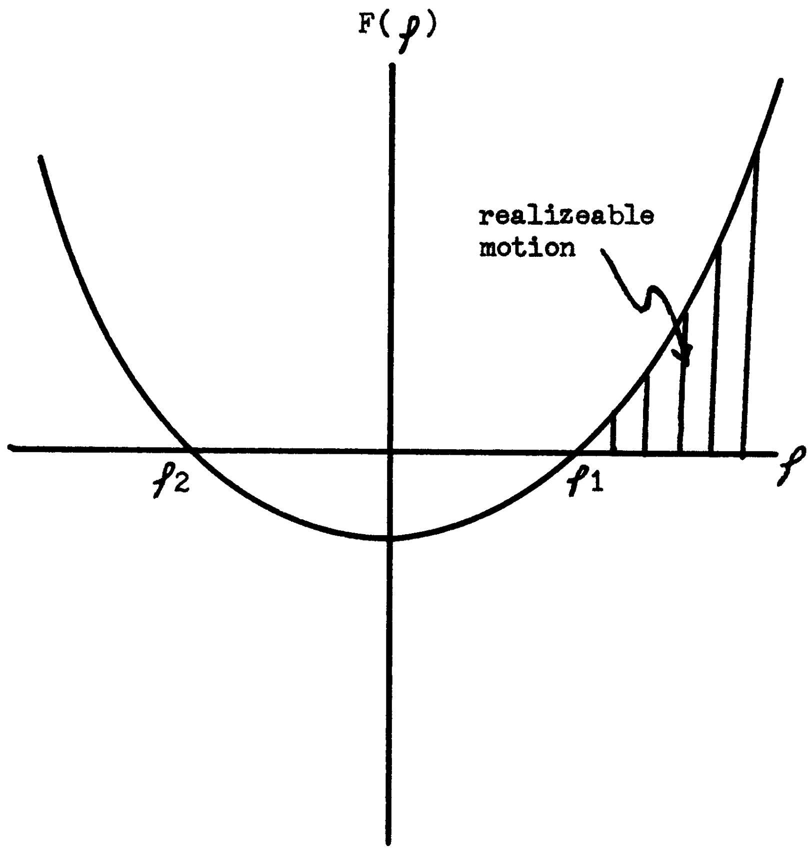

FIGURE 2.3 Sketch of F(ρ) for $\alpha_3 = 0$

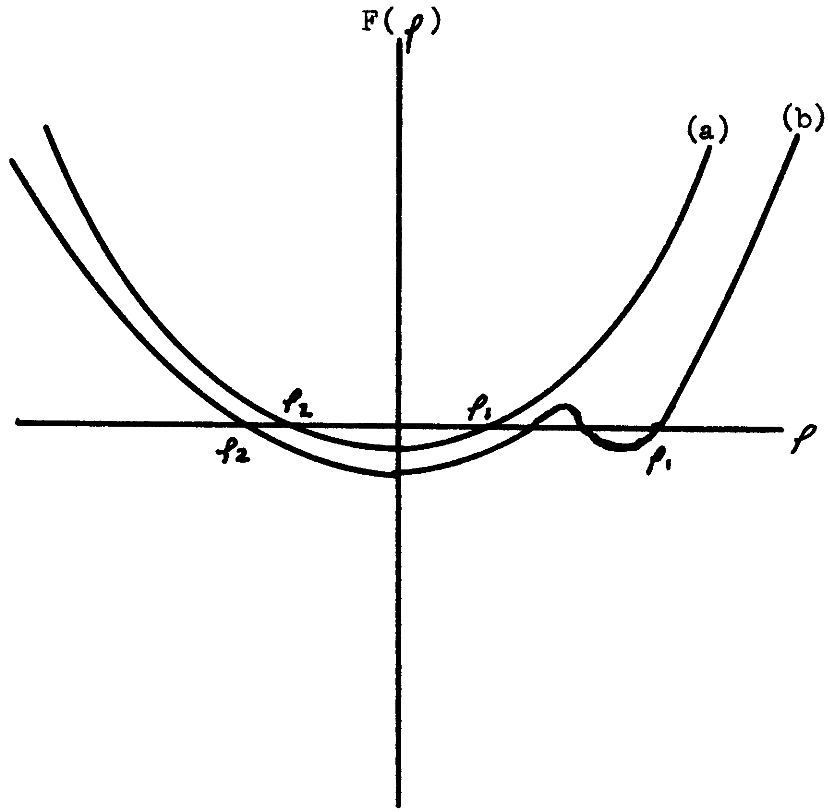

FIGURE 2.4 Sketch of F(ρ) for $\alpha_3 \neq 0$

According to (2.12) we can then set

$$\tilde{k} = -\alpha_2{}^2 \tag{2.16}$$

where

$$\alpha_2{}^2 \geq \alpha_3{}^2 \tag{2.17}$$

Having done this we can take α_2 as our new Jacobi constant. The equations (2.5), (2.6), and (2.7) then become

$$\frac{\partial S}{\partial \alpha_1} = t + \beta_1 = \int_{\rho_1}^{\rho} \pm \rho^2 F^{-\frac{1}{2}}(\rho)d\rho + c^2 \int_0^{\eta} \pm \eta^2 G^{-\frac{1}{2}}(\eta)d\eta \tag{2.18a}$$

$$\frac{\partial S}{\partial \alpha_2} = \beta_2 = \alpha_2 \int_{\rho_1}^{\rho} \mp F^{-\frac{1}{2}}(\rho)d\rho + \alpha_2 \int_0^{\eta} \pm G^{-\frac{1}{2}}(\eta)d\eta \tag{2.18b}$$

$$\frac{\partial S}{\partial \alpha_3} = \phi - \beta_3 = c^2\alpha_3 \int_{\rho_1}^{\rho} \mp (\rho^2+c^2)^{-1}F^{-\frac{1}{2}}(\rho)d\rho + \alpha_3\int_0^{\eta} \pm(1-\eta^2)^{-1}G^{-\frac{1}{2}}(\eta)d\eta \tag{2.18c}$$

where

$$F(\rho) = c^2\alpha_3{}^2 + (\rho^2+c^2)(-\alpha_2{}^2+2\mu\rho+2\alpha_1\rho^2) \tag{2.19}$$

and

$$G(\eta) = -\alpha_3{}^2 + (1-\eta^2)(\alpha_2{}^2+2\alpha_1c^2\eta^2) \tag{2.20}$$

The α's and β's are the Jacobi constants, with α_1, the energy, greater than zero for hyperbolic motion and α_3, the polar component of angular momentum, greater than or less than zero for direct or retrograde trajectories respectively. As $c \to 0$ and we approach Keplerian motion the separation constant α_2 reduces to the total angular momentum, $-\beta_1$ to the time of perigee passage, β_2 to the argument of perigee ω, and β_3 to the longitude of the node Ω. (These claims will be justified in Section 7)

Since terms with c^2 in them represent the deviations from a conic trajectory (which we expect to be small) the appropriate zeroes, ρ_1 and ρ_2, of $F(\rho)$ will approximately be equal to the zeroes of

$$f(\rho) = (-\alpha_2{}^2+2\mu\rho+2\alpha_1\rho^2) \tag{2.21}$$

Since $\alpha_1 \geq 0$, this tells us that one zero will be positive, the other negative (approaching $-\infty$ as α_1 approaches zero); which is just what we found in our investigation of $F(\rho)$. The positive zero has been labeled ρ_1; the negative zero, ρ_2. We will discover shortly that the other two roots of $F(\rho)$ are imaginary, except for nearly equatorial orbits.

Our ultimate objective is to solve (2.18), (2.19), and (2.20) for ρ, η, ϕ as functions of time. To do this we must first solve (2.18a) and (2.18b) for $\rho(t)$ and $\eta(t)$ and substitute into (2.18c) to find $\phi(t)$. This requires evaluating the six integrals which we shall find possible through the introduction of certain uniformizing variables, and the ability to factor the quartic polynomials $F(\rho)$ and $G(\eta)$.

CHAPTER 3

THE ORBITAL ELEMENTS a_0, e_0, i_0, β_1, β_2, β_3

For Keplerian hyperbolic motion ($c = 0$) the two roots of $f(\rho)$ would be

$$\rho_1 = r_1 = a_0(e_0-1) \tag{3.1}$$

$$\rho_2 = r_2 = -a_0(e_0+1) \tag{3.2}$$

where r_1 would be the perigee radius and r_2 an unphysical quantity. For future reference we solve for a_0 and e_0 from the above. Thus

$$a_0 = -\frac{(r_1+r_2)}{2} \tag{3.1a}$$

$$e_0 = -\frac{(r_1-r_2)}{r_1+r_2} \tag{3.2a}$$

Here we have taken

$$a_0 \equiv \frac{1}{2}\,\mu\alpha_1{}^{-1} \tag{3.3}$$

and

$$e_0{}^2 \equiv 1 + 2\alpha_1\alpha_2{}^2\mu^{-2} \tag{3.4}$$

For our spheroidal solution with $c \neq 0$ we can still define the constants a_0 and e_0 as above as well as another constant

$$i_0 \equiv \cos^{-1}(\alpha_3/\alpha_2) \tag{3.5}$$

Thus the constants a_0, e_0, i_0, β_1, β_2, β_3 would be one possible set of orbital elements. The corresponding semi-latus rectum p_0 would then be

$$p_0 = a_0(e_0^2 - 1) \tag{3.6}$$

so that

$$\alpha_2^2 = \mu p_0 \tag{3.7}$$

To find a_0, e_0, and i_0 we first need α_1, α_2, and α_3. These constants may be determined from the initial conditions. From Vinti [2]

$$\alpha_1 = \tfrac{1}{2}u_i^2 - \mu\rho_i(\rho_i^2 + c^2\eta_i^2)^{-1} \tag{3.8}$$

$$\alpha_2^2 = (1-\eta_i^2)^{-1}[(\rho_i^2 + c^2\eta_i^2)^2\dot{\eta}_i^2 + \alpha_3^2 - 2\alpha_1 c^2\eta_i^2(1-\eta_i^2)] \tag{3.9}$$

$$\alpha_3 = r_i^2\cos^2\theta_i\dot{\phi}_i = X_i\dot{Y}_i - Y_i\dot{X}_i \tag{3.10}$$

where the subscript i denotes initial values and u is the speed. So from the initial coordinates and their derivatives we can determine the α's and thus find a_0, e_0, and i_0. Numerical values for the α's would allow numerical factoring of $F(\rho)$ in the form

$$F(\rho) = 2\alpha_1(\rho-\rho_1)(\rho-\rho_2)(\rho^2+A\rho+B) \tag{3.11}$$

We can, however get an analytical solution for $\rho_1+\rho_2$, $\rho_1\rho_2$, A, and B by following the procedure of Vinti [2] and equating corresponding powers of ρ in (3.11) and (2.19). Thus we find

$$\rho^3: \quad \rho_1+\rho_2-A = -\mu\alpha_1^{-1} = -2a_0 \tag{3.12}$$

$$\rho^2: \quad B+\rho_1\rho_2-(\rho_1+\rho_2)A = c^2-\tfrac{1}{2}\alpha_2^2\alpha_1^{-1} = c^2-a_0p_0 \tag{3.13}$$

$$\rho^1: \quad (\rho_1+\rho_2)B-\rho_1\rho_2A = -\mu c^2\alpha_1^{-1} = -2a_0c^2 \tag{3.14}$$

$$\rho^0: \qquad \rho_1\rho_2 B = -\frac{1}{2}c^2(\alpha_2{}^2-\alpha_3{}^2)\alpha_1{}^{-1} = -a_0 p_0 c^2 \sin^2 i_0 \tag{3.15}$$

We can define

$$k_0 \equiv c^2/p_0{}^2 \equiv \left(\frac{r_e}{p_0}\right)^2 J_2 \tag{3.16}$$

$$x \equiv (e_0{}^2-1)^{\frac{1}{2}} \tag{3.17}$$

$$y \equiv \alpha_3/\alpha_2 = \cos i_0 \tag{3.18}$$

and then solve (3.12) through (3.15) for $\rho_1\rho_2$, $\rho_1 + \rho_2$, A, and B by assuming that each is a power series in k_0

$$\rho_1 + \rho_2 \equiv \sum_{n=0}^{\infty} b_{1_n} k_0{}^n \tag{3.19}$$

$$\rho_1\rho_2 \equiv \sum_{n=0}^{\infty} b_{2_n} k_0{}^n \tag{3.20}$$

$$A \equiv \sum_{n=0}^{\infty} b_{3_n} k_0{}^n \tag{3.21}$$

$$B \equiv \sum_{n=0}^{\infty} b_{4_n} k_0{}^n \tag{3.22}$$

and solving for the coefficients. For sufficiently small k_0 the series will converge rapidly (the range of validity of this assumption will be investigated at the end of this section) and the solution to $O(k_0{}^2)$ will be sufficiently accurate. The series solution, which is carried out in Appendix A, results in:

$$\rho_1 + \rho_2 = \frac{-2p_0}{x^2}\left[1 + x^2y^2k_0 + x^2y^2k_0{}^2(8y^2+3x^2y^2-2x^2-4)+ \cdots\right] \tag{3.23}$$

$$\rho_1\rho_2 = \frac{-p_0{}^2}{x^2}\left[1-y^2k_0(4+x^2) + k_0{}^2y^2(12x^2-20x^2y^2-32y^2+16 +x^4-x^4y^2)+ \cdots\right] \tag{3.24}$$

$$A = -2k_0p_0y^2 \left[1 + k_0(8y^2+3x^2y^2-2x^2-4)+ \cdots\right] \tag{3.25}$$

$$B = k_0p_0{}^2(1-y^2) \left[1 + k_0(4y^2+x^2y^2)+ \cdots\right] \tag{3.26}$$

Note that

$$A^2 - 4B \simeq -4k_0p_0{}^2 \left[1-y^2+k_0(4y^2-5y^4+x^2y^2-x^2y^4)\right] \tag{3.27}$$

which is positive for $\sin^2 i_0 < k_0/[1+k_0(6+x^2)]$ so that for equatorial or very nearly equatorial orbits we have case (b) of Figure 2.4.

We can define the constants a and e in terms of the two real roots of $F(\rho)$. By analogy to Keplerian motion, equations (3.1), (3.2), (3.1a), and (3.2a) we define

$$\rho_1 \equiv a(e-1) \tag{3.28}$$

$$\rho_2 \equiv -a(e+1) \tag{3.29}$$

so that

$$a \equiv -\frac{(\rho_1+\rho_2)}{2} \tag{3.30}$$

and

$$e \equiv -\frac{\rho_1-\rho_2}{\rho_1+\rho_2} \tag{3.31}$$

the corresponding semi-latus rectum is

$$p \equiv a(e^2-1) = -\frac{\rho_1\rho_2}{a} \tag{3.32}$$

The constants a, e, and p will occur in the evaluation of the ρ-integrals. We can calculate them in terms of a_0, e_0, p_0, and i_0 to the second order in k_0 as follows:

From (3.30)

$$\frac{a}{a_0} = - \frac{(\rho_1+\rho_2)}{2a}$$

Substituting for $\rho_1+\rho_2$ from (3.23) yields

$$\frac{a}{a_0} = 1 + k_0x^2y^2 + k_0{}^2x^2y^2(8y^2+3x^2y^2-2x^2-4)+ \cdots \tag{3.33}$$

Then from (3.32) and (3.30)

$$\frac{p}{p_0} = - \frac{\rho_1\rho_2}{ap_0} = \frac{2\rho_1\rho_2}{p_0(\rho_1+\rho_2)}$$

Substituting for $\rho_1\rho_2$ and $\rho_1+\rho_2$ from (3.24) and (3.23) and using the binomial expansion theorem we find

$$\frac{p}{p_0} = 1-2k_0y^2(2+x^2) + k_0{}^2y^2(16x^2-24x^2y^2-32y^2+16 +3x^4-2x^4y^2)+ \cdots \tag{3.34}$$

or equivalently

$$\frac{p_0}{p} = 1 + 2k_0y^2(2+x^2) + k_0{}^2y^2(-3x^4+6x^4y^2-16x^2 +40x^2y^2-16+48y^2)+ \cdots \tag{3.35}$$

Then since

$$\frac{e^2-1}{e_0{}^2-1} = \frac{p}{p_0}\ \frac{a_0}{a}$$

we find

$$\frac{e^2-1}{e_0{}^2-1} = 1 + k_0y^2(-3x^2-4) + k_0{}^2y^2(5x^4-2x^4y^2 +20x^2-28x^2y^2+16-32y^2)+ \cdots \tag{3.36}$$

and thus

$$\left(\frac{e^2-1}{e_0{}^2-1}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\, k_0 y^2(-3x^2-4) + \frac{1}{8}\, k_0{}^2 y^2(20x^4 -17x^4y^2+80x^2-136x^2y^2+64- 144y^2)+ \cdots \tag{3.37}$$

We can calculate e in terms of e_0 by noting

$$e \equiv \left[(e^2-1) + 1\right]^{\frac{1}{2}} \tag{3.38}$$

substituting for (e^2-1) from (3.36) and taking the square root we find that, to first order in k_0

$$e = \left[(e_0{}^2-1)[1+k_0y^2(-3x^2-4)+ \cdots]+1\right]^{\frac{1}{2}}$$

so that

$$e = \left[e_0{}^2 - x^2y^2k_0(3x^2+4)+ \cdots\right]^{\frac{1}{2}} \tag{3.39}$$

We shall later regard a and e as two of the final orbital elements, since they are part of a set from which the orbit can be calculated immediately. The above equations show how to find a, e, and related quantities, without iteration, from those quantities a_0, e_0, and i_0 which are obtainable immediately from the initial conditions.

The quartic $G(\eta)$ may be put in the factored form

$$G(\eta) \equiv (\alpha_2{}^2-\alpha_3{}^2)\ \eta^4(\eta^{-2}-\eta_0{}^{-2})(\eta^{-2}-\eta_2{}^{-2}) \tag{3.40}$$

On comparing this with (2.20) we find that $\eta_0{}^{-2}$ and $\eta_2{}^{-2}$ are the roots of

$$(\alpha_2{}^2-\alpha_3{}^2)\eta^{-4} + (2\alpha_1c^2-\alpha_2{}^2)\eta^{-2} - 2\alpha_1c^2 = 0 \tag{3.41}$$

Solving this for η^{-2} we get two roots and label them as follows:

$$\eta_0{}^{-2} = \frac{1}{2}\,(\alpha_2{}^2-2\alpha_1c^2)(\alpha_2{}^2-\alpha_3{}^2)^{-1}(1+W^{\frac{1}{2}}) \tag{3.42}$$

and

$$\eta_2^{-2} = \frac{1}{2}\,(\alpha_2{}^2-2\alpha_1c^2)(\alpha_2{}^2-\alpha_3{}^2)^{-1}(1-W^{\frac{1}{2}}) \tag{3.43}$$

where

$$W \equiv 1 + 8\alpha_1c^2(\alpha_2{}^2-\alpha_3{}^2)(\alpha_2{}^2-2\alpha_1c^2)^{-2} \tag{3.44}$$

In order to find $\eta_0{}^2$ and $\eta_2{}^2$ in terms of a_0, e_0, p_0, and i_0 we substitute for the α's in the above equations according to (3.3) through (3.7). Thus

$$W = 1 + \frac{4k_0x^2(1-y^2)}{(1-k_0x^2)^2} \tag{3.45}$$

$$\eta_0{}^2 = \frac{2(1-y^2)}{(1-k_0x^2)(1+W^{\frac{1}{2}})} \tag{3.46}$$

and

$$\eta_2{}^2 = \frac{2(1-y^2)}{(1-k_0x^2)(1-W^{\frac{1}{2}})} \tag{3.47}$$

Substituting W from (3.45) into (3.46) and (3.47) and performing the required binomial expansions we find

$$\eta_0 = \sin i_0 \left[1 + \frac{1}{2}\,k_0x^2y^2 + \frac{1}{8}\,k_0^2x^4y^2(7y^2-4)\,\cdots\right] \tag{3.48}$$

$$(1-\eta_0{}^2)^{-\frac{1}{2}} = |\sec i_0|\,\Big[1 + \frac{1}{2}\,k_0x^2(1-y^2) + \frac{1}{8}\,k_0x^4(1-y^2)(5y^2-1) + \cdots\Big] \tag{3.49}$$

$$\eta_2{}^{-2} = -k_0x^2(1 + k_0x^2y^2 + \cdots) \tag{3.50}$$

$$(\eta_0/\eta_2)^2 = -k_0x^2(\sin^2 i_0)(1 + 2k_0x^2y^2 + \cdots) \tag{3.51}$$

Note that $\eta_2{}^{-2} \approx -k_0x^2$, so that $\eta_2{}^2$ is negative and η_2 is imaginary. We shall retain the notation η_2^2 for the present, recalling that it is actually a negative quantity, since we shall have no occasion to deal with the quantity η_2 alone.

For the above series expansions in powers of k_0 to be valid we must have

$$k_0 x^2 = \frac{c^2}{a_0 p_0} = \frac{2\alpha_1 c^2}{\alpha_2^2} << 1 \tag{3.52}$$

which justifies our earlier restriction in (2.15) and implies that we must have

$$\frac{c^2}{ap} = \frac{c^2}{-\rho_1 \rho_2} << 1 \tag{3.53}$$

Or, since $|\rho_2| > |\rho_1|$, we should like to keep

$$\frac{c^2}{\rho_1^2} << 1 \tag{3.54}$$

Since ρ_1 is the value of ρ-perigee, if our trajectory or its continuation through the planet does not pass too near the surface

$$\rho^2 = c^2 \tag{3.55}$$

then (3.52) will be valid. In x, y, z coordinates this surface is an oblate spheroid as in Figure 3.1. This small forbidden zone (recall $c \approx 210$ km) in the center of the Earth should not appreciably limit the scope of problems to which this spheroidal method may be applied. Few space vehicles are fired directly from Earth into hyperbolic orbit and even fewer of these on paths whose extensions would have passed through this zone. The only practical case which readily comes to mind of a

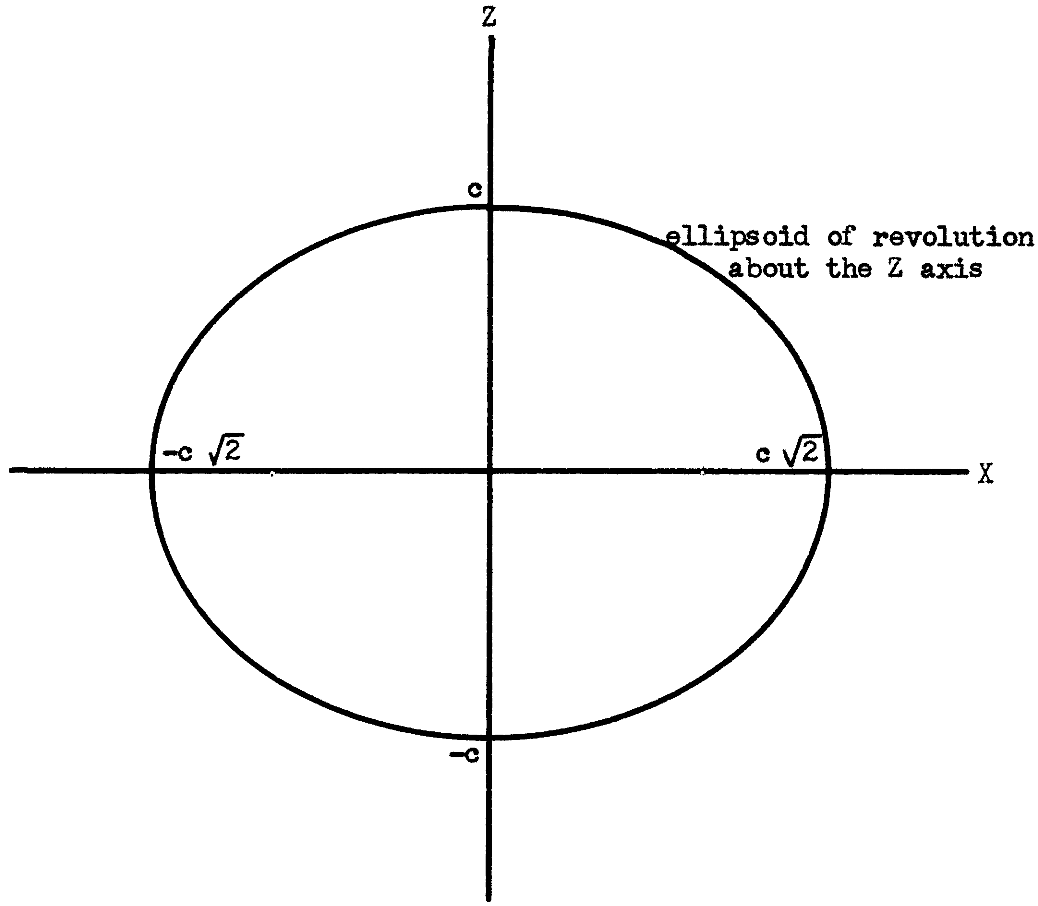

FIGURE 3.1 Forbidden Zone for Convergence of Series Solutions

trajectory which might pass through this zone is a meteor on a collision course with the Earth. Certainly any trajectory which does not intersect the Earth will cause no trouble.

CHAPTER 4

THE ORBITAL ELEMENTS a, e, I, β_1, β_2, β_3

If we put $G(\eta)$ in the factored form

$$G(\eta) = -2\alpha_1 c^2(\eta_0^2 - \eta^2)(\eta_2^2 - \eta^2) \tag{4.1}$$

then on comparing this to (3.40) and equating coefficients of equal powers of η we find

$$\eta_0^2 + \eta_2^2 = 1 - \frac{\alpha_2^2}{2\alpha_1 c^2} = 1 - \frac{a_0 p_0}{c^2} \tag{4.2}$$

$$\eta_0^2 \eta_2^2 = -\frac{(\alpha_2^2 - \alpha_3^2)}{2\alpha_1 c^2} = -\frac{a_0 p_0 \sin^2 i_0}{c^2} \tag{4.3}$$

We now define I by

$$I \equiv \sin^{-1}\eta_0 \qquad \cos I > 0 \quad \text{for direct orbits} \tag{4.4}$$

$$< 0 \quad \text{for retrograde orbits}$$

Such a definition is allowed since rationalizing the denominator of (3.46) yields

$$\eta_0^2 = -\left(\frac{1-k_0x^2}{2k_0x^2}\right) + \frac{1}{2k_0x^2}\left[(1+k_0x^2)^2 - 4k_0x^2y^2\right]^{\frac{1}{2}}$$

It then follows that $\eta_0^2 \leq 1$, with the equality occuring for polar trajectories.

If we now substitute equations (3.30), (3.31), and (3.32) into equations (3.12) through (3.15) we find

$$-2a - A = -2a_0 \tag{4.5}$$

$$B - ap + 2aA = c^2 - a_0p_0 \tag{4.6}$$

$$-2aB + apA = -2a_0c^2 \tag{4.7}$$

$$-apB = -a_0p_0c^2\sin^2 i_0 \tag{4.8}$$

If we now assume a, e, and η_0 are known (i.e. regard a, e, I, β_1, β_2, β_3 as orbital elements), then we have six equations in the six unknowns a , e , i , A, B, and η_2^2. With the orbital elements a, e, I, β_1, β_2, β_3 it turns out that the system can be solved exactly. This fact was first pointed out by Izsac (1960) and later utilized by Vinti (1961) in his spheroidal solution for bounded motion.

To proceed with the solution we first solve for η_2^2 from (4.2) and (4.3)

$$\eta_2^2 = -\frac{a_0p_0}{c^2}\frac{\sin^2 i_0}{\eta_0^2} = 1 - \frac{a_0p_0}{c^2} - \eta_0^2 \tag{4.9}$$

solving now for $\sin^2 i_0$ we find

$$\frac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2} = \sin^2 i_0 = \eta_0^2 - \frac{c^2\eta_0^2(1-\eta_0^2)}{a_0p_0} \tag{4.10}$$

Inserting this into (4.8)

$$-apB = c^2\eta_0^2(-a_0p_0 + c^2) - c^4\eta_0^4 \tag{4.11}$$

If we now use (4.6) to eliminate $c^2-a_0p_0$ from (4.11) we find

$$\frac{c^4\eta_0^4 - apB}{c^2\eta_0^2} = B - ap + 2aA \tag{4.12}$$

Eliminating a_0 between equations (4.5) and (4.7) yields

$$-2a - A = \frac{-2aB}{c^2} + \frac{apA}{c^2} \tag{4.13}$$

Equations (4.12) and (4.13) are now in terms of A and B only. Their simultaneous solution results in

$$A = \frac{-2ac^2(1-\eta_0{}^2)(ap + c^2\eta_0{}^2)}{(ap+c^2)(ap + c^2\eta_0{}^2) + 4a^2c^2\eta_0{}^2} \tag{4.14}$$

$$B = c^2\eta_0{}^2 \left[\frac{4a^2c^2 + (ap+c^2\eta_0{}^2)(c^2+ap)}{(ap+c^2)(ap+c^2\eta_0{}^2) + 4a^2c^2\eta_0{}^2}\right] \tag{4.15}$$

Then from (4.14) and (4.5)

$$\frac{\mu}{2a\alpha_1} \equiv \frac{a_0}{a} = 1 + \frac{A}{2a} = 1 - \frac{c^2(1-\eta_0{}^2)(ap+c^2\eta_0{}^2)}{(ap+c^2)(ap+c^2\eta_0{}^2) + 4a^2c^2\eta_0{}^2} \tag{4.16}$$

From (3.3), (3.7), (4.15) and (4.11)

$$\frac{\alpha_2{}^2}{2\alpha_1} \equiv a_0p_0 = c^2(1-\eta_0{}^2) + ap\left[\frac{(ap+c^2)(ap+c^2\eta_0{}^2)+4a^2c^2}{(ap+c^2)(ap+c^2\eta_0{}^2)+4a^2c^2\eta_0{}^2}\right] \tag{4.17}$$

These last two equations allow the determination of α_2 from the set of orbital elements a, e, and I. Putting η_0 = sin I into (4.10) yields

$$\alpha_3 = \alpha_2 \cos I \left[1 + \frac{c^2\eta_0{}^2}{a_0p_0}\right]^{\frac{1}{2}} \tag{4.18}$$

So that (4.16), (4.17) and (4.18) determine the α's, or equivalently, the elements a_0, e_0, i_0. Finally to find $\eta_2{}^2$ we combine (4.3) and (4.8) to get

$$\eta_2{}^2 = -\frac{Bap}{c^4\eta_0{}^2} \tag{4.19}$$

and then substitute for B from (4.15) to get

$$\eta_2{}^2 = -\frac{ap}{c^2}\left[\frac{(ap+c^2)(ap+c^2\eta_0{}^2) + 4a^2c^2}{(ap+c^2)(ap+c^2\eta_0{}^2) + 4a^2c^2\eta_0{}^2}\right] \qquad (4.20)$$

Thus equations (4.14) through (4.20) yield the required unknowns when the orbital elements are taken to be a, e, and I. Use of these orbital elements allows factoring the quartic polynomials $F(\rho)$ and $G(\eta)$ and facilitates evaluating the ρ and η integrals.

If we define a new oblateness parameter

$$k \equiv \frac{c^2}{p^2} = \frac{re^2}{p^2} J_2 \qquad (4.21)$$

corresponding to the orbital elements a, e, and I, then we can easily show that, at least to first order, the results obtained in this section agree with those of section 3.

Taking first order terms of (3.25), (3.26), (4.14), and (4.15), we find

$$A \approx -2k_0p_0\cos^2 i_0 \approx -2kp\cos^2 I \qquad (4.22)$$

$$B \approx k_0p_0{}^2\sin^2 i_0 \approx kp^2\sin^2 I \qquad (4.23)$$

To first order (3.33) and (4.16) become

$$\frac{a_0}{a} \approx 1 - k_0\cos^2 i_0(e_0{}^2-1) \approx 1 - k\cos^2 I(e^2-1) \qquad (4.24)$$

Solving (4.17) for p_0/p to first order and using (4.24) to eliminate a_0 and then comparing results with the first order portion of (3.35), we find

$$\frac{p_0}{p} \approx 1 + 2k_0\cos^2 i_0(e_0{}^2+1) \approx 1 + 2k\cos^2 I(e^2+1) \qquad (4.25)$$

Then, since $\frac{e_0{}^2-1}{e^2-1} = \frac{p_0}{p}\ \frac{a}{a_0}$ we find to first order

$$\frac{e_0{}^2-1}{e^2-1} \approx 1 + k_0\cos^2 i_0(3e_0{}^2+1) \approx 1 + k\cos^2 I(3e^2+1) \tag{4.26}$$

From (4.10)

$$\frac{\sin^2 i_0}{\sin^2 I} = 1 - \frac{c^2\cos^2 I}{a_0 p_0} \approx 1 - k(e^2-1)\cos^2 I \tag{4.27}$$

to first order. Taking the square root of (4.27) and comparing with the first order inverse of (3.48) gives

$$\frac{\sin i_0}{\sin I} \approx 1 - \frac{1}{2}\, k_0(e_0{}^2-1)\cos^2 i_0 \approx 1 - \frac{1}{2}\, k(e^2-1)\cos^2 I \tag{4.28}$$

Comparing the first order portions of (3.50) and (4.20) gives

$$\eta_2{}^{-2} \approx -k_0(e_0{}^2-1) \approx -k(e^2-1) \tag{4.29}$$

Then from (3.51) and the previous equation, we find

$$(\eta_0/\eta_2)^2 \approx -k_0(e_0{}^2-1)\sin^2 i_0 \approx -k(e^2-1)\sin^2 I \tag{4.30}$$

Thus the results of Sections 3 and 4 agree to first order.

We have seen that, of $G(\eta)$'s four roots, two are real, $\eta = \pm\sqrt{\eta_0{}^2}$, while two are imaginary, $\eta = \pm\sqrt{\eta_2{}^2}$ (recalling that $\eta_2{}^2 < 0$). With $G(\eta)$ in the form of (4.1) we can see that for $\alpha_1 > 0$ $G(\eta)$ will approach $-\infty$ for η approaching either $+\infty$ or $-\infty$. Furthermore at $\eta = 0$, $G(\eta) = -2\alpha_1 c^2\eta_0{}^2\eta_2{}^2 = \alpha_2{}^2-\alpha_3{}^2$, so that a sketch of $G(\eta)$ can be drawn as in Figure 4.1.

Again, according to the Kinetic equation, only the portion of the curve $G(\eta) \geq 0$ represents allowable motion. Thus, the real motion must occur in the interval between the two hyperboloids

$$-\eta_0 \leq \eta \leq +\eta_0 \tag{4.31}$$

where $\eta_0{}^2 \leq 1$

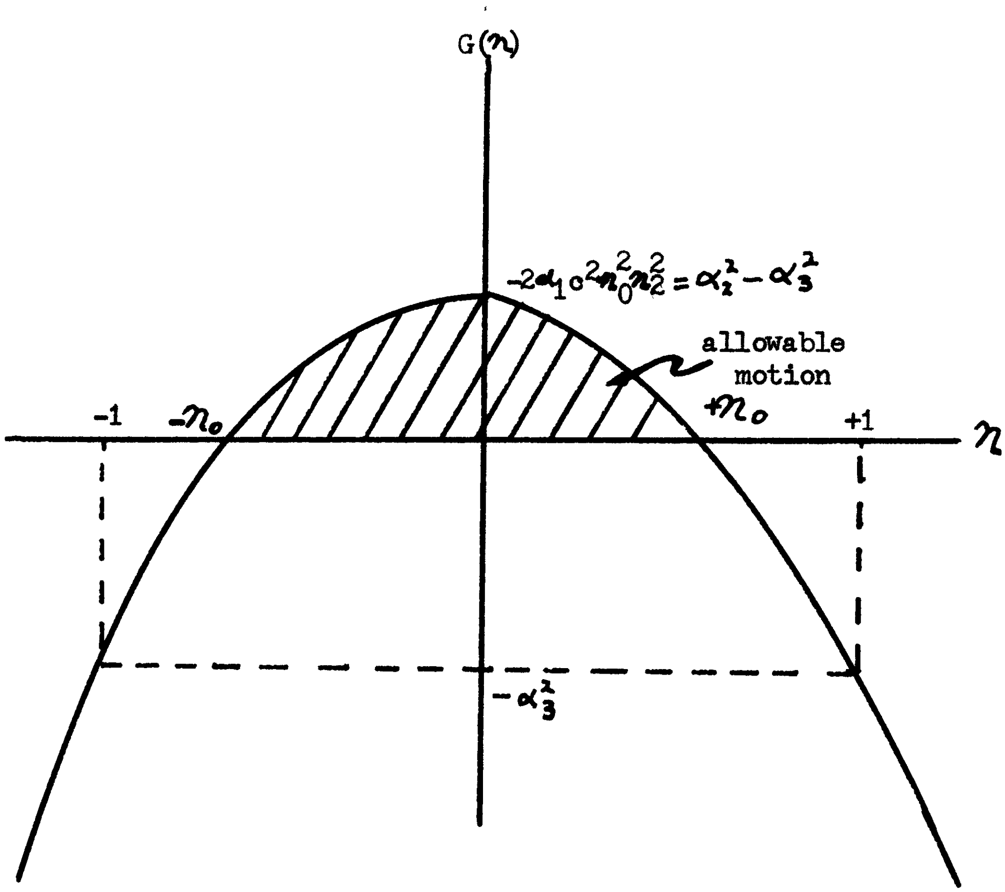

FIGURE 4.1 Sketch of G(η)

The following two sections will be devoted to evaluating the ρ and η integrals in terms of the orbital elements a, e, I, and certain uniformizing variables H, f, and ψ, corresponding to the hyperbolic anomaly, the true anomaly, and the argument of latitude respectively in Keplerian hyperbolic motion.

As opposed to elliptic motion, where the quantities a, e, and I might be determined by following the orbit for many revolutions and applying some sort of least-squares process, the "one-shot" nature of hyperbolic motion will probably require the use of a_0, e_0, and i_0, which can be determined from a set of initial conditions, as orbital elements. To find a, e, and I, one must then numerically factor $F(\rho)$. This is accomplished through order $J_2{}^2$ by means of equations (3.23) through (3.26) and equations (3.42) and (3.43). Once a, e, and I are thus known, we can insert observations into the formulae resulting from equations (2.18) to find the β's. (see Section 7.5) Thus we can always find the orbital elements a, e, I, β_1, β_2, β_3. From this point on, in the evaluation of the integrals, we shall assume that a, e, $\eta_0 \equiv \sin I$, β_1, β_2, and β_3 are known and give the solution in terms of these quantities.

CHAPTER 5

THE ρ-INTEGRALS

In the Kinetic equations (2.18) we define

$$R_1 \equiv \int_{\rho_1}^{\rho} \pm\, \rho^2\, F(\rho)^{-\frac{1}{2}} d\rho \qquad (5.1)$$

$$R_2 \equiv \int_{\rho_1}^{\rho} \pm\, F(\rho)^{-\frac{1}{2}} d\rho \qquad (5.2)$$

$$R_3 \equiv \int_{\rho_1}^{\rho} \pm\, (\rho^2+c^2)^{-1}\, F(\rho)^{-\frac{1}{2}} d\rho \qquad (5.3)$$

where the + sign is to be used for positive $d\rho$ and the - sign for negative $d\rho$. With $F(\rho)$ given by (3.11), A and B by (4.14) and (4.15), and ρ_1 by (3.28), we then write $F^{-\frac{1}{2}}(\rho)$ as

$$F(\rho)^{-\frac{1}{2}} = (2\alpha_1)^{-\frac{1}{2}}\, (\rho-\rho_1)^{-\frac{1}{2}}\, (\rho-\rho_2)^{-\frac{1}{2}}\, \rho^{-1}(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}} \qquad (5.4)$$

If we define

$$b_1 \equiv -\frac{A}{2} \qquad (5.5)$$

$$b_2 \equiv \sqrt{B} \qquad (5.6)$$

$$\lambda \equiv \frac{b_1}{b_2} \qquad (5.7)$$

$$h \equiv \frac{b_2}{\rho} \qquad (5.8)$$

then

$$(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}} = (1-2\lambda h+h^2)^{-\frac{1}{2}} \tag{5.9}$$

In this form we recognize the generating function for the Legendre polynomials. Thus we can write

$$(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(\lambda) \tag{5.10}$$

where the $P_n(\lambda)$ are the Legendre polynomials. According to Hobson [4] such replacement is valid as long as

$$h < 1 \quad \text{for } \lambda \leq 1 \tag{5.11}$$

$$h < \lambda - (\lambda^2-1)^{\frac{1}{2}} \quad \text{for } \lambda > 1 \tag{5.12}$$

The conditions under which (5.11) is satisfied are investigated in Appendix B. The resulting limiting surface, within which we require the extension of the trajectory not to pass, is somewhat less restrictive than the limiting surface discussed previously in Section 3.

Putting (5.8) and (5.10) into (5.4) we find

$$F(\rho)^{-\frac{1}{2}} = (2\alpha_1)^{-\frac{1}{2}} \sum_{n=0}^{\infty} b_2{}^n P_n(\lambda)\rho^{-1-n} [(\rho-\rho_1)(\rho-\rho_2)]^{-\frac{1}{2}} \tag{5.13}$$

On putting this into (5.1) and evaluating the first two terms of the summation, we find

$$(2\alpha_1)^{\frac{1}{2}}R_1 = \int_{\rho_1}^{\rho} \rho^2(\rho^{-1}+b_2\lambda\rho^{-2})[(\rho-\rho_1)(\rho-\rho_2)]^{-\frac{1}{2}}(\pm d\rho)$$

$$+ \sum_{n=2}^{\infty} b_2{}^n P_n(\lambda) \int_{\rho_1}^{\rho} \rho^{1-n} [(\rho-\rho_1)\ (\rho-\rho_2)]^{-\frac{1}{2}}(\pm d\rho) \tag{5.14}$$

or equivalently

$$(2\alpha_1)^{\frac{1}{2}} R_1 = \int_{\rho_1}^{\rho} (\rho+b_1)[(\rho-\rho_1)(\rho-\rho_2)]^{-\frac{1}{2}} (\pm d\rho)$$

$$+ \sum_{n=2}^{\infty} b_2{}^n P_n(\lambda) \int_{\rho_1}^{\rho} \rho^{1-n} [(\rho-\rho_1)(\rho-\rho_2)]^{-\frac{1}{2}} (\pm d\rho) \quad (5.15)$$

We can eliminate the plus or minus signs in the above integrals by introducing the uniformizing variables H and f, analagous respectively to the hyperbolic anomaly and true anomaly in Keplerian hyperbolic motion. We thus define H and f by

$$\rho = a(e \cosh H-1) = \frac{p}{1+e\cos f} \quad (5.16)$$

and the requirement that they always increase with time. Here H goes from $-\infty$ on the inbound assymptote ($\rho = \infty$, $t = -\infty$) to $+\infty$ on the outbound assymptote ($\rho = \infty$, $t = \infty$). From (5.16) we find

$$d\rho = ae \sinh H \, dh \quad (5.17)$$

Using (5.16), (5.17), $\rho_1 = a(e-1)$, and $\rho_2 = -a(e+1)$ we find that

$$[(\rho-\rho_1)(\rho-\rho_2)]^{-\frac{1}{2}} (\pm d\rho) = \pm\frac{\sinh H}{|\sinh H|} dH = dH \quad (5.18)$$

Again from (5.16)

$$d\rho = \frac{pe \sin f \, df}{(1+e \cos f)^2} \quad (5.19)$$

Using (5.19) and (5.16) we find

$$[(\rho-\rho_1)(\rho-\rho_2)]^{-\frac{1}{2}} (\pm d\rho) = (e^2-1)^{\frac{1}{2}} (1+e\cos f)^{-1} \, df \quad (5.20)$$

5.1 The R_1 Integral

If we substitute (5.18) into the first term of (5.15) and substitute (5.20) into the remaining terms of the series we find

$$(2\alpha_1)^{\frac{1}{2}}R_1 = \int_0^H (ae \cosh H - a + b_1)dH$$

$$+ \sum_{n=2}^{\infty} b_2{}^n \, P_n(\lambda) \int_0^f \left(\frac{p}{1+e\cos f}\right)^{1-n}(e^2-1)^{\frac{1}{2}}(1+e\cos f)^{-1}df \qquad (5.21)$$

which becomes

$$(2\alpha_1)^{\frac{1}{2}}R_1 = a(e\sinh H - H) + b_1 H$$

$$+ \, p(e^2-1)^{\frac{1}{2}} \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda) \int_0^f (1+e\cos f)^{n-2}df \qquad (5.22)$$

where we have replaced the lower limits of the integrals by zero, since at ρ_1, $H = f = 0$. If we define

$$S_1 \equiv \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda) \int_0^f (1+e\cos f)^{n-2}df \qquad (5.23)$$

or equivalently

$$S_1 \equiv \left(\frac{b_2}{p}\right)^2 \sum_{m=0}^{\infty} \left(\frac{b_2}{p}\right)^m P_{m+2}(\lambda) \int_0^f (1+e\cos f)^m \, df \qquad (5.24)$$

then we should like to investigate the convergence of this series. This is done in Appendix C for both of the cases $\lambda \leq 1$ and $\lambda > 1$. The resulting limiting surface, within which we require the extension of the trajectory not to pass, is found to be the same as the limiting surface for allowing the replacement of $(1 + A\rho^{-1} + B\rho^{-2})^{-\frac{1}{2}}$ by the Legendre series. Thus we are certain S_1 will converge as long as the trajectory does not pass within the surface of Figure 3.1.

To evaluate R_1 we need to evaluate S_1 and to do this we first separate it into a part proportional to f and a periodic part. Using the method of Vinti [2] in his treatment of the bounded case we define

$$f_m(f) \equiv \int_0^f (1+e\cos f)^m \, df \tag{5.25}$$

In Appendix D we notice that $f_m(f) - (\frac{f}{2\pi})f_m(2\pi)$ is an odd function of f of period 2π. Then, from (D.9) $f_m(2\pi) = 2f_m(\pi)$, so that from (D.12)

$$f_m \equiv \int_0^f (1+e\cos f)^m \, df = \frac{1}{\pi} f \int_0^\pi (1+e\cos f)^m \, df + \sum_{j=1}^{m} c_{mj} \sin j f \tag{5.26}$$

We have thus separated f_m into periodic and proportional parts. We will later observe that to evaluate the periodic portions of the ρ-integrals to order k^2 it will only be necessary to consider values of m up to 4 in the summation of (5.26). The proportional term of (5.26), however, requires more analysis.

We can obtain a more convenient expression for the proportional term of (5.26) by noting that

$$\int_0^\pi (z + \sqrt{z^2-1} \cos f)^m \, df = \pi \, P_m(z) \tag{5.27}$$

a relation which, according to Madelung [5] is valid for all z including imaginary values. Here the $P_m(z)$ are again the Legendre polynomials

$$P_m(z) \equiv \frac{2^{-m}}{m!} \frac{d^m}{dz^m} (z^2-1)^m \tag{5.28}$$

If in (5.27) we substitute

$$z = -\frac{i}{\sqrt{e^2-1}} \tag{5.29}$$

we obtain

$$\pi P_m[-i(e^2-1)^{-\frac{1}{2}}] = \int_0^\pi \left[-\frac{i}{\sqrt{e^2-1}} - i\left(\frac{e^2}{e^2-1}\right)^{\frac{1}{2}} \cos f\right]^m df$$

$$= \left(-\frac{i}{\sqrt{e^2-1}}\right)^m \int_0^\pi (1+e \cos f)^m \, df \tag{5.30}$$

so that

$$\int_0^\pi (1+e \cos f)^m \, df = \pi(i \sqrt{e^2-1})^m \, P_m \left(-\frac{i}{\sqrt{e^2-1}}\right) \tag{5.31}$$

$$= \pi \, T_m \, (\sqrt{e^2-1}) \tag{5.32}$$

where we have defined

$$T_m \, (\sqrt{e^2-1}) \equiv (i \sqrt{e^2-1})^m \, P_m \left(-\frac{i}{\sqrt{e^2-1}}\right) \tag{5.33}$$

The first few of these polynomials $T_m(x)$ are given in Table 5.1 along with the corresponding Legendre functions.

The polynomials $T_n(x)$ can easily be calculated from $T_0(x) = 1$, $T_1(x) = 1$, and the recursion formula

$$T_m(x) = [(2m-1) \, T_{m-1}(x) + (m-1)x^2 T_{m-2}(x)]/m \tag{5.34}$$

Now, substituting (5.32) into (5.26), we have in terms of $T_n(\sqrt{e^2-1})$

$$\int_0^f (1+e \cos f)^m \, df = f \, T_m(\sqrt{e^2-1}) + \sum_{j=1}^{m} c_{mj} \sin jf \tag{5.35}$$

We then find the coefficients c_{mj} by choosing an m, expanding $(1+e \cos f)^m$, integrating, and comparing to the right hand side of (5.35). Thus for $m = 1$

$$\int_0^f (1 + e \cos f) \, df = f + e \sin f \tag{5.36}$$

Then since $T_1(x) = 1$ we must have $c_{11} = e$ and all the other $c_{1j} = 0$.

TABLE 5.1

THE POLYNOMIALS $T_m(x)$

m	$P_m(-\frac{i}{x})$	$T_m(x)$
0	1	1
1	$-\frac{i}{x}$	1
2	$\frac{1}{2}(-\frac{3}{x^2} - 1)$	$\frac{1}{2}(3 + x^2)$
3	$\frac{1}{2}(\frac{5i}{x^3} + \frac{3i}{x})$	$\frac{1}{2}(5 + 3x^2)$
4	$\frac{1}{8}(\frac{35}{x^4} + \frac{30}{x^2} + 3)$	$\frac{1}{8}(35 + 30x^2 + 3x^4)$
5	$\frac{1}{8}(-\frac{63i}{x^5} - \frac{70i}{x^3} - \frac{15i}{x})$	$\frac{1}{8}(63 + 70x^2 + 15x^4)$
6	$\frac{1}{16}(-\frac{231}{x^6} - \frac{315}{x^4} - \frac{105}{x^2} - 5)$	$\frac{1}{16}(231 + 315x^2 + 105x^4 + 5x^6)$

Applying this method for m = 0 to m = 4 we compile Table 5.2. Substituting (5.35) into (5.23) would then give us S_1. Let us consider first the portion of S_1 proportional to f. Thus from (5.35) and (5.23) we write

$$p(e^2-1)^{\frac{1}{2}}(S_1)_{prop.} = p(e^2-1)^{\frac{1}{2}} f \sum_{n=2}^{\infty} (\frac{b_2}{p})^n P_n(\lambda) T_{n-2}[(e^2-1)^{\frac{1}{2}}] \quad (5.37)$$

$$\equiv A_1 f \quad (5.38)$$

where we have defined

$$A_1 \equiv p(e^2-1)^{\frac{1}{2}} \sum_{n=2}^{\infty} (\frac{b_2}{p})^n P_n(\lambda) T_{n-2}(\sqrt{e^2-1}) \quad (5.39)$$

then since $T_m(\sqrt{e^2-1}) \leq (1+e)^m$ our previous investigation of the convergence of S_1 shows the rapid convergence of A_1.

If we now consider the periodic part of S_1, we find from (5.35) and (5.24)

$$p(e^2-1)^{\frac{1}{2}} (S_1)_{per.} = (\frac{b_2}{p})^2 \sum_{m=1}^{\infty} (\frac{b_2}{p})^m P_{m+2}(\lambda) \sum_{j=1}^{m} c_{mj} \sin jf \quad (5.40)$$

$$\equiv \sum_{j=1}^{\infty} A_{1_j} \sin jf \quad (5.41)$$

where we have defined by (5.41)

$$A_{1_j} = p(e^2-1)^{\frac{1}{2}} \left[(\frac{b_2}{p})^3 P_3(\lambda)\, c_{1_j} + (\frac{b_2}{p})^4 P_4(\lambda)\, c_{2_j} + (\frac{b_2}{p})^5 P_5(\lambda)\, c_{3_j} + \cdots\right]$$

If in (5.42) we substitute the values of the $P_n(b_1/b_2)$ and the c_{mj} and retain only those terms through order k^2 [recalling from (B.3) and (B.4) that $b_1 = O(k)$, $b_2 = O(k^{\frac{1}{2}})$] we find

TABLE 5.2

THE COEFFICIENTS c_{mj}

m \ j	1	2	3	4
0				
1	e			
2	$2e$	$\frac{e^2}{4}$		
3	$3e + \frac{3e^3}{4}$	$\frac{3e^2}{4}$	$\frac{e^3}{12}$	
4	$4e + 3e^3$	$\frac{3e^2}{2} + \frac{e^4}{4}$	$\frac{e^3}{3}$	$\frac{e^4}{32}$

$$A_{11} = \frac{3(e^2-1)^{\frac{1}{2}}}{4p^3}\left[-2b_1b_2{}^2p + b_2{}^4\right]e \tag{5.43}$$

$$A_{12} = \frac{3(e^2-1)^{\frac{1}{2}}\, b_2{}^4e^2}{32p^3} \tag{5.44}$$

The other A_{1_j} are of order k^3 or higher. Finally, substituting (5.41) and (5.38) back into (5.22) there results

$$(2\alpha_1)^{\frac{1}{2}}R_1 = a(e \sinh H - H) + b_1H + A_1f + \sum_{j=1}^{2} A_{1_j} \sin jf \tag{5.45}$$

with A_1, A_{11}, and A_{12} given by (5.39), (5.43) and (5.44) respectively.

5.2 The R_2 Integral

From (5.2) the R_2 integral is

$$R_2 = \int_{\rho_1}^{\rho} \pm F(\rho)^{-\frac{1}{2}}\, d\rho \tag{5.2}$$

If into this we insert (5.13) for $F(\rho)^{-\frac{1}{2}}$ and then use (5.20) to put the result in terms of the uniformizing variable f we find

$$(2\alpha_1)^{\frac{1}{2}}R_2 = \frac{(e^2-1)^{\frac{1}{2}}}{p}\sum_{n=0}^{\infty}\left(\frac{b_2}{p}\right)^n P_n(\lambda)\int_0^f (1+e\cos f)^n\, df \tag{5.46}$$

We then define

$$S_2 \equiv \sum_{n=0}^{\infty}\left(\frac{b_2}{p}\right)^n P_n(\lambda)\int_0^f (1+e\cos f)^n\, df \tag{5.47}$$

We should like to investigate the convergence of S_2. Since cos f cannot exceed 1, we may write

$$|S_2| \leq f\sum_{n=0}^{\infty}\left(\frac{b_2}{p}\right)^n P_n(\lambda)\,(1+e)^n \tag{5.48}$$

Evaluating the first two terms of the series we find

$$|S_2| \le f + \frac{b_1(1+e)f}{p} + \left[\frac{b_2(1+e)}{p}\right]^2 f \sum_{m=0}^{\infty} \left(\frac{b_2}{p}\right)^m P_{m+2}(\lambda)(1+e)^m \qquad (5.49)$$

On comparing the third term of (5.49) with equation (C.1) of Appendix C, we see that S_2 will converge whenever S_1 converges. We saw in Appendix C that we were assured of S_1 (and thus S_2) converging as long as the conditions derived in Appendix B, for the replacement of $(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}}$ by the Legendre series, were met.

To evaluate R_2 we proceed as we did with R_1 and separate it into parts proportional to f and periodic in f. We substitute (5.35) into (5.46) to get

$$(2\alpha_1)^{\frac{1}{2}} R_2 = A_2 f + \sum_{j=1}^{\infty} A_{2_j} \sin jf \qquad (5.50)$$

where we have defined

$$A_2 \equiv \frac{(e^2-1)^{\frac{1}{2}}}{p} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda)\, T_n(\sqrt{e^2-1}) \qquad (5.51)$$

and

$$A_{2_j} \equiv \frac{(e^2-1)^{\frac{1}{2}}}{p} \left[\frac{b_2}{p} P(\lambda)\, c_{1_j} + \left(\frac{b_2}{p}\right)^2 P_2(\lambda)\, c_{2_j} + \cdots\right] \qquad (5.52)$$

Evaluating the A_{2_j} as before and keeping only the terms through order k^2 we find

$$A_{21} = \frac{(e^2-1)^{\frac{1}{2}}e}{p} \left[\frac{b_1}{p} + \frac{3b_1{}^2-b_2{}^2}{p^2} - \frac{9b_1b_2{}^2}{2p^3}\left(1+\frac{e^2}{4}\right) + \frac{3b_2{}^4}{8p^4}(4+3e^2)\right] \qquad (5.53)$$

$$A_{22} = \frac{(e^2-1)^{\frac{1}{2}}e^2}{p} \left[\frac{3b_1{}^2-b_2{}^2}{8p^2} - \frac{9b_1b_2{}^2}{8p^3} + \frac{3b_2{}^4}{16p^4}\left(3+\frac{e^2}{2}\right)\right] \qquad (5.54)$$

$$A_{23} = \frac{(e^2-1)^{\frac{1}{2}}e^3}{8p} \left[-\frac{b_1b_2{}^2}{p^3} + \frac{b_2{}^4}{p^4}\right] \qquad (5.55)$$

$$A_{24} = \frac{3(e^2-1)^{\frac{1}{2}} b_2{}^4 e^4}{256p^5} \tag{5.56}$$

The remaining A_{2_j}'s are of order k^3 or higher. Thus we can rewrite (5.50) as

$$(2\alpha_1)^{\frac{1}{2}} R_2 = A_2 f + \sum_{j=1}^{4} A_{2_j} \sin jf \tag{5.57}$$

where A_2, A_{21}, A_{22}, A_{23}, and A_{24} are given by (5.51) and (5.53) through (5.56). Note that the convergence of A_2 is assured by the convergence of S_2.

5.3 The R_3 Integral

We should now like to evaluate

$$R_3 \equiv \int_{\rho_1}^{\rho} \pm (\rho^2+c^2)^{-1} F(\rho)^{-\frac{1}{2}} d\rho \tag{5.3}$$

Applying the binomial expansion theorem to $(\rho^2+c^2)^{-1}$ results in

$$(\rho^2+c^2)^{-1} = \rho^{-2} \sum_{j=0}^{\infty} (-1)^j c^{2j} \rho^{-2j}$$

$$= \left(\frac{1+e\cos f}{p}\right)^2 \sum_{j=0}^{\infty} (-1)^j \left(\frac{c}{p}\right)^{2j} (1+e\cos f)^{2j} \tag{5.58}$$

since $\rho \equiv \frac{p}{1+e\cos f}$. Substituting (5.58), (5.13), and (5.20) into (5.3) yields

$$(2\alpha_1)^{\frac{1}{2}} R_3 = \frac{(e^2-1)^{\frac{1}{2}}}{p} \int_0^f \sum_{j=0}^{\infty} (-1)^j \left(\frac{c}{p}\right)^{2j} (1+e\cos f)^{2j} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda)(1+e\cos f)^{n+2} df \tag{5.59}$$

Let us consider the two series within the integrand separately. We define

$$S_{31} \equiv \sum_{j=0}^{\infty} (-1)^j \left(\frac{c}{p}\right)^{2j} (1+e\cos f)^{2j} \tag{5.60}$$

The series S_{31} is a geometric series with first term = 1 and a ratio between successive terms of

$$r_{31} \equiv -\left(\frac{c}{p}\right)^2 (1+e\cos f)^2 = -\frac{c^2}{\rho^2} \tag{5.61}$$

Clearly S_{31} will converge as long as $|r_{31}| < 1$ which requires

$$\rho > c \tag{5.62}$$

Thus $\rho = c$ is the limiting surface within which actual motion must not take place if S_{31} is to converge. Since the substance of the Earth effectively prohibits all actual motion in this region, S_{31} will always converge.

We next define the series

$$S_{32} \equiv \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda)\, (1+e\cos f)^{n+2} \tag{5.63}$$

If $\lambda \leq 1$ then $|P_n(\lambda)| \leq 1$ and so

$$|S_{32}| \leq (1+e\cos f)^2 \sum_{n=0}^{\infty} \left(\frac{b_2}{\rho}\right)^n \tag{5.64}$$

We can consider (5.64) to be a geometric series and thus require the ratio between successive terms to be less than one.

$$r_{32} \equiv \frac{b_2}{\rho} < 1 \tag{5.65}$$

or equivalently we require

$$\rho > b_2 \simeq c \sin I \tag{5.66}$$

The condition $\rho > c \sin I$ will always be satisfied since actual motion cannot occur within the Earth. The approximation $b_2 \simeq c \sin I$, however, is only valid under the restrictions discussed at the end of Section 3.

For $\lambda > 1$ our restriction on the ratio between successive terms becomes

$$r_{32} = \frac{2b_1}{\rho} < 1 \tag{5.67}$$

This restriction will certainly be satisfied if $\rho_1 > 2b_1$, a condition which we imposed in Appendix B in order that $(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}}$ be the Legendre polynomial generating function. Thus no additional restrictions have been required in order that S_{32} converge. We have now established the integrand of (5.59) as a product of two absolutely convergent series (assuming our trajectory meets the conditions previously imposed) and, consequently, [2] it is equal to the absolutely convergent series formed by summing the products of the individual terms. Since this resulting series is uniformly convergent we may integrate it term by term.

Accordingly we can rewrite (5.59) as

$$2(\alpha_1)^{\frac{1}{2}}\, R_3 = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \int_0^f \sum_{m=0}^{\infty} D_m (1+e\cos f)^{m+2}\, df \tag{5.68}$$

where

$$D_m = \sum d_j \delta_{n'} \tag{5.69}$$

and the summation takes place over all the integers j and n' such that

$$m = 2j + n' \tag{5.70}$$

In (5.69) we have defined

$$d_j \equiv (-1)^j \left(\frac{c}{p}\right)^{2j} \tag{5.71}$$

and

$$\delta_{n'} \equiv (\frac{b_2}{p})^{n'} \, P_{n'}(\lambda) \tag{5.72}$$

Then, since we can integrate (5.68) term by term we rewrite it as

$$(2\alpha_1)^{\frac{1}{2}} R_3 = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \sum_{m=0}^{\infty} D_m \int_0^f (1+e\cos f)^{m+2} df \tag{5.73}$$

Before proceeding with the evaluation of R_3 let us investigate the convergence of our new series

$$S_{33} \equiv \sum_{m=0}^{\infty} D_m \int_0^f (1+e\cos f)^{m+2} df \tag{5.74}$$

We first take the case $\lambda \leq 1$, which excludes only near-equatorial trajectories. If m is even, then n' must also be even, so we set $m = 2i$, $n' = 2n$, and $j = i-n$. Thus

$$D_{2_i} = \sum_{n=0}^{i} d_{i-n} \, \delta_{2_n} = \sum_{n=0}^{i} (-1)^{i-n} (\frac{c}{p})^{2i-2n} (\frac{b_2}{p})^{2n} P_{2_n}(\lambda) \tag{5.75}$$

Then since $|P_{2_n}(\lambda)| \leq 1$ for $\lambda \leq 1$

$$|D_{2_i}| \leq (\frac{c}{p})^{2i} \sum_{n=0}^{i} (\frac{b_2}{c})^{2n} \tag{5.76}$$

If the extension of the trajectory does not pass through the region defined in Section 3, then $b_2 \simeq c \sin I$ and

$$|D_{2_i}| \leq k^i \sum_{n=0}^{i} (\sin^2 I)^n \leq k^i (i+1) \tag{5.77}$$

If, on the other hand, m is odd, then $m = 2i+1$, $n' = 2n+1$, and $j = i-n$. Thus

$$D_{2i+1} = \sum_{n=0}^{i} d_j \; \delta_{2n+1} = \sum_{n=0}^{i} (-1)^{i-n} (\frac{c}{p})^{2i-2n} (\frac{b_2}{p})^{2n+1} P_{2n+1}(\lambda) \qquad (5.78)$$

and

$$|D_{2i+1}| \leq (\frac{c}{p})^{2i} \frac{b_2}{p} \sum_{n=0}^{i} (\frac{b_2}{c})^{2n} \qquad (5.79)$$

If we again take $b_2 \simeq c \sin I$

$$|D_{2i+1}| \leq k^{i+\frac{1}{2}} \sin I \sum_{n=0}^{i} (\sin^2 I)^n \leq k^{i+\frac{1}{2}} (i+1) \sin I \qquad (5.80)$$

If we now consider $\lambda > 1$, the equatorial and near-equatorial trajectories, then by (C.10) $|(\frac{b_2}{p})^n P_n(\lambda)| \leq (\frac{2b_1}{p})^n$. If m is even, we then find

$$|D_{2i}| \leq (\frac{c}{p})^{2i} \sum_{n=0}^{i} (\frac{2b_1}{c})^{2n} \qquad (5.81)$$

Again assuming the trajectory's extension does not pass through the region defined in Section 3, we can set $b_1 \simeq kp\cos^2 I$, so that

$$|D_{2i}| \leq k^i \sum_{n=0}^{i} (4k\cos^4 I)^n \leq k^i \sum_{n=0}^{i} (4k)^n \qquad (5.82)$$

Then by finding the sum of this infinite geometric progression we must have

$$|D_{2i}| \leq k^i \left[\frac{1 - (4k)^i}{1 - 4k}\right] \qquad (5.83)$$

Finally, if m is odd, we find

$$|D_{2i+1}| \leq (\frac{c}{p})^{2i} \frac{2b_1}{p} \sum_{n=0}^{i} (\frac{2b_1}{c})^{2n} \leq 2k^{i+1} \cos^2 I \sum_{n=0}^{i} (4k)^n \qquad (5.84)$$

Again evaluating the sum of the infinite geometric progression and setting $\cos^2 I \simeq 1$ for near-equatorial trajectories

$$|D_{2_{i+1}}| \leq 2k^{i+1}\left[\frac{1-(4k)^i}{1-4k}\right] \qquad (5.85)$$

We now separate S_{33} into an even series and an odd series, S_e and S_o respectively. Thus for $\lambda \leq 1$

$$|S_e| \leq f(e+1)^2 \sum_{i=0}^{\infty} k^i\ (i+1)(1+2e)^{2i} \qquad (5.86)$$

and

$$|S_o| \leq k^{\frac{1}{2}}\sin I\ (1+e)^3\ f \sum_{i=0}^{\infty} (i+1)\ [(k(e+1)^2]^i \qquad (5.87)$$

Then since $\sum_0^\infty x^i = \frac{1}{1-x}$ and $\sum_0^\infty ix^i = \frac{x}{(1-x)^2}$ we can see that $\sum_0^\infty (i+1)x^i = \frac{1}{(1-x)^2}$. Applying this to (5.86) and (5.87) yields

$$|S_e| \leq \frac{f(e+1)^2}{[1-k(e+1)^2]^2} \qquad (5.88)$$

and

$$|S_o| \leq \frac{k^{\frac{1}{2}}\sin I(1+e)^3 f}{[1-k(e+1)^2]^2} \qquad (5.89)$$

so that

$$|S_{33}| \leq \frac{(e+1)^2\ f[1+k^{\frac{1}{2}}\sin I(e+1)]}{[1-k(e+1)^2]^2} \qquad (5.90)$$

For $\lambda > 1$ S_e and S_o become

$$|S_e| \leq \frac{(e+1)^2 f}{1-4k} \sum_{i=0}^{\infty} [k(e+1)^2]^i - [4k^2(e+1)^2]^i =$$

$$\frac{(e+1)^2 f}{1-4k}\left[\frac{1}{1-k(e+1)^2} - \frac{1}{1-4k^2(e+1)^2}\right] \qquad (5.91)$$

and

$$|S_o| \leq \frac{2k(e+1)^3 f}{1 - 4k} \sum_{i=0}^{\infty} [k(e+1)^2]^i - [4k^2(e+1)^2]^i =$$

$$\frac{2k(e+1)^3 f}{1 - 4k} \left[\frac{1}{1 - k(e+1)^2} - \frac{1}{1 - 4k^2(e+1)^2}\right] \tag{5.92}$$

Adding these even and odd series together yields

$$|S_{33}| \leq \frac{k(e+1)^4 f}{[1 - k(e+1)^2][1 - 2k(e+1)]} \tag{5.93}$$

From (5.90) and (5.93) it is clear that S_{33} will converge as long as $k(e+1)^2 < 1$ or equivalently $\frac{c^2}{\rho_1{}^2} < 1$. This requirement for the convergence of S_{33} is exactly the same as that discussed in Section 3.

The values of D_m can be found from equations (5.69) through (5.72), and are shown in Table 5.3 to order k^2.

These D_m can easily be calculated from the first two values and the recursion formula

$$D_j = -\left(\frac{c}{p}\right)^2 D_{j-2} + \left(\frac{b_2}{p}\right)^j P_j(b_1/b_2) \tag{5.94}$$

To evaluate R_3 we separate it into a part proportional to f and a part periodic in f. Thus we rewrite (5.73) as

$$(2\alpha_1)^{\frac{1}{2}} R_3 = A_3 f + \sum_{j=1}^{\infty} A_{3_j} \sin jf \tag{5.95}$$

where according to (5.35) we must have

$$A_3 = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \sum_{m=0}^{\infty} D_m T_{m+2} (\sqrt{e^2-1}) \tag{5.96}$$

and

$$A_{3_j} = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \sum_{m=0}^{\infty} D_m c_{m+2,j} \tag{5.97}$$

TABLE 5.3

VALUES OF D_m TO ORDER k^2

m	Order	D_m
0	k^0	1
1	k	b_1/p
2	k	$(b_2/p)^2\ P_2(b_1/b_2) - (c/p)^2$
3	k^2	$(b_2/p)^3\ P_3(b_1/b_2) - (c/p)^2(b_1/p)$
4	k^2	$(b_2/p)^4\ P_4(b_1/b_2) - (c/p)^2(b_2/p)^2\ P_2(b_1/b_2) + (\frac{c}{p})^4$

Since R_3 is multiplied by $c^2 = kp^2$ in the Kinetic equation (2.18c) we need only to find the periodic terms through order k in order to have the final solution correct to order k^2. Thus from (5.97) we find

$$A_{31} = \frac{(e^2-1)^{\frac{1}{2}}e}{p^3} \left[2 + \frac{b_1}{p}\left(3 + \frac{3e^2}{4}\right) - \left(\frac{b_2{}^2}{2p^2} + \frac{c^2}{p^2}\right)(4 + 3e^2)\right] \tag{5.98}$$

$$A_{32} = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \left[\frac{e^2}{4} + \frac{3b_1e^2}{4p} - \left(\frac{b_2{}^2}{2p^2} + \frac{c^2}{p^2}\right)\left(\frac{3e^2}{2} + \frac{e^4}{4}\right)\right] \tag{5.99}$$

$$A_{33} = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \left[\frac{b_1e^3}{12p} - \left(\frac{b_2{}^2}{2} + c^2\right)\frac{e^3}{3p^2}\right] \tag{5.100}$$

and

$$A_{34} = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \left[-\frac{e^4}{32p^2}\left(\frac{b_2{}^2}{2} + c^2\right)\right] \tag{5.101}$$

Since the other A_{3_j} are of order k^2 or higher we can rewrite (5.95) as

$$(2\alpha_1)^{\frac{1}{2}} R_3 = A_3 f + \sum_{j=1}^{4} A_{3_j} \sin jf \tag{5.102}$$

with A_3 and A_{3_j} as given previously.

CHAPTER 6

THE η-INTEGRALS

In the Kinetic equations (2.18) we define the three η integrals

$$N_1 \equiv \int_0^\eta \pm\eta^2\, G^{-\frac{1}{2}}(\eta)\, d\eta \tag{6.1}$$

$$N_2 \equiv \int_0^\eta \pm G^{-\frac{1}{2}}(\eta)\, d\eta \tag{6.2}$$

$$N_3 \equiv \int_0^\eta \pm(1-\eta^2)^{-1}\, G^{-\frac{1}{2}}(\eta)\, d\eta \tag{6.3}$$

where the upper sign is to be used for positive dη and the lower sign for negative dη, and with $G(\eta)$ given by (3.40), $\eta_2{}^2$ by (4.20), $\alpha_2{}^2$ by (4.16) and (4.17), and $(\alpha_2{}^2-\alpha_3{}^2)$ by (4.10). Following Vinti's example for bounded motion we introduce the uniformizing variable ψ defined by

$$\psi = \sin^{-1}(\eta/\eta_0) \tag{6.4}$$

and the requirement that ψ always increase with time. For the case of Keplerian motion c = 0 and (6.4) becomes

$$\sin\psi = \frac{\eta}{\eta_0} \rightarrow \frac{\sin\theta}{\sin i} \tag{6.5}$$

so that ψ reduces to the angle between the line of nodes and the radius vector to the satellite, commonly referred to as the argument of latitude.

Inserting (6.4) into (3.40) yields

$$G(\eta) = (\alpha_2{}^2-\alpha_3{}^2)\cos^2\psi(1 + n^2\sin^2\psi) \tag{6.6}$$

where we have defined

$$n^2 \equiv -\frac{\eta_0{}^2}{\eta_2{}^2} = O(k) \tag{6.7}$$

Notice that n is a real positive number whose magnitude is less than one, since $|\eta_2{}^2| > 1 > \eta_0{}^2$. Thus we find

$$\pm G(\eta)^{-\frac{1}{2}}d\eta = (\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\eta_0(1+n^2\sin^2\psi)^{-\frac{1}{2}}\, d\psi \tag{6.8}$$

6.1 The N_1 Integral

Inserting (6.8) and (6.4) into (6.1) results in

$$N_1 = (\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\,\eta_0{}^3\int_0^\psi(1+n^2\sin^2\psi)^{-\frac{1}{2}}\sin^2\psi\, d\psi \tag{6.9}$$

To have the final result correct to $O(k^2)$ we shall only need to evaluate N_1 to $O(k)$ since it is multiplied by c^2 in the Kinetic equations. Use of the relation

$$-n^2(1+n^2\sin^2\psi)^{-\frac{1}{2}}\sin^2\psi \equiv (1+n^2\sin^2\psi)^{-\frac{1}{2}} - (1+n^2\sin^2\psi)^{\frac{1}{2}} \tag{6.10}$$

enables us to separate (6.9) into parts which can be more easily integrated.

$$N_1 = -(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\frac{\eta_0{}^3}{n^2}\left[\int_0^\psi\frac{d\psi}{\sqrt{1+n^2\sin^2\psi}} - \int_0^\psi\sqrt{1+n^2\sin^2\psi}\, d\psi\right] \tag{6.11}$$

The integrals are now in the form of Elliptic integrals of the first and second kinds, with imaginary moduli. To avoid using imaginary moduli, we employ variable transformations. If in the first integral of (6.11) we set $\psi = \gamma - \frac{\pi}{2}$, $d\psi = d\gamma$, $\cos\psi = \sin\gamma$, we find

$$\int_0^\psi\frac{d\psi}{\sqrt{1+n^2\sin^2\psi}} = \frac{1}{\sqrt{n^2+1}}\int_{\pi/2}^\gamma\frac{d\gamma}{\sqrt{1-k_1{}^2\sin^2\gamma}} \tag{6.12}$$

where

$$k_1{}^2 = \frac{n^2}{n^2+1} \tag{6.12a}$$

so that

$$\int_0^{\psi} \frac{d\psi}{\sqrt{1+n^2\sin^2\psi}} = \frac{1}{\sqrt{n^2+1}} \left[\int_0^{\gamma} \frac{d\gamma}{\sqrt{1-k_1{}^2\sin^2\gamma}} - \int_0^{\pi/2} \frac{d\gamma}{\sqrt{1-k_1{}^2\sin^2\gamma}}\right] \tag{6.13}$$

$$= \frac{1}{\sqrt{n^2+1}} [F(\gamma,k_1) - K(k_1)] \tag{6.14}$$

where $F(\gamma,k_1)$ is the elliptic integral of the first kind of amplitude γ and modulus k_1, and $K(k_1)$ is the complete elliptic integral of the first kind of modulus k_1.

Applying the same transformation to the second integral of (6.11) yields

$$\int_0^{\psi} \sqrt{1 + n^2\sin^2\psi}\, d\psi = \sqrt{1 + n^2} \int_{\pi/2}^{\gamma} \sqrt{1 - k_1{}^2\sin^2\gamma}\, d\gamma \tag{6.15}$$

$$= \sqrt{1 + n^2} \left[\int_0^{\gamma} \sqrt{1 - k_1{}^2\sin^2\gamma}\, d\gamma - \int_0^{\pi/2} \sqrt{1-k_1{}^2\sin^2\gamma}\, d\gamma\right] \tag{6.16}$$

$$= \sqrt{1 + n^2} [E(\gamma, k_1) - E(k_1)] \tag{6.17}$$

where $E(\gamma, k_1)$ and $E(k_1)$ are the elliptic integral of the second kind and the complete elliptic integral of the second kind, respectively.

We now seek to express each elliptic integral as a part proportional to γ and a Fourier series of the form $\sum_1^{\infty} B_n \sin 2n\gamma$. We first note [6] that

$$F(\gamma + m\pi,\ k_1) = 2m\ K(k_1) + F(\gamma,\ k_1) \tag{6.18}$$

so that the function $F(\gamma,\ k_1) - \frac{2}{\pi} K(k_1)\gamma$ is periodic in γ with period π. Furthermore, since it is an odd function of γ, we can expand it in a Fourier series using only the sines of even multiples of γ. Thus we write

$$F(\gamma,\ k_1) = \frac{2}{\pi}\ \gamma K(k_1) + \sum_{m=1}^{\infty} F_{k_1 m}\ \sin 2m\gamma \tag{6.19}$$

To calculate the Fourier coefficients we note the definition of $F(\gamma,\ k_1)$

$$F(\gamma,\ k_1) \equiv \int_0^{\gamma} (1 - k_1{}^2\sin^2\gamma)^{-\frac{1}{2}}\ d\gamma \tag{6.20}$$

and then differentiate both (6.19) and (6.20) with respect to γ to get

$$(1 - k_1{}^2\sin^2\gamma)^{-\frac{1}{2}} = \frac{2}{\pi}\ K(k_1) + 2\sum_{m=1}^{\infty} m\ F_{k_1 m}\ \cos 2m\gamma \tag{6.21}$$

which is in the form $f(x) = A_0 + \sum_1^{\infty} A_n \cos \frac{n\pi x}{L}$. Thus by Hildebrand [7] $A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\ dx$ or solving for the $F_{k_1 m}$, we find

$$F_{k_1 m} = \frac{2}{m\pi}\int_0^{\pi/2} (1 - k_1{}^2\sin^2\gamma)^{-\frac{1}{2}}\cos 2m\gamma\ d\gamma \tag{6.22}$$

To evaluate these Fourier coefficients we follow Vinti's example [2] and expand the radical using the relation

$$(1 - k_1{}^2\sin^2\gamma)^{-\frac{1}{2}} = 1 + \sum_{\ell=1}^{\infty} \frac{(2\ell)!\ k_1{}^{2\ell}\sin^{2\ell}\gamma}{2^{2\ell}(\ell!)^2} \tag{6.23}$$

which can easily be proved by noting that the right hand side generates the binomial expansion of the left hand side. Substituting (6.23) into

(6.22) yields

$$F_{k_1 m} = \frac{2}{\pi m} \sum_{\ell=1}^{\infty} \frac{(2\ell)!\ k_1{}^{2\ell}}{2^{2\ell}\ (\ell!)^2} \int_0^{\pi/2} \sin^{2\ell}\gamma\ \cos 2m\gamma\ d\gamma \tag{6.24}$$

However, for $\ell \geq 1$

$$\sin^{2\ell}\gamma = \frac{(2\ell)!}{(\ell!)^2 2^{2\ell}} + (-1)^{\ell}\ 2^{1-2\ell} \sum_{j=0}^{\ell-1} (-1)^j\ \frac{(2\ell)!}{(2\ell-j)!j!} \cos(2\ell-2j)\gamma \tag{6.25}$$

This relation can be proved by expanding $[\frac{1}{2i}(e^{i\gamma}-e^{-i\gamma})]^{2\ell}$ binomially and rearranging terms so as to form the summation of (6.25). We also have the identity

$$\cos[(2\ell-2j)\gamma]\ \cos 2m\gamma = \frac{1}{2}\cos[(2\ell+2m-2j)\gamma] + \frac{1}{2}\cos[(2\ell-2m-2j)\gamma] \tag{6.26}$$

With (6.25) and (6.26) substituted into (6.24) we see that the integral of the first term of (6.25) is zero for the limits $0 \rightarrow \frac{\pi}{2}$ and the terms of (6.26) will yield a zero integral unless either $j = \ell + m$ or $j = \ell - m$. However, (6.26) tells us that $j < \ell$ so that only the term corresponding to $j = \ell - m$ contributes to the integral. Thus

$$\int_0^{\pi} \sin^{2\ell}\gamma\ \cos 2m\gamma\ d\gamma = \frac{(-1)^m\ (2\ell)!\ \frac{\pi}{2}}{2^{2\ell}\ (\ell+m)!\ (\ell-m)!} \tag{6.27}$$

for $m \leq \ell$. Substituting this into (6.24) yields

$$F_{k_1 m} = \frac{(-1)^m}{m} \sum_{\ell=m}^{\infty} \frac{(2\ell)!^2\ k_1{}^{2\ell}}{2^{4\ell}(\ell!)^2(\ell+m)!(\ell-m)!} \tag{6.28}$$

from which we can see that $F_{k_1 m} = O(k_1{}^{2m}) = O(n^{2m}) = O(k^m)$. Consequently for accuracy up to $O(k^2)$ in the periodic part we need only take $F_{k_1 1}$ and $F_{k_1 2}$. To $O(k^2)$ these are found to be

$$F_{k_1 1} = -\frac{1}{8}k_1^2 - \frac{3}{32}k_1^4 \tag{6.29}$$

$$F_{k_1 2} = \frac{3\,k_1^4}{256} \tag{6.30}$$

Substituting these back into (6.19) yields

$$F(\gamma,\,k_1) = \frac{2\gamma}{\pi}\;K(k_1) - \frac{k_1^2}{8}(1 + \frac{3}{4}k_1^2)\sin 2\gamma + \frac{3k_1^4}{256}\sin 4\gamma + \cdots \tag{6.31}$$

We now transform back to our original variable ψ using $\gamma = \psi + \frac{\pi}{2}$, $\sin 2\gamma = -\sin 2\psi$, and $\sin 4\gamma = \sin 4\psi$, so that

$$F(\psi,\,k_1) - K(k_1) = \frac{2}{\pi}\,\psi\;K(k_1) + \frac{k_1^2}{8}(1 + \frac{3}{4}k_1^2)\,\sin 2\psi + \frac{3k_1^4}{256}\sin 4\psi + \cdots \tag{6.32}$$

We can now use the same method to handle the elliptic integral $E(\gamma,\,k_1)$. Thus $E(\gamma,\,k_1) - \frac{2}{\pi}\,\gamma\;E(k_1)$ is odd in γ and periodic in γ with period π, so that

$$E(\gamma,\,k_1) \equiv \int_0^{\gamma} (1 - k_1^2\sin^2\gamma)^{\frac{1}{2}}d\gamma = \frac{2}{\pi}E(k_1)\gamma + \sum_{\ell=1}^{\infty} E_{k_1 m}\sin 2m\gamma \tag{6.33}$$

As before we differentiate (6.33) with respect to γ to get

$$(1 - k_1^2\sin^2\gamma)^{\frac{1}{2}} = \frac{2}{\pi}\;\gamma\;E(k_1) + 2\sum_{m=1}^{\infty} m\;E_{k_1 m}\cos 2m\gamma \tag{6.34}$$

and evaluate the Fourier coefficients as

$$E_{k_1 m} = \frac{2}{m\pi}\int_0^{\pi/2} (1 - k_1^2\sin^2\gamma)^{\frac{1}{2}}\,\cos 2m\gamma\;d\gamma \tag{6.35}$$

Binomial expansion then yields

$$(1 - k_1^2\sin^2\gamma)^{\frac{1}{2}} = 1 - \sum_{\ell=1}^{\infty} \frac{(2\ell-2)!\;k_1^{2\ell}\sin\gamma}{2^{2\ell-1}\;\ell!\;(\ell-1)!} \tag{6.36}$$

and insertion of (6.36) into (6.35) thus gives

$$E_{k_1 m} = -\frac{2}{m\pi} \sum_{\ell=1}^{\infty} \frac{(2\ell-2)!\; k_1^{2\ell}}{2^{2\ell-1}\; \ell!\; (\ell-1)!} \int_0^{\pi/2} \sin^{2\ell}\gamma \cos 2m\gamma \, d\gamma \qquad (6.37)$$

since the integral of the first term of (6.36) is zero. Then by equation (6.27)

$$E_{k_1 m} = \frac{(-1)^{m+1}}{m} \sum_{\ell=m}^{\infty} \frac{(2\ell-2)!\; (2\ell)!\; k_1^{2\ell}}{2^{4\ell-1}\; \ell!\; (\ell-1)!\; (\ell+m)!\; (\ell-m)!} \qquad (6.38)$$

so that $E_{k_1 m} = O(k_1^{2m}) = O(n^{2m}) = O(k^m)$. For accuracy to $O(k^2)$ we need use only the first two Fourier coefficients. They are

$$E_{k_1 1} = \frac{k_1^2}{8} + \frac{k_1^4}{32} + \cdots \qquad (6.39)$$

and

$$E_{k_1 2} = -\frac{k_1^4}{256} + \cdots \qquad (6.40)$$

On substituting (6.39) and (6.40) into (6.33), we find

$$E(\gamma,k_1) = \frac{2}{\pi}\,\gamma\, E(k_1) + \left(\frac{k_1^2}{8} + \frac{k_1^4}{32}\right) \sin 2\gamma - \frac{k_1^4}{256} \sin 4\gamma + \cdots \qquad (6.41)$$

Transformation back to our original variable ψ then shows that

$$E(\psi,\, k_1) - E(k_1) = \frac{2}{\pi}\,\psi E(k_1) - \left(\frac{k_1^2}{8} + \frac{k_1^4}{32}\right)\sin 2\psi - \frac{k_1^4}{256} \sin 4\psi + \cdots \qquad (6.42)$$

If we now substitute (6.14), (6.17), (6.32), and (6.41) back into (6.11), we find, after some manipulation, to $O(k)$

$$N_1 = -(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\, \eta_0^3 \left[B_1\psi + \frac{1}{8}(2-n^2)\sin 2\psi + \frac{n^2}{64} \sin 4\psi\right] \qquad (6.43)$$

where

$$B_1 \equiv \frac{k_1}{n^3}\left[\frac{2}{\pi}\,[K(k_1) - E(k_1)] - n^2(\tfrac{2}{\pi})\,E(k_1)\right] \tag{6.44}$$

$$= \frac{k_1^3}{n^3}[\tfrac{1}{2} + \tfrac{3}{16}k_1^2 + \tfrac{15}{128}k_1^4 + \cdots] - \frac{k_1}{n}[1 - \tfrac{1}{4}k_1^2 - \tfrac{3}{64}k_1^4 + \cdots] \tag{6.45}$$

We then transform B_1 back to our original variable n by using

$$\frac{k_1^3}{n^3} = (1+n^2)^{-\frac{3}{2}} = 1 - \frac{3}{2}\,n^2 + \frac{15}{8}\,n^4 - \cdots \tag{6.46}$$

and

$$\frac{k_1}{n} = (1+n^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}\,n^2 + \frac{3}{8}\,n^4 - \cdots \tag{6.47}$$

Substituting (6.47) and (6.46) into B_1 yields to $O(k^2)$

$$B_1 = -\frac{1}{2} + \frac{3}{16}\,n^2 - \frac{15}{128}\,n^4 + \cdots \tag{6.48}$$

N_1 is now given by (6.43) and (6.48).

6.2 The N_2 Integral

The N_2 integral is given by

$$N_2 \equiv \int_0^{\eta} \pm\, G^{-\frac{1}{2}}(\eta)\;d\eta \tag{6.2}$$

or by (6.8)

$$N_2 = (\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\,\eta_0 \int_0^{\psi} (1+n^2\sin^2\psi)^{-\frac{1}{2}}\,d\psi \tag{6.49}$$

With use of (6.14) and (6.32), this becomes

$$N_2 = (\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\,\frac{\eta_0}{\sqrt{1+n^2}}\left[\frac{2}{\pi}\,\psi\,K(k_1) + \frac{k_1^2}{8}(1 + \tfrac{3}{4}k_1^2)\sin 2\psi + \frac{3k_1^4}{256}\,\sin 4\psi + \cdots\right] \tag{6.50}$$

or, with $k_1^2 = n^2-n^4 + \cdots$ in the periodic terms,

$$N_2 = (\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\,\eta_0[B_2\psi + \tfrac{1}{8}(n^2 - \tfrac{3}{4}n^4)\sin 2\psi + \frac{3n^4}{256}\sin 4\psi + \cdots] \tag{6.51}$$

where

$$B_2 \equiv \frac{1}{\sqrt{1+n^2}}\,\frac{2}{\pi}K(k_1) = \frac{k_1}{n}[1 + \tfrac{1}{4}k_1{}^2 + \tfrac{9}{64}k_1{}^4 + \cdots] \tag{6.52}$$

Transforming B_2 back to the parameter n by means of (6.47) yields

$$B_2 = 1 - \tfrac{1}{4}n^2 + \frac{9n^4}{64} - \ldots \tag{6.53}$$

Thus N_2 is given by (6.51) and (6.53).

6.3 The N_3 Integral

Inserting (3.40) into (6.3) yields N_3 in the form

$$(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}\,N_3 = \int_0^{\eta} \pm(1-\eta^2)^{-1}(1-\eta^2/\eta_0{}^2)^{-\frac{1}{2}}(1-\eta^2/\eta_2{}^2)^{-\frac{1}{2}}d\eta \tag{6.54}$$

Since $|\eta^2/\eta_2{}^2| \leq |\eta_0{}^2/\eta_2{}^2| = n^2 = O(k)$ we can expand $(1-\eta^2/\eta_2{}^2)^{-\frac{1}{2}}$ by the binomial theorem. Thus

$$(1-\eta^2/\eta_2{}^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}(m!)^2}(\eta^2/\eta_2{}^2)^m \tag{6.55}$$

where $\eta^2/\eta_2{}^2$ is negative. Substituting (6.55) into (6.54), we find

$$(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}\,N_3 = \sum_{m=0}^{\infty} \frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}\,L_m \tag{6.56}$$

where

$$L_m \equiv \int_0^{\eta} \pm(1-\eta^2)^{-1}(1-\eta^2/\eta_0{}^2)^{-\frac{1}{2}}\,\eta^{2m}\,d\eta \tag{6.57}$$

for m=0

$$L_0 \equiv \int_0^{\eta} \pm(1-\eta^2)^{-1}(1-\eta^2/\eta_0{}^2)^{-\frac{1}{2}}\,d\eta \tag{6.58}$$

We then notice that $\eta^2/(1-\eta^2) = 1/(1-\eta^2) - 1$, $\eta^4/(1-\eta^2) = \eta^2/(1-\eta^2) - \eta^2$, $\eta^6/(1-\eta^2) = \eta^4/(1-\eta^2) - \eta^4$, etc., so that we can write

$$\frac{\eta^{2m}}{1-\eta^2} = \frac{1}{1-\eta^2} - \sum_{n=0}^{m-1} \eta^{2n} \tag{6.59}$$

for $m \geq 1$. Use of this relation in (6.57) yields

$$L_m = \int_0^\eta \pm(1-\eta^2)^{-1}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}}\, d\eta - \int_0^\eta \pm \sum_{n=0}^{m-1} \eta^{2n}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}}\, d\eta \tag{6.60}$$

$$= L_0 - \sum_{n=0}^{m-1} L_{1_n} \tag{6.61}$$

where

$$L_{1_n} \equiv \int_0^\eta \pm\eta^{2n}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}}\, d\eta \tag{6.62}$$

To evaluate L_0 we first rewrite it as

$$L_0 = \int_0^\eta \pm\eta^{-3}(\eta^{-2}-1)^{-1}(\eta^{-2}-\eta_0^{-2})^{-\frac{1}{2}}\, d\eta \tag{6.63}$$

and then introduce the new variable χ, defined by

$$\tan \chi \equiv (1-\eta_0^2)^{\frac{1}{2}} \tan \psi = |\cos I| \tan \psi \tag{6.64}$$

We also require that χ and ψ keep in step; that is, whenever ψ is a multiple of $\pi/2$, χ and ψ are equal. Using this new variable we find

$$\csc^2\psi = 1+\cos^2 I \cot^2\chi \tag{6.65}$$

so that

$$\eta^{-2} = \eta_0^{-2} \csc^2\psi = 1 + \cot^2 I \csc^2\chi \tag{6.66}$$

Differentiating (6.66) yields

$$\eta^{-3}d\eta = \cot^2 I \csc^2\chi \cot \chi \, d\chi \tag{6.67}$$

We note that

$$\begin{aligned} \cot\chi \, d\chi &> 0 \quad \text{for } d\eta > 0 \\ \cot\chi \, d\chi &< 0 \quad \text{for } d\eta < 0 \end{aligned} \tag{6.68}$$

Then, substituting (6.67) and $\eta=\eta_0\sin\psi$ into (6.63), we find

$$L_0 = \int_0^\chi \pm|\tan I|\cot\chi|\tan\chi|d\chi = \int_0^\chi |\tan I|d\chi \tag{6.69}$$

by (6.68). Thus L_0 is given by

$$L_0 = |\tan I|\chi = \eta_0(1-\eta_0{}^2)^{-\frac{1}{2}}\chi \tag{6.70}$$

For Keplerian hyperbolic motion (i.e. c=0) we see from (6.66)

$$\eta^{-2} \rightarrow \csc^2\theta = 1 + \cot^2 I \csc^2\chi \tag{6.71}$$

which becomes

$$\sin\chi \rightarrow \tan\theta/|\tan I| \tag{6.72}$$

or

$$\chi \rightarrow \phi - \Omega \tag{6.73}$$

for direct orbits. (For retrograde orbits $\dot{\phi}<0$ and since $\dot{\chi}>0$ always, one must then reverse the sign.) Thus, for c=0, χ becomes the projection of the argument of latitude on the equator.

We then evaluate the L_{1_n} integrals by substituting $\eta=\eta_0\sin\psi$ in (6.62). Thus

$$L_{1_n} = \eta_0{}^{2n+1} \int_0^\psi \sin^{2n}\psi \, d\psi \tag{6.74}$$

For n=0, we can easily integrate (6.74) to get

$$L_{10} = \eta_0\psi \tag{6.75}$$

To facilitate evaluating the L_{1_n} for n>0 we note that

$$\sin^{2n}\psi = \frac{(2n)!}{2^{2n}(n!)^2} + 2^{1-2n}\sum_{j=1}^{n}\frac{(-1)^j(2n)!}{(n+j)!(n-j)!}\cos 2j\psi \tag{6.76}$$

a relation which can easily be proved by applying the binomial expansion theorem to $[(e^{i\psi}-e^{-i\psi})/2i]^{2n}$ and noting that the resulting series generates the right hand side of (6.76). Inserting (6.76) into (6.74) and performing the integration yields

$$L_{1n} = \frac{\eta_0^{2n+1}(2n)!\psi}{2^{2n}(n!)^2} + \frac{\eta_0^{2n+1}}{2^{2n}}\sum_{j=1}^{n}\frac{(-1)^j(2n)!\sin 2j\psi}{(n+j)!(n-j)!\,j} \tag{6.77}$$

for n>0. If we now substitute (6.61), (6.70), (6.75), and (6.77) back into (6.56), our expression for N_3, we find

$$(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}} N_3 = \eta_0(1-\eta_0{}^2)^{-\frac{1}{2}}\times\sum_{m=0}^{\infty}\frac{(2m)!\eta_2{}^{-2m}}{2^{2m}(m!)^2} - \eta_0\psi\sum_{m=1}^{\infty}\frac{(2m)!\eta_2{}^{-2m}}{2^{2m}(m!)^2}$$

$$-\psi\sum_{m=2}^{\infty}\frac{(2m)!\eta_2{}^{-2m}}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\frac{\eta_0{}^{2n+1}(2n)!}{2^{2n}(n!)^2} \tag{6.78}$$

$$-\sum_{m=2}^{\infty}\frac{(2m)!\eta_2{}^{-2m}}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\frac{\eta_0{}^{2n+1}}{2^{2n}}\sum_{j=1}^{n}\frac{(-1)^j(2n)!\sin 2j\psi}{(n+j)!(n-j)!\,j}$$

Letting $\eta^2=1$ in (6.55) results in

$$(1-\eta_2{}^{-2})^{-\frac{1}{2}} = \sum_{m=0}^{\infty}\frac{(2m)!\eta_2{}^{-2m}}{2^{2m}(m!)^2} = 1 + \sum_{m=1}^{\infty}\frac{(2m)!\eta_2{}^{-2m}}{2^{2m}(m!)^2} \tag{6.79}$$

so that we can rewrite (6.78) as

$$(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}\, N_3 = \eta_0(1-\eta_0{}^2)^{-\frac{1}{2}}(1-\eta_2{}^{-2})^{-\frac{1}{2}}\chi - \chi\Big[\eta_0[(1-\eta_2{}^{-2})^{-\frac{1}{2}}$$

$$-1] + \eta_0 \sum_{m=2}^{\infty} \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{\eta_0{}^{2n}(2n)!}{2^{2n}(n!)^2}\, \eta_2{}^{-2m}\Big] \tag{6.80}$$

$$- \eta_0 \sum_{m=2}^{\infty} \frac{(2m)!\,\eta_2{}^{-2m}}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{\eta_0{}^{2n}}{2^{2n}} \sum_{j=1}^{n} \frac{(-1)^j(2n)!\sin 2j\psi}{(n+j)!(n-j)!j}$$

or in abbreviated form

$$(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}\, N_3 = \eta_0[(1-\eta_0{}^2)^{-\frac{1}{2}}(1-\eta_2{}^{-2})^{-\frac{1}{2}}\chi + B_3\psi$$

$$+ \sum_{s=1}^{\infty} B_{3_s} \sin 2s\psi] \tag{6.81}$$

where we have defined

$$B_3 \equiv 1 - (1-\eta_2{}^{-2})^{-\frac{1}{2}} - \sum_{m=2}^{\infty} \gamma_m\, \eta_2{}^{-2m} \tag{6.82}$$

$$\gamma_m \equiv \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{\eta_0{}^{2n}(2n)!}{2^{2n}(n!)^2} \tag{6.83}$$

and

$$\sum_{s=1}^{\infty} B_{3_s} \sin 2s\psi \equiv - \sum_{m=2}^{\infty} \frac{(2m)!\,\eta_2{}^{-2m}}{2^{2m}\,(m!)^2} \sum_{n=1}^{m-1} \frac{\eta_0{}^{2n}}{2^{2n}} \sum_{j=1}^{n} \frac{(-1)^j(2n)!\,\sin 2j\psi}{(n+j)!(n-j)!j} \tag{6.84}$$

To solve for B_{3s} we note that the functions $\sin 2j\psi$ are orthogonal. Suppose we multiply (6.84) by $\sin 2s'\psi$ and integrate from 0 to $\frac{\pi}{2}$. We can then do away with the summation over j and set $j = s'$ for the coefficient $B_{3s}{}'$. Since j must be less than or equal to n, our lower bound on n becomes s'. Similarly, n must be less than m, so the lower bound on m becomes $s' + 1$. Thus

$$B_{3_s} = - \sum_{m=s+1}^{\infty} \frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2} \sum_{n=s}^{m-1} \frac{\eta_0^{2n}\eta_2^{-2n}(-1)^s(2n)!}{(n+s)!(n-s)!s} \tag{6.85}$$

where we have dropped the prime from the s. Since $B_{3_s} = O(\eta_2^{-2(s+1)})$ $= O(k^{s+1})$ we need only the B_{31} terms in order to have the sine terms of N_3 accurate to order k^2. Setting s = 1 in (6.85) yields to $O(k^2)$

$$B_{31} = \frac{3}{32}\, \eta_0^2 \eta_2^{-4} + \cdots \tag{6.86}$$

Thus for our purposes (6.81) becomes

$$(\alpha_2^2 - \alpha_3^2)^{\frac{1}{2}} N_3 = \eta_0[(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}} + B_3\psi$$
$$+ B_{31}\sin 2\psi + \cdots] \tag{6.87}$$

with B_3 and B_{31} given by (6.82), (6.83), and (6.86).

Let us now investigate the convergence of the series $\sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m}$. Since

$$\frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{(2n-1)}{2n} \leqq \frac{1}{2} \tag{6.88}$$

for $n \geqq 1$, then if we sum m-1 of such terms we must have

$$\sum_{n=1}^{m-1} \frac{(2n)!\eta_0^{2n+1}}{2^{2n}(n!)^2} \leqq \sum_{n=1}^{m-1} \frac{(2n)!}{2^{2n}(n!)^2} \leqq \frac{m-1}{2} \tag{6.89}$$

Also, for $m \geqq 2$

$$\frac{(2m)!}{2^{2m}(m!)^2} \leqq \frac{3}{8} \tag{6.90}$$

Thus by (6.89) and (6.90)

$$\gamma_m \equiv \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{(2n)!\eta_0^{2n}}{2^{2n}(n!)^2} \leqq \frac{3}{8}\left(\frac{m-1}{2}\right) = \frac{3}{16}(m-1) \tag{6.91}$$

for $m \geqq 2$. Thus

$$\sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m} \leqq \sum_{m=2}^{\infty} \frac{3}{16} (m-1) \eta_2^{-2m} \tag{6.92}$$

The ratio of two successive terms is

$$r_4 = \frac{m}{m-1} \eta_2^{-2} \leqq 2\eta_2^{-2} \tag{6.93}$$

Since $\eta_2^{-2} = O(k)$, the series $\sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m}$ converges rapidly.

Consider now the convergence of the Fourier series

$$\left| \sum_{s=1}^{\infty} B_{3_s} \sin 2s\psi \right| \leqq \sum_{s=1}^{\infty} |B_{3_s}| \leqq \sum_{m=2}^{\infty} \frac{(2m)!\,|\eta_2^{-2m}|}{2^{2m}(m!)^2} \sum_{n=1}^{m-1}$$

$$\sum_{j=1}^{n} \frac{2^{-2n}(2n)!}{(n+j)!(n-j)!} \tag{6.94}$$

But, for $n \geqq 0$

$$\frac{(2n)!}{2^{2n}(n+s)!(n-s)!} \leqq \frac{(2n)!}{2^{2n}(n!)} \leqq 1 \tag{6.95}$$

so that

$$\sum_{n=1}^{m-1} \sum_{j=1}^{n} \frac{(2n)!}{2^{2n}(n+s)!(n-s)!} < \sum_{n=1}^{m-1} n < m^2 \tag{6.96}$$

and

$$\left| \sum_{s=1}^{\infty} B_{3_s} \sin 2s\psi \right| < \sum_{m=2}^{\infty} \frac{m^2 (2m)!\,|\eta_2^{-2m}|}{2^{2m}\,(m!)^2} < \sum_{m=2}^{\infty} m^2 |\eta_2^{-2m}| \tag{6.97}$$

by (6.95). Then the ratio between successive terms must be less than

$$r_5 = \frac{(m+1)^2}{m^2} |\eta_2^{-2}| \leqq \frac{9}{4} |\eta_2^{-2}| \tag{6.98}$$

for $m \geqq 2$. Again since $\eta_2^{-2} = O(k)$ the series converges rapidly.

Referring back to our equation for N_3 (6.87), it is now clear that all the terms are well-behaved, except for the χ term which seems to become infinite for $\eta_0 = \pm 1$ (ie $I = \frac{\pi}{2}, \frac{3\pi}{2}$, etc.) These apparent singular points represent polar trajectories. N_3, however, occurs only in the third Kinetic equation (2.18c) where it is multiplied by α_3, the polar component of angular momentum, which reduces to zero for a polar orbit.

Let us investigate $\alpha_3 N_3$ as $\eta_0^2 \to 1$. For this case the χ term of N_3 will be by far the largest, so for a polar trajectory we can write

$$\alpha_3 N_3 = \frac{\alpha_3}{(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}} \eta_0(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi + \text{higher order terms} \tag{6.99}$$

or since $\alpha_2^2-\alpha_3^2 \to \alpha_2^2$

$$\alpha_3 N_3 \simeq \frac{\alpha_3}{|\alpha_2|} \eta_0(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi \tag{6.100}$$

But from (4.9) and (4.10) we find

$$\frac{\alpha_2^2-\alpha_3^2}{\alpha_2^2} = \eta_0^2 + \sin^2 i_0 \ \eta_2^{-2}(1-\eta_0^2) \tag{6.101}$$

so that

$$\frac{\alpha_3^2}{\alpha_2^2} = 1 - \eta_0^2 - \sin^2 i_0 \ \eta_2^{-2}(1-\eta_0^2) \tag{6.102}$$

or since $\sin^2 i_0 \simeq 1$ for polar trajectories

$$\frac{|\alpha_3|}{\alpha_2} \simeq (1-\eta_2^{-2})^{\frac{1}{2}}(1-\eta_0^2)^{\frac{1}{2}} \tag{6.103}$$

Inserting (6.103) into (6.100) we find

$$\alpha_3 N_3 = (\text{sgn}\alpha_3)\chi + \text{higher order terms} \tag{6.104}$$

Thus $\alpha_3 N_3$ is well-behaved for a polar trajectory as long as χ is well-behaved. We can plot χ versus ψ from equation (6.64) and the requirement that χ equal ψ at every multiple of $\frac{\pi}{2}$.

1. For $\eta_0{}^2 = 0$ (equatorial trajectory), $\tan\chi = \tan\psi$

2. For $\eta_0{}^2 = \frac{3}{4}$, $\tan\chi = \frac{1}{2}\tan\psi$

3. For $\eta_0{}^2 = 1$, $\tan\chi = 0$

Thus χ as a function of ψ can be plotted as in Figure 6.1. Whenever the trajectory passes over a pole, $\psi = \frac{\pi}{2}, \frac{3\pi}{2}$, etc., with $\dot{\psi} > 0$, then χ jumps by π. If the trajectory passes over a pole with $\dot{\psi} < 0$ then χ drops by π. Since for a polar orbit $\alpha_3 N_3$ is given by (6.104), it clear from (2.18c) that the right ascension,ϕ, also jumps or drops by π, in accord with the jump or drop in χ.

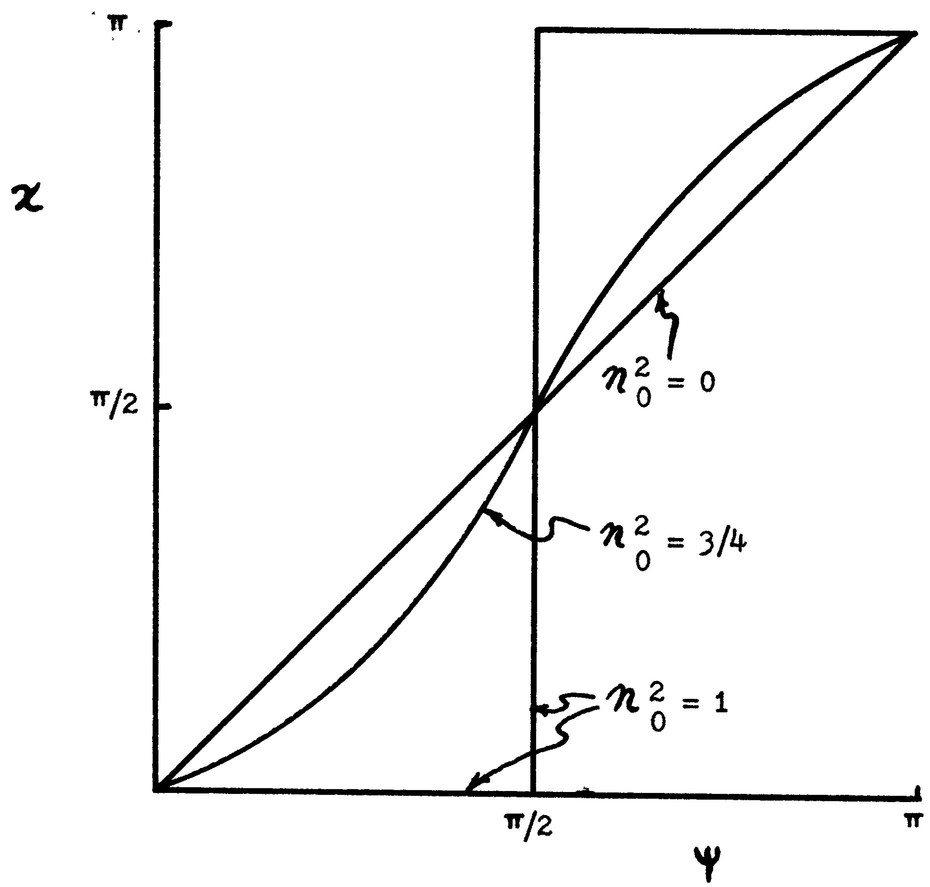

FIGURE 6.1 χ Versus Ψ for Various Values of n_0^2

CHAPTER 7

SOLUTION OF THE KINETIC EQUATIONS

7.1 Summary of Integrals

Before proceeding to the solution of the Kinetic equations, let us assemble the results of the two preceeding sections

$$(2\alpha_1)^{\frac{1}{2}}R_1 = a(e \sinh H - H) + b_1 H + A_1 f + A_{11}\sin f + A_{12}\sin 2f \qquad (5.45)$$

$$(2\alpha_1)^{\frac{1}{2}}R_2 = A_2 f + A_{21}\sin f + A_{22}\sin 2f + A_{23}\sin 3f + A_{24}\sin 4f \qquad (5.57)$$

$$(2\alpha_1)^{\frac{1}{2}}R_3 = A_3 f + A_{31}\sin f + A_{32}\sin 2f + A_{33}\sin 3f + A_{34}\sin 4f \qquad (5.102)$$

$$N_1 = -(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}} \eta_0{}^3\left[B_1\psi + \frac{1}{8}(2-n^2)\sin 2\psi + \frac{n^2}{64}\sin 4\psi\right] \qquad (6.43)$$

$$N_2 = (\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}} \eta_0\left[B_2\psi + \frac{1}{8}(n^2-\frac{3}{4}n^4)\sin 2\psi + \frac{3n^4}{256}\sin 4\psi\right] \qquad (6.51)$$

$$N_3 = (\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}} \eta_0\left[(1-\eta_0{}^2)^{-\frac{1}{2}}(1-\eta_{\bar{2}}{}^2)^{-\frac{1}{2}}\chi + B_3\psi + B_{31}\sin 2\psi\right] \qquad (6.87)$$

where

$$A_1 = p(e^2-1)^{\frac{1}{2}} \sum_{n=2}^{\infty} (\frac{b_2}{p})^n P_n(\lambda)\ T_{n-2}(\sqrt{e^2-1}) \qquad (5.39)$$

$$A_{11} = \frac{3(e^2-1)^{\frac{1}{2}}}{4p^3}\left[-2b_1 b_2{}^2 p + b_2{}^4\right]e \qquad (5.43)$$

$$A_{12} = \frac{3(e^2-1)^{\frac{1}{2}} b_2{}^4 e^2}{32p^3} \tag{5.44}$$

$$A_2 = \frac{(e^2-1)^{\frac{1}{2}}}{p} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda)\, T_n(\sqrt{e^2-1}) \tag{5.51}$$

$$A_{21} = \frac{(e^2-1)^{\frac{1}{2}} e}{p} \left[\frac{b_1}{p} + \frac{3b_1{}^2-b_2{}^2}{p^2} - \frac{9b_1 b_2{}^2}{2p^3}\left(1+\frac{e^2}{4}\right) + \frac{3b_2{}^4}{8p^4}(4+3e^2)\right] \tag{5.53}$$

$$A_{22} = \frac{(e^2-1)^{\frac{1}{2}} e^2}{p} \left[\frac{3b_1{}^2-b_2{}^2}{8p^2} - \frac{9b_1 b_2{}^2}{8p^3} + \frac{3b_2{}^4}{16p^4}\left(3+\frac{e^2}{2}\right)\right] \tag{5.54}$$

$$A_{23} = \frac{(e^2-1)^{\frac{1}{2}} e^3}{8p} \left[-\frac{b_1 b_2{}^2}{p^3} + \frac{b_2{}^4}{p^4}\right] \tag{5.55}$$

$$A_{24} = \frac{3(e^2-1)^{\frac{1}{2}}\, b_2{}^4\, e^4}{256p^5} \tag{5.56}$$

$$A_3 = \frac{(e^2-1)^{\frac{1}{2}}}{p} \sum_{m=0}^{\infty} D_m\, T_{m+2}(\sqrt{e^2-1}) \tag{5.96}$$

$$A_{31} = \frac{(e^2-1)^{\frac{1}{2}} e}{p^3} \left[2 + \frac{b_1}{p}\left(3 + \frac{3e^2}{4}\right) - \left(\frac{b_2{}^2}{2p^2} + \frac{c^2}{p^2}\right)(4+3e^2)\right] \tag{5.98}$$

$$A_{32} = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \left[\frac{e^2}{4} + \frac{3b_1 e^2}{4p} - \left(\frac{b_2{}^2}{2p^2} + \frac{c^2}{p^2}\right)\left(\frac{3e^2}{2} + \frac{e^4}{4}\right)\right] \tag{5.99}$$

$$A_{33} = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \left[\frac{b_1 e^3}{12p} - \left(\frac{b_2{}^2}{2} + c^2\right) \frac{e^3}{3p^2}\right] \tag{5.100}$$

$$A_{34} = \frac{(e^2-1)^{\frac{1}{2}}}{p^3} \left[- \frac{e^4}{32p^2}\left(\frac{b_2{}^2}{2} + c^2\right)\right] \tag{5.101}$$

$$B_1 = -\frac{1}{2} + \frac{3}{16}n^2 - \frac{15}{128}n^4 \tag{6.48}$$

$$B_2 = 1 - \frac{1}{4}n^2 + \frac{9}{64}n^4 \tag{6.53}$$

$$B_3 = 1 - (1-\eta_2^{-2})^{-\frac{1}{2}} - \sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m} \tag{6.82}$$

$$\gamma_m = \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{\eta_0^{2n}(2n)!}{2^{2n}(n!)^2} \tag{6.83}$$

$$B_{31} = \frac{3}{32} \eta_0^2 \eta_2^{-4} \tag{6.86}$$

$$n^2 = -\frac{\eta_0^2}{\eta_2^2} \tag{6.7}$$

7.2 The ρ and η Kinetic Equations

Our method of solving the Kinetic equations will be to solve for ρ and η from the first two Kinetic equations and then use these results to solve the third Kinetic equation for ϕ. Before doing this it is desirable to obtain some relationships between f and H. From $\rho = a(e \cosh H-1) = a(e^2-1)/(1+e \cos f)$ we find

$$\cos f = \frac{e - \cosh H}{e \cosh H - 1} \tag{7.1}$$

$$\sin f = \frac{\sqrt{e^2-1} \sinh H}{e \cosh H - 1} \tag{7.2}$$

$$\cosh H = \frac{e + \cos f}{1 + e \cos f} \tag{7.3}$$

$$\sinh H = \frac{\sqrt{e^2-1} \sin f}{1 + e \cos f} \tag{7.4}$$

$$\tan \frac{f}{2} = \left(\frac{e+1}{e-1}\right)^{\frac{1}{2}} \tanh \frac{H}{2} \tag{7.5}$$

so that one anomaly is uniquely determined by the value of the other. The plus sign occurs in (7.2) and (7.4) because both $\dot{f}$ and $\dot{H}$ are positive for all t.

Substituting the appropriate ρ and η integrals into the first two Kinetic equations, (2.18a) and (2.18b), results in

$$t + \beta_1 = (2\alpha_1)^{-\frac{1}{2}} [b_1 H + a(e\sinh H - H) + A_1 f + A_{11}\sin f + A_{12}\sin 2f] - c^2(\alpha_2{}^2 - \alpha_3{}^2)^{-\frac{1}{2}} \eta_0{}^3 [B_1\psi + \frac{1}{8}(2-n^2)\sin 2\psi + \frac{n^2}{64}\sin 4\psi] + \text{periodic terms } O(k^3) \qquad (7.6)$$

$$\beta_2/\alpha_2 = -(2\alpha_1)^{-\frac{1}{2}} [A_2 f + A_{21}\sin f + A_{22}\sin 2f + A_{23}\sin 3f + A_{24}\sin 4f] + (\alpha_2{}^2 - \alpha_3{}^2)^{-\frac{1}{2}} \eta_0 [B_2\psi + \frac{1}{8}(n^2 - \frac{3}{4}n^4)\sin 2\psi + \frac{3n^4}{256}\sin 4\psi] + \text{periodic terms } O(k^3) \qquad (7.7)$$

Our aim is to solve the above two equations for f, H, and ψ using the relations between f and H derived previously. With f, H, and ψ we can then find $\rho = a(e\cosh H - 1) = \frac{a(e^2-1)}{1+e\cos f}$ and $\eta = \eta_0 \sin\psi$.

To solve for f, H, and ψ we first set

$$f = f_0 + f_1 + f_2 \qquad (7.8)$$

$$H = H_0 + H_1 + H_2 \qquad (7.9)$$

$$\psi = \psi_0 + \psi_1 + \psi_2 \qquad (7.10)$$

where f_0, H_0, and ψ_0 are the solutions to the zeroth order portion of (7.6) and (7.7) (ie the part of the equations remaining after all terms of order J_2 or higher have been dropped.) Similarly, $f_0 + f_1$, $H_0 + H_1$, and $\psi_0 + \psi_1$ are solutions to the equations after the terms of order $J_2{}^2$ or higher have been dropped. Finally, f_2, H_2, and ψ_2 complete the solution to an accuracy of $J_2{}^2$. Thus f_0, H_0, ψ_0 are of order $J_2{}^0$; f_1, H_1, ψ_1 of order $J_2{}^1$; and f_2, H_2, ψ_2 of order $J_2{}^2$. We are essentially solving for f, H, and ψ as series solutions in powers of J_2.

Listing the coefficients of (7.6) and (7.7) according to their order in J_2 we find

$J_2{}^0$: a, $(2\alpha_1)^{-\frac{1}{2}}$, $(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}$, e, η_0, B_1, B_2, A_2

$J_2{}^1$: A_1, A_{21}, n^2, b_1, c^2, A_{22}, $b_2{}^2$

$J_2{}^2$: A_{11}, A_{12}, A_{23}, A_{24}

For the zeroth order solution we set $H = H_0$, $f = f_0$, $\psi = \psi_0$ and drop all terms of $O(J_2)$ or higher. Equations (7.6) and (7.7) become

$$(t+B_1) = (2\alpha_1)^{-\frac{1}{2}}[a(e\sinh H_0-H_0)] \tag{7.11}$$

and

$$\beta_2/\alpha_2 = -(2\alpha_1)^{-\frac{1}{2}} A_2 f_0 + \eta_0(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}} B_2\psi_0 \tag{7.12}$$

Equation (7.11) is just Kepler's equation in hyperbolic form. We can assume that it has been solved by some convenient method to yield the value of H_0. We then find f_0 from our previously derived anomaly relations, e.g.

$$\tan \frac{f_0}{2} = (\frac{e+1}{e-1})^{\frac{1}{2}} \tanh \frac{H_0}{2} \tag{7.13}$$

Using this value of f_0 we can solve for ψ_0 from (7.12) in the form

$$\psi_0 = (\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}} \eta_0{}^{-1}B_2{}^{-1} [\beta_2/\alpha_2 + (2\alpha_1)^{-\frac{1}{2}} A_2 f_0] \tag{7.14}$$

Proceeding to the first order solution we set $f = f_0 + f_1$, $H = H_0 + H_1$, $\psi = \psi_0 + \psi_1$ and drop all terms of $O(J_2{}^2)$ or higher in (7.6) and (7.7). Thus we find

$$t + \beta_1 = (2\alpha_1)^{-\frac{1}{2}} [b_1 H_0 + ae\sinh(H_0+H_1) - a(H_0+H_1)$$
$$+ A_1 f_0] - c^2 \eta_0{}^3 (\alpha_2{}^2 - \alpha_3{}^2)^{-\frac{1}{2}} [B_1 \psi_0 + \frac{1}{4} \sin 2\psi_0] \qquad (7.15)$$

$$\beta_2/\alpha_2 = -(2\alpha_1)^{-\frac{1}{2}} [A_2(f_0+f_1) + A_{21}\sin f_0 + A_{22}\sin 2f_0]$$
$$+ \eta_0(\alpha_2{}^2 - \alpha_3{}^2)^{-\frac{1}{2}} [B_2(\psi_0+\psi_1) + \frac{n^2}{8} \sin 2\psi_0] \qquad (7.16)$$

But

$$\sinh(H_0+H_1) = \sinh H_0 + \sinh H_0(\cosh H_1 - 1) + \cosh H_0 \sinh H_1 \qquad (7.17)$$

$$= \sinh H_0 + \frac{H_1{}^2}{2} \sinh H_0 + H_1 \cosh H_0 \qquad (7.18)$$

to order $J_2{}^2$. Substituting (7.18) into (7.15) and subtracting (7.11), we find

$$0 = (2\alpha_1)^{-\frac{1}{2}} [b_1 H_0 + ae \frac{H_1{}^2}{2} \sinh H_0 + aeH_1 \cosh H_0 - aH_1$$
$$+ A_1 f_0] - c^2 \eta_0{}^3 (\alpha_2{}^2 - \alpha_3{}^2)^{-\frac{1}{2}} [B_1 \psi_0 + \frac{1}{4} \sin 2\psi_0] \qquad (7.19)$$

which can be put in the form

$$H_1{}^2 (\frac{ae\sinh H_0}{2}) + H_1(ae \cosh H_0 - a) + b_1 H_0 + A_1 f_0$$
$$- c^2 \eta_0{}^3 (2\alpha_1)^{\frac{1}{2}} (\alpha_2{}^2 - \alpha_3{}^2)^{-\frac{1}{2}} (B_1 \psi_0 + \frac{1}{4} \sin 2\psi_0) = 0 \qquad (7.20)$$

Equation (7.20) is a quadratic in H_1, and thus yields two solutions for H_1. Since H_1 is small we seek that solution of (7.20) which is closest to the solution of (7.20) with the first term set to zero. That solution is

$$H_1 = H_1{}'(1 - \frac{e\sinh H_0}{2(e\cosh H_0 - 1)} H_1{}') \qquad (7.21)$$

where

$$h_1' = \frac{c^2\eta_0{}^3(2\alpha_1)^{\frac{1}{2}}(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}[B_1\psi_0+\frac{1}{4}\sin2\psi_0]-b_1H_0-A_1f_0}{a(e\ \cosh H_0-1)} \qquad (7.22)$$

we then find f_1 from our anomaly relations, e.g.

$$\tan\frac{(f_0+f_1)}{2} = \left(\frac{e+1}{e-1}\right)^{\frac{1}{2}} \tanh\frac{(H_0+H_1)}{2} \qquad (7.23)$$

Subtracting (7.12) from (7.16) and solving for ψ_1 results in

$$\psi_1 = -B_2{}^{-1}\frac{n^2}{8}\sin2\psi_0 + B_2{}^{-1}(2\alpha_1)^{-\frac{1}{2}}(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}\eta_0{}^{-1}[A_2f_1$$
$$+ A_{21}\sin f_0 + A_{22}\sin2f_0] \qquad (7.24)$$

Finally, for the second order solution we set $f = f_0 + f_1 + f_2$, $H = H_0 + H_1 + H_2$, $\psi = \psi_0 + \psi_1 + \psi_2$ and drop only those terms of order $J_2{}^3$ or higher in (7.6) and (7.7). There results

$$t + \beta_1 = (2\alpha_1)^{-\frac{1}{2}}[b_1(H_0+H_1) + ae\sinh(H_0+H_1+H_2)$$
$$- a(H_0+H_1+H_2) + A_1(f_0+f_1) + A_{11}\sin f_0$$
$$+ A_{12}\sin2f_0] - c^2\eta_0{}^3(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}[B_1(\psi_0+\psi_1)$$
$$+ \frac{1}{4}(\sin2\psi_0+2\psi_1\cos2\psi_0) - \frac{n^2}{8}\sin2\psi_0 + \frac{n^2}{64}\sin4\psi_0] \qquad (7.25)$$

and

$$\beta_2/\alpha_2 = -(2\alpha_1)^{-\frac{1}{2}}[A_2(f_0+f_1+f_2) + A_{21}(\sin f_0 + f_1\cos f_0)$$
$$+ A_{22}(\sin2f_0 + 2f_1\cos2f_0) + A_{23}\sin3f_0$$
$$+ A_{24}\sin4f_0] + \eta_0(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}[B_2(\psi_0+\psi_1+\psi_2)$$
$$+ \frac{n^2}{8}(\sin2\psi_0 + 2\psi_1\cos2\psi_0) - \frac{3n^4}{32}\sin2\psi_0 + \frac{3n^4}{256}\sin4\psi_0] \qquad (7.26)$$

But to order $J_2{}^2$

$$\sinh(H_0+H_1+H_2) = \sinh(H_0+H_1) + H_2\cosh H_0 \qquad (7.27)$$

If we substitute this into (7.25), and subtract (7.15) from the result, we find

$$0 = (2\alpha_1)^{-\frac{1}{2}}[b_1H_1 + aeH_2\ \cosh H_0 - aH_2 + A_1f_1 + A_{11}\sin f_0 + A_{12}\sin 2f_0] - c^2\eta_0{}^3(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}[B_1\psi_1 + \frac{\psi_1}{2}\cos 2\psi_0 - \frac{n^2}{8}\sin 2\psi_0 + \frac{n^2}{64}\sin 4\psi_0] \tag{7.28}$$

or solving for H_2

$$H_2 = (ae\cosh H_0 - a)^{-1}\left[-b_1H_1 - A_1f_1 - A_{11}\sin f_0 - A_{12}\sin 2f_0 + c^2\eta_0{}^3(2\alpha_1)^{\frac{1}{2}}(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}[B_1\psi_1 + \frac{\psi_1}{2}\cos 2\psi_0 - \frac{n^2}{8}\sin 2\psi_0 + \frac{n^2}{64}\sin 4\psi_0]\right] \tag{7.29}$$

We then solve for f_2 from the anomaly relations, e.g.

$$\tan\frac{(f_0+f_1+f_2)}{2} = (\frac{e+1}{e-1})^{\frac{1}{2}}\tanh\frac{(H_0+H_1+H_2)}{2} \tag{7.30}$$

To find ψ_2 we subtract (7.16) from (7.26) leaving

$$0 = -(2\alpha_1)^{-\frac{1}{2}}[A_2f_2 + A_{21}f_1\cos f_0 + A_{22}2f_1\cos 2f_0 + A_{23}\sin 3f_0 + A_{24}\sin 4f_0] + \eta_0(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}[B_2\psi_2 + \frac{n^2}{8}2\psi_1\cos 2\psi_0 - \frac{3n^4}{32}\sin 2\psi_0 + \frac{3n^4}{256}\sin 4\psi_0] \tag{7.31}$$

Solving for ψ_2 we find

$$\psi_2 = B_2{}^{-1}(-\frac{n^2}{4}\psi_1\cos 2\psi_0 + \frac{3n^4}{32}\sin 2\psi_0 - \frac{3n^4}{256}\sin 4\psi_0) + B_2{}^{-1}\eta_0{}^{-1}(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}(2\alpha_1)^{-\frac{1}{2}}[A_2f_2 + A_{21}f_1\cos f_0 + A_{22}2f_1\cos 2f_0 + A_{23}\sin 3f_0 + A_{24}\sin 4f_0] \tag{7.32}$$

This completes the solution, through order J_2^2, for f, H, and ψ, and thus for the spheroidal coordinates ρ and η.

7.3 The Right Ascension ϕ

According to the final Kinetic equation (2.18c) and the definitions of R_3 and N_3 we have

$$\phi = \beta_3 - c^2\alpha_3 R_3 + \alpha_3 N_3 \tag{7.33}$$

Substituting above for R_3 and N_3 from (5.102) and (6.87) respectively we find

$$\begin{aligned}\phi = \beta_3 &+ \alpha_3(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\eta_0[(1-\eta_0{}^2)^{-\frac{1}{2}}(1-\eta_2{}^{-2})^{-\frac{1}{2}}\chi \\ &+ B_3\psi + \frac{3}{32}\eta_0{}^2\eta_2{}^{-4}\sin 2\psi] - c^2\alpha_3(2\alpha_1)^{-\frac{1}{2}}[A_3 f \\ &+ A_{31}\sin f + A_{32}\sin 2f + A_{33}\sin 3f + A_{34}\sin 4f]\end{aligned} \tag{7.34}$$

where ψ and f are now known and χ can be found from $\tan\chi = (1-\eta_0{}^2)^{\frac{1}{2}}\tan\psi$. Thus the final spheroidal coordinate ϕ can be found from (7.34)

From Figure 6.1 it is clear that χ may vary quite rapidly as the satellite passes over or near a pole. We can avoid such difficult calculations, however, by expressing the position of the satellite in rectangular coordinates. The relations between rectangular and spheroidal coordinates are given by (2.1) and (2.2). From (7.34) it is clear that ϕ may be expressed as

$$\phi = \Omega' + \kappa\chi \tag{7.35}$$

where

$$\begin{aligned}\Omega' \equiv \beta_3 &+ \alpha_3(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\eta_0[B_3\psi + \frac{3}{32}\eta_0{}^2\eta_2{}^{-4}\sin 2\psi] \\ &- c^2\alpha_3(2\alpha_1)^{-\frac{1}{2}}[A_3 f + \sum_{j=1}^{4} A_{3_j}\sin jf]\end{aligned} \tag{7.36}$$

which is well-behaved near the poles and $\kappa\chi$ is that part of ϕ which varies rapidly near the poles. By definition η_0 is positive so that we may write

$$\kappa = |\kappa| \text{ sgn } \alpha_3 \tag{7.37}$$

and thus find

$$\kappa^2 = \alpha_3{}^2(\alpha_2{}^2-\alpha_3{}^2)^{-1}\eta_0{}^2(1-\eta_0{}^2)^{-1}(1-\eta_2{}^{-2})^{-1} \tag{7.38}$$

$$= \frac{\alpha_3{}^2\ \eta_0{}^2\ \eta_2{}^2}{(\alpha_2{}^2-\alpha_3{}^2)(\eta_0{}^2+\eta_2{}^2-1-\eta_0{}^2\eta_2{}^2)} \tag{7.39}$$

Substituting for $\eta_0{}^2\eta_2{}^2$ and $\eta_0{}^2 + \eta_2{}^2$ from (4.3) and (4.2) respectively we find

$$\kappa^2 = 1 \tag{7.40}$$

so that

$$\kappa = \text{sgn}\alpha_3 = \pm 1 \tag{7.41}$$

for direct or retrograde motion, respectively, so that ϕ will either always increase or always decrease. Thus

$$\phi = \Omega' + \chi \text{sgn } \alpha_3 \tag{7.42}$$

From the relation $\tan \chi = (1-\eta_0{}^2)^{\frac{1}{2}} \tan \psi$ and the requirement that $\dot{\chi}$ and $\dot{\psi}$ are always positive, we find

$$\sin\chi = \frac{(1-\eta_0{}^2)^{\frac{1}{2}} \sin\psi}{\sqrt{1-\eta_0{}^2 \sin^2\psi}} \tag{7.43}$$

and

$$\cos\chi = \frac{\cos\psi}{\sqrt{1-\eta_0{}^2 \sin^2\psi}} \tag{7.44}$$

so that

$$\exp i\chi = \frac{\cos\psi + i\sqrt{1-\eta_0^2}\,\sin\psi}{\sqrt{1-\eta_0^2\,\sin^2\psi}} \tag{7.45}$$

If we insert (7.45) and (7.42) into (2.1) using the relation $\operatorname{sgn}\alpha_3(1-\eta_0^2)^{\frac{1}{2}} = \operatorname{sgn}\alpha_3|\cos I| = \cos I$ we find

$$X + iY = (\rho^2+c^2)^{\frac{1}{2}}\,[\cos\psi + i\cos I\,\sin\psi]\,\exp i\Omega' \tag{7.46}$$

or separately

$$X = (\rho^2+ c^2)^{\frac{1}{2}}[\cos\psi\,\cos\Omega' - \cos I\,\sin\psi\,\sin\Omega'] \tag{7.47}$$

and

$$Y = (\rho^2+c^2)^{\frac{1}{2}}[\cos I\,\sin\psi\,\cos\Omega' + \sin\Omega'\,\cos\psi] \tag{7.48}$$

also from (2.2)

$$Z = \rho\eta \tag{7.49}$$

These expressions for X, Y, and Z cover all cases and do not involve calculating ϕ. Thus they contain no singularities or rapidly varying quantities, causing no trouble for polar or nearly polar trajectories. For a strictly polar trajectory $\cos I \to 0$, $\alpha_3 \to 0$, so that $\Omega' = \beta_3$ and

$$X + iY = (\rho^2+c^2)^{\frac{1}{2}}\,\cos\psi\,\exp i\beta_3 \tag{7.50}$$

If we did want to calculate ϕ for a nearly polar trajectory, it would be best calculated by

$$\exp i\chi = \frac{\cos\psi + i|\cos I|\,\sin\psi}{\sqrt{\cos^2\psi + \cos^2 I\,\sin^2\psi}} \tag{7.51}$$

and (7.36). Using (7.51) for χ avoids the difference of two nearly equal terms in the denominator.

7.4 The Velocity Components

Differentiating (7.46) with respect to time yields

$$\dot{X} + i\dot{Y} = [\frac{\rho\dot{\rho}}{\rho^2+c^2} + i\dot{\Omega}'](X+iY) + (\rho^2+c^2)^{\frac{1}{2}}(-\sin\psi + i\cos I\ \cos\psi)\dot{\psi}\ \exp i\Omega' \tag{7.52}$$

or separately

$$\dot{X} = \frac{\rho\ \dot{\rho}}{\rho^2+c^2} X - Y\dot{\Omega}' + (\rho^2+c^2)^{\frac{1}{2}}(-\sin\psi\ \cos\Omega' - \cos I\ \cos\psi\ \sin\Omega')\dot{\psi} \tag{7.53}$$

$$\dot{Y} = \frac{\rho\ \dot{\rho}}{\rho^2+c^2} Y + X\dot{\Omega}' + (\rho^2+c^2)^{\frac{1}{2}}(-\sin\psi\ \sin\Omega' + \cos I\ \cos\psi\ \cos\Omega')\dot{\psi} \tag{7.54}$$

Then from (7.49) we obtain

$$\dot{Z} = \dot{\rho}\eta + \rho\dot{\eta} = \dot{\rho}\eta + \eta_0\rho\ \cos\psi\ \dot{\psi} \tag{7.55}$$

Clearly, the three velocity components are well-behaved for all trajectories. To find $\dot{\rho}$ we first combine equations (10.1) and (13.1) of Vinti [1] to get

$$\frac{\partial S}{\partial \xi} = c^2\frac{(\rho^2+c^2\eta^2)}{(\rho^2+c^2)}\ \dot{\xi} \tag{7.56}$$

where S is the Hamilton-Jacobi function and

$$\xi = \frac{\rho}{c} \tag{7.57}$$

Then from equation (53.1) of Vinti [1] we find

$$\frac{\partial S}{\partial \xi} = \frac{\pm\sqrt{F(\rho)}}{c(\xi^2+1)} \tag{7.58}$$

Combining (7.56), (7.57), and (7.58) yields

$$\dot{\rho} = \frac{\pm\sqrt{F(\rho)}}{(\rho^2+c^2\eta^2)} \tag{7.59}$$

If we take $F(\rho)$ in the form of (3.11) and insert $\rho_1 = a(e-1)$, $\rho_2 = -a(e+1)$, and $\rho = a(e \cosh H-1)$ we find

$$F(\rho) = 2\alpha_1(ae\sinh H)^2 \ (\rho^2+A\rho+B) \tag{7.60}$$

Since $\dot{\rho} = ae \sinh H \ \dot{H}$ and $\dot{H} > 0$ for all t, $\sinh H \gtrless 0$ accordingly as $\dot{\rho} \gtrless 0$, respectively. The $\pm|\sinh H|$ resulting from (7.59) and (7.60) thus reduces to $\sinh H$ and thus

$$\dot{\rho} = (2\alpha_1)^{\frac{1}{2}} \frac{ae\sinh H(\rho^2+A\rho+B)^{\frac{1}{2}}}{(\rho^2+c^2\eta^2)} \tag{7.61}$$

We then proceed to find $\dot{\eta}$ by a similar method.

Combining equations (13.2) and (10.2) of Vinti [1] yields

$$\frac{\partial S}{\partial \eta} = \frac{\rho^2+c^2\eta^2}{1-\eta^2} \ \dot{\eta} \tag{7.62}$$

But from (53.2) of the same paper

$$\frac{\partial S}{\partial \eta} = \pm\frac{\sqrt{G(\eta)}}{(1-\eta^2)} \tag{7.63}$$

so that

$$\dot{\eta} = \pm \frac{\sqrt{G(\eta)}}{\rho^2+c^2\eta^2} \tag{7.64}$$

If we use $G(\eta)$ in the form of (3.40) and insert $\eta = \eta_0\sin\psi$ and $n^2 = -\frac{\eta_0^2}{\eta_2^2}$, there results

$$G(\eta) = (\alpha_2^2-\alpha_3^2)(1+n^2\sin^2\psi)\cos^2\psi \tag{7.65}$$

Substituting (7.65) for $G(\eta)$ in (7.64) results in

$$\dot{\eta} = \frac{(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}(1+n^2\sin^2\psi)^{\frac{1}{2}}\cos\psi}{\rho^2+c^2\eta^2} \tag{7.66}$$

The $\pm|\cos\psi|$ resulting from (7.64) becomes $\cos\psi$, since $\cos\psi \gtrless 0$ accordingly as $\dot{\eta} \gtrless 0$, respectively. Then, since $\dot{\eta} = \eta_0\cos\psi\dot{\psi}$ we can solve for $\dot{\psi}$ as

$$\dot{\psi} = \frac{\dot{\eta}}{\eta_0\cos\psi} = \frac{(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}(1+n^2\sin^2\psi)^{\frac{1}{2}}}{\eta_0(\rho^2+c^2\eta^2)} \tag{7.67}$$

To find $\dot{\Omega}'$ we differentiate (7.36). Thus

$$\dot{\Omega}' = \alpha_3(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\ \eta_0[B_3 + \frac{3}{16}\eta_0{}^2\eta_2{}^{-4}\cos 2\psi]\dot{\psi}$$
$$- c^2\alpha_3(2\alpha_1)^{-\frac{1}{2}}\ [A_3 + \sum_{j=1}^{4} jA_{3_j}\cos jf]\dot{f} \tag{7.68}$$

To solve for $\dot{f}$ we differentiate the relation $\rho = p(1+e\cos f)^{-1}$. Solving for $\dot{f}$ we find

$$\dot{f} = \frac{p\dot{\rho}}{\rho^2 e\sin f} \tag{7.69}$$

Using the anomaly connection (7.2) to substitute for sinf yields

$$\dot{f} = \frac{\dot{\rho}}{\rho\sinh H}\frac{\sqrt{e^2-1}}{e} \tag{7.70}$$

Finally substituting for $\dot{\rho}$ from (7.61)

$$\dot{f} = \frac{a\sqrt{e^2-1}\ (2\alpha_1)^{\frac{1}{2}}\ (\rho^2+A\rho+B)^{\frac{1}{2}}}{\rho(\rho^2+c^2\eta^2)} \tag{7.71}$$

Thus equations (7.53), (7.54), (7.55), (7.61), (7.66), (7.67), (7.68), and (7.71) form a complete algorithm for the velocity vector in X, Y, Z, space.

7.5 Determining the β's From an Initial State Vector

Up to this point we have merely assumed the β's to be known constants without specifying how they might be calculated. In this section, therefore, we consider determining the values of the β's given an initial state vector of the satellite.

We have seen previously that from this initial state vector the α's can be calculated according to equations (3.8), (3.9), and (3.10). Using these values of the α's we can then calculate the orbital elements a_0, e_0, and i_0 from equations (3.3), (3.4), and (3.5). This allows us to calculate values for the set of orbital elements a, e, and $\eta_0 = \sin I$ from equations (3.33), (3.36), and (3.48). We can determine the initial values of the anomalies H and f from the equation, $\rho_i = a(e^2-1)/(1+e\cos f_i)$, and the anomaly connections (viz. equation (7.5)). The initial values of ψ and χ can be calculated from $\eta_i = \eta_0 \sin\psi_i$ and $\tan\chi_i = (1-\eta_0{}^2)^{\frac{1}{2}}\tan\psi_i$. We now have sufficient information to evaluate all the terms in the Kinetic equations (7.6), (7.7), and (7.34), excluding the β's. Thus we can solve for the β's numerically from these three Kinetic equations. Note that for nearly polar orbits, where our calculation of χ_i depends on the difference of two nearly equal numbers, χ_i would be better calculated by (7.5.)

In summary, we have seen that the β's like the α's, can be determined from an initial state vector.

7.6 Solution of the Kinetic Equations as c Approaches Zero

Setting c = 0 in the Kinetic equations (7.6), (7.7), and (7.34) should reduce them to the case of simple Keplerian hyperbolic motion. In this manner we can not only provide a rough check on our results, but also obtain some insight into what the constants β_1, β_2, and β_3 represent.

Dropping all terms of order k or higher in (7.6), (7.7), and (7.34) results in

$$t + \beta_1 = \frac{a}{\sqrt{2\alpha_1}}\,(e\sinh H - 1) \tag{7.72}$$

$$\beta_2/\alpha_2 = -(2\alpha_1)^{-\frac{1}{2}}\,\frac{\sqrt{e^2-1}}{p}\,f + \eta_0(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\,\psi \tag{7.73}$$

$$\phi = \beta_3 + \alpha_3(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}} \frac{\eta_0}{\sqrt{1-\eta_0{}^2}} \chi \tag{7.74}$$

But for $c = 0$ we know $a = a_0$, $e = e_0$, and $I = i_0$ so that

$$2\alpha_1 \rightarrow \frac{\mu}{a} \tag{7.75}$$

$$\eta = \sin I \sin\psi \rightarrow \sin\theta \tag{7.76}$$

$$(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}} \rightarrow \alpha_2 \sin I \tag{7.77}$$

Using these relations in (7.72), (7.73), and (7.74) we reduce the Kinetic equations to

$$t + \beta_1 = (\frac{a^3}{\mu})^{\frac{1}{2}}(e\sinh H-H) \tag{7.78}$$

$$\beta_2 = \psi - f = \sin^{-1}(\frac{\sin\theta}{\sin I}) - f \tag{7.79}$$

$$\phi = \beta_3 + \chi = \beta_3 + \sin^{-1}(\frac{\tan\theta}{\tan I}) \tag{7.80}$$

The corresponding Keplerian relations are

$$t - \tau = (\frac{a^3}{\mu})^{\frac{1}{2}}(e\sinh H-H) \tag{7.81}$$

$$\omega = \psi - f = \sin^{-1} (\frac{\sin\theta}{\sin I}) - f \tag{7.82}$$

$$\phi = \Omega + \chi = \Omega + \sin^{-1} (\frac{\tan\theta}{\tan I}) \tag{7.83}$$

These angles are defined in Figure 7.1.

Upon comparing the reduced Kinetic equations (7.78), (7.79), and (7.80), to the corresponding Keplerian relations, it is apparent that to agree identically we must have

$$-\beta_1 \rightarrow \tau \quad = \text{time of perigee passage} \tag{7.84}$$

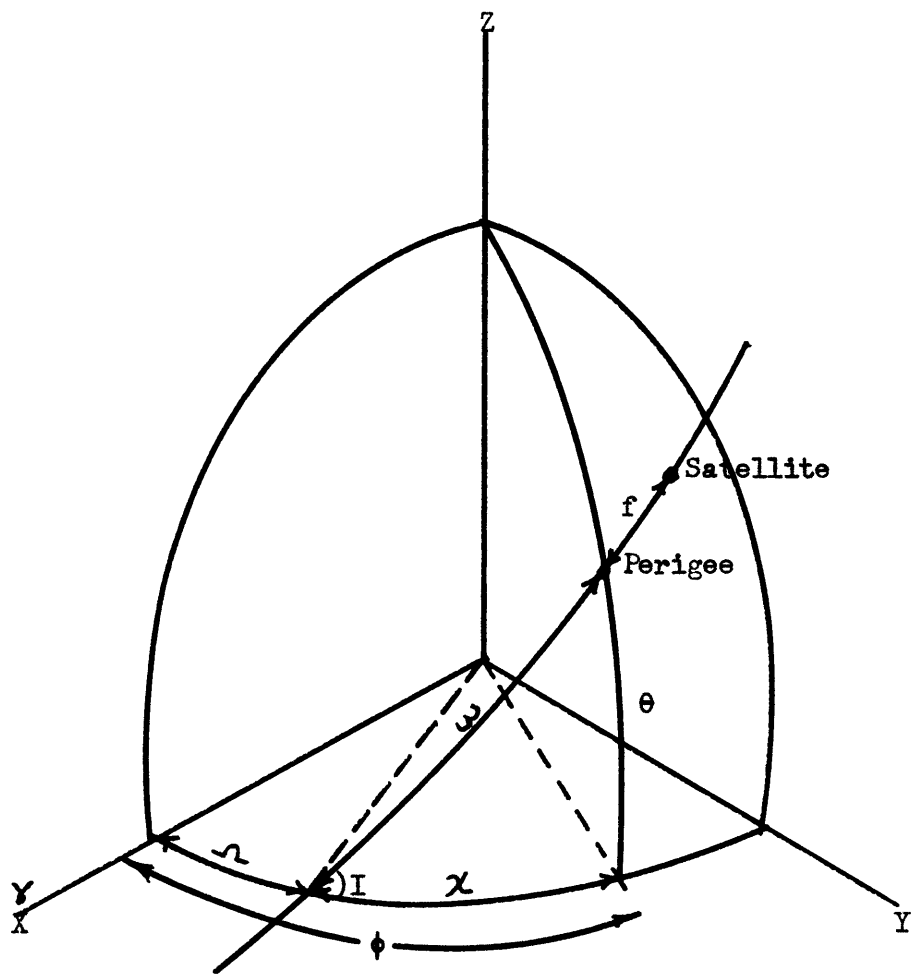

FIGURE 7.1 Definition of Angles on the Celestial Sphere

$$\beta_2 \rightarrow \omega = \text{argument of perigee} \tag{7.85}$$

$$\beta_3 \rightarrow \Omega = \text{longitude of the node} \tag{7.86}$$

as c goes to zero. We have also seen that f, I, and χ reduce to the true anomaly, inclination, and projection of the argument of latitude onto the equator, respectively, as c goes to zero.

CHAPTER 8

ASYMPTOTIC MOTION

In this section we consider the analytic determination of one of the asymptotes of the spheroidal "hyperbola," either incoming or outgoing, given the other asymptote. Such a solution would find application in calculating swingby maneuvers about an oblate body. Thus given a desired outgoing asymptote, for example, we could determine the necessary incoming conditions.

The incoming asymptote is defined by $\rho \to r \to \infty$, $t \to -\infty$, while the outgoing asymptote is defined by $\rho \to r \to \infty$, $t \to \infty$. From (2.2), these conditions imply $\eta \to \sin\theta$ on the asymptotes. Setting $\rho \to \infty$ in (7.61), (7.66), (7.67), (7.68), and (7.71) results in

$$\dot{\rho} \to \dot{r} \to (2\alpha_1)^{\frac{1}{2}} \tag{8.1}$$

$$\dot{\eta} \to 0 \tag{8.2}$$

$$\dot{\psi} \to 0 \tag{8.3}$$

$$\dot{f} \to 0 \tag{8.4}$$

$$\dot{\Omega}' \to 0 \tag{8.5}$$

Note that these equations are true on both asymptotes. Inserting (8.1) through (8.5) into (7.53), (7.54), and (7.55) we find

$$\dot{X} \to \frac{X\dot{r}}{r} \tag{8.6}$$

$$\dot{Y} \to \frac{Y\dot{r}}{r} \tag{8.7}$$

$$\dot{Z} \to \dot{r}\eta \tag{8.8}$$

But by (7.47) and (7.48)

$$X \to r(\cos\psi \cos\Omega' - \cos I \sin\psi \sin\Omega') \tag{8.9}$$

$$Y \to r(\cos I \sin\psi \cos\Omega' + \sin\Omega' \cos\psi) \tag{8.10}$$

Substituting (8.9) and (8.10) into (8.6) and (8.7) and using $\eta = \sin I \sin\psi$ in (8.8) yields

$$\dot{X} \to \dot{r}(\cos\psi \cos\Omega' - \cos I \sin\psi \sin\Omega') \tag{8.11}$$

$$\dot{Y} \to \dot{r}(\cos I \sin\psi \cos\Omega' + \sin\Omega' \cos\psi) \tag{8.12}$$

$$\dot{Z} \to \dot{r} \sin I \sin\psi \tag{8.13}$$

Given a state vector on an asymptote we can most easily calculate the α's using the modified forms of (3.8), (3.9), and (3.10)

$$\alpha_1 = \frac{1}{2}\,\dot{r}_i^{\,2} = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) \tag{8.14}$$

$$\alpha_3 = X_i\dot{Y}_i - Y_i\dot{X}_i = [(r_{ax}^{\,2}+r_{ay}^{\,2})(\dot{X}^2+\dot{Y}^2)]^{\frac{1}{2}} \tag{8.15}$$

$$\alpha_2^{\,2} = \frac{[r_i(\dot{Z}_i-\dot{r}_i\eta_i)]^2 + \alpha_3^{\,2} - 2\alpha_1 c^2\eta_i^{\,2}(1-\eta_i^{\,2})}{(1-\eta_i^{\,2})} \tag{8.16}$$

$$= (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)(r_{ax}^{\,2} + r_{ay}^{\,2} + r_{az}^{\,2}) - c^2\dot{Z}^2$$

where $\dot{X}\underline{i} + \dot{Y}\underline{j} + \dot{Z}\underline{k}$ is now the hyperbolic excess velocity vector and $\underline{r}_a = r_{ax}\underline{i} + r_{ay}\underline{j} + r_{az}\underline{k}$ is the vector from the origin to the aiming point, i.e. the point on the extension of the asymptote closest to the origin. With these values of the α's we can then determine the orbital elements a_0, e_0, i_0, and the elements a, e, I using the method of the previous section. Then equations (8.11), (8.12), and (8.13) allow us to find ψ_i and Ω'_i. Since $\dot{r}$ and I will be the same for both asymptotes, the problem of determining one asymptote given the other reduces to the problem of determining the $\Delta\Omega'$ and $\Delta\psi$ caused by a swingby of the oblate body. Here we have defined

$$\psi_{out} = \psi_{in} + \Delta\psi \tag{8.17}$$

$$\Omega'_{out} = \Omega'_{in} + \Delta\Omega' \tag{8.18}$$

To solve for $\Delta\Omega'$ and $\Delta\psi$ we first consider the outgoing asymptote. Setting $t \to \infty$ in (7.11) yields

$$H_{0_{out}} = \infty \tag{8.19}$$

and thus by (7.13)

$$f_{0_{out}} = \cos^{-1}(-\frac{1}{e}) \equiv f_a \tag{8.20}$$

Inserting this value of f_0 into (7.14) results in

$$\psi_{0_{out}} = \frac{(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}}{n_0 B_2}\left[\frac{\beta_2}{\alpha_2} + (2\alpha_1)^{-\frac{1}{2}} A_2 f_a\right] \tag{8.21}$$

Then by (7.22)

$$H'_{1_{out}} = \frac{-2b_1 H_0}{ae\ e^{H_0}} = \frac{-2b_1}{ae}\ \frac{1}{1+\frac{H_0}{2!}+\frac{H_0{}^2}{3!}+\cdots} = 0 \tag{8.22}$$

so that by (7.21)

$$H_{1_{out}} = 0 \tag{8.23}$$

Proceeding similarly with the remaining equations of Section 7.2 we find

$$f_{1_{out}} = 0 \tag{8.24}$$

$$\psi_{1_{out}} = -B_2^{-1} \frac{n^2}{8} \sin 2\psi_{0_{out}} + B_2^{-1}(2\alpha_1)^{-\frac{1}{2}} \frac{(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}}{n_0} [A_{21}\sin f_a + A_{22}\sin 2f_a] \tag{8.26}$$

$$H_{2_{out}} = 0 \tag{8.27}$$

$$f_{2_{out}} = 0 \tag{8.28}$$

$$\psi_{2_{out}} = B_2^{-1}(- \frac{n^2}{4} \psi_{1_{out}} \cos 2\psi_{0_{out}} + \frac{3n^4}{32} \sin 2\psi_{0_{out}} - \frac{3n^4}{256} \sin 4\psi_{0_{out}}) + B_2^{-1} n_0^{-1}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}(2\alpha_1)^{-\frac{1}{2}} [A_{23}\sin 3f_{0_{out}} + A_{24}\sin 4f_{0_{out}}] \tag{8.29}$$

Thus

$$\psi_{out} = \psi_{0_{out}} + \psi_{1_{out}} + \psi_{2_{out}} = \frac{(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}}{B_2 n_0} [\beta_2/\alpha_2 + (2\alpha_1)^{-\frac{1}{2}}(A_2 f_a + A_{21}\sin f_a + A_{22}\sin 2f_a + A_{23}\sin 3f_a + A_{24}\sin 4f_a)] + B_2^{-1}[- \frac{n^2}{8} \sin 2\psi_{0_{out}} - \frac{n^2}{4} \psi_{1_{out}} \cos 2\psi_{0_{out}} + \frac{3n^4}{32} \sin 2\psi_{0_{out}} - \frac{3n^4}{256} \sin 4\psi_{0_{out}}] \tag{8.30}$$

and by (7.36)

$$\Omega'_{out} = \beta_3 + \alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}} n_0[B_3\psi_{out} + \frac{3}{32} n_0^2 n_2^{-4} \sin 2\psi_{out}] - c^2\alpha_3(2\alpha_1)^{-\frac{1}{2}}[A_3 f_a + A_{31}\sin f_a + A_{32}\sin 2f_a + A_{33}\sin 3f_a + A_{34}\sin 4f_a] \tag{8.31}$$

Proceeding now to the incoming asymptote we set $t \to -\infty$ in (7.11), resulting in

$$H_{0_{in}} = -\infty \tag{8.32}$$

so that by (7.13)

$$f_{0_{in}} = \cos^{-1}(-\frac{1}{e}) \tag{8.33}$$

Since $f_{0_{in}}$ cannot equal $f_{0_{out}}$ we must have

$$f_{0_{in}} = -f_{0_{out}} = -f_a \tag{8.34}$$

Clearly equations (8.21) through (8.31) can be applied just as well to the incoming asymptote if we replace all "out" subscripts by "in" subscripts and substitute $-f_a$ for f_a. Performing the above replacements in (8.30) and then subtracting the result from (8.30) yields

$$\begin{aligned}\Delta\psi \equiv \psi_{out} - \psi_{in} = {}& \frac{2(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}(2\alpha_1)^{-\frac{1}{2}}}{n_0 B_2}[A_2 f_a \\ &+ A_{21}\sin f_a + A_{22}\sin 2f_a + A_{23}\sin 3f_a + A_{24}\sin 4f_a] \\ &+ B_2{}^{-1}[-\frac{n^2}{8}\sin(2\psi_0+2\Delta\psi_0) + \frac{n^2}{8}\sin 2\psi_0 - \frac{n^2}{4}(\psi_1+\Delta\psi_1)\cos(2\psi_0+2\Delta\psi_0) \\ &+ \frac{n^2}{4}\psi_1\cos 2\psi_0 + \frac{3n^4}{32}\sin(2\psi_0+2\Delta\psi_0) - \frac{3n^4}{32}\sin 2\psi_0 \\ &- \frac{3n^4}{256}\sin(4\psi_0+4\Delta\psi_0) + \frac{3n^4}{256}\sin 4\psi_0]\end{aligned} \tag{8.35}$$

where we have dropped the in-out subscripts in favor of letting the incoming conditions be represented by ψ_0, ψ_1, and the outgoing conditions by $\psi_0 + \Delta\psi_0$, $\psi_1 + \Delta\psi_1$. If in (8.35) we expand the sin(a+b) and cos(a+b) terms and substitute $\sin n\psi_0 = \sin n\psi_{in} - n\psi_1\cos n\psi_{in}$ and $\cos n\psi_0 = \cos n\psi_{in} + n\psi_1\sin n\psi_{in}$ we find that, to order $J_2{}^2$, (8.35) becomes

$$\Delta\psi = 2(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}(2\alpha_1\ )^{-\frac{1}{2}}\ \eta_0{}^{-1}B_2^{-1}\ [A_2f_a + A_{21}\sin f_a$$
$$+ A_{22}\sin 2f_a + A_{23}\sin 3f_a + A_{24}\sin 4f_a]$$
$$+B_2{}^{-1}[-\frac{n^2}{8}\ (\sin 2(\psi_{in}+\Delta\psi_0) - \sin 2\psi_{in}) - \frac{n^2}{4}\ \Delta\psi_1\cos 2(\psi_{in}+\Delta\psi_0)$$
$$+ \frac{3n^4}{32}(\sin 2(\psi_{in}+\Delta\psi_0) - \sin 2\psi_{in}) - \frac{3n^4}{256}(\sin 4(\psi_{in}+\Delta\psi_0) - \sin 4\psi_{in})] \qquad (8.36)$$

where ψ_{in} can be determined from the incoming asymptote as discussed earlier in this section. If in (8.21) we replace f_a by $-f_a$, change the subscripts to "in", and subtract the result from the original equation, we find

$$\Delta\psi_0 = \frac{2A_2(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}(2\alpha_1)^{-\frac{1}{2}}\ f_a}{\eta_0 B_2} \qquad (8.37)$$

Performing the same operations on (8.26) and keeping only first order terms results in

$$\Delta\psi_1 = -B_2{}^{-1}\ \frac{n^2}{8}\ [\sin 2(\psi_{in}+\Delta\psi_0) - \sin 2\psi_{in}]$$
$$+2B_2{}^{-1}\eta_0{}^{-1}(2\alpha_1)^{-\frac{1}{2}}(\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}}\ [A_{21}\sin f_a + A_{22}\sin 2f_a] \qquad (8.38)$$

Thus since f_a and ψ_{in} can be found for the incoming asymptote, we can then calculate $\Delta\psi_0$, $\Delta\psi_1$, and finally $\Delta\psi$ from (8.37), (8.38), and (8.36) respectively. Then, by (8.31), $\Delta\Omega'$ can be determined from

$$\Delta\Omega' = \alpha_3(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}}\ \eta_0[B_3\Delta\psi + \frac{3}{32}\ \eta_0{}^2\eta_2{}^{-4}(\sin 2(\psi_{in}+\Delta\psi)$$
$$- \sin 2\psi_{in})] - c^2\alpha_3(2\alpha_1)^{-\frac{1}{2}}\ 2[A_3f_a + A_{31}\sin f_a$$
$$+ A_{32}\sin 2f_a + A_{33}\sin 3f_a + A_{34}\sin 4f_a] \qquad (8.39)$$

Having then determined $\psi_{out} = \psi_{in} + \Delta\psi$ and $\Omega'_{out} = \Omega'_{in} + \Delta\Omega'$ we can then proceed to calculate the outgoing velocity component and thus the outgoing asymptote, from (8.11), (8.12), and (8.13).

Notice that equations (8.36), (8.37), (8.38), and (8.39) can be used equally well to find the incoming asymptote if given the outgoing asymptote. In this case we merely replace f_a by $-f_a$ and ψ_{in} by $\psi_{out} - \Delta\psi_0$. The resulting $\Delta\psi$ and $\Delta\Omega'$ will be such that $\psi_{in} = \psi_{out} + \Delta\psi$ and $\Omega'_{in} = \Omega'_{out} + \Delta\Omega'$.

CHAPTER 9

THE "PARABOLIC" TRAJECTORY

9.1 Definition and Problem Formulation

In this section we consider the special case of trajectories which have $\alpha_1 = 0$. These trajectories are the analog of parabolic motion in the Keplerian formulation. Setting $\alpha_1 = 0$ in (2.6) we find

$$F(\rho) = c^2\alpha_3^2 + (\rho^2+c^2)(\tilde{k}+2\mu\rho) \tag{9.1}$$

to have $F(\rho) = 0$ we must have $\tilde{k} + 2\mu\rho = -c^2\alpha_3^2(\rho^2+c^2)^{-1}$, or for $\tilde{k} = 0$

$$\rho^3 + \rho c^2 + \frac{c^2\alpha_3^2}{2\mu} = 0 \tag{9.2}$$

which will have one negative real root, which we will label ρ_1, and two imaginary roots. For $\tilde{k} = 0$ we can see from (9.1) that the slope of $F(\rho)$ will always be positive. Also from (9.1) it is clear that increasing $\tilde{k}$ from zero allows ρ_1 to be even more negative, while decreasing $\tilde{k}$ from zero forces ρ_1 to increase, finally becoming positive. The critical value of $\tilde{k}$ at which this transition occurs is seen to be

$$\tilde{k}_{crit} = -\alpha_3^2 \tag{9.3}$$

From the above information we can sketch $F(\rho)$ as in Figure 9.1. For physically realizable motion we must have $F(\rho) > 0$, so that for perigee, ρ_1, to be positive $\tilde{k}$ must be negative and greater in magnitude

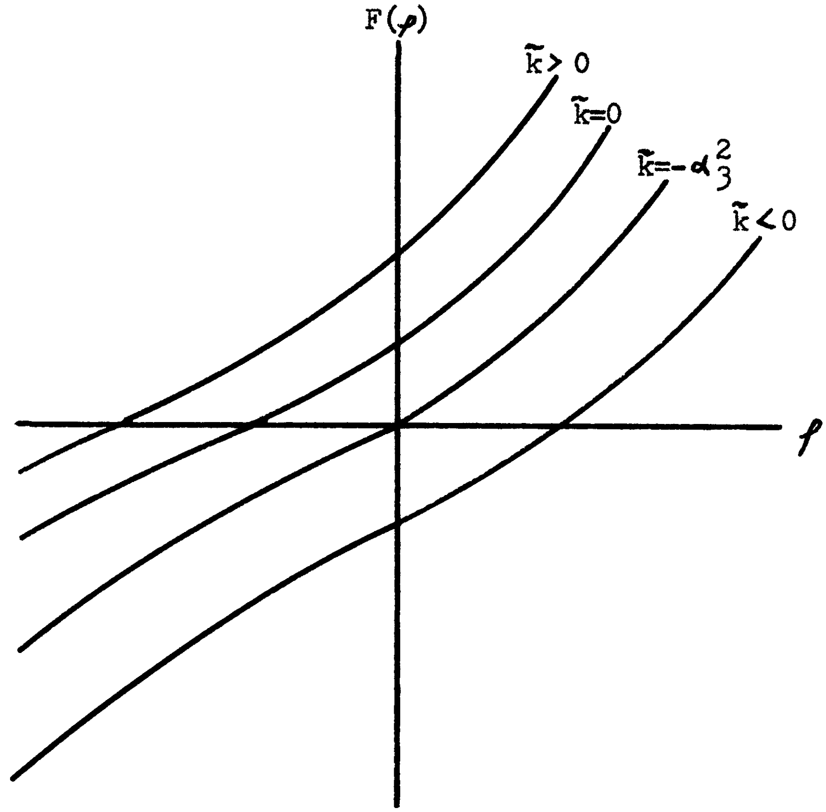

FIGURE 9.1 F(φ) Versus φ for α_1=0

than α_3^2. Thus, as before, we can define

$$\tilde{k} \equiv -\alpha_3^2 \tag{9.4}$$

where

$$\alpha_2^2 \geq \alpha_3^2 \tag{9.5}$$

Setting $\alpha_1 = 0$ in (2.20) results in

$$G(\eta) = -\alpha_3^2 + (1-\eta^2)\alpha_2^2 \tag{9.6}$$

If $\pm\eta_0$ are the roots of $G(\eta) = 0$ then

$$\eta_0 = \frac{(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}}{\alpha_2} \tag{9.7}$$

According to (9.5) both roots are real. Then, since $G(\eta) = -\infty$ for $\eta = \infty$ or $\eta = -\infty$ we can sketch $G(\eta)$ as in Figure 9.2. The motion again takes place between $-\eta_0$ and $+\eta_0$.

9.2 The Orbital Elements

According to (3.3), (3.4), and (3.5) the orbital elements a_0, e_0, and i_0 become

$$a_0 \rightarrow \infty \tag{9.8}$$

$$e_0 \rightarrow 1 \tag{9.9}$$

$$i_0 \equiv \cos^{-1}(\alpha_3/\alpha_2) \tag{9.10}$$

and by (3.7)

$$p_0 = \frac{\alpha_2^2}{\mu} \tag{9.11}$$

Note that these are the same values we would obtain from setting $\alpha_1 = -0$

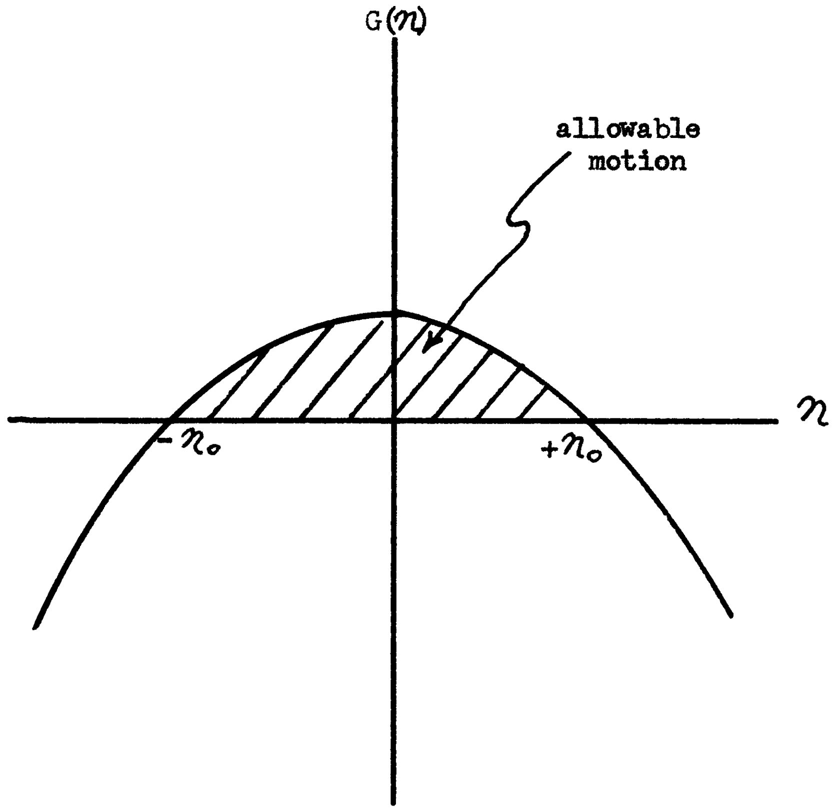

FIGURE 9.2 G(η) Versus η for α_1=0

in the corresponding orbital element definitions for the bounded case. (Vinti [2] equations (3.3), (3.4), and (3.5).) Inserting (9.4) in (9.1) results in

$$F(\rho) = c^2\alpha_3^2 + (\rho^2+c^2)(-\alpha_2^2+2\mu\rho) \tag{9.12}$$

If we put $F(\rho)$ in the form

$$F(\rho) = 2\mu(\rho-\rho_1)(\rho^2+A\rho+B) \tag{9.13}$$

we find, upon comparing coefficients of ρ in (9.12) and (9.13)

$$\alpha_2^2 = 2\mu(\rho_1-A) = \mu p_0 \tag{9.14}$$

$$c^2 = B - \rho_1 A \tag{9.15}$$

$$c^2(\alpha_2^2-\alpha_3^2) = 2\mu\rho_1 B \tag{9.16}$$

With k_0 and y defined as before we then seek a series solution of the form

$$\rho_1 = \sum_{n=0}^{\infty} b_{1_n} k_0^n \tag{9.17}$$

$$A = \sum_{n=0}^{\infty} b_{2_n} k_0^n \tag{9.18}$$

$$B = \sum_{n=0}^{\infty} b_{3_n} k_0^n \tag{9.19}$$

Inserting (9.17), (9.18), and (9.19) into (9.14), (9.15), and (9.16) and solving for the b_{ij} we find to order k_0^2

$$\rho_1 = \frac{p_0}{2}\,[1 - 4y^2k_0 + 16y^2k_0^2(1-2y^2)] \tag{9.20}$$

$$A = -2p_0y^2k_0[1 - 4k_0(1-2y^2)] \tag{9.21}$$

$$B = p_0^2(1-y^2)\ k_0[1 + 4y^2k_0] \tag{9.22}$$

For nearly equatorial trajectories (9.13) will have three positive real roots. Note that (9.21) and (9.22) correspond exactly to the A and B equations for the bounded case (Vinti [2] equations (3.19) and (3.20)) with $x = (1-e_0^2)^{\frac{1}{2}} = 0$. They also correspond exactly to (3.25) and (3.26) of the present paper with $x = (e_0^2-1) = 0$.

Since $a_0 \to \infty$ for "parabolic" motion we find it more convenient to choose as orbital elements p_0, e_0, and i_0. The elements p, e, and I are then defined by

$$p \equiv 2\rho_1 \tag{9.23}$$

$$\eta_0 \equiv \sin I \tag{9.24}$$

$$e \equiv e_0 = 1 \tag{9.25}$$

Combining (9.20) with (9.23) results in

$$p/p_0 = 1 - 4y^2k_0 + 16y^2k_0^2(1-2y^2) + \cdots \tag{9.26}$$

and from (9.7) and (9.10)

$$\eta_0 = \sin I = \frac{(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}}{\alpha_2} = \sin i_0 \tag{9.27}$$

Thus the elements I and i_0 are identical for this case. Equations (9.25), (9.26), and (9.27), which define the relationship between the two sets of orbital elements, can be obtained alternately from the corresponding bounded (Vinti [2]: (3.32), (3.26), and (3.40)) or unbounded (present paper: (3.39), (3.34), and (3.48)) relations by considering the limiting case $e_0 = 1$. Having shown that the "parabolic" orbital elements p_0, e_0, i_0, or p, e, I can be derived as the limiting case of either the bounded or unbounded equations, we can now proceed to obtain the values of the ρ and η integrals for the case $\alpha_1 = 0$ from those previously derived for the cases $\alpha_1 < 0$ or $\alpha_1 > 0$. The next two subsections are devoted to showing that the two sets of "parabolic" equations of motion so derived are identical.

9.3 The "Parabolic" Solution as Derived From the Unbounded Solution

In this subsection we seek to determine the "parabolic" ρ and η integrals, and thus the Kinetic equations, as the limiting case, $\alpha_1 \to 0$, of our results for unbounded motion. From (5.45) and the definition of a_0 we find

$$R_1 = \left(\frac{a_0}{\mu}\right)^{\frac{1}{2}} [b_1 H - aH + ae\sinh H + A_1 f + A_{11}\sin f + A_{12}\sin 2f] \qquad (9.28)$$

However, substituting $x^2 \equiv e_0{}^2 - 1 = 0$ into (3.33) we find that $a = a_0$ so that (9.28) can be rewritten as

$$R_1 = \left(\frac{p}{\mu}\right)^{\frac{1}{2}} \left[\frac{1}{\sqrt{e^2-1}} [b_1 H - aH + ae\sinh H]_1 + \frac{1}{\sqrt{e^2-1}} [A_1 f + A_{11}\sin f + A_{12}\sin 2f]\right] \qquad (9.29)$$

Consider first the term in brackets with the subscript 1. From the anomaly relation (7.4) we have

$$\sinh H = \frac{\sqrt{e^2-1}\,\sin f}{1+e\cos f} \qquad (7.4)$$

and so

$$H = \sinh^{-1} \frac{\sqrt{e^2-1}\,\sin f}{1+e\cos f} \qquad (9.30)$$

However, the series expansion of $\sinh^{-1}$ is

$$\sinh^{-1}u = u - \frac{u^3}{6} + \frac{3u^5}{40} + \cdots \tag{9.31}$$

so that (9.30) may be rewritten as

$$H = (e^2-1)^{\frac{1}{2}} \left[\frac{\sin f}{1+e\cos f} - \frac{1}{6}\frac{(e^2-1)\sin^3 f}{(1+e\cos f)^3} + \cdots\right] \tag{9.32}$$

If we then write the subscripted bracket term in the form

$$\left(\frac{p}{\mu}\right)^{\frac{1}{2}} (e^2-1)^{-\frac{1}{2}} [\ldots]_1 = \left(\frac{p^3}{\mu}\right)^{\frac{1}{2}}(e^2-1)^{-\frac{1}{2}}\left[\frac{b_1}{p} H - \frac{H}{(e^2-1)} + \frac{e\sinh H}{(e^2-1)}\right] \tag{9.33}$$

and insert into it (9.3) and (7.4) for H and sinhH, respectively, we find

$$\left(\frac{p}{\mu}\right)^{\frac{1}{2}}(e^2-1)^{-\frac{1}{2}} [\ldots]_1 = \left(\frac{p^3}{\mu}\right)^{\frac{1}{2}} \Bigg[\frac{b_1}{p} \left[\frac{\sin f}{1+e\cos f} - \frac{1}{6}\frac{(e^2-1)\sin^3 f}{(1+e\cos f)^3} + \cdots\right]$$

$$+ \frac{\sin f}{(e+1)(1+e\cos f)} + \frac{1}{6}\frac{\sin^3 f}{(1+e\cos f)^3} + \cdots\Bigg] \tag{9.34}$$

$$= \left(\frac{p^3}{\mu}\right)^{\frac{1}{2}} \left[\frac{b_1}{p}\tan\frac{f}{2} + \frac{1}{2}\tan\frac{f}{2} + \frac{1}{6}\tan^3\frac{f}{2}\right] \tag{9.35}$$

in the limit as $e \to 1$. With this result we can rewrite (9.29) as

$$R_1 = \left(\frac{p^3}{\mu}\right)^{\frac{1}{2}}\Big[\left(\frac{b_1}{p} + \frac{1}{2}\right)\tan\frac{f}{2} + \frac{1}{6}\tan^3\frac{f}{2} + A_1' f$$

$$+ A_{11}'\sin f + A_{12}'\sin 2f\Big] \tag{9.36}$$

where by the use of (5.39), (5.43), and (5.44) we have defined

$$A_1' = \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda)\, T_{n-2}(0) \tag{9.37}$$

$$A_{11}' = \frac{3}{4p^4}(-2b_1b_2{}^2p + b_2{}^4) \tag{9.38}$$

$$A_{12}' = \frac{3b_2{}^4}{32p^4} \tag{9.39}$$

Again using the fact that $a = a_0$ in the limit, we can write (5.57) in the form

$$R_2 = \left(\frac{p}{\mu}\right)^{\frac{1}{2}} \frac{1}{\sqrt{e^2-1}} \left[A_2 f + \sum_{j=1}^{4} A_{2_j} \sin jf\right] \tag{9.40}$$

or, dividing the coefficients by $\sqrt{e^2-1}$ and taking the limit as $e \to 1$

$$R_2 = \left(\frac{p}{\mu}\right)^{\frac{1}{2}} \left[A_2' f + \sum_{j=1}^{4} A_{2_j}' \sin jf\right] \tag{9.41}$$

where, with the use of (5.51) and (5.53) through (5.56)

$$A_2' = p^{-1} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda)\, T_n(0) \tag{9.42}$$

$$A_{21}' = \frac{b_1}{p^2} + \frac{3b_1^2 - b_2^2}{p^3} - \frac{45}{8}\frac{b_1 b_2^2}{p^4} + \frac{21}{8}\frac{b_2^4}{p^5} \tag{9.43}$$

$$A_{22}' = \frac{3b_1^2 - b_2^2}{8p^3} - \frac{9}{8}\frac{b_1 b_2^2}{p^4} + \frac{21}{32}\frac{b_2^4}{p^4} \tag{9.44}$$

$$A_{23}' = -\frac{b_1 b_2^2}{8p^4} + \frac{b_2^4}{8p^5} \tag{9.45}$$

$$A_{24}' = \frac{3b_2^4}{256p^5} \tag{9.46}$$

In a similar fashion the R_3 integral (5.102) becomes

$$R_3 = \left(\frac{p}{\mu}\right)^{\frac{1}{2}} \left[A_3' f + \sum_{j=1}^{4} A_{3_j}' \sin jf\right] \tag{9.47}$$

where from (5.96) through (5.101)

$$A_3' = p^{-3} \sum_{m=0}^{\infty} D_m\, T_{m+2}(0) \tag{9.48}$$

$$A_{31}' = p^{-3}\left[2 + \frac{15}{4}\frac{b_1}{p} - 7\left(\frac{b_2^2}{2p^2} + \frac{c^2}{p^2}\right)\right] \tag{9.49}$$

$$A_{32}' = p^{-3}\left[\frac{1}{4} + \frac{3}{4}\frac{b_1}{p} - \frac{7}{4}\left(\frac{b_2^2}{2p^2} + \frac{c^2}{p^2}\right)\right] \tag{9.50}$$

$$A_{33}' = \frac{b_1}{12p^4} - \frac{1}{3p^5}\left(\frac{b_2^2}{2} + c^2\right) \tag{9.51}$$

$$A_{34}' = -\frac{1}{32p^5}\left(\frac{b_2^2}{2} + c^2\right) \tag{9.52}$$

In evaluating the η integrals we first note that setting $x^2 \equiv e_0{}^2 - 1 = 0$ in (3.51) results in

$$n^2 = -\eta_0{}^2/\eta_2{}^2 = 0 \tag{9.53}$$

for the "parabola." Inserting (9.53) and (9.7) into (6.43) and (6.48) results in

$$N_1 = \frac{\alpha_2{}^2 - \alpha_3{}^2}{\alpha_2{}^3}\left(\frac{\psi}{2} - \frac{1}{4}\sin 2\psi\right) \qquad \text{exact} \tag{9.54}$$

Similarly (6.51) and (6.53) become

$$N_2 = \frac{\psi}{\alpha_2} \qquad \text{exact} \tag{9.55}$$

To calculate N_3 we note, in addition, that from (3.50) $\eta_2{}^{-2} = 0$ for the "parabola," so that (6.87) becomes

$$N_3 = \frac{\chi}{\alpha_3} \qquad \text{exact} \tag{9.56}$$

It is interesting to notice that all the η integrals are exact. Substituting these integrals into the Kinetic equations (2.18) results in

$$t + \beta_1 = \left(\frac{p^3}{\mu}\right)^{\frac{1}{2}}\left[\left(\frac{b_1}{p} + \frac{1}{2}\right)\tan\frac{f}{2} + \frac{1}{6}\tan^3\frac{f}{2} + A_1'f + \sum_{j=1}^{2} A_{1j}'\sin jf + \text{periodic terms } O(k^3)\right] + c^2\frac{(\alpha_2{}^2 - \alpha_3{}^2)}{\alpha_2{}^3}\left[\frac{\psi}{2} - \frac{1}{4}\sin 2\psi\right] \tag{9.57}$$

$$\beta_2 = -\alpha_2(\frac{p}{\mu})^{\frac{1}{2}} [A_2{}'f + \sum_{j=1}^{4} A_{2_j}{}'\sin jf + \text{periodic terms } O(k^3)]+\psi \tag{9.58}$$

$$\phi - \beta_3 = -c^2\alpha_3(\frac{p}{\mu})^{\frac{1}{2}} [A_3{}'f + \sum_{j=1}^{4} A_{3_j}{}' \sin jf + \text{periodic terms } O(k^3)]+\chi \tag{9.59}$$

Setting $c \rightarrow 0$ in the above Kinetic equations and comparing to the corresponding Keplerian equations provides one method of checking our solution. Thus dropping all terms of $O(k)$ or higher from (9.57), (9.58), and (9.59) we find

$$t + \beta_1 = (\frac{p^3}{\mu})^{\frac{1}{2}} (\frac{1}{2} \tan\frac{f}{2} + \frac{1}{6} \tan^3\frac{f}{2}) \tag{9.60}$$

$$\beta_2 = \psi - f = \sin^{-1} (\frac{\sin\theta}{\sin i}) - f \tag{9.61}$$

$$\phi - \beta_3 = \chi = \sin^{-1} (\frac{\tan\theta}{\tan i}) \tag{9.62}$$

which are indeed the Keplerian equations of motion for the parabolic trajectory.

9.4 The "Parabolic" Solution as Derived From the Bounded Solution

In this subsection we will show that the "parabolic" solution obtained as the limiting case of Vinti's [2] bounded spheroidal solution is identical to the "parabolic" solution derived in the previous subsection. This, of course, must be the case if both the solutions for the bounded and unbounded cases are to be correct. Since we have previously shown that the orbital elements p, e, i and p_0, e_0, i_0 have the same value in the limit of "parabolic" motion whether derived from the bounded or unbounded case, we can retain the notation found in Vinti's equations without ambiguity. Furthermore, since Vinti's variable v is defined exactly the same as f in the present paper, we may make the substitution $v \rightarrow f$ in Vinti's equations.

From V(5.30)* and V(3.3) the R_1 integral becomes

$$R_1 = \left(\frac{a_0}{\mu}\right)^{\frac{1}{2}}[b_1E + aE - ae\sin E + A_1f + A_{11}\sin f + A_{12}\sin 2f] \tag{9.63}$$

Dividing V(3.26) by V(3.28) yields

$$\frac{a}{a_0} = \frac{p/p_0}{(1-e^2)/(1-e_0^2)}$$
$$= \frac{1 + 2k_0y^2(x^2-2)+k_0^2y^2(3x^4-2x^4y^2-16x^2+24x^2y^2+16-32y^2)}{1 + k_0y^2(3x^2-4)+k_0^2y^2(5x^4-2x^4y^2-20x^2+28x^2y^2+16-32y^2)} \tag{9.64}$$

Setting $x^2 \equiv 1-e_0^2 = 0$ in the above relation yields

$$a = a_0 \tag{9.65}$$

for the "parabola." Using this fact we can rewrite (9.63) as

$$R_1 = \left(\frac{p}{\mu}\right)^{\frac{1}{2}} (1-e^2)^{-\frac{1}{2}}[b_1E + aE - ae\sin E + A_1f + A_{11}\sin f + A_{12}\sin 2f] \tag{9.66}$$

If we then solve V(8.1a) and V(8.16) for sinE we find

$$\sin E = \frac{(1-e^2)^{\frac{1}{2}}\sin f}{1+e\cos f} \tag{9.67}$$

or, equivalently

$$E = \sin^{-1}\left[\frac{(1-e^2)^{\frac{1}{2}}\sin f}{1+e\cos f}\right] \tag{9.68}$$

where $\sin^{-1}u$ may be written as the series expansion

$$\sin^{-1}u = u + \frac{u^3}{6} + \frac{3u^5}{40} + \cdots \tag{9.69}$$

If we insert (9.67) and (9.68) into (9.66), using the above series

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* Equations preceeded by a V indicate their origin in Vinti [2].

expansion for $\sin^{-1}$, perform the indicated division by $(1-e^2)^{\frac{1}{2}}$, and then take the limit as $e \to 1$ we find that R_1 reduces to

$$R_1 = (\frac{p}{\mu})^{\frac{1}{2}} [(\frac{b_1}{p} + \frac{1}{2}) \tan\frac{f}{2} + \frac{1}{6} \tan^3\frac{f}{2} + A_1'f + A_{11}'\sin f + A_{12}'\sin 2f] \tag{9.70}$$

where

$$A_1' = \sum_{n=2}^{\infty} (\frac{b_2}{p})^n P_n(\lambda)\ R_{n-2}(0) \tag{9.71}$$

and A_{11}' and A_{12}' are identical to the coefficients derived previously and given in (9.38) and (9.39). Further investigation reveals the fact that $R_n(0) = T_n(0)$, so that (9.70) and (9.37) are identical. A comparison of (9.36) and (9.70) then shows the R_1 derived from Vinti's equations to be the same as the R_1 derived previously from the unbounded case.

From V(5.35) the R_2 integral may be expressed as

$$R_2 = (\frac{p}{\mu})^{\frac{1}{2}} (1-e^2)^{-\frac{1}{2}} [A_2 f + \sum_{j=1}^{4} A_{2_j} \sin jf] \tag{9.72}$$

which, upon dividing the coefficients by $(1-e^2)^{\frac{1}{2}}$ and then setting $e \to 1$, reduces immediately to equations (9.41) through (9.46) of the previous subsection. Thus the R_2 integrals as obtained from the limiting cases of the two solutions are identical.

The R_3 may be written from V(5.60) as

$$R_3 = (\frac{p}{\mu})^{\frac{1}{2}} (1-e^2)^{-\frac{1}{2}} [A_3 f + \sum_{j=1}^{4} A_{3_j} \sin jf] \tag{9.73}$$

The same manipulations used in reducing R_2 allow us to reduce R_3 to equations (9.48) through (9.52) of the previous subsection. Thus, in the limit of "parabolic" motion, the R_3 integrals are identical.

In finding the limiting values of the η integrals we first note that for $x^2 \equiv (1-e^2) = 0$ (3.42) and (3.43) become

$$q^2 \equiv (\eta_0/\eta_2)^2 = 0 = \eta_2^{-2} \tag{9.74}$$

Using the above relation along with (9.7) in Vinti's N_1, N_2, and N_3 integrals (V(6.38) through V(6.41) and V(6.64), V(6.65), and V(6.70)) permits us to reduce them to the same η integrals obtained previously ((9.54), (9.55), and (9.56)) from the unbounded case.

We have thus shown that, for the spheroidal problem, the solutions for bounded motion proposed by Vinti [2] and the unbounded motion given in the present paper reduce to the same "parabolic" solution, as given by (9.57), (9.58), and (9.59), in the limiting case $\alpha_1 = 0$.

CHAPTER 10

SUMMARY: ALGORITHM FOR SATELLITE POSITION AND VELOCITY

In this section we summarize the results of previous chapters by presenting an algorithm for calculating the position and velocity vectors of a satellite on an unbounded trajectory using the spheroidal method of solution. We assume that the constants μ, r_e, and J_2, which characterize the gravitating body, are given and, in addition, the spheroidal orbital elements a, e, I, β_1, β_2, and β_3 are known. This case is entirely general since we have seen in section 7.5 how one could calculate these elements from a set of initial conditions. Thus with μ, r_e, J_2, a, e, I, β_1, β_2, and β_3 known we compute once for each trajectory

$$c^2 = r_e^2 J_2 \tag{2.4}$$

$$\eta_0 = \sin I \tag{4.4}$$

$$p = a(e^2-1) > 0 \tag{3.32}$$

$$D = (ap+c^2)(ap+c^2\eta_0^2) + 4a^2c^2\eta_0^2 \tag{10.1}$$

$$N = (ap+c^2)(ap+c^2\eta_0^2) + 4a^2c^2 \tag{10.2}$$

$$A = -2ac^2(1-\eta_0^2)(ap+c^2\eta_0^2)/D \quad < 0 \tag{4.14}$$

$$B = c^2\eta_0^2 N/D \quad > 0 \tag{4.15}$$

$$\eta_{\bar{2}}^{2} = -\frac{c^2 D}{apN} < 0 \tag{4.20}$$

$$b_1 = -A/2 \tag{5.5}$$

$$b_2 = B^{\frac{1}{2}} > 0 \tag{5.6}$$

$$a_0 = a + A/2 = a - b_1 < a \tag{4.5}$$

$$p_0 = [c^2(1-\eta_0^2) + apN/D]/a_0 \tag{4.17}$$

$$\alpha_2 = (p_0\mu)^{\frac{1}{2}} \tag{3.7}$$

$$\alpha_3 = \alpha_2 \cos I(1 + \frac{c^2\eta_0^2}{a_0 p_0})^{\frac{1}{2}} \tag{4.18}$$

$$n^2 = -\eta_0^2\eta_{\bar{2}}^{-2} > 0 \tag{6.7}$$

$$\alpha_2' = (\alpha_2^2 - \alpha_3^2)^{\frac{1}{2}}/\eta_0 = \alpha_2[1 - \frac{c^2(1-\eta_0^2)}{a_0 p_0}]^{\frac{1}{2}} \tag{10.3}$$

$$\lambda = b_1/b_2 \tag{5.7}$$

The Legendre Polynomials are found from

$$P_0(\lambda) = 1 \tag{10.4}$$

$$P_1(\lambda) = \lambda \tag{10.5}$$

and for m>1

$$P_m(\lambda) = [(2m-1)\lambda P_{m-1}(\lambda) - (m-1)P_{m-2}(\lambda)]/m \tag{10.6}$$

$$x^2 = e^2 - 1$$

The polynomials $T_m(x)$ are found from

$$T_0(x) = 1 \tag{10.7}$$

$$T_1(x) = 1 \tag{10.8}$$

and for m>1

$$T_m(x) = [(2m-1)T_{m-1}(x) + (m-1)x^2T_{m-2}(x)]/m \tag{5.34}$$

$$A_1 = px \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda)T_{n-2}(x) \tag{5.39}$$

$$A_2 = \frac{x}{p} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n(\lambda)T_n(x) \tag{5.51}$$

$$D_0 = 1 \tag{10.9}$$

$$D_1 = b_1/p \tag{10.10}$$

for n>1

$$D_n = -\left(\frac{c}{p}\right)^2 D_{n-2} + \left(\frac{b_2}{p}\right)^n P_n(\lambda) \tag{5.94}$$

then

$$A_3 = \frac{x}{p^3} \sum_{m=0}^{\infty} D_m T_{m+2}(x) \tag{5.96}$$

$$B_1' = -B_1 = 1/2 - 3n^2/16 + 15n^4/128 \tag{6.48}$$

$$B_2 = 1 - n^2/4 + 9n^4/64 \tag{6.53}$$

$$\gamma_m = \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{\eta_0^{2n}(2n)!}{2^{2n}(n!)^2} \tag{6.83}$$

$$B_3 = 1 - (1-\eta_2^{-2})^{-\frac{1}{2}} - \sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m} \tag{6.82}$$

$$A_{11} = (3ex/4p^3)(-2b_1b_2^2p + b_2^4) \tag{5.34}$$

$$A_{12} = 3xb_2^4e^2/32p^3 \tag{5.44}$$

$$A_{21} = \frac{xe}{p}\left[\frac{b_1}{p} + \frac{3b_1^2-b_2^2}{p^2} - \frac{9b_1b_2^2}{2p^3}\left(1+\frac{e^2}{4}\right) + \frac{3b_2^4}{8p^4}(4+3e^2)\right] \tag{5.53}$$

$$A_{22} = \frac{xe^2}{p}\left[\frac{3b_1^2-b_2^2}{8p^2} - \frac{9b_1b_2^2}{8p^3} + \frac{3b_2^4}{16p^4}(3+e^2/2)\right] \tag{5.54}$$

$$A_{23} = \frac{xe^3}{8p}\left[\frac{-b_1b_2^2}{p^3} + \frac{b_2^4}{p^4}\right] \tag{5.55}$$

$$A_{24} = \frac{3xb_2^4e^4}{256p^5} \tag{5.56}$$

$$A_{31} = \frac{xe}{p^3}\left[2 + \frac{b_1}{p}\left(3+\frac{3e^2}{4}\right) - \left(\frac{b_2^2}{2p^2} + \frac{c^2}{p^2}\right)(4+3e^2)\right] \tag{5.98}$$

$$A_{32} = \frac{x}{p^3}\left[\frac{e^2}{4} + \frac{3e^2b_1}{4p} - \left(\frac{b_2^2}{2p^2} + \frac{c^2}{p^2}\right)\left(\frac{3e^2}{2} + \frac{e^4}{4}\right)\right] \tag{5.99}$$

$$A_{33} = \frac{x}{p^3}\left[\frac{b_1e^3}{12p} - \frac{e^3}{3p^2}\left(\frac{b_2^2}{2} + c^2\right)\right] \tag{5.100}$$

$$A_{34} = -\frac{e^4x}{32p^5}\left(\frac{b_2^2}{2} + c^2\right) \tag{5.101}$$

$$z_1 = (2\alpha_1)^{-\frac{1}{2}} = \left(\frac{a_0}{\mu}\right)^{\frac{1}{2}} \tag{10.11}$$

Then for each time t at which the satellite's position and velocity are desired we calculate

solve for H_0:

$$t + \beta_1 = z_1a(e \sinh H_0 - H_0) \tag{7.11}$$

then

$$f_0 = 2 \tan^{-1}\left[\left(\frac{e+1}{e-1}\right)^{\frac{1}{2}} \tanh (H_0/2)\right] \tag{7.13}$$

$$\psi_0 = \frac{\alpha_2'}{B_2} [\ \beta_2/\alpha_2 + z_1 A_2 f_0\ ] \quad (7.14)$$

$$H_1' = \frac{[(c^2 n_0^2/z_1\alpha_2')(-B_1'\psi_0 + \frac{1}{4}\sin 2\psi_0) - b_1 H_0 - A_1 f_0]}{a(e \cosh H_0 - 1)} \quad (7.22)$$

$$H_1 = H_1' [1 - H_1' e \sinh H_0/2(e \cosh H_0 - 1)] \quad (7.21)$$

$$f_1 = 2 \tan^{-1} [(\frac{e+1}{e-1})^{\frac{1}{2}} \tanh (\frac{H_0+H_1}{2})] - f_0 \quad (7.23)$$

$$\psi_1 = - \frac{n^2 \sin 2\psi_0}{8B_2} + \frac{z_1\alpha_2'}{B_2} [\ A_2 f_1 + A_{21} \sin f_0 + A_{22} \sin 2f_0] \quad (7.24)$$

$$H_2 = [\ -b_1 H_1 - A_1 f_1 - A_{11}\sin f_0 - A_{12}\sin 2f_0 + \frac{c^2 n_0^2}{z_1\alpha_2'} (\ -B_1'\psi_1 + \frac{1}{2}\psi_1\cos 2\psi_0 - \frac{n^2}{8}\sin 2\psi_0 + \frac{n^2}{64} \sin 4\psi_0)]/a(e \cosh H_0 - 1) \quad (7.29)$$

$$H = H_0 + H_1 + H_2 \quad (10.12)$$

$$f_2 = 2 \tan^{-1} [(\frac{e+1}{e-1})^{\frac{1}{2}} \tanh H/2] - f_0 - f_1 \quad (7.30)$$

$$f = f_0 + f_1 + f_2 \quad (10.13)$$

$$\psi_2 = B_2^{-1} (-\frac{n^2}{4} \psi_1\cos 2\psi_0 + \frac{3n^4}{32}\sin 2\psi_0 - \frac{3n^4}{256}\sin 4\psi_0) + \alpha_2' z_1 B_2^{-1} (\ A_2 f_2 + A_{21} f_1 \cos f_0 + A_{22} 2 f_1 \cos 2f_0 + A_{23}\sin 3f_0 + A_{24}\sin 4f_0) \quad (7.32)$$

$$\psi = \psi_0 + \psi_1 + \psi_2 \quad (10.14)$$

$$\rho = a(e \cosh H - 1) \quad (5.16)$$

$$\eta = \eta_0 \sin\psi \tag{6.4}$$

$$\Omega' = \beta_3 + \frac{\alpha_3}{\alpha_2'}\left(B_3\psi + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right) - c^2\alpha_3 z_1 \left(A_3 f + \sum_{j=1}^{4} A_{3_j}\sin jf \right) \tag{7.36}$$

The components of the position vector are given by

$$X = (\rho^2 + c^2)^{\frac{1}{2}} [\cos\psi\cos\Omega' - \cos I\sin\psi\sin\Omega'] \tag{7.47}$$

$$Y = (\rho^2 + c^2)^{\frac{1}{2}} [\cos I\sin\psi\cos\Omega' + \sin\Omega'\cos\psi] \tag{7.48}$$

$$Z = \rho\eta \tag{7.49}$$

The components of the velocity vector are found from

$$\dot{f} = \frac{ax}{\rho z_1} \frac{(\rho^2 + A\rho + B)^{\frac{1}{2}}}{(\rho^2 + c^2\eta^2)} \tag{7.71}$$

$$\dot{\rho} = \rho^2 e\dot{f}\sin f/p \tag{7.69}$$

$$\dot{\psi} = \alpha_2' (1 + n^2\sin^2\psi)^{\frac{1}{2}}/(\rho^2 + c^2\eta^2) \tag{7.67}$$

$$\dot{\Omega}' = \frac{\alpha_3}{\alpha_2'} (B_3 + \frac{3}{16}\eta_0^2\eta_2^{-4}\cos 2\psi)\dot{\psi} - c^2\alpha_3 z_1(A_3 + \sum_{j=1}^{4} jA_{3_j}\cos jf)\dot{f} \tag{7.68}$$

$$\dot{X} = \frac{\rho\dot{\rho}}{\rho^2+c^2}X - Y\dot{\Omega}' + (\rho^2+c^2)^{\frac{1}{2}}(-\sin\psi\cos\Omega' - \cos I\cos\psi\sin\Omega')\dot{\psi} \tag{7.53}$$

$$\dot{Y} = \frac{\rho\dot{\rho}}{\rho^2+c^2} Y + X\dot{\Omega}' + (\rho^2+c^2)^{\frac{1}{2}}(-\sin\psi\sin\Omega' + \cos I\cos\psi\cos\Omega')\dot{\psi} \tag{7.54}$$

$$\dot{Z} = \dot{\rho}\eta + \eta_0\rho\cos\psi\,\dot{\psi} \tag{7.55}$$

This completes the algorithm for determining, at any time t, the position and velocity vector of a satellite on an unbounded trajectory. The next chapter deals with a computer program which compares the results obtained from this algorithm to ordinary Keplerian hyperbolic motion.

CHAPTER 11

NUMERICAL COMPARISON OF THE ORBITS, WITH AND WITHOUT OBLATENESS, FOR GIVEN INITIAL CONDITIONS

In the previous chapter we summarized the algorithm that one would use to calculate the time history of the position and velocity vectors for a satellite on an unbounded trajectory given the oblateness parameter J_2 of the gravitating body and the spheroidal orbital elements, a, e, I, β_1, β_2, and β_3. Using the state vector calculated for t = 0, we can then rework the problem for $J_2 = 0$ by the application of simple conic formulae. We thus calculate the Keplerian orbital elements a_s, e_s, I_s, τ, ω, Ω* and the time history of the trajectory the satellite would traverse if the Earth were a perfect sphere. I will refer to this latter trajectory as the Keplerian or spherical trajectory. A flow chart for the process is shown in Figure 11.1

In this manner we can compare the trajectories for an oblate and a spherical planet for the same initial conditions and determine the magnitude of the deviations as functions of the energy, perigee distance, and inclination of the trajectory. At this point let us derive a set of Keplerian relations suitable for the numerical examples to be considered.

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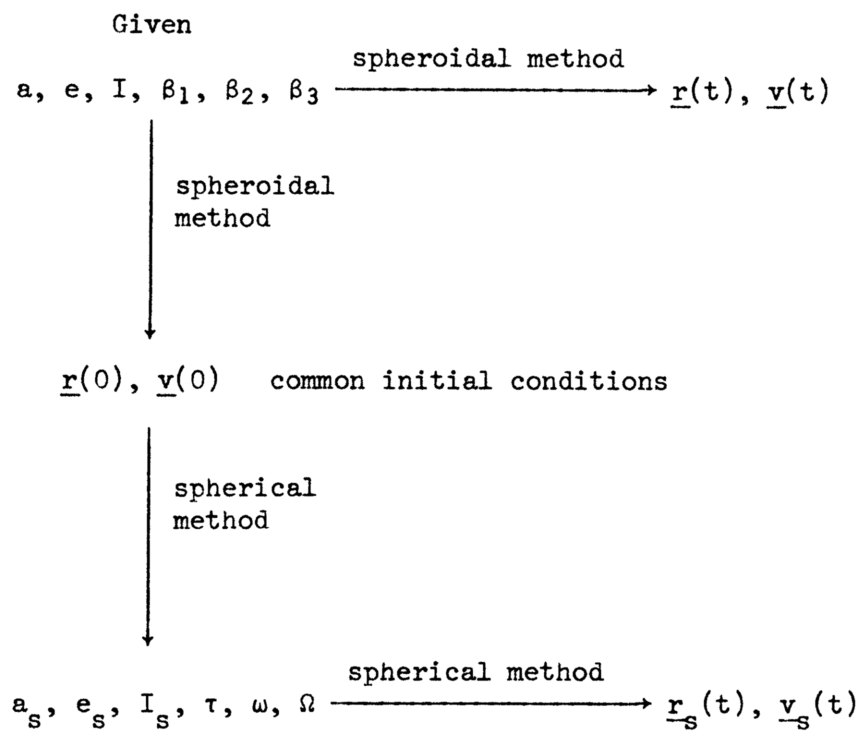
* τ = time of perigee passage

ω = argument of perigee

Ω = longitude of ascending node or, if there is only a descending node, longitude of descending node -180°.

FIGURE 11.1

TRAJECTORY COMPARISON FLOW CHART



11.1 The Keplerian Trajectory From an Initial State Vector

In this section we assume that $\underline{r}_0$ and $\underline{v}_0$, the initial position and velocity, vectors are given and should like to determine the spherical (i.e. $J_2 = 0$) orbital elements a_s, e_s, I_s, τ, ω, and Ω as well as the time history of the trajectory. Battin [8] has been used as a reference in the following derivations.

We can find the element a_s from the energy equation as

$$a_s = \left(\frac{v_0^2}{\mu} - \frac{2}{r_0}\right)^{-1} \tag{11.1}$$

which is greater than zero for hyperbolic motion. We then find the angular momentum from

$$\underline{h} = \underline{r}_0 \times \underline{v}_0 \tag{11.2}$$

which allows us to obtain the eccentricity from

$$e_s^2 = 1 + \frac{h^2}{\mu a_s} \tag{11.3}$$

We then solve for the inclination by dotting a unit vector in the $\underline{h}$ direction with a unit vector in the z direction.

$$I_s = \cos^{-1}\left(\frac{\underline{h}}{h} \cdot \underline{i}_z\right) \tag{11.4}$$

We then calculate the unit vector in the direction of perigee

$$\underline{i}_p = \frac{1}{e_s}\left(\frac{1}{a_s} + \frac{1}{r_0}\right)\underline{r}_0 - \frac{1}{\mu e_s}(\underline{r}_0 \cdot \underline{v}_0)\underline{v}_0 \tag{11.5}$$

and the unit vector along the line of nodes

$$\underline{i}_n = \frac{\underline{i}_z \times \underline{h}}{|\underline{i}_z \times \underline{h}|} \tag{11.6}$$

The longitude of the node and argument of perigee are then

$$\Omega = \cos^{-1}(\underline{i}_n \cdot \underline{i}_x) \tag{11.7}$$

$$\omega = \cos^{-1}(\underline{i}_n \cdot \underline{i}_p) \tag{11.8}$$

The initial value of the true anomaly is

$$\frac{f_{s0}}{2} = \frac{1}{2}\tan^{-1}\frac{\frac{h}{r_0}\,\underline{r}_0 \cdot \underline{v}_0}{\frac{h^2}{r_0} - \mu} \tag{11.9}$$

which allows us to calculate the initial value of the hyperbolic anomaly

$$H_{s0} = 2\tanh^{-1}\left[\left(\frac{e-1}{e+1}\right)^{\frac{1}{2}} \tan\frac{f_{s0}}{2}\right] \tag{11.10}$$

Since the initial time is $t_0 = 0$ we finally solve for the time of perigee passage from Kepler's equation

$$\tau = \left(\frac{a_s^3}{\mu}\right)(H_{s0} - e_s \sinh H_{s0}) \tag{11.11}$$

Having thus found the spherical orbital elements we then calculate the position and velocity vectors for each time t from the following set of equations. First, we obtain H_s as a solution of the Kepler equation

$$\left(\frac{\mu}{a_s^3}\right)^{\frac{1}{2}}(t-\tau) = e_s \sinh H_s - H_s \tag{11.12}$$

and then the position and velocity vectors can be found directly from Battin [8] as

$$\underline{r}_s(t) = \left[1 - \frac{a_s}{r_0}\left[\cosh(H_s - H_{s0}) - 1\right]\right]\underline{r}_0 + \left[t - \frac{\sinh(H_s - H_{s0}) - (H_s - H_{s0})}{\left(\frac{\mu}{a_s^3}\right)^{\frac{1}{2}}}\right]\underline{v}_0 \tag{11.13}$$

$$\underline{v}_s(t) = -\frac{(\mu a)^{\frac{1}{2}}}{r_s r_0} \sinh(H_s - H_{s0})\underline{r}_0$$

$$+ \left[1 - \frac{a_s}{\bar{r}_s}\left[\cosh(H_s - H_{0s}) - 1\right]\right]\underline{v}_0 \qquad (11.14)$$

This completes the algorithm for finding the spherical orbital elements and state vector given an initial position and velocity vector.

11.2 The Computer Program

In order to compare results derived from the spheroidal method with results of the ordinary Keplerian equations a computer program, embodying the flow chart of Figure 11.1, was written in Fortran IV computer language for use on the IBM 360 machine. The program, which is shown in Appendix E, accepts values of the oblateness parameter J_2 and the spheroidal orbital elements a, e, I, β_1, β_2, and β_3 as input. Using the spheroidal algorithm summarized in Chapter 10 it then computes the position and velocity vectors and the spheroidal true anomaly and spheroidal hyperbolic anomaly for the desired values of time. The number of time steps required as well as the amount of the time step are specified as inputs to the program. On the basis of the initial position and velocity vectors thus calculated the program then determines the Keplerian orbital elements and the spherical position and velocity vectors for the same values of time using the algorithm derived in the previous section. Notice that the spherical algorithm is invalid for inclinations close to 0 or 180° since the orbital elements ω and Ω lose their meaning. The spheroidal portion of the program, however, is valid for all inclinations.

It is appropriate at this time to explain certain details of the actual numerical computation performed by the program. The physical units utilized are those of the canonical Vanguard system. In this system the unit of length is the equatorial radius of the Earth (taken to be 6378.388 kilometers) and the gravitational constant G is taken to be unity. The value of the Vanguard unit of time is found to be 806.832 seconds and μ, the product of the gravitational constant and the Earth's mass, is unity.

For the purpose of this thesis arithmetic calculation was performed in the single-precision mode, since higher numerical accuracy was unwarranted to show the effects of oblateness (to be justified in the following section). Only minor program changes are required to utilize the double-precision mode and thus obtain more accurate results.

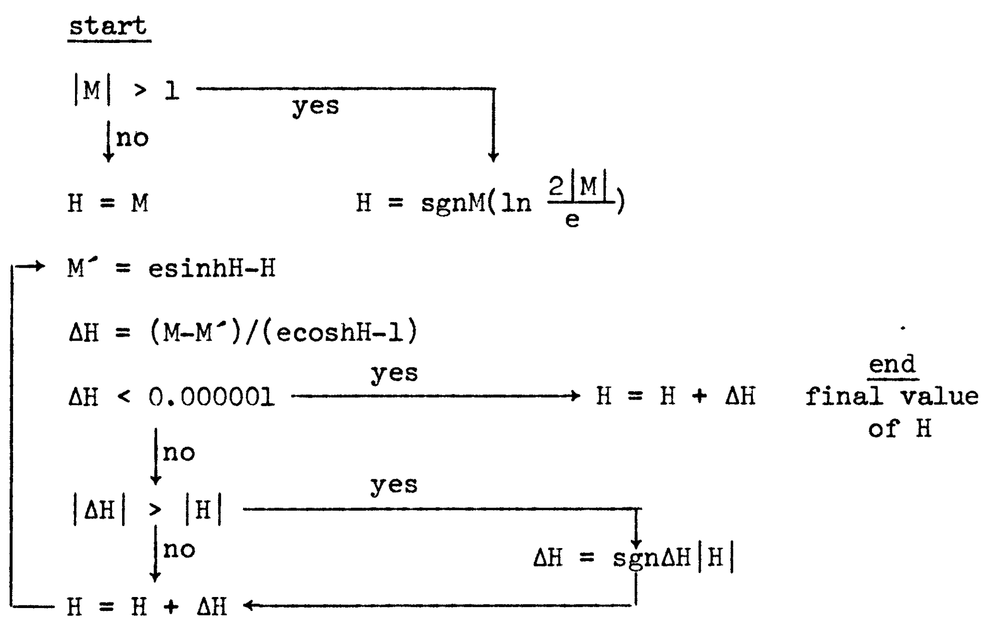
Both the spheroidal algorithm and the spherical algorithm involved the solution of a Kepler-type equation. The iterative sequence found in statements 141 ff. and 242 ff. of the program in Appendix E was quite sufficient for my purposes. A flow chart of this routine is shown in Figure 11.2. M represents the "mean anomaly" and H the "hyperbolic anomaly."

Using the method depicted in Figure 11.2 and single-precision values for the cosh and sinh functions ΔH was found to converge quite rapidly (within ten cycles for all cases tested) to a limiting value well below the required 10^{-6}. Once ΔH had reached this limiting value further iterations could not decrease it. Thus the program allows 20 cycles for ΔH to become less that 10^{-6}, after which it prints out a warning and proceeds with the calculations using the most recent value for H. The twenty cycle limit was never exceeded in any of the many trajectory calculations performed by the author.

The calculation of the spheroidal coefficients A_1, A_2, and A_3 involving convergent infinite series was also done by an iterative procedure. As soon as the ratio of the nth term of the series to the sum of the

FIGURE 11.2

FLOW CHART FOR SOLUTION OF KEPLER-TYPE EQUATION



previous n-1 terms is less than 10^{-7} we consider the calculation complete. Convergence of the series is normally so rapid that if the criterion has not been met by the 10th term, the program prints out a warning and proceeds with the calculation using the most recent value of the coefficient.

11.3 Results of the Computer Comparison

The following six pages are samples of computer output for several unbounded trajectories. The value of J_2 used is that of the Earth and R and V are the magnitudes of the position and velocity vectors, respectively. The subscripts X, Y, and Z indicate components of the vectors along the X, Y, and Z axes. The H and F are the hyperbolic and true anomalies, respectively. An asterisk following an H or F indicates an anomaly is defined by spheroidal coordinates, otherwise the spheroidal and spherical results may be compared directly.

Since the deviations between the spheroidal and spherical trajectories are considerable, it would appear that the single-precision mode of computation is justified for the accuracy required in this analysis. One might well ask, however, just how much of the observed deviation is due to computer roundoff or truncation error in the two different schemes of computation. In order to answer this question the program was run for a number of different trajectories (including the six example trajectories of the text) with the oblateness parameter J_2 set equal to zero. If all calculations were performed perfectly, i.e. no roundoff or truncation error, we would expect the spheroidal and spherical trajectories to be identical. The deviations observed in this case, which can be taken as an indication of roundoff and truncation error, in no case exceeded 3 percent of their value for the J_2 or Earth and were, in fact, of the opposite sign. Thus the deviation of the spheroidal trajectory

TRAJECTORY NUMBER 1

GIVEN J2= 0.00108228
A=1.1000 E=2.0000 I=0.3490
BETA1=0.0000 BETA2=0.5240 BETA3=0.5240

SPHEROIDAL SOLUTION

TIME	RX VX	RY VY	RZ VZ	R V	H* F*
0.0	0.566089	0.924758	0.188184	1.100475	-0.000003
	-1.387759	0.749889	0.489112	1.651495	-0.000005
3.0	-3.395325	1.513356	1.094563	3.875118	1.456193
	-1.173351	0.019983	0.219765	1.193920	1.645056
6.0	-6.738613	1.509605	1.701618	7.112193	1.991857
	-1.074125	-0.012813	0.191375	1.091115	1.842113
9.0	-9.895065	1.458208	2.259665	10.254010	2.324736
	-1.034828	-0.020096	0.181930	1.050890	1.916740
12.0	-12.964940	1.393302	2.797706	13.336350	2.568580
	-1.013591	-0.022801	0.177214	1.029218	1.956570

SPHERICAL SOLUTION

THE SPHERICAL ELEMENTS ARE
AS= 0.10988500E 01ES= 0.20014740E 01 IS= 0.34903580E 00
OMGA= 0.52437630E 00 W= 0.52368600E 00TAU= 0.47857000E-04

TIME	RX VX	RY VY	RZ VZ	R V	H F
0.0	0.566089	0.924758	0.188184	1.100475	-0.000041
	-1.387759	0.749889	0.489112	1.651495	-0.000072
3.0	-3.395430	1.514508	1.095820	3.876016	1.456511
	-1.173496	0.020446	0.220269	1.194164	1.644723
6.0	-6.739287	1.512132	1.704369	7.114026	1.992273
	-1.074340	-0.012361	0.191866	1.091407	1.841753
9.0	-9.896434	1.462075	2.263873	10.256800	2.325204
	-1.035069	-0.019653	0.182413	1.051203	1.916367
12.0	-12.967060	1.398487	2.803353	13.340140	2.569081
	-1.013847	-0.022364	0.177692	1.029543	1.956187

TRAJECTORY NUMBER 2

GIVEN J2= 0.00108228
A=1.1000 E=2.0000 I=0.7850
BETA1=0.0000 BETA2=0.5240 BETA3=0.5240

SPHEROIDAL SOLUTION

TIME	RX VX	RY VY	RZ VZ	R V	H* F*
0.0	0.630098 -1.221494	0.814009 0.462048	0.388986 1.010767	1.100429 1.651418	-0.000012 -0.000020
3.0	-3.022916 -1.098582	0.869859 -0.109111	2.262918 0.454493	3.874980 1.193880	1.456170 1.645044
6.0	-6.159676 -1.009013	0.509516 -0.125223	3.518383 0.395792	7.111976 1.091072	1.991831 1.842107
9.0	-9.126260 -0.972926	0.130345 -0.126955	4.672523 0.376263	10.253680 1.050845	2.324707 1.916736
12.0	-12.013070 -0.953292	-0.250594 -0.126889	5.785276 0.366510	13.335890 1.029172	2.568548 1.956566

SPHERICAL SOLUTION

THE SPHERICAL ELEMENTS ARE
AS= 0.10992500E 01ES= 0.20010660E 01 IS= 0.78505620E 00
OMGA= 0.52428190E 00 W= 0.52398870E 00TAU= 0.20632300E-03

TIME	RX VX	RY VY	RZ VZ	R V	H F
0.0	0.630098 -1.221494	0.814009 0.462048	0.388986 1.010767	1.100429 1.651418	-0.000179 -0.000310
3.0	-3.022852 -1.098558	0.869202 -0.109358	2.264124 0.454950	3.875486 1.194055	1.456303 1.644755
6.0	-6.159600 -1.009025	0.508173 -0.125444	3.520961 0.396245	7.113091 1.091272	1.992039 1.841820
9.0	-9.126246 -0.972956	0.128342 -0.127169	4.676439 0.376709	10.255430 1.051059	2.324953 1.916447
12.0	-12.013170 -0.953333	-0.253214 -0.127097	5.790545 0.366954	13.338310 1.029394	2.568818 1.956275

TRAJECTORY NUMBER 3

GIVEN J2= 0.00108228
A=1.1000 E=2.0000 I=1.2220
BETA1=0.0000 BETA2=0.5240 BETA3=0.5240

SPHEROIDAL SOLUTION

TIME	RX VX	RY VY	RZ VZ	R V	H* F*
0.0	0.730835 -0.959983	0.639702 0.009388	0.517206 1.343605	1.100382 1.651341	-0.000021 -0.000036
3.0	-2.436705 -0.980796	-0.143623 -0.312648	3.009354 0.604602	3.874837 1.193839	1.456146 1.645032
6.0	-5.248076 -0.906431	-1.066208 -0.302474	4.679511 0.526534	7.111744 1.091029	1.991803 1.842099
9.0	-7.915515 -0.875401	-1.962250 -0.295460	6.214892 0.500555	10.253320 1.050798	2.324676 1.916729
12.0	-10.513910 -0.858291	-2.841522 -0.291026	7.695239 0.487581	13.335410 1.029122	2.568515 1.956561

SPHERICAL SOLUTION

THE SPHERICAL ELEMENTS ARE
AS= 0.10996510E 01ES= 0.20006610E 01 IS= 0.12220350E 01
OMGA= 0.52413610E 00 W= 0.52429810E 00TAU= 0.36519970E-03

TIME	RX VX	RY VY	RZ VZ	R V	H F
0.0	0.730835 -0.959983	0.639702 0.009388	0.517206 1.343605	1.100382 1.651341	-0.000316 -0.000548
3.0	-2.437115 -0.980871	-0.144968 -0.313140	3.009115 0.604440	3.874960 1.193947	1.456094 1.644787
6.0	-5.248691 -0.906495	-1.068955 -0.302927	4.678831 0.526393	7.112162 1.091140	1.991803 1.841887
9.0	-7.916330 -0.875468	-1.966333 -0.295898	6.213802 0.500424	10.254070 1.050915	2.324700 1.916527
12.0	-10.514920 -0.858361	-2.846901 -0.291456	7.693762 0.487456	13.336500 1.029243	2.568555 1.956363

TRAJECTORY NUMBER 4

GIVEN J2= 0.00108228
A=1.0500 E=2.0000 I=0.7850
BETA1=0.0000 BETA2=0.5240 BETA3=0.5240

SPHEROIDAL SOLUTION

TIME	RX / VX	RY / VY	RZ / VZ	R / V	H* / F*
0.0	0.601483	0.777045	0.371301	1.050449	-0.000013
	-1.250238	0.472894	1.034563	1.690279	-0.000022
3.0	-3.111749	0.807484	2.253247	3.925828	1.507944
	-1.113500	-0.114662	0.457108	1.209121	1.669758
6.0	-6.295903	0.434515	3.521371	7.226840	2.048168
	-1.025612	-0.128701	0.401046	1.108728	1.856736
9.0	-9.313935	0.045717	4.692794	10.429460	2.383293
	-0.990537	-0.129982	0.382412	1.069718	1.927282
12.0	-12.254580	-0.344045	5.824722	13.572780	2.628592
	-0.971516	-0.129762	0.373100	1.048754	1.964874

SPHERICAL SOLUTION

THE SPHERICAL ELEMENTS ARE
AS= 0.10492100E 01ES= 0.20011770E 01 IS= 0.78506150E 00
OMGA= 0.52430930E 00 W= 0.52399180E 00TAU= 0.21146880E-03

TIME	RX / VX	RY / VY	RZ / VZ	R / V	H / F
0.0	0.601483	0.777045	0.371301	1.050449	-0.000197
	-1.250238	0.472894	1.034563	1.690279	-0.000340
3.0	-3.111678	0.806755	2.254622	3.926412	1.508100
	-1.113479	-0.114930	0.457627	1.209324	1.669440
6.0	-6.295843	0.433022	3.524283	7.228118	2.048406
	-1.025631	-0.128944	0.401555	1.108959	1.856420
9.0	-9.313970	0.043499	4.697219	10.431470	2.383573
	-0.990576	-0.130217	0.382916	1.069963	1.926964
12.0	-12.254740	-0.346939	5.830664	13.575550	2.628898
	-0.971567	-0.129991	0.373602	1.049007	1.964554

TRAJECTORY NUMBER 5

GIVEN J2= 0.00108228
A=2.2000 E=1.5000 I=0.7850
BETA1=0.0000 BETA2=0.5240 BETA3=0.5240

SPHEROIDAL SOLUTION

TIME	RX VX	RY VY	RZ VZ	R V	H* F*
0.0	0.630104 -1.115030	0.814000 0.421767	0.388996 0.922701	1.100428 1.507500	-0.000009 -0.000020
6.0	-5.343512 -0.831162	-0.025309 -0.223529	2.647197 0.221752	5.963336 0.888802	1.555257 1.938237
12.0	-10.029410 -0.746901	-1.350449 -0.217564	3.841183 0.184822	10.824390 0.799597	2.049596 2.091274
18.0	-14.399210 -0.713591	-2.638868 -0.212266	4.909046 0.172768	15.440190 0.764276	2.360538 2.150681
24.0	-18.621550 -0.695401	-3.901099 -0.208695	5.925914 0.166772	19.927270 0.744949	2.590429 2.182991

SPHERICAL SOLUTION

THE SPHERICAL ELEMENTS ARE
AS= 0.21974100E 01ES= 0.15007840E 01 IS= 0.78507830E 00
OMGA= 0.52427880E 00 W= 0.52402680E 00TAU= 0.24921790E-03

TIME	RX VX	RY VY	RZ VZ	R V	H F
0.0	0.630104 -1.115030	0.814000 0.421767	0.388996 0.922701	1.100428 1.507500	-0.000153 -0.000341
6.0	-5.343531 -0.831221	-0.026653 -0.223736	2.650151 0.222274	5.964672 0.889040	1.555812 1.937881
12.0	-10.029990 -0.747020	-1.352983 -0.217755	3.847201 0.185325	10.827380 0.799876	2.050305 2.090891
18.0	-14.400590 -0.713737	-2.642560 -0.212454	4.918029 0.173260	15.444970 0.764576	2.361329 2.150282
24.0	-18.623870 -0.695562	-3.905912 -0.208882	5.937845 0.167259	19.933940 0.745261	2.591276 2.182581

TRAJECTORY NUMBER 6

GIVEN J2= 0.00108228
A=0.5500 E=3.0000 I=0.7850
BETA1=0.0000 BETA2=0.5240 BETA3=0.5240

SPHEROIDAL SOLUTION

TIME	RX VX	RY VY	RZ VZ	R V	H* F*
0.0	0.630090 -1.410536	0.814021 0.533570	0.388974 1.167149	1.100429 1.906972	-0.000014 -0.000020
3.0	-3.596646 -1.321800	1.244234 0.044549	2.873891 0.699152	4.768986 1.495978	1.838666 1.596584
6.0	-7.463285 -1.266719	1.350074 0.030250	4.897921 0.659249	9.028451 1.428320	2.444382 1.742521
9.0	-11.228040 -1.245694	1.435085 0.027011	6.853011 0.645939	13.232240 1.403466	2.812149 1.795166
12.0	-14.946950 -1.234573	1.513955 0.025737	8.779881 0.639280	17.400840 1.390507	3.077891 1.822491

SPHERICAL SOLUTION

THE SPHERICAL ELEMENTS ARE
AS= 0.54973170E 00ES= 0.30017540E 01 IS= 0.78502850E 00
OMGA= 0.52428810E 00 W= 0.52396020E 00TAU= 0.16015890E-03

TIME	RX VX	RY VY	RZ VZ	R V	H F
0.0	0.630090 -1.410536	0.814021 0.533570	0.388974 1.167149	1.100429 1.906972	-0.000196 -0.000278
3.0	-3.596594 -1.321776	1.243654 0.044329	2.874928 0.699535	4.769419 1.496130	1.838610 1.596347
6.0	-7.463190 -1.266710	1.348841 0.030039	4.900095 0.659629	9.029368 1.428483	2.444357 1.742301
9.0	-11.227930 -1.245693	1.433243 0.026806	6.856337 0.646319	13.233670 1.403637	2.812138 1.794950
12.0	-14.946850 -1.234576	1.511510 0.025535	8.784347 0.639658	17.402800 1.390679	3.077888 1.822278

from the Keplerian trajectory as calculated by the program is conservative and can be assumed to be accurate to about 3 parts in 100. For more critical investigations higher accuracy can be obtained, of course, by using higher precision computation.

Let us now turn our attention to the magnitude of the deviation caused by the oblateness as a function of the orbital geometry. Since we have assumed the Earth to be axially symmetric we expect the amount of deviation caused by J_2 to depend most heavily on the inclination, energy, and ρ-perigee distance of the trajectory. The closer the inclination to 0 or 180^o the larger should be the effect of the oblateness. Low energy trajectories, since they spend more time close to the oblate planet, should be affected more than high energy trajectories. Similarly, a trajectory with a low ρ-perigee (recall ($\rho_1 \equiv a(e-1)$)), should be affected more than a trajectory which does not pass so close to the planet (other factors being equal). To test these contentions the twelve trajectories shown in Table 11.1 (six of which were shown earlier as examples) were used as input to the computer program. For all twelve cases β_1, β_2, and β_3 were taken to be 0, 0.542, and 0.542 respectively. Figures 11.3 through 11.6 show the position and velocity deviations from the spherical trajectory as functions of time for these twelve cases. Position deviation was calculated as $r_{spherical} - r_{spheroidal}$ and velocity deviation as $v_{spherical} - v_{spheroidal}$. Note that the velocity deviation curves approach an asymptote as the satellite approaches asymptotic motion.

We first note that within each of the four trajectory classifications increasing the inclination decreases the amount of the deviations. Comparing trajectory 1A to 3A, 1B to 3B, and 1C to 3C it is clear that for the same inclination and ρ-perigee decreasing the energy of the orbit

TABLE 11.1

TRAJECTORIES USED TO COMPARE THE EFFECTS OF J_2

Number	a	e	I	Classification
1A	1.1	2.0	0.349	
1B	"	"	0.785	"standard" trajectory
1C	"	"	1.222	
2A	1.05	2.0	0.349	
2B	"	"	0.785	lower ρ-perigee, but higher energy
2C	"	"	1.222	
3A	2.2	1.5	0.349	
3B	"	"	0.785	lower energy, same ρ-perigee
3C	"	"	1.222	
4A	.55	3.0	0.349	
4B	"	"	0.785	higher energy, same ρ-perigee
4C	"	"	1.222	

FIGURE 11.3 Position Deviation from a Keplerian Hyperbolic Trajectory

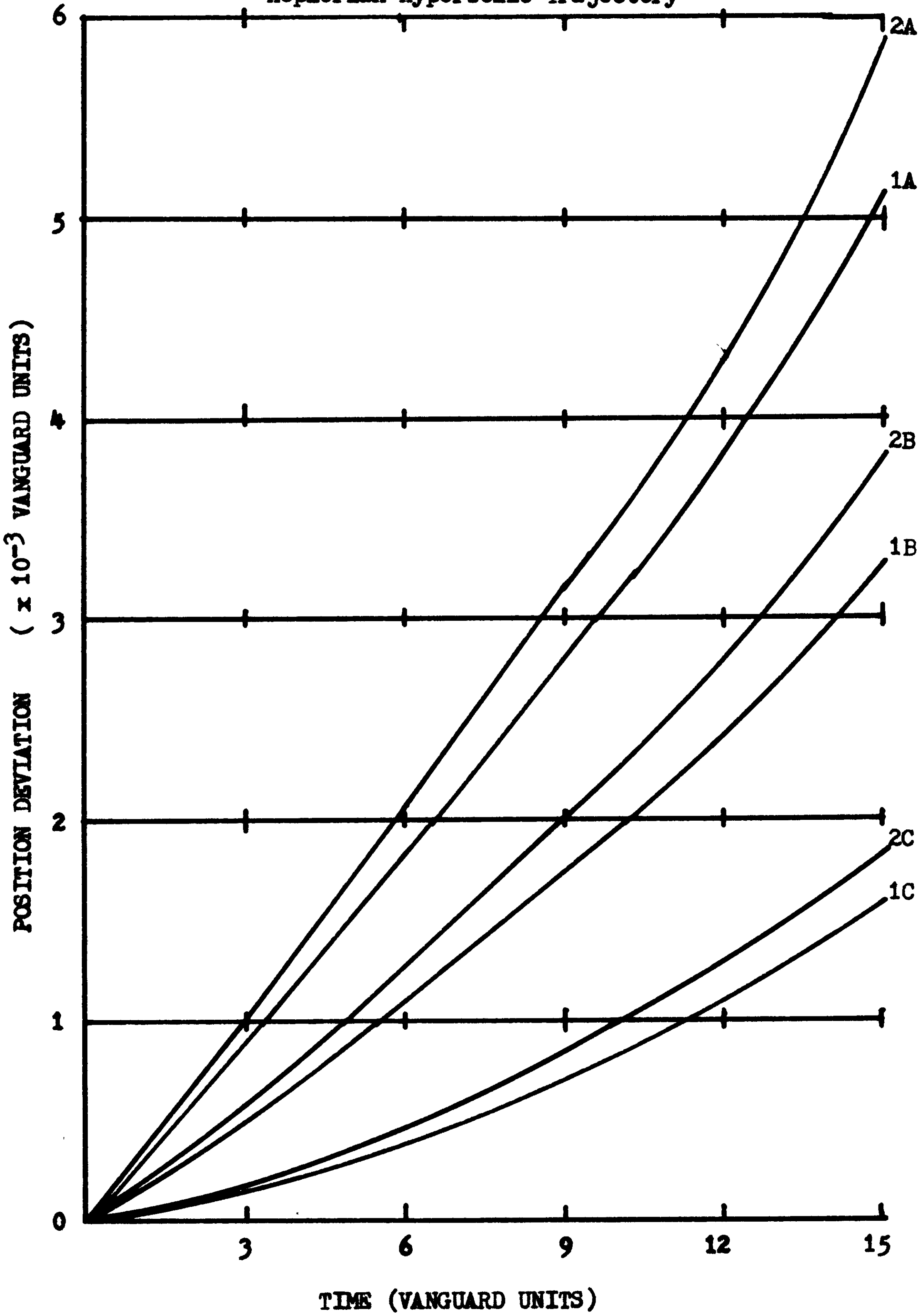


FIGURE 11.4 Position Deviation from a

Keplerian Hyperbolic Trajectory

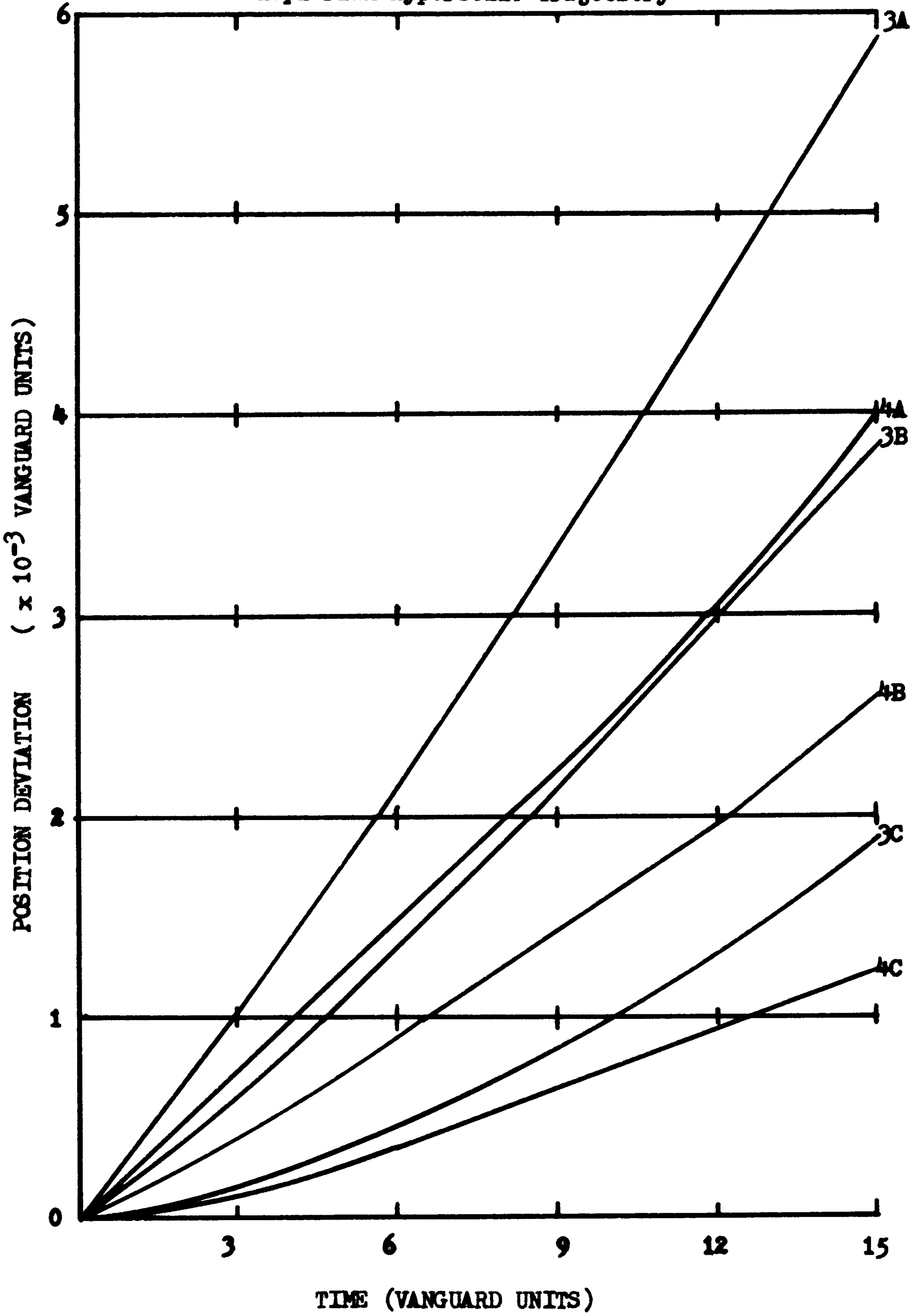


FIGURE 11.5 Velocity Deviation from a Keplerian Hyperbolic Trajectory

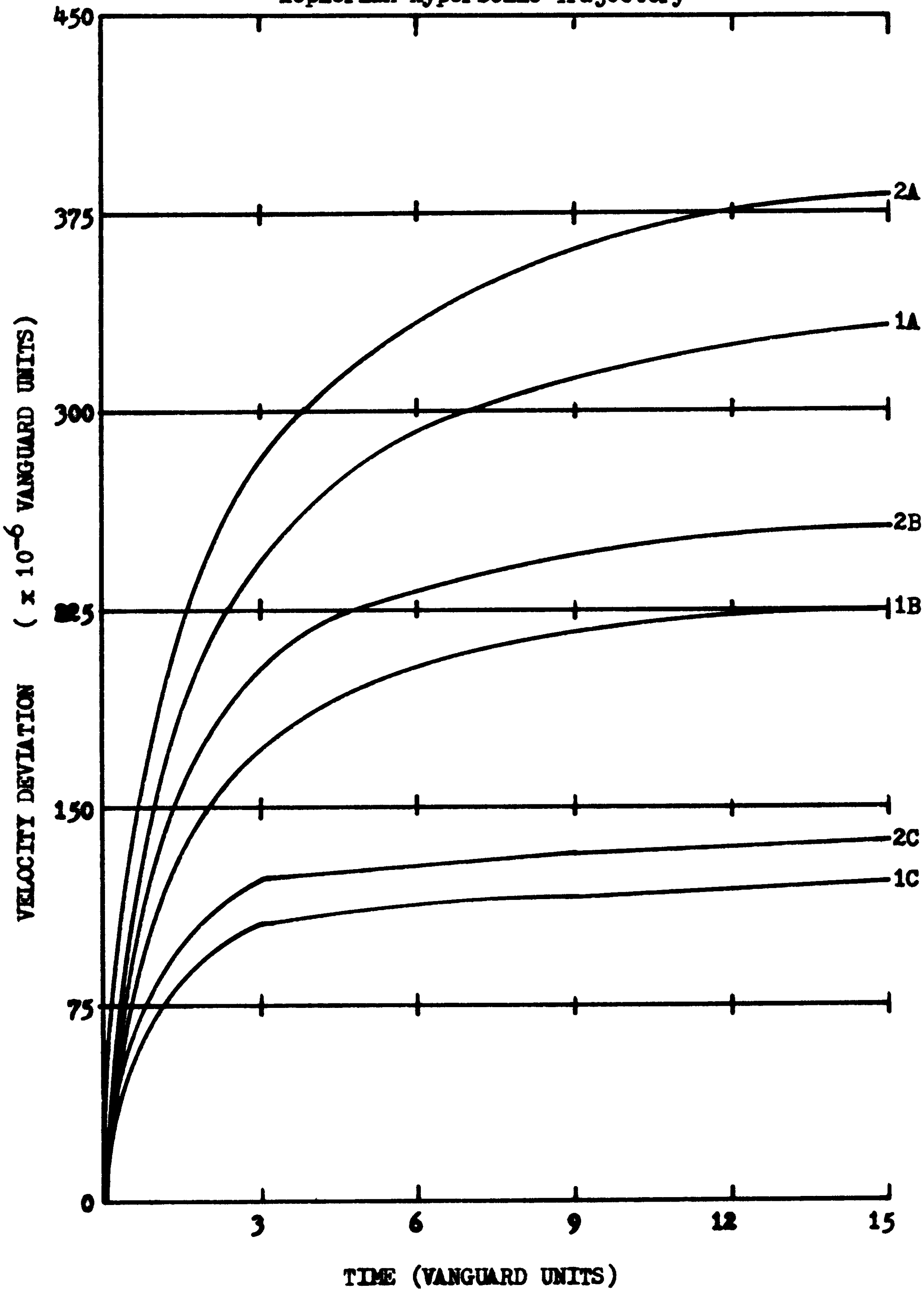
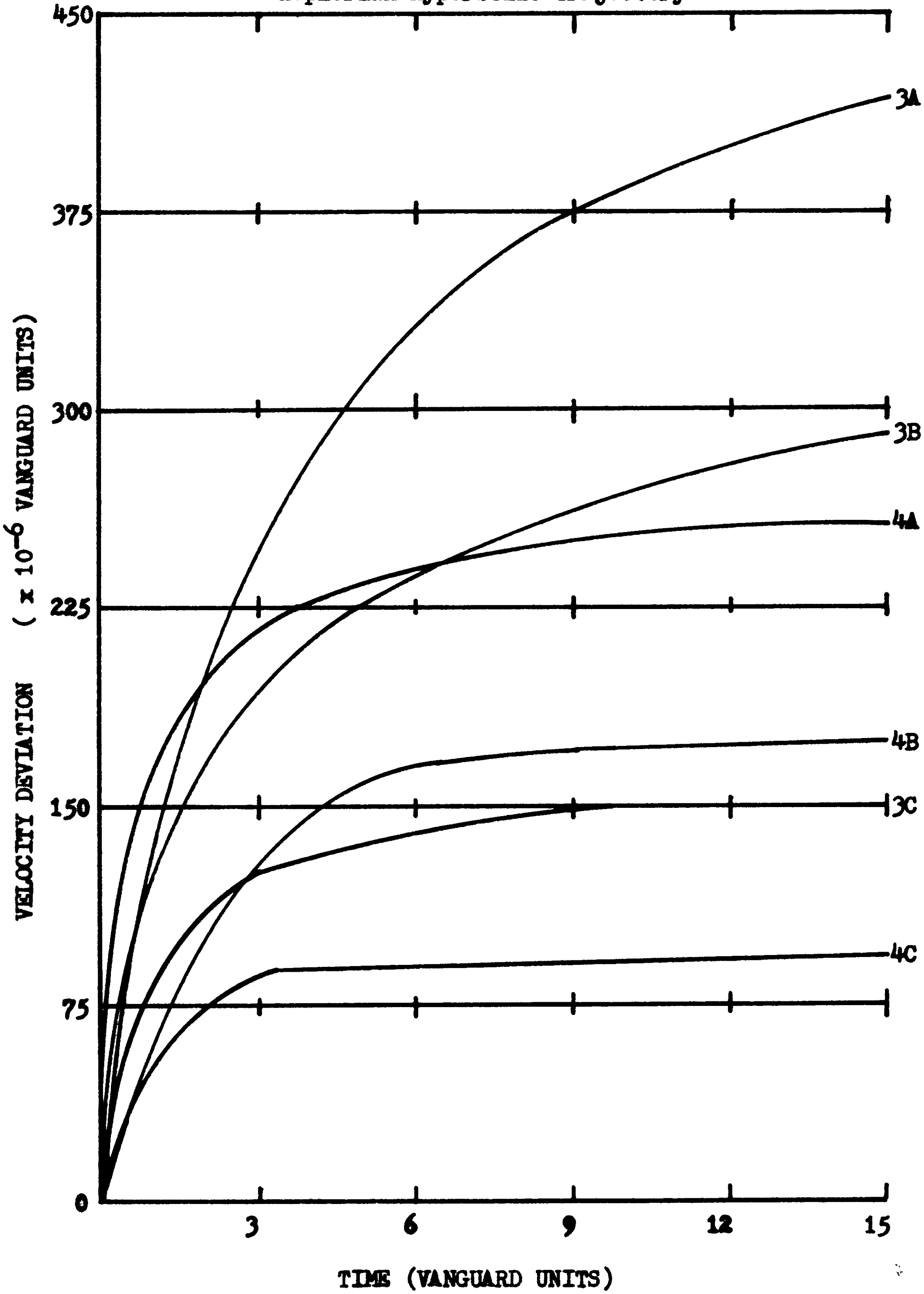


FIGURE 11.6 Velocity Deviation from a

Keplerian Hyperbolic Trajectory



increases the deviations. Conversely comparing 1A to 4A, 1B to 4B, and 1C to 4C, we find that the higher energy orbit has smaller deviations. To determine the effect of ρ-perigee distance on the amount of deviation we compare 1A to 2A, 1B to 2B, and 1C to 2C. Note that even though the class 2 trajectories have a slightly higher energy, their lower ρ-perigee distances cause them to have higher deviations than the corresponding "standard" trajectories. We have thus numerically confirmed the following three points

1. Increasing the inclination (toward 90 degrees) decreases the effect of J_2.

2. Increasing the energy of the trajectory decreases the effect of J_2.

3. Increasing the ρ-perigee distance decreases the effect of J_2.

Several trajectories with perigees well within the Earth were put on the computer to test the convergence of the involved series. According to the investigation conducted at the end of Chapter 3 we expect difficulty with the series as ρ-perigee, ρ_1, approaches c ($c \approx 0.0325$ for Earth). Indeed, when ρ_1 was decreased to 0.1 the coefficients A_1 and A_3 failed to converge rapidly enough to satisfy the criterion of the program. As expected, the effect of J_2 on these low-perigee trajectories was quite marked. For $\rho_1 = 0.3$ (where all the series still converged) differences between the spherical and spheroidal results in the second significant figure were quite common.

CHAPTER 12

CONCLUSION

12.1 Applications and Advantages of the Spheroidal Solution for Unbounded Orbits

We have seen that the spheroidal method provides a means of calculating accurately the trajectory of an unbounded satellite for cases where the oblateness of the gravitating body is the most important force, other than the normal inverse-square force, acting on the satellite. In cases where forces arising from atmospheric drag, electromagnetic fields, or other bodies must be accounted for, the spheroidal method yields an accurate reference orbit upon which perturbation techniques may be used. Actual computation with the spheroidal method is rapid and no difficulties are encountered with poles or inclinations.

The scope of problems to which the spheroidal method may be applied is, however, limited by the convergence of the series involved. For trajectories whose extensions pass too close to the center of the Earth, i.e. within the small limiting surface derived in Chapter 3, we cannot be certain that the series used in the solution will converge. Because of the rather small size of this zone and the physical rarity of unbounded trajectories whose extensions might pass through it, the applicability of the spheroidal method is not significantly limited.

Cases in which the spheroidal solution for unbounded orbits could be fruitfully applied are quite numerous. Some examples are listed below.

1. Calculating escape trajectories for orbiting space vehicles.
2. Determining the paths of incoming meteors.
3. Finding the required incoming asymptote to achieve a desired outgoing asymptote (or vice-versa) on a swingby of an oblate planet.
4. Calculating incoming trajectories to an oblate destination planet.

12.2 Areas for Future Research

There are several areas of spheroidal theory toward which future research effort should be devoted. One of these is the comparison of the effects of J_2 predicted by the present solution for unbounded motion to effects predicted by other methods of analysis. Hori [9] has developed a solution using the Von Ziepel approach which includes the first order perturbation of the oblateness of the Earth on a satellite in hyperbolic motion. Sauer [10] uses the variation of orbital elements approach to obtain expressions for the perturbation of each orbital element in terms of the values of other elements and J_2. The differences in the forms of the final results as obtained by each of the three methods make analytical comparisons between them difficult. For this reason the application of the three methods to a set of numerical data is perhaps the easiest way to compare them.

The solution for unbounded orbits presented in the present paper is for a gravitational potential which includes all of the second zonal harmonic and more than half of the fourth zonal harmonic. Vinti [11, 12], however, has developed a form for the potential in spheroidal coordinates which allows for the inclusion of the third zonal harmonic J_3 as well and has extended his spheroidal solution for bounded orbits accordingly. A similar extension for the present unbounded case would be valuable to spheroidal theory.

In addition to the research indicated above some effort should be applied toward combining Vinti's spheroidal solution for bounded orbits [2] with the present solution for unbounded orbits. The determination of such a universally applicable solution should be facilitated by the many similarities of the two cases and the work done in Chapter 9. Here it was shown that solutions for both the unbounded and bounded cases reduced to the same "parabolic" trajectory.

APPENDIX A

SERIES SOLUTION FOR A, B, $\rho_1+\rho_2$, $\rho_1\rho_2$

From equations 3.12 through 3.18 we have

$$\rho_1 + \rho_2 - A = -2a_o \tag{A.1}$$

$$B + \rho_1\rho_2 - (\rho_1+\rho_2)A = c^2 - a_o p_o = k_o p_o^2 - a_o p_o \tag{A.2}$$

$$(\rho_1+\rho_2)B - \rho_1\rho_2 A = -2a_o c^2 = -2a_o p_o^2 k_o \tag{A.3}$$

$$\rho_1\rho_2 B = -a_o p_o c^2 \sin^2 i_o = -a_o p_o^3 k_o (1-y^2) \tag{A.4}$$

We then assume series solutions in the form of (3.19) through (3.22)

$$\rho_1 + \rho_2 = \sum_{n=0}^{\infty} b_{1_n} k_o^n \tag{3.19}$$

$$\rho_1\rho_2 = \sum_{n=0}^{\infty} b_{2_n} k_o^n \tag{3.20}$$

$$A = \sum_{n=0}^{\infty} b_{3_n} k_o^n \tag{3.21}$$

$$B = \sum_{n=0}^{\infty} b_{4_n} k_o^n \tag{3.22}$$

Inserting (3.19) through (3.22) into (A.1) through (A.4) and equating coefficients of like powers of k_o we find

$$b_{3_o} = b_{1_o} + 2a_o$$

$$b_{1_n} = b_{3_n} \quad \text{for } n>0$$

$$b_{2_o} b_{4_o} = 0$$

$$b_{21} b_{4_o} + b_{2_o} b_4 = -a_o p_o^3 (1-y^2)$$

$$(b_2 b_4)_n = 0 \quad \text{for } n>1 \qquad (A.5)$$

$$b_{2_o} - b_{1_o}(b_{1_o} - 2a_o) = -a_o p_o$$

$$b_{41} + b_{21} - b_{1_o} b_{31} - b_{11} b_{3_o} = p_o^2$$

$$b_{4_n} + b_{2_n} - (b_1 b_3)_n = 0$$

$$(b_1 b_4)_n - (b_2 b_3)_n = 0 \quad \text{for } n>1$$

$$b_{1_o} b_{4_o} - b_{2_o} b_{3_o} = 0$$

$$b_{1_o} b_{41} + b_{11} b_{4_o} - b_{2_o} b_{31} - b_{21} b_{3_o} = -2a_o p_o^2$$

Since for the Keplerian case (c=0) we know A=B=0, we choose $b_{3_o}=b_{4_o}=0$ in the present case. Solution of the equations (A.5) is then straight forward and yields (in the order in which they may be determined)

$$b_{3_o} = b_{4_o} = 0$$

$$b_{1_o} = -2a_o$$

$$b_{2_o} = -a_o p_o$$

$$b_{41} = p_o^2 (1-y^2)$$

$$b_{31} = -2p_o y^2$$

$$b_{21} = a_o p_o y^2 (4+x^2) \qquad (A.6)$$

$$b_{11} = -2p_0y^2$$

$$b_{42} = x^2(1-y^2)(4+x^2)\ a_0p_0y^2$$

$$b_{32} = 2p_0(2x^2y^2-3x^2y^4-8y^4+4y^2)$$

$$b_{12} = b_{32}$$

$$b_{22} = -4a_0p_0(3x^2y^2-5x^2y^4-8y^4+4y^2+\frac{1}{4}x^4y^2-\frac{1}{4}x^4y^4)$$

Equations (3.23) through (3.26) of the text then follow.

APPENDIX B

RANGE OF VALIDITY OF LEGENDRE EXPANSION

According to Hobson [4] we may use

$$(1 - 2\lambda h + h^2)^{-\frac{1}{2}} = \sum_{h=0}^{\infty} h^2 Pn(\lambda) \qquad (B.1)$$

as long as $h \leq 1$ and $\lambda \leq 1$, or for $\lambda > 1$ if

$$h < \lambda - (\lambda^2-1)^{\frac{1}{2}} \qquad (B.2)$$

By the definitions of b_1 and b_2 and equations (4.22) and (4.23) we see that to first order

$$b_1 \simeq kp \cos^2 I \qquad (B.3)$$

$$b_2 \simeq k^{\frac{1}{2}} p\sin I \qquad (B.4)$$

Case 1 $\lambda \leq 1$

To have $\lambda \leq 1$ we must have

$$\frac{b_1}{b_2} \simeq k^{\frac{1}{2}} \cos^2 I \csc I < 1$$

or equivalently

$$\tan^2 I + \tan^4 I > k$$

or

$$2\tan^2 I > (1+4k)^{\frac{1}{2}} - 1$$

to order k this becomes

$$\tan^2 I > k$$

from Vinti [2] this occurs provided

$$I_c < I < 180^\circ - I_c \tag{B.5}$$

where

$$I_c = 1^\circ\ 54' \tag{B.6}$$

In this case we require

$$h = \frac{b_2}{\rho} \simeq \frac{k^{\frac{1}{2}} p\,|\sin I|}{\rho} < 1$$

this will occur at all points along the trajectory provided it occurs at ρ-perigee, thus we ask

$$\frac{k^{\frac{1}{2}} p\,|\sin I|}{\rho_1} < 1$$

or

$$\rho_1 > k^{\frac{1}{2}} p\,|\sin I|$$

or

$$a(e-1) > k^{\frac{1}{2}} a(e-1)(e+1)\,|\sin I|$$

or

$$k^{\frac{1}{2}} (e+1)\,|\sin I| < 1$$

or

$$\frac{c}{p}(e+1)|\sin I| < 1$$

or

$$\rho_1 > c|\sin I| \tag{B.7}$$

Case 2 $\lambda > 1$

In this region we require

$$h < \lambda - (\lambda^2-1)^{\frac{1}{2}} \tag{B.2}$$

but

$$\frac{b_1}{\rho} = \frac{b_1}{b_2}\frac{b_2}{\rho} = \lambda h$$

so that (B.2) is satisfied if and only if

$$\frac{b_1}{\rho} < \lambda^2 - \lambda(\lambda^2-1)^{\frac{1}{2}} = g(\lambda) \tag{B.8}$$

but Vinti [3] shows that for $\lambda \geq 1$

$$\frac{1}{2} < g(\lambda) \leq 1 \tag{B.9}$$

so the condition

$$\frac{b_1}{\rho} < \frac{1}{2} \tag{B.10}$$

is sufficient to satisfy (B.2). The above equation will be satisfied at all points in the trajectory provided it is satisfied at ρ-perigee. Thus we require

$$\frac{b_1}{\rho_1} < \frac{1}{2} \tag{B.11}$$

from (4.14)

$$b_1 = \frac{ac^2(1-\eta_0{}^2)(ap+c^2\eta_0{}^2)}{(ap+c^2)(ap+c^2\eta_0{}^2) + 4a^2c^2\eta_0{}^2} \tag{B.12}$$

so that

$$\frac{b_1}{\rho_1} < \frac{ac^2(1-\eta_0{}^2)(ap+c^2\eta_0{}^2)}{\rho_1(ap+c^2)(ap+c^2\eta_0{}^2)} = \frac{ac^2\cos^2 I}{\rho_1(ap+c^2)} \tag{B.13}$$

so it will be sufficient to have

$$\frac{2ac^2\cos^2 I}{\rho_1} < (ap+c^2) \tag{B.14}$$

or

$$\frac{2c^2\cos^2 I}{p\rho_1} - \frac{c^2}{ap} < 1$$

or

$$\frac{2c^2\cos^2 I - c^2(e-1)}{p\rho_1} < 1$$

or

$$\rho_1{}^2(e+1) > 2c^2\cos^2 I - c^2(e-1) \tag{B.15}$$

There will be no trouble satisfying this for $e \geq 3$. The worst case occurs for the "parabola" where $e = 1$. Here (B.15) becomes

$$\rho_1 > c|\cos I|, \quad \text{for} \qquad e = 1 \tag{B.16}$$

Conditions (B.7) and (B.16) are, at worst, no more restrictive than the limitations discussed at the end of Chapter 3. Thus for all trajectories whose extensions do not pass within the surface of Figure 3.1 we may certainly replace $(1-2\lambda h+h^2)^{-\frac{1}{2}}$ by the Legendre Polynomial series.

APPENDIX C

INVESTIGATION OF THE CONVERGENCE OF S_1

We should like to investigate the convergence of the series involved in the evaluation of R_1

$$S_1 = \left(\frac{b_2}{p}\right)^2 \sum_{m=0}^{\infty} \left(\frac{b_2}{p}\right)^m P_{m+2}(\lambda) \int_0^f (1+e\cos f)^m \, df \qquad (5.24)$$

clearly since cosf never exceeds 1 we can say

$$|S_1| \leq \left(\frac{b_2}{p}\right)^2 \sum_{m=0}^{\infty} \left(\frac{b_2}{p}\right)^m P_{m+2}(\lambda) \int_0^f (1+e)^m \, df$$

$$\leq \left(\frac{b_2}{p}\right)^2 \sum_{m=0}^{\infty} \left(\frac{b_2}{p}\right)^m P_{m+2}(\lambda) \, (1+e)^m f \qquad (C.1)$$

Case 1: $\lambda \leq 1$

For this case

$$|P_n(\lambda)| \leq 1 \qquad (C.2)$$

so that

$$|S_1| \leq \left(\frac{b_2}{p}\right)^2 f \sum_{m=0}^{\infty} \left[\frac{b\,(e+1)}{p}\right]^m \qquad (C.3)$$

If the ratio

$$r_1 = \frac{b_2(e+1)}{p} < 1 \qquad (C.4)$$

then the summation of (C.3) can be considered an infinite geometric progression with first term of 1 and a ratio r_1. In this case we would have

$$\sum_{m=0}^{\infty} \left[\frac{b_2(1+e)}{p}\right]^m = \frac{1}{1 - \frac{b_2(e+1)}{p}} \tag{C.5}$$

and so

$$|S_1| \leq \frac{\left(\frac{b_2}{p}\right)^2 f}{1 - \frac{b_2(e+1)}{p}} \tag{C.6}$$

Consequently we should like to know the conditions under which (C.4) is true. Substituting $b_2 \simeq k^{\frac{1}{2}} p\sin I$ into (C.4) we are thus requiring

$$\left|\frac{k^{\frac{1}{2}}p \sin I \ (e+1)}{p}\right| < 1$$

or

$$\frac{c|\sin I| \ (e+1)}{\rho_1(e+1)} < 1$$

$$\rho_1 > c|\sin I| \tag{C.7}$$

<u>Case 2:</u> $\lambda > 1$

According to Vinti [3] we can write for this case

$$P_n(\lambda) = \frac{1}{\pi}\int_0^{\pi} [\lambda + (\lambda^2-1)^{\frac{1}{2}} \cos x]^n \, dx \tag{C.8}$$

where x is a dummy variable. Then

$$\left(\frac{b_2}{p}\right)^n P_n(\lambda) = \frac{1}{\pi}\int_0^{\pi}[b_1 p^{-1} + p^{-1}(b_1{}^2-b_2{}^2)^{\frac{1}{2}} \cos x]^n \, dx \tag{C.9}$$

so that

$$|(\frac{b_2}{p})^n P_n(\lambda)| \leq [\frac{b_1}{p} + \frac{b_1}{p}(1-\lambda^{-2})^{\frac{1}{2}}]^n \leq (\frac{2b_1}{p})^n \tag{C.10}$$

since we are now considering $\lambda > 1$. Thus S_1 becomes

$$|S_1| \leq \sum_{n=2}^{\infty} (\frac{2b_1}{p})^n \int_0^f (1+e\cos f)^{n-2}\, df$$

again since cosf cannot exceed 1

$$|S_1| \leq f\,(\frac{2b_1}{p})^2 \sum_{m=0}^{\infty} [\frac{2b_1(e+1)}{p}]^m \tag{C.11}$$

If we now define the ratio

$$r_2 \equiv \frac{2b_1(e+1)}{p} \tag{C.12}$$

then the summation of (C.11) can be considered an infinite geometric progression of sum

$$\sum_{m=0}^{\infty} r_2{}^m = \frac{1}{1-r_2} = \frac{1}{1-\frac{2b_1(e+1)}{p}} \tag{C.13}$$

as long as we require that

$$r_2 = \frac{2b_1(1+e)}{p} = \frac{2b_1}{\rho_1} < 1 \tag{C.14}$$

or

$$\frac{b_1}{\rho_1} < \frac{1}{2} \tag{C.15}$$

but this is the same condition we were required to satisfy in order to replace $(1+A_\rho{}^{-1}+B_\rho{}^{-2})^{-\frac{1}{2}}$ by the Legendre series $\sum_{n=0}^{\infty} h^n P_n(\lambda)$. In Appendix B we saw that this condition reduced to

$$\rho_1{}^2(e+1) > 2c^2\cos^2 I - c^2(e-1) \tag{C.16}$$

so that for $e \gtrsim 3$, the series always converges.

Thus comparing (C.7) and (C.16) to the results of Appendix B we see that the requirements for S_1 to converge are identical to the requirements on ρ_1 in order that $(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}}$ be the generating function for the Legendre polynomials.

For a trajectory which does not hit the Earth, the series converges faster than a geometric series of ratio

$$\frac{c}{\rho_1} = \frac{c}{r_e} = \sqrt{J_2} \approx \frac{1}{30} \qquad (C.17)$$

APPENDIX D

INVESTIGATION OF $f_m(f) \equiv \int_0^f (1+e\cos f)^m \, df$

In determining the R_1 integral we found it necessary to evaluate

$$f_m(f) \equiv \int_0^f (1+e\cos x)^m \, dx \tag{D.1}$$

where x is a dummy variable. We also define

$$g_m(f) \quad \frac{1}{2\pi} f_m(2\pi) f \tag{D.2}$$

We should first like to prove that $f_m(f)-g_m(f)$ is an odd function of f. Noting that

$$f_m(-f) = \int_0^{-f} (1+e\cos x)^m \, dx$$

if in the above equation we replace $x = -y$, then $dx = -dy$, $\cos x = \cos y$, and the limits of integration become $0 \to f$. Thus

$$f_m(-f) = -\int_0^f (1+e\cos y)^m \, dy = -f_m(f) \tag{D.3}$$

Therefore $f_m(f)$ is an odd function. By its definition g_m is an odd function since

$$g_m(-f) = \frac{1}{2\pi} f_m(2\pi)(-f) = -g_m(f) \tag{D.4}$$

As a consequence of (D.3) and (D.4) the function $f_m(f)-g_m(f)$ must be odd. To find the period of $f_m(f)-g_m(f)$ we first note

$$f_m(f+2\pi) = \int_0^{2\pi} (1+e\cos x)^m \, dx + \int_{2\pi}^{2\pi+f} (1+e\cos x)^m \, dx \tag{D.5}$$

If, in the second of these integrals we replace $x = y+2\pi$, then $dx = dy$, $\cos x = \cos y$, and the limits of integration become $0 \rightarrow f$ so that (D.5) becomes

$$f_m(f+2\pi) = \int_0^{2\pi} (1+e\cos x)^m \, dx + \int_0^{f} (1+e\cos y)^m \, dy$$

so that

$$f_m(f+2\pi) = f_m(2\pi) + f_m(f) \tag{D.6}$$

we also note that

$$g_m(f+2\pi) = \frac{1}{2\pi} f_m(2\pi)(f+2\pi) = g_m(f) + f_m(2\pi) \tag{D.7}$$

Thus by (D.6) and (D.7)

$$f_m(f+2\pi) - g_m(f+2\pi) = f_m(f) - g_m(f) \tag{D.8}$$

So that $f_m(f) - g_m(f)$ has been established to be odd and of period 2π. Then we see that

$$f_m(2\pi) = \int_0^{\pi} (1+e\cos x)^m \, dx + \int_{\pi}^{2\pi} (1+e\cos x)^m \, dx$$

If in the second integral we set $x = 2\pi-y$, so that $dx = -dy$, $\cos x = -\cos y$ and the limits of integration become $\pi \rightarrow 0$

$$f_m(2\pi) = \int_0^{\pi} (1+e\cos x)^m \, dx - \int_{\pi}^{0} (1+e\cos y)^m \, dy = 2\, f_m(\pi) \tag{D.9}$$

Using (D.9) we can write

$$g_m(f) = \frac{f}{\pi} f_m(\pi) \tag{D.10}$$

Then since $f_m(f)-g_m(f)$ is odd of period 2π

$$f_m(f) - \frac{f}{\pi} f_m(\pi) = \sum_{j=1}^{m} c_{mj} \sin jf \tag{D.11}$$

or equivalently

$$f_m(f) \equiv \int_0^f (1+e\cos f)^m \, df = \frac{f}{\pi} \int_0^\pi (1+e\cos f)^m \, df + \sum_{j=1}^{m} c_{mj} \sin jf \tag{D.12}$$

The fourier series in (D.12) terminates at $j = m$, because the integrand of (D.1) may be expressed as a trigonometric polynomial in $\cos jx$, with $j \leq m$.

APPENDIX E

THE COMPUTER PROGRAM

The following Fortran IV computer program was run as problem number M6973 at the MIT Computation Center to generate data used in the text on the comparison between the spheroidal solution and a normal Keplerian trajectory. Inputs to the program are the oblateness parameter of the gravitating body (J_2), the spheroidal orbital elements (a, e, I, β_1, β_2, β_3), the number of time steps N for which the state vector is desired, the amount DT of each time step, and the number of different trajectories NTRAJ included in the data. Output is as shown in Chapter 11; formatting allows cutting to standard $8\frac{1}{2}$ by 11 sheets. Units are assumed to be those of the canonical Vanguard system discussed in the text.

It may be of some value to note that the six example trajectories of Chapter 11 required a total of 4.3 seconds execution time on the IBM 360. The small computation time required for such a large increase in accuracy is one of the main advantages of the spheroidal method.

EXPLANATION OF PARAMETERS

Parameters used in the computer programs are related to the parameters used in the text as follows:

Input Parameters

C2 = $c^2 = J_2$

N = number of time steps

DT = amount of each time step

NTRAJ = number of different trajectories to be computed

A, E, I, BETA1, BETA2, BETA3 = spheroidal orbital elements

Other Parameters

NUO = η_0	PL(N) = P_n Legendre polynomials
NUO2= $\eta_0{}^2$	T(N) = T_n
X2 = $x^2 = e^2-1$	D(N) = D_n
NU22 = $\eta_2{}^2$	GN = γ_n
BS1 = b_1	TIME = t
BS2 = b_2	HO = H_0, etc.
AO = a_0	FO = f_0, etc.
PO = p_0	PSIO = ψ_0, etc.
N2 = n^2	NU = η
A1 = A_1, etc	RX = X, etc.
A11 = A_{11}, etc.	VX = $\dot{X}$, etc.
B1P = B_1'	FDT = $\dot{f}$
B2 = B_2, etc.	RHODT = $\dot{\rho}$
OMGA = Ω	PSIDT = $\dot{\psi}$
W = ω	OMGADT = $\dot{\Omega}'$
ROX = r_{0_x}, etc.	OMGAP = Ω'

```
$JOB            HYP,KP=29,TIME=4,PAGES=100
C      ***********************************************************
C      THIS PROGRAM ACCEPTS VALUES FOR THE SPHEROIDAL ORBITAL ELEMENTS-
C      (A,E,I,BETA1,BETA2,BETA3) AND A VALUE OF J2.  FROM THIS IT COMPUTES,
C      USING THE SPHEROIDAL SOLN, THE STATE VECTOR (RX,RY,RZ,VX,VY,VZ),
C      THE MAGNITUDE OF THE POSITION AND VELOCITY VECTORS (R,V),
C      AND THE VALUES OF THE SPHEROIDAL ANOMALIES (H*,F*), AS FUNCTIONS
C      OF TIME.  THEN, ON THE BASIS OF THE POS AND VEL VECTORS AT
C      TIME=ZERO, IT COMPUTES, USING THE SPHERICAL SOLN, THE SPHERICAL
C      ORBITAL ELEMENTS (AS,ES,IS,OMGA=LONG OF NODE,W=ARG OF PER,TAU),
C      THE STATE VECTOR, THE MAGN OF THE POS AND VEL VECTORS, AND THE
C      VALUES OF THE SPHERICAL ANOMALIES (H,F), AS FUNCTIOS OF TIME.
C      LIMITATIONS....GOOD ONLY FOR UNBOUND TRAJECTORIES AND
C                     INCLINATIONS NOT TOO CLOSE TO 0 OR 180 DEGREES
C      PHYSICAL UNITS EMPLOYED ARE THOSE OF THE CANONICAL VANGUARD SYSTEM
C           UNIT OF LENGTH=EQUATORIAL RADIUS= 6378.388 KILOMETERS
C           UNIT OF TIME= 806.832 SECONDS
C           MU= 1
C      ***********************************************************
       DIMENSION PL(11),T(11),D(9)
       REAL I,NU0,NU02,NF,NU22,N2,LAMB,M,MN,H
       REAL H0,H1P,H1,H2,NU,OMGAP,OMGADT
       REAL HX,HY,HZ,NUM4,IS,NUM1,IZX,IZY,IZZ,INX,INY,OMGA,H0
       REAL NUM2,HYP,NUM5,NUM6,NUM7,NUM8
   300 FORMAT (6F10.4)
   301 FORMAT (E13.5,I3,F6.3)
   307 FORMAT (' A1 DID NOT CONVERGE')
   308 FORMAT (' A2 DID NOT CONVERGE')
   309 FORMAT (' A3 DID NOT CONVERGE')
   310 FORMAT (' ***KEPLER EQUATION TOOK 20 CYCLES***')
   321 FORMAT (I3)
   350 FORMAT (//36X,'TRAJECTORY NUMBER ',I2)
   351 FORMAT (36X,20(1H_))
   352 FORMAT (/16X,'GIVEN  J2=',F11.8)
   353 FORMAT (24X,'A=',F6.4,7X,'E=',F6.4,7X,'I=',F6.4)
   354 FORMAT (20X,'BETA1=',F6.4,3X,'BETA2=',F6.4,3X,'BETA3=',F6.4/)
   355 FORMAT (36X,'SPHEROIDAL SOLUTION')
   356 FORMAT (36X,19(1H_))
   357 FORMAT (/16X,'TIME',5X,'RX',9X,'RY',9X,'RZ',9X,'R',10X,'H*')
   358 FORMAT (25X,'VX',9X,'VY',9X,'VZ',9X,'V',10X,'F*')
   359 FORMAT (16X,60(1H_))
   360 FORMAT (16X,F4.1,5(1X,F10.6))
   361 FORMAT (20X,5(1X,F10.6)/)
   362 FORMAT (37X,'SPHERICAL SOLUTION')
   363 FORMAT (37X,18(1H_))
   364 FORMAT (/16X,'THE SPHERICAL ELEMENTS ARE')
   365 FORMAT (18X,'AS=',E16.8,'ES=',E16.8,' IS=',E16.8)
   366 FORMAT (16X,'OMGA=',E16.8,' W=',E16.8,'TAU=',E16.8)
   367 FORMAT (/16X,'TIME',5X,'RX',9X,'RY',9X,'RZ',9X,'R',10X,'H')
   368 FORMAT (25X,'VX',9X,'VY',9X,'VZ',9X,'V',10X,'F')
       READ (5,321) NTRAJ
       DO 999 NTRA=1,NTRAJ
C              START OF SPHEROIDAL SOLUTION
       READ (5,300) A,E,I,BETA1,BETA2,BETA3
       READ (5,301) C2,N,DT
```

```
      WRITE (6,350) NTRA
      WRITE (6,351)
      WRITE (6,352) C2
      WRITE (6,353) A,E,I
      WRITE (6,354) BETA1,BETA2,BETA3
      WRITE (6,355)
      WRITE (6,356)
      WRITE (6,357)
      WRITE (6,358)
      WRITE (6,359)
      NUO=SIN(I)
      NUO2=NUO**2
      X2=E**2-1.0
      X=SQRT(X2)
      P=A*X2
      DF=(A*P+C2)*(A*P+C2*NUO2)+4.0*(A**2)*C2*NUO2
      NF=(A*P+C2)*(A*P+C2*NUO2)+4.0*(A**2)*C2
      AF=(-2.0*A*C2*(1.0-NUO2)*(A*P+C2*NUO2))/DF
      BF=C2*NUO2*NF/DF
      NU22=-(C2*DF)/(A*P*NF)
      BS1=-AF/2.0
      BS2=SQRT(BF)
      AO=A-BS1
      PO=(C2*(1.0-NUO2)+A*P*NF/DF)/AO
      ALPHA2=SQRT(PO)
      ALPHA3=ALPHA2*COS(I)*SQRT(1.0+C2*NUO2/(AO*PO))
      N2=ABS(NUO2*NU22)
      P2=P**2
      P3=P**3
      P4=P**4
      ALPH2=ALPHA2*SQRT(1.0-C2*(1.0-NUO2)/(AO*PO))
      LAMB=BS1/BS2
C              CALCULATION OF LEGENDRE POLYNOMIALS,PL(N)
      PL(1)=1.0
      PL(2)=LAMB
      DO 401 K=3,11
      J=K-1
      PL(K)=((2*J-1)*LAMB*PL(K-1)-(J-1)*PL(K-2))/J
  401 CONTINUE
C              CALCULATION OF POLYNOMIALS, T(N)
      T(1)=1.0
      T(2)=1.0
      T(3)=(3.0+X2)/2.0
      DO 402 K=4,11
      J=K-1
      T(K)=((2*J-1)*T(K-1)+(J-1)*X2*T(K-2))/J
  402 CONTINUE
      Z2=BS2/P
      Z3=C2/(P**2)
C              CALCULATION OF A1
      A1=P*X*(Z2**2)*PL(3)
      DO 403 J=3,10
      DA1=P*X*(Z2**J)*PL(J+1)*T(J-1)
      A1=A1+DA1
      IF (ABS(DA1/A1) .LT. .0000001) GO TO 404
```

```
  403 CONTINUE
      WRITE (6,307)
  404 CONTINUE
C               CALCULATION OF A2
      A2=X/P
      DO 405 J=1,10
      DA2=(X/P)*(Z2**J)*PL(J+1)*T(J+1)
      A2=A2+DA2
      IF (ABS(DA2/A2) .LT. .0000001) GO TO 406
  405 CONTINUE
      WRITE (6,308)
  406 CONTINUE
C               CALCULATION OF D(N)
      D(1)=1.0
      D(2)=Z2*LAMB
      FAC=Z2
      DO 407 K=3,9
      FAC=Z2*FAC
      D(K)=-Z3*D(K-2)+PL(K)*FAC
  407 CONTINUE
C               CALCULATION OF A3
      A3=X*T(3)/P3
      DO 408 K=2,9
      DA3=X*D(K)*T(K+2)/P3
      A3=A3+DA3
      IF (ABS(DA3/A3) .LT. .0000001) GO TO 409
  408 CONTINUE
      WRITE (6,309)
  409 CONTINUE
C               CALCULATION OF B1 PRIME AND B2
      B1P=.5-3.0*N2/16.0+15.0*N2**2/128.0
      B2=1.0-N2/4.0+9.0*N2**2/64.0
C               CALCULATION OF THE GAMMAS AND B3
      G2=3.0*NU02/16.0
      G3=5.0*(3.0*NU02/4.0+1.0)*NU02/32.0
      G4=35.0*((5.0*NU02/8.0+.75)*NU02+1.0)*NU02/256.0
      G5=63.0*(((35.0*NU02/64.0+5.0/8.0)*NU02+.75)*NU02+1.0)*NU02/512.0
      B3=1.0-1.0/SQRT(1.0-NU22)-G2*NU22**2-G3*NU22**3-G4*NU22**4-G5*
     1NU22**5
C               CALCULATION OF THE A(IJ)
      BF2=BF**2
      BS12=BS1**2
      E2=E**2
      A11=(.75*X*E/P3)*(-2.0*BS1*BF*P+BF2)
      A12=3.0*X*BF2*E2/(32.0*P3)
      A21=(X*E/P)*(BS1/P+(3.0*BS12-BF)/P2-4.5*BS1*BF*(1.0+E2/4.0)/P3
     1+3.0*BF2*(4.0+3.0*E2)/(8.0*P4))
      A22=(X*E2/P)*((3.0*BS12-BF)/(8.0*P2)-9.0*BS1*BF/(8.0*P3)
     1+3.0*BF2*(3.0+E2/2.0)/(16.0*P4))
      A23=(X*E2*E/(8.0*P4))*(-BS1*BF+BF2/P)
      A24=3.0*BF2*E2**2/(256.0*P3*P2)
      Z5=(BF/2.0+C2)/P2
      E3=E**3
      E4=E2**2
      A31=(X*E/P3)*(2.0+BS1*(3.0  .+.75*E2)/P-Z5*(4.0+3.0*E2))
```

```
      A32=(X/P3)*(E2/4.0+BS1*.75*E2/P-Z5*(1.5*E2+E4/4.0))
      A33=(X/P3)*(BS1*E3/(12.0*P)-Z5*E3/3.0)
      A34=(X/P3)*(-Z5*E4/32.0)
      Z1=SQRT(A0)
      TIME=0.0
C               CALCULATIONS FOR EACH TIME STEP
      DO 498 L=1,N
      M=(TIME+BETA1)/(Z1*A)
C               SOLVING KEPLERS EQUATION
      INDEX=0
      IF (ABS(M) .GT. 1.0) GO TO 452
      H0=M
  453 MN=E*SINH(H0)-H0
      INDEX=INDEX+1
      DELT=(M-MN)/(E*COSH(H0)-1.0)
      IF (ABS(DELT) .LT. .000001) GO TO 454
      IF (INDEX .GT. 20) GO TO 457
      IF (ABS(DELT) .GT. ABS(H0)) GO TO 455
  456 H0=H0+DELT
      GO TO 453
  452 Z=ALOG(ABS(M)*2.0/E)
      H0=SIGN(Z,M)
      GO TO 453
  455 DELT=DELT*ABS(H0)/ABS(DELT)
      GO TO 456
  457 WRITE (6,310)
  454 H0=H0+DELT
      F0=2.0*ATAN(SQRT((E+1.0)/(E-1.0))*TANH(H0/2.0))
      PSI0=ALPH2*(BETA2/ALPHA2+Z1*A2*F0)/B2
      H1P=((C2*NU02)*(-B1P*PSI0+.25*SIN(2.0*PSI0))/(Z1*ALPH2)
     1-BS1*H0-A1*F0)/(A*(E*COSH(H0)-1.0))
      H1=H1P*(1.0-H1P*E*SINH(H0)/(2.0*(E*COSH(H0)-1.0)))
      F1=2.0*ATAN(SQRT((E+1.0)/(E-1.0))*TANH((H0+H1)/2.0))-F0
      PSI1=-N2*SIN(2.0*PSI0)/(8.0*B2)+(Z1*ALPH2/B2)*(A2*F1+A21*SIN(F0)
     1+A22*SIN(2.0*F0))
      H2=(-BS1*H1-A1*F1-A11*SIN(F0)-A12*SIN(2.0*F0)+(C2*NU02)/(Z1*ALPH2)
     1*(-B1P*PSI1+PSI1*COS(2.0*PSI0)/2.0-N2*SIN(2.0*PSI0)/8.0
     1+N2*SIN(4.0*PSI0)/64.0))/(A*(E*COSH(H0)-1.0))
      H=H0+H1+H2
      F=2.0*ATAN(SQRT((E+1.0)/(E-1.0))*TANH(H/2.0))
      F2=F-F0-F1
      PSI2=(-N2*PSI1*COS(2.0*PSI0)/4.0+3.0*(N2**2)*SIN(2.0*PSI0)/32.0
     1-3.0*(N2**2)*SIN(4.0*PSI0)/256.0+ALPH2*Z1*(A2*F2+A21*F1*COS(F0)
     1+A22*2.0*F1*COS(2.0*F0)+A23*SIN(3.0*F0)+A24*SIN(4.0*F0)))/B2
      PSI=PSI0+PSI1+PSI2
      RHO=A*(E*COSH(H)-1.0)
      NU=NU0*SIN(PSI)
      Z6=A31*SIN(F)+A32*SIN(2.0*F)+A33*SIN(3.0*F)+A34*SIN(4.0*F)
      OMGAP=BETA3+ALPHA3/ALPH2*(B3*PSI+3.0*NU02*(NU22**2)*SIN(2.0*
     1PSI)/32.0)-C2*ALPHA3*Z1*(A3*F+Z6)
C               THE POSITION AND VELOCITY VECTORS
      RX=SQRT(RHO**2+C2)*(COS(PSI)*COS(OMGAP)
     1-COS(I)*SIN(PSI)*SIN(OMGAP))
      RY=SQRT(RHO**2+C2)*(COS(I)*SIN(PSI)*COS(OMGAP)
     1+SIN(OMGAP)*COS(PSI))
```

```
      RZ=RHO*NU
      FDT=A*X*SQRT(RHO**2+AE*RHO+BE)/(RHO*Z1*(RHO**2+C2*(NU**2)))
      RHODT=(RHO**2)*E*SIN(F)*FDT/P
      PSIDT=ALPH2*SQRT(1.0+N2*(SIN(PSI))**2)/(RHO**2+C2*(NU**2))
      Z7=A31*COS(F)+2.0*A32*COS(2.0*F)+3.0*A33*COS(3.0*F)
     1+4.0*A34*COS(4.0*F)
      OMGADT=ALPHA3*PSIDT/ALPH2*(B3+3.0*NUO2*(NU22**2)*
     1COS(2.0*PSI)/16.0)-C2*ALPHA3*Z1*(A3+Z7)*FDT
      VX=RHO*RHODT*RX/(RHO**2+C2)-OMGADT*RY+SQRT(RHO**2+C2)*(-SIN(PSI)
     1*COS(OMGAP)-COS(I)*COS(PSI)*SIN(OMGAP))*PSIDT
      VY=RHO*RHODT*RY/(RHO**2+C2)+OMGADT*RX+SQRT(RHO**2+C2)*(-SIN(PSI)
     1*SIN(OMGAP)+COS(I)*COS(PSI)*COS(OMGAP))*PSIDT
      VZ=RHODT*NU+NUO*RHO*COS(PSI)*PSIDT
      R=SQRT(RX**2+RY**2+RZ**2)
      V=SQRT(VX**2+VY**2+VZ**2)
      IF (L .GT. 1) GO TO 459
      ROX=RX
      ROY=RY
      ROZ=RZ
      VOX=VX
      VOY=VY
      VOZ=VZ
  459 CONTINUE
      WRITE (6,360) TIME,RX,RY,RZ,R,H
      WRITE (6,361) VX,VY,VZ,V,F
      TIME=TIME+DT
  498 CONTINUE
C                CALCULATION OF SPHERICAL ELEMENTS
C                BEWARE OF I=0 IN CALC OF OMGA AND W
      WRITE (6,362)
      WRITE (6,363)
      WRITE (6,364)
      RO=SQRT(ROX**2+ROY**2+ROZ**2)
      VO2=VOX**2+VOY**2+VOZ**2
      VO=SQRT(VO2)
      AS=1.0/(VO2-2.0/RO)
      HX=ROY*VOZ-ROZ*VOY
      HY=ROZ*VOX-ROX*VOZ
      HZ=ROX*VOY-ROY*VOX
      H=SQRT(HX**2+HY**2+HZ**2)
      NUM4=ROX*VOX+ROY*VOY+ROZ*VOZ
      ES=SQRT((H**2/RO-1.0)**2+(H*NUM4/RO)**2)
      IS=ARCOS(HZ/H)
      NUM1=(1.0/AS+1.0/RO)/ES
      NUM2=NUM4/ES
      IZX=NUM1*ROX-NUM2*VOX
      IZY=NUM1*ROY-NUM2*VOY
      IZZ=NUM1*ROZ-NUM2*VOZ
      INX=-HY/SQRT(HX**2+HY**2)
      INY=HX/SQRT(HX**2+HY**2)
      OMGA=ARCOS(INX)
      Z=INX*IZX+INY*IZY
      IF (ABS(Z) .GE. 1.0) GO TO 207
      W=ARCOS(Z)
      GO TO 208
```

```
  207 W=0.0
  208 FO=ATAN(H*NUM4/(H**2-RO))
      X=SQRT(ES**2-1.0)*SIN(FO)/(1.0+ES*COS(FO))
      Y=X+SQRT(X**2+1.0)
      HO=ALOG(Y)
      TAU=SQRT(AS**3)*(HO-ES*SINH(HO))
      TIME=0.0
      WRITE (6,365) AS,ES,IS
      WRITE (6,366) OMGA,W,TAU
      WRITE (6,367)
      WRITE (6,368)
      WRITE (6,359)
      WRITE (6,360) TIME,ROX,ROY,ROZ,RO,HO
      WRITE (6,361) VOX,VOY,VOZ,VO,FO
C        CALCULATION OF R AND V FOR ANY T
      DO 201 J=2,N
      TIME=TIME+DT
C        SOLVING KEPLERS EQUATION
      INDEX=0
      M=(TIME-TAU)/SQRT(AS**3)
      IF (ABS(M) .GT. 1.0) GO TO 202
      HYP=M
  203 MN=ES*SINH(HYP)-HYP
      INDEX=INDEX+1
      DELT=(M-MN)/(ES*COSH(HYP)-1.0)
      IF (ABS(DELT) .LT. .000001) GO TO 204
      IF (INDEX .GT. 20) GO TO 217
      IF (ABS(DELT) .GT. ABS(HYP)) GO TO 205
  206 HYP=HYP+DELT
      GO TO 203
  202 Z=ALOG(ABS(M)*2.0/ES)
      HYP=SIGN(Z,M)
      GO TO 203
  205 DELT=DELT*ABS(HYP)/ABS(DELT)
      GO TO 206
  217 WRITE (6,310)
  204 HYP=HYP+DELT
      NUM5=1.0-AS*(COSH(HYP-HO)-1.0)/RO
      NUM6=TIME-(SINH(HYP-HO)-(HYP-HO))*SQRT(AS**3)
      RX=NUM5*ROX+NUM6*VOX
      RY=NUM5*ROY+NUM6*VOY
      RZ=NUM5*ROZ+NUM6*VOZ
      R=SQRT(RX**2+RY**2+RZ**2)
      NUM7=-SQRT(AS)*SINH(HYP-HO)/(R*RO)
      NUM8=1.0-AS*(COSH(HYP-HO)-1.0)/R
      VX=NUM7*ROX+NUM8*VOX
      VY=NUM7*ROY+NUM8*VOY
      VZ=NUM7*ROZ+NUM8*VOZ
      V=SQRT(VX**2+VY**2+VZ**2)
      F=2.0*ATAN(SQRT((ES+1.0)/(ES-1.0))*TANH(HYP/2.0))
      WRITE (6,360) TIME,RX,RY,RZ,R,HYP
      WRITE (6,361) VX,VY,VZ,V,F
  201 CONTINUE
  999 CONTINUE
      RETURN
```

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