Kähler Structures on Cotangent Bundles of Real Analytic Riemannian Manifolds

by

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Abstract

We show in many homogeneous cases that given a real analytic Riemannian manifold M, there is a unique complex structure on a neighborhood of M in T^*M that turns T^*M into a Kähler manifold whose Kähler form is the standard symplectic form, and such that the Kähler metric restricted to the tangent bundle of the zero section is the original metric on M. This complex structure is characterized by the conditions

1. $\operatorname{Im} \overline{\partial}\phi_o = \alpha_o$
2. the standard involution of T^*M is an antiholomorphic map.

Here α_o is the canonical one form and ϕ_o is the quadratic function on T^*M associated with the metric.

We show that the function $u = \sqrt{\phi_o}$ is a solution of the complex homogeneous Monge-Ampère equation (away from the zero section). This gives rise to a Monge-Ampère exhaustion of T^*M near M. We explore the geometric properties of the Kähler metric and the Monge-Ampere foliation.

We give an explicit description of this complex structure in the case of compact Lie groups and Riemannian locally symmetric spaces of the "compact type". In these cases we find that the complex structure is globally defined, instead of only locally near the zero section.

Thesis Supervisor: Victor W. Guillemin
Title: Professor of Mathematics

Acknowledgment

I would like to thank my advisor, Prof. Victor Guillemin, for his patience, encouragement, enthusiasm, and love of mathematics. I have been extremely lucky to be able to work with him for the past four years.

Retrospective

Having spent six of the best years of my life at MIT (both as a graduate and undergraduate student), this thesis represents the culmination and the end of my formative years. During this time I have met many wonderful people who have been very special to me. If there is any value in this work, then it is solely because of the generosity and love of the people I have known.

I would like to thank Claudia, and my Mother and Father, for the support, encouragement, and love they have given me over the years.

Contents

Chapter 1

Background Material

1.1 Complex and Kähler Manifolds

A complex manifold Ω is a smooth manifold whose coordinate functions map into $\mathbb{C}^n$, and such that if ϕ_1 and ϕ_2 are two coordinate functions on an open set in Ω, then $\phi_2 \circ \phi_1^{-1}$ is a holomorphic map. Complex manifolds can be identified with real analytic manifolds of even dimension by identifying the range of the coordinate functions in $\mathbb{C}^n$ with open sets in $\mathbb{R}^n$. If the coordinate function ϕ is given by

$$\phi(\zeta) = (x^1(\zeta), y^1(\zeta), \ldots, x^n(\zeta), y^n(\zeta))$$

then the tangent vectors $\partial/\partial x^1, \partial/\partial y^1, \ldots, \partial/\partial x^n, \partial/\partial y^n$ evaluated at ζ_o form a basis for the real tangent space $T_{\zeta_o}\Omega$. At each point ζ_o of Ω there is an endomorphism J of $T_{\zeta_o}\Omega$ defined by

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$$

$$J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}$$

It turns out that this endomorphism is canonically defined, i.e., does not depend on the choice of coordinates. See for example Helgason [10], lemma 1.1, chapter VIII. J is called the complex structure operator, or just the complex structure.

An almost complex manifold is a $2n$ real dimensional smooth manifold carrying a type $(1,1)$ tensor field J (i.e. each J_ζ is an endomorphism of $T_\zeta\Omega$) such that $J^2 = -Id$. It is well known that an almost complex manifold is a complex manifold if and only if it satisfies the integrability condition

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0 \tag{1.1}$$

where X and Y are smooth vector fields on M. This theorem is due to Newlander-Nirenberg, and the tensor N is known as the Nijenhuis tensor. It is easy to see that if Ω is a complex manifold with complex structure J and $F\colon\Omega \to \Omega'$ is a diffeomorphism, then the complex structure $F_*J = dF \circ J \circ dF^{-1}$ is integrable and turns Ω' into a complex manifold.

Let Ω be a complex manifold of complex dimension n, and let $T_{\mathbf{C}}\Omega$ denote the complexified tangent bundle of Ω, $T\Omega \otimes \mathbf{C}$. The complex structure induces a splitting of $T_{\mathbf{C}}\Omega$ into the $+\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J, each eigenspace having complex dimension n. The (complex valued) vector fields on Ω with values in the $+\sqrt{-1}$ eigenspace are called type $(1,0)$ vector fields, and those with values in the $-\sqrt{-1}$ eigenspace are called type $(0,1)$. A local basis over $C^\infty(\Omega)$ for the type $(1,0)$ vector fields is

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\,\frac{\partial}{\partial y^i}\right), \quad i = 1,\ldots,n$$

and a local basis for the type $(0,1)$ vector fields is

$$\frac{\partial}{\partial \overline{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\,\frac{\partial}{\partial y^i}\right), \quad i = 1,\ldots,n.$$

This also induces a decomposition of the exterior derivative d into $d = \partial + \overline{\partial}$, where $\overline{\partial}$ annihilates the type $(1,0)$ vector fields, and ∂ annihilates the type $(0,1)$ vector fields. In local coordinates, if f is a smooth function on Ω, then

$$\begin{aligned}\partial f &= \sum_{i=1}^{n} \frac{\partial f}{\partial z^i} dz^i \\ \overline{\partial} f &= \sum_{i=1}^{n} \frac{\partial f}{\partial \overline{z}^i} d\overline{z}^i.\end{aligned}$$

Since $dx^i \circ J = -dy^i$ and $dy^i \circ J = dx^i$, it is easy to check that this can be written invariantly as

$$\begin{aligned}\partial f &= \frac{1}{2}(df - \sqrt{-1}\, df \circ J) \\ \overline{\partial} f &= \frac{1}{2}(df + \sqrt{-1}\, df \circ J)\end{aligned}$$

It will be important for us to note that if f is a smooth function on Ω and V is a vector field, then

$$\operatorname{Im} \overline{\partial} f(V) = \frac{1}{2} df(JV).$$

If Ω_1 and Ω_2 are complex manifolds and $F\colon \Omega_1 \to \Omega_2$ is a smooth map, then F is called holomorphic if it can be expressed in complex local coordinates as a holomorphic map. It is easy to check that if F is a holomorphic map, then dF satisfies

$$dF \circ J_1 = J_2 \circ dF \tag{1.2}$$

where J_1, J_2 are the complex structures on Ω_1 and Ω_2, respectively. Conversely, if F is a smooth map satisfying 1.2, then the Cauchy-Riemann equations imply that F is a holomorphic map.

A Kähler manifold is a complex manifold Ω which is also a symplectic manifold with the property that if J is the complex structure and ω is the symplectic form, then for all vector fields V, W on Ω,

1. $\omega(JV, JW) = \omega(V, W)$.

2. The symmetric form $b(V,W) = \omega(V,JW)$ is positive definite.

The first condition implies that b is symmetric. The symmetric form b is a Riemannian metric on Ω, which we will refer to as the Kähler metric associated with the Kähler form ω.

1.2 Totally Real Submanifolds

Let Ω be a complex manifold of complex dimension n, and let M be a real submanifold of real dimension n. We say that M is a totally real submanifold of Ω is there exist near every point m of M a neighborhood $\mathcal{O}$ of m in Ω and a complex coordinate system $z^i = x^i + \sqrt{-1}\, y^i$ on $\mathcal{O}$ such that

$$M \cap \mathcal{O} = \{\zeta \in \mathcal{O} \colon y^1(\zeta) = \cdots = y^n(\zeta) = 0\}.$$

It is clear that if M is totally real, then it is a real analytic manifold. An important result is that diffeomorphisms of totally real submanifolds extend uniquely to holomorphic maps of complex neighborhoods.

Lemma 1.2.1 *Let M and M' be totally real submanifolds of complex manifolds Ω and Ω', and let $F \colon M \to M'$ be a real analytic diffeomorphism. Then there are neighborhoods $\mathcal{O}$ and $\mathcal{O}'$ of M and M' and a unique holomorphic map $\tilde{F} \colon \mathcal{O} \to \mathcal{O}'$ extending F.*

Proof. We will prove the local existence and uniqueness of $\tilde{F}$; then by patching together the local representations we get a well defined holomorphic map extending F.

Let $\mathcal{U}$ be a neighborhood of 0 in $\mathbb{R}^n$, and let $F \colon \mathcal{U} \to \mathcal{U}'$ be a real analytic diffeomorphism. The coordinate components of F are real analytic functions on $\mathbb{R}^n$, and can be analytically continued to a neighborhood of $\mathcal{U}$ in $\mathbb{C}^n$. This defines a local holomorphic extension of F. To see that it is unique, suppose there are two such extensions, $\tilde{F}$ and

$\tilde{G}$. Then for all multi-indices α,

$$(\frac{\partial}{\partial x})^{\alpha}(\tilde{F} - \tilde{G})|_{\mathcal{U}} = 0.$$

Now since $\tilde{F} - \tilde{G}$ is an analytic function,

$$(\frac{\partial}{\partial x})^{\alpha}(\tilde{F} - \tilde{G}) = (\frac{\partial}{\partial z})^{\alpha}(\tilde{F} - \tilde{G}).$$

This means that $\tilde{F} - \tilde{G}$ vanishes to infinite order in the z variables on $\mathcal{U}$, so it must be identically zero. □

Lemma 1.2.2 *Let M be a totally real submanifold of a complex manifold Ω. Then there is a neighborhood $\mathcal{O}$ of M in Ω and a unique antiholomorphic involution $\sigma: \mathcal{O} \to \mathcal{O}$ such that M is the fixed point set of σ.*

Proof. It is enough to prove local existence and uniqueness. For the local existence, let $\mathcal{O}$ be a coordinate neighborhood with coordinate functions $z^i = x^i + \sqrt{-1}\, y^i$. Let $\sigma(x + \sqrt{-1}\, y) = x - \sqrt{-1}\, y$. For the uniqueness, suppose there were two such involutions, σ and σ'. Their composition would then be a holomorphic map whose fixed point set is M. By lemma 1.2.1, there is a *unique* holomorphic map extending the identity diffeomorphism of M. This map must be the identity, which proves the uniqueness. □

Lemma 1.2.3 *Let M and M' be totally real submanifolds of complex manifolds Ω and Ω', let $F: M \to M'$ be a diffeomorphism, and let σ, σ' be the conjugations of M and M' described in lemma 1.2.2. If $\tilde{F}$ is the unique holomorphic extension of F given in lemma 1.2.1, then $\tilde{F} \circ \sigma = \sigma' \circ \tilde{F}$.*

Proof. The map $\sigma' \circ \tilde{F} \circ \sigma$ is a holomorphic map which is equal to the diffeomorphism F on M, so by the uniqueness part of lemma 1.2.1 it must be equal to $\tilde{F}$. □

If M is a real analytic manifold, it is known that M can be embedded as a totally real submanifold of a complex manifold Ω.

Theorem 1.2.4 (Bruhat-Whitney) *Every real analytic manifold M can be embedded as a totally real submanifold of a complex manifold Ω. This embedding is unique in the sense that if $\iota_1: M \to \Omega_1$ and $\iota_2: M \to \Omega_2$ are two such embeddings, then there is a neighborhood $\mathcal{O}_1$ of M in Ω_1 and a neighborhood $\mathcal{O}_2$ of M in Ω_2 and a bijective holomorphic map $\Phi: \mathcal{O}_1 \to \mathcal{O}_2$ such that $\iota_2 = \Phi \circ \iota_1$.*

Proof. See Bruhat-Whitney [2]. □

1.3 The Cotangent Bundle Structure

Let M be a smooth manifold, and let T^*M be the cotangent bundle of M. It is well known that T^*M has a canonical symplectic structure, that is, there is a nondegenerate, closed two from ω_o on T^*M. The form ω_o is exact, in the sense that it is (minus one times) the exterior derivative of a one form α_o. The one form α_o is canonically defined by, for $\xi \in T^*M$ and $V \in T_\xi(T^*M)$,

$$\alpha_o(\xi)(V) = \xi(d\pi_{T^*M}(V))$$

where $\pi_{T^*M}: T^*M \to M$ is the cotangent projection. It is easy to see that M is a Lagrangian submanifold of T^*M, since α_o vanishes on M. The following important result, due to B. Kostant, S. Sternberg, and A. Weinstein, shows that this is locally in some sense a standard model for Lagrangian submanifolds of symplectic manifolds.

Theorem 1.3.1 *Let Ω be a symplectic manifold whose symplectic form is exact and equal to $-d\alpha$. Let M be a Lagrangian submanifold of Ω such that $\alpha = 0$ on $T\Omega|_M$. Then there is a neighborhood $\mathcal{O}$ of M in Ω, a neighborhood $\mathcal{O}_o$ of M in T^*M, and a unique diffeomorphism $\Psi: \mathcal{O} \to \mathcal{O}_o$ such that*

$$\begin{aligned} \Psi^*\alpha_o &= \alpha \\ \Psi \circ \imath &= \imath_o \end{aligned}$$

where $\imath$, $\imath_o$ are the inclusions of M in Ω, T^*M respectively.

Proof. This follows from propositions 3.1 and 3.2 in chapter V of Guillemin and Sternberg [7]. To explain how this follows, we need some terminology. A polarization of a symplectic manifold Ω, with symplectic form ω, is a smooth assignment of a Lagrangian subspace of $T_\zeta\Omega$ to each $\zeta \in \Omega$ in such a way that this assignment is integrable. If $\mathcal{G}$ is a polarization transversal to a Lagrangian submanifold M, then in a neighborhood of M there is a unique one form β such that

1. $d\beta = \omega$

2. for $V \in \mathcal{G}$, $\beta(V) = 0$

3. $\beta|_M = 0$.

We say that β is the one form associated with the polarization $\mathcal{G}$. See [7], page 228 for details.

Proposition 3.2 in [7] says that there is a unique polarization defined on Ω near M which is transversal to M and whose one form is α. Proposition 3.1 in [7] says that there is a symplectic diffeomorphism Ψ of a neighborhood of M in Ω with a neighborhood of M in T^*M which carries the leaves of $\mathcal{G}$ into the standard cotangent fibration of T^*M over M. Examining the proof of this proposition, we see that we may assume that Ψ is the identity on M. Let $\tilde{\alpha} = (\Psi^{-1})^*\alpha$. Then $\tilde{\alpha} - \alpha$ is closed, and is locally equal to df for some function f. Since $\tilde{\alpha} - \alpha$ vanishes on M, we may assume f is constant on M. Note that $\tilde{\alpha} - \alpha$ vanishes on the tangent space to the fibers of T^*M over M. This shows that f is constant along the fibers, hence f must be a constant function and df equal to zero.

It remains to show that the diffeomorphism Ψ is unique. This is a standard argument (see for example Abraham and Marsden [1], exercise 3.2F), which we will now sketch. Suppose Ψ' is another such diffeomorphism. Let $\Phi = \Psi' \circ \Psi^{-1}$. Then Φ is the identity on M, and preserves the canonical one form α_o on T^*M. Since Φ preserves α_o, it preserves the vector field Ξ_o defined by $\iota(\Xi_o)d\alpha_o = \alpha_o$. This implies that Φ is a fiber mapping.

Since Φ is the identity on M, we conclude that $\pi_{T^*M} \circ \Phi = \pi_{T^*M}$, where π_{T^*M} is the cotangent projection. Now if $V \in T_{\xi_x}(T^*M)$, we have

$$\begin{aligned}(\Phi^*\alpha_o)(\xi_x)(V) &= \alpha_o(\Phi(\xi_x))(d\Phi(V)) \\ &= \Phi(\xi_x)(d(\pi_{T^*M} \circ \Phi)(V)) \\ &= \Phi(\xi_x)(d\pi_{T^*M}(V)).\end{aligned}$$

On the other hand, since $\Phi^*\alpha_o = \alpha_o$,

$$\Phi(\xi_x)(d\pi_{T^*M}(V)) = \xi_x(d\pi_{T^*M}(V)).$$

Since π_{T^*M} is a submersion, we conclude that $\xi_x = \Phi(\xi_x)$. □

The cotangent bundle of M carries an antisymplectic involution σ_o, given by $\sigma_o(\xi) = -\xi$. If Ω carries an antisymplectic involution σ such that $\sigma^*\alpha = -\alpha$, it is not hard to see that Ψ preserves this involution.

Proposition 1.3.2 *Suppose Ω, Ψ are as in theorem 1.3.1, and suppose Ω has an involution σ such that $\sigma^*\alpha = -\alpha$. Then $\sigma_o \circ \Psi = \Psi \circ \sigma$.*

Proof. Let $\tilde{\sigma} = \sigma_o \circ \Psi \circ \sigma \circ \Psi^{-1}$. Then $\tilde{\sigma}$ preserves the canonical one form α_o, and is the identity on M. The proof of theorem 1.3.1 implies that $\tilde{\sigma}$ is the identity. □

Suppose Ω is a Kähler manifold with Kähler form ω, M is a Lagrangian submanifold of Ω, and ϕ is a smooth function on a neighborhood of M in Ω such that

1. $\omega = \sqrt{-1}\partial\bar{\partial}\phi$

2. $\phi = d\phi = 0$ on M.

We will then say that ϕ is a defining phase function for M. It is known that if M is Lagrangian we can always find such a defining phase function.

Theorem 1.3.3 *If M is a connected Lagrangian submanifold of a Kähler manifold Ω, then there is a neighborhood $\mathcal{O}$ of M in Ω and a unique defining phase function ϕ on $\mathcal{O}$ for M.*

Proof. See Guillemin and Sternberg [6]. □

Conversely, we have the following observation.

Lemma 1.3.4 *Let M be a totally real submanifold of a complex manifold Ω, and let ϕ be a real valued function on Ω such that $\phi = d\phi = 0$ on M, and $\partial^2\phi/\partial y^\alpha \partial y^\beta$ is a positive definite matrix when evaluated on M. Then the two form $\sqrt{-1}\partial\overline{\partial}\phi$ is a Kähler form on a neighborhood of M in Ω.*

Proof. Note that $\sqrt{-1}\partial\overline{\partial}\phi = -d\,\mathrm{Im}\,\overline{\partial}\phi$, so it is closed, and will be nondegenerate on a neighborhood of M (still denoted by Ω) if the matrix $\partial^2\phi/\partial y^\alpha \partial y^\beta$ is positive definite on M. This shows that $\sqrt{-1}\partial\overline{\partial}\phi$ is a symplectic form on Ω. To show that $\sqrt{-1}\partial\overline{\partial}\phi(JX, JY)$ is equal to $\sqrt{-1}\partial\overline{\partial}\phi(X, Y)$, we use a standard expression for the exterior derivative of a one form (see for example Abraham and Marsden [1], page 121, line 6 on table 2.4.1). If X, Y are vector fields on M, then

$$d\,\mathrm{Im}\,\overline{\partial}\phi(JX, JY) = -\frac{1}{2}JX(Y\phi) + \frac{1}{2}JY(X\phi) - \frac{1}{2}J[JX, JY]\phi. \tag{1.3}$$

Using the vanishing of the Nijenhuis tensor (see equation 1.1 on page 7), we can write

$$\begin{aligned} -\frac{1}{2}JX(Y\phi) &= \frac{1}{2}[Y, JX]\phi - Y\,\mathrm{Im}\,\overline{\partial}\phi(X) \\ \frac{1}{2}JY(X\phi) &= \frac{1}{2}[JY, X]\phi + X\,\mathrm{Im}\,\overline{\partial}\phi(Y). \end{aligned}$$

Putting this into equation 1.3 gives

$$\begin{aligned} d\,\mathrm{Im}\,\overline{\partial}\phi(JX, JY) &= X\,\mathrm{Im}\,\overline{\partial}\phi(Y) - Y\,\mathrm{Im}\,\overline{\partial}\phi(X) - \mathrm{Im}\,\overline{\partial}\phi([X, Y]) \\ &= d\,\mathrm{Im}\,\overline{\partial}\phi(X, Y). \end{aligned}$$

This shows that the form

$$b(X,Y) = d\,\mathrm{Im}\,\overline{\partial}\phi(X, JY)$$

is symmetric. To see that it is positive definite, note that

$$-d\,\mathrm{Im}\,\overline{\partial}\phi|_M = \frac{1}{2}\frac{\partial^2\phi}{\partial y^\alpha \partial y^\beta} dx^\alpha \wedge dy^\beta.$$

This shows that b is positive definite when evaluated on M, so it must be positive definite in a neighborhood of M in Ω. □

Remark. Let M, Ω, ϕ be as above. Note that if we set $\alpha = \mathrm{Im}\,\overline{\partial}\phi$, then α vanishes on M, and so M is a Lagrangian submanifold of Ω with respect to the symplectic form $\omega = -d\,\mathrm{Im}\,\overline{\partial}\phi$. We can now apply theorem 1.3.1 to conclude that there is a unique diffeomorphism Ψ of a neighborhood of M in Ω with a neighborhood of M in T^*M such that Ψ is the identity on M, and if α_o is the canonical one form on T^*M, then $\Psi^*\alpha_o = \alpha$. We will use this procedure to construct Kähler structures on cotangent bundles, near the zero section.

Chapter 2

Overview and General Theory

2.1 Overview

Let M be a compact real analytic manifold. By theorem 1.2.4 it is possible to embed M as a totally real submanifold of a complex manifold Ω. By choosing a strictly plurisubharmonic exhaustion function ϕ near M, vanishing to second order on M, it is possible to consider Ω as a Kähler manifold with Kähler form $\sqrt{-1}\partial\overline{\partial}\phi$ and M as a Lagrangian submanifold. Then the Kostant-Sternberg-Weinstein theorem gives a unique symplectic identification of Ω with a neighborhood of M in T^*M. See chapter 1 for details. So it is possible to regard T^*M, at least in a neighborhood of M, as a Kähler manifold whose Kähler form is the standard symplectic form. This identification has been very useful in the study of Toeplitz operators by L. Boutet de Monvel and V. Guillemin; see for example [14] and the survey article [8]. However, this identification is not in any way canonical. It depends strongly on the choice of Bruhat-Whitney embedding and exhaustion function.

The goal of this thesis is to show that if we are given one addition piece of data, a real analytic Riemann metric g on M, then there is associated with (M, g) a canonical embedding of M as a totally real submanifold of a complex manifold. This (canonically determined) complex manifold is in fact a Kähler manifold, and the Kähler structure is intimately connected with the symplectic structure of T^*M. Our result, obtained jointly with my thesis advisor Prof. V. Guillemin, is the following. We have been able to prove it in enough homogeneous cases to perhaps allow us to call it a theorem, although we have not proved it in full generality.

Theorem 2.1.1 *Let M be a compact real analytic manifold equipped with a real analytic Riemannian metric g. Then there is a neighborhood Ω_o of M in T^*M and a unique integrable complex structure J_o on Ω_o such that if α_o is the canonical one form on T^*M, σ_o is the standard involution of T^*M, and ϕ_o is the quadratic function on T^*M associated with the metric g, then*

$$\operatorname{Im} \overline{\partial}\phi_o = \alpha_o$$

and σ_o is an antiholomorphic map, i.e. $\sigma_o^* J_o = -J_o$. This complex structure turns Ω_o into a Kähler manifold whose Kähler form is the standard symplectic form on T^*M.

This theorem describes a completely natural and canonical complex structure on T^*M (near the zero section). It displays a previously unknown connection between the Riemannian structure, the symplectic structure, and the complex geometry of T^*M. The symplectic structure of the cotangent bundle of any smooth manifold is completely canonical and fixed. Given a real analytic Riemann metric on M one obtains a unique complex structure as in theorem 2.1.1. Conversely, given a complex structure J such that $\operatorname{Im} \overline{\partial}\phi = \alpha_o$ for some real analytic function ϕ we can define a Riemann metric g on M by, for X and Y vector fields on M,

$$g(X,Y) = \omega_o(X, JY).$$

If ϕ is a *quadratic* function, i.e. it satisfies the partial differential equation

$$\Xi_o\phi = 2\phi$$

where Ξ_o is the radial vector field on T^*M defined intrinsically by $\iota(\Xi_o)d\alpha_o = \alpha_o$, then the uniqueness part of theorem 2.1.1 implies that $J_g = J$. We postpone the proof until the end of section 2.2.

This theorem may be formulated in several seemingly different but equivalent ways, each of which adds a new insight into the complex geometry of T^*M near the zero section. We will show that theorem 2.1.1 is equivalent to the following result. Choose a Bruhat-Whitney embedding of M into a complex manifold Ω. Then there is a unique real analytic function ϕ on a neighborhood of M in Ω such that

1. $\phi = d\phi = 0$ on M.

2. If σ is the complex conjugation about M then $\sigma^*\phi = \phi$.

3. Set $\alpha = \operatorname{Im} \overline{\partial}\phi$, $\omega = -d\alpha$. If X and Y are vector fields on M, then

$$\omega(X, JY) = g(X, Y)$$

   where J is the complex structure operator on Ω.

4. Define a vector field Ξ in terms of ϕ by $\iota(\Xi)\omega = -\alpha$. Then ϕ satisfies the "Monge-Ampère type" equation

$$\Xi\phi = 2\phi. \tag{2.1}$$

Conditions 1 and 3 are initial conditions for ϕ. They say that ϕ is a defining phase function for M in the Kähler manifold (Ω, ω), and that the associated Kähler metric extends the given metric g on M. Condition 2 is needed to establish the uniqueness of the solution ϕ via a formal power series argument. Condition 4 is the heart of the matter; it says that when ϕ is pulled back to T^*M via the Kostant-Sternberg-Weinstein identification, it is a quadratic function. Equation 2.1 can be written in local holomorphic coordinates as

$$\phi_\alpha \phi^\alpha = 2\phi$$

where $\phi_\alpha = \frac{\partial\phi}{\partial z^\alpha}$, and ϕ^α is defined by $\phi_{\overline{\beta}} = \sqrt{-1}\phi_{\alpha\overline{\beta}}\phi^\alpha$. We call this an equation of

"Monge-Ampère type" because it involves the determinant of the matrix $\phi_{\alpha\overline{\beta}}$.

We will show that on $\Omega\backslash M$, $u = \phi^{\frac{1}{2}}$ satisfies the homogeneous complex Monge-Ampère equation

$$(\partial\overline{\partial}u)^n = 0$$

(n-fold wedge product of $\partial\overline{\partial}u$ with itself), where $n = dim_{\mathbf{C}}\Omega = dim_{\mathbf{R}}M$. Thus we may view Ω, and hence T^*M near the zero section, as a Stein manifold with center M (in the terminology of P. M. Wong [25]) equipped with a Monge-Ampère exhaustion. Such manifolds, called Monge-Ampère manifolds in [25], have been studied by several authors and much is known about their geometry. See Wong [23], [25], [24] and D. Burns [3]. In section 2.4 we will try to survey the most interesting results about such manifolds.

We have not as yet been able to give a proof of theorem 2.1.1 in complete generality. We have been able to reduce the problem to solving locally a single equation of "Monge-Ampère type". We have a formal power series solution of the problem, but the convergence of this series is not yet established. We have been able to prove theorem 2.1.1 in many homogeneous situations, and for the compact Riemannian symmetric spaces. These results extend those of Wong in [25], who essentially proved theorem 2.1.1 for the rank one symmetric spaces (albeit in a different context). Explicit constructions will be given in chapter 4.

For the compact Riemannian symmetric spaces we have been able to show that the complex structure described in 2.1.1 exists *globally* on all of T^*M, not just on a neighborhood of M. We show that T^*M is isomorphic to a complex homogeneous space, and a very explicit description of the complex structure operator is given.

2.2 Formulation of the Problem on a Complex Manifold

In this section we will prove the equivalence of theorem 2.1.1 with the problem of solving a certain partial differential equation of "Monge-Ampère type". The virtue of this approach is that we have to solve only a single equation, rather than the overdetermined system of equations for the matrix entries of the complex structure operator. In addition we obtain a clearer description of the canonical complex structure on T^*M.

Theorem 2.2.1 *Let (M, g) be a compact real analytic Riemannian manifold with a real analytic metric g. Then the following are equivalent:*

1. *There is a neighborhood Ω_o of M in T^*M and a unique integrable complex structure J_o on Ω_o such that if α_o is the canonical one form on T^*M, σ_o is the standard involution of T^*M, and ϕ_o is the quadratic function on T^*M associated with the metric g, then*

$$\operatorname{Im} \overline{\partial}\phi_o = \alpha_o$$

   and σ_o is an antiholomorphic map, i.e. $\sigma_o^* J_o = -J_o$.

2. *There exists a Bruhat-Whitney embedding of M into a complex manifold Ω and a unique real valued, real analytic function ϕ on a neighborhood of M in Ω such that*

   (a) *$\phi = d\phi = 0$ on M.*

   (b) *If σ is the complex conjugation about M, then $\sigma^*\phi = \phi$.*

   (c) *Set $\alpha = \operatorname{Im} \overline{\partial}\phi$ and $\omega = -d\alpha$. If X and Y are vector fields on M and J is the complex structure on Ω, then*

$$\omega(X, JY) = g(X, Y).$$

   (d) *Define a vector field Ξ by $\iota(\Xi)\omega = -\alpha$. Then*

$$\Xi\phi = 2\phi.$$

Proof. It is obvious that 1 implies 2 (the uniqueness follows from the uniqueness of defining phase functions; condition 2c follows from the expression for J_o given in lemma 3.3.1), so we must prove the other direction. The strategy is to establish the local existence of the complex structure J_o; having done that we will show that it is uniquely determined locally. This will enable is to "globalize" by piecing together the local complex structures so obtained.

Let $\mathcal{O}$ be a holomorphic coordinate patch on Ω with holomorphic coordinates $z^1, \ldots, z^n$ where ϕ exists and such that

$$\mathcal{O} \cap M = \{\operatorname{Im} z^1 = \cdots = \operatorname{Im} z^n = 0\}.$$

Let $\mathcal{U} = \mathcal{O} \cap M$. Notice that condition c implies that by shrinking $\mathcal{O}$ we may assume that ω is nondegenerate, so that $(\mathcal{O}, \omega)$ is a symplectic manifold. Since ϕ vanishes to second order on M, M is a Lagrangian submanifold of $\mathcal{O}$. By the Kostant-Sternberg-Weinstein theorem there is a neighborhood (still denoted $\mathcal{O}$) of $\mathcal{U}$, a neighborhood $\mathcal{O}_o$ of $\mathcal{U}$ in $T^*\mathcal{U}$ and a unique diffeomorphism Φ from $\mathcal{O}$ to $\mathcal{O}_o$ such that

$$\Phi^* \alpha_o = \alpha,$$

and $\Phi|_{\mathcal{U}}$ is the identity map. Equip $\mathcal{O}_o$ with the pushforward via Φ of the complex structure on $\mathcal{O}$, which we denote by J_o. This complex structure is clearly integrable. Since σ is an antiholomorphic map on $\mathcal{O}$ and $\Phi^* \sigma_o = \sigma$ it is clear that σ_o is an antiholomorphic map on $\mathcal{O}_o$.

We now show that $\operatorname{Im} \overline{\partial} \phi_o = \alpha_o$. It is clear from the construction that $\alpha_o = \operatorname{Im} \overline{\partial}(\phi \circ \Phi^{-1})$. We need to prove that in terms of the cotangent coordinates ξ_i corresponding to $x^i = \operatorname{Re} z^i$ that $\phi \circ \Phi^{-1}(x, \xi) = g^{ij} \xi_i \xi_j$. Since the vector field Ξ is defined by $\iota(\Xi)\omega = -\alpha$ it is easy to see that $\Phi_* \Xi$ is defined by $\iota(\Phi_* \Xi)\omega_o = -\alpha_o$. This means that $\Phi_* \Xi$ is the radial vector field on T^*M, and in coordinates

$$\Phi_* \Xi(x, \xi) = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}.$$

Then

$$\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}(\phi \circ \Phi^{-1}) = d\phi(\Xi) \circ \Phi^{-1} = 2(\phi \circ \Phi^{-1}).$$

It follows easily that $\phi \circ \Phi^{-1}$ is homogeneous of order 2 in ξ. Since $\phi \circ \Phi^{-1}$ is a smooth function, Taylor's formula shows that it must be a quadratic function in ξ. To determine which quadratic function it is, we must examine the fiber Hessian of $\phi \circ \Phi^{-1}$ along the zero section. Since $\Phi|_{\mathcal{U}}$ is the identity it is clear from conditions a and c above that

$$\frac{\partial^2(\phi \circ \Phi^{-1})}{\partial \xi_\alpha \partial \xi_\beta}|_{\xi=0} = 2g_{ij}(x)\frac{\partial}{\partial \xi_\alpha}(y^i \circ \Phi^{-1})\frac{\partial}{\partial \xi_\beta}(y^j \circ \Phi^{-1})|_{\xi=0}. \tag{2.2}$$

Since ϕ vanishes to second order on $\mathcal{U}$ we have

$$\omega_o|_{\xi=0} = (\Phi^{-1})^*\omega|_{\xi=0} = \frac{1}{2}(\frac{\partial^2 \phi}{\partial y^\alpha \partial y^\beta} \circ \Phi^{-1})d(x^\beta \circ \Phi^{-1}) \wedge d(y^\alpha \circ \Phi^{-1})|_{\xi=0}.$$

Using condition c again and noticing that $\frac{\partial}{\partial x^i}(y^\alpha \circ \Phi^{-1})|_{\xi=0} = 0$ since Φ preserves $\mathcal{U}$ and $y^\alpha \equiv 0$ on $\mathcal{U}$ gives

$$\omega_o|_{\xi=0} = g_{\alpha\beta}\frac{\partial}{\partial \xi_j}(y^\alpha \circ \Phi^{-1})dx^\beta \wedge d\xi_j|_{\xi=0} + g_{\alpha\beta}\frac{\partial}{\partial \xi_i}(x^\beta \circ \Phi^{-1})\frac{\partial}{\partial \xi_j}(y^\alpha \circ \Phi^{-1})d\xi_i \wedge d\xi_j|_{\xi=0}.$$

Since $\omega_o = dx^i \wedge d\xi_i$ this means that

$$\frac{\partial}{\partial \xi_i}(y^\gamma \circ \Phi^{-1})|_{\xi=0} = g^{i\gamma}.$$

Putting this into equation 2.2 shows that $\phi \circ \Phi^{-1} = \phi_o$. This proves the local existence of the complex structure described in 1 above.

Now we claim that this complex structure is unique. Suppose there are two such complex structures, J_o and J_o', on $\mathcal{O}_o$ such that $\text{Im}\,\overline{\partial}\phi_o = \alpha_o = \text{Im}\,\overline{\partial}'\phi_o$ (the prime means with respect to the complex structure J_o'), and the standard involution σ_o is an antiholomorphic map. By lemma 1.2.1 we can find a diffeomorphism f of a (possibly smaller) neighborhood $\mathcal{O}_o$ of $\mathcal{U}$ in $T^*\mathcal{U}$ onto itself such that

$$f_* J' = J$$

(i.e. $df \circ J' \circ df^{-1} = J$) and which is the identity on $\mathcal{U}$. We will show that f preserves the quadratic function ϕ_o.

On each of the complex manifolds $(\mathcal{O}_o, J_o)$ and $(\mathcal{O}_o, J_o')$ we pose the "Monge-Ampère" problem in 2 of the statement of theorem 2.2.1 and find unique solutions ϕ and ϕ'. Since $\operatorname{Im} \overline{\partial}\phi_o = \operatorname{Im} \overline{\partial}' \phi_o = \alpha_o$ we see that ϕ_o is the unique solution to both problems (condition 2c follows from lemma 3.3.1). So we must have $\phi = \phi' = \phi_o$. To show that f preserves ϕ_o it suffices to show that $\phi' = f^*\phi$, i.e. that $f^*\phi$ is the unique solution for the problem given in 2 for the complex structure J_o'. First we must check that $f^*\phi$ is invariant under the complex conjugation about $\mathcal{U}$ with respect to the complex structure J_o'. This conjugation must be the standard involution σ_o of $T^*\mathcal{U}$ by the uniqueness part of lemma 1.2.2. Since ϕ is invariant under σ_o by hypothesis, we must check that $f \circ \sigma_o = \sigma_o \circ f$. Consider the map $F = f^{-1} \circ \sigma_o \circ f \circ \sigma_o$. It is easy to see that F is a biholomorphism of the complex manifold $(\mathcal{O}_o, J_o)$ which is the identity on $\mathcal{U}$. By the uniqueness part of lemma 1.2.1, F must be the identity. This proves that $f^*\phi$ is invariant under σ_o.

Next we check that $f^*\phi$ satisfies 2c above. Set $\alpha' = \operatorname{Im} \overline{\partial}' f^*\phi$ and $\omega' = -d\alpha'$. Then

$$\operatorname{Im} \overline{\partial}' f^*\phi = f^* \operatorname{Im} \overline{\partial}\phi$$

and so $\omega' = f^*\omega$. Since f is the identity on $\mathcal{U}$, for any vector field X tangent to $\mathcal{U}$ we have (on $\mathcal{U}$) that $df(X) = X$. Then for any vector fields X and Y on $\mathcal{U}$ and any $x \in \mathcal{U}$ we have

$$\omega'(X, J'Y)(x) = \omega(df(X), df(J'Y))(f(x))$$

$$= \omega(X, Jdf(Y))(x)$$

$$= \omega(X, JY)(x) = g(X, Y)(x).$$

This shows that $f^*\phi$ satisfies condition 2c above.

It is clear that $f^*\phi = df^*\phi = 0$ on $\mathcal{U}$. It remains to check that if we define a vector field Ξ' by $\iota(\Xi')d\alpha' = \alpha'$ then $\Xi' f^*\phi = 2f^*\phi$. Since $\alpha' = f^*\alpha$ we have

$$f^*\iota(f_*\Xi')d\alpha = f^*\alpha.$$

This shows that $f_*\Xi' = \Xi$. Then

$$\Xi'(f^*\phi) = df^{-1}(\Xi \circ f)(f^*\phi)$$

$$= d\phi(\Xi \circ f) = 2f^*\phi.$$

This shows that $f^*\phi$ is the unique solution to the problem given in 2 for the complex manifold $(\mathcal{O}_o, J_o')$, so that $f^*\phi = \phi_o$.

Since f preserves ϕ_o it also preserves $\alpha_o = \mathrm{Im}\,\overline{\partial}\phi_o$. By the uniqueness part of the Kostant-Sternberg-Weinstein theorem, this forces f to be the identity. Then $J_o = J_o'$ and we have established the local uniqueness. We can now cover M by a finite number of open sets in T^*M, each carrying a complex structure as in part 1 of theorem 2.2.1. By the uniqueness they agree on overlaps. This defines a complex structure on a neighborhood of M in T^*M which has the properties described in 1. □

We can now prove the result stated in section 2.1. We want to show that if J is a complex structure on T^*M near M such that $\mathrm{Im}\,\overline{\partial}\phi = \alpha_o$ for some quadratic function ϕ, then J is equal to the complex structure J_g corresponding to the metric $g(\cdot,\cdot) = \omega_o(\cdot, J\cdot)$. Note that ϕ is the solution of the "Monge-Ampère type" problem for the metric g with respect to the complex structure J. Then the proof of theorem 2.2.1 shows that there is a unique symplectic diffeomorphism Φ which is the identity on M, carrying $\mathrm{Im}\,\overline{\partial}\phi$ into α_o, and such that $J_g = \Phi_* J$. But since $\mathrm{Im}\,\overline{\partial}\phi = \alpha_o$, Φ must be the identity map (see the uniqueness part of the proof of theorem 1.3.1).

2.3 The Complex Homogeneous Monge-Ampère Equation

The complex homogeneous Monge-Ampère equation is

$$det\ \frac{\partial^2 u}{\partial z^i \partial \overline{z}^j} = 0$$

where u is a twice differentiable function and $z^1, \ldots, z^n$ is a local holomorphic coordinate system. This can be written invariantly as

$$(\partial\overline{\partial}u)^n = 0$$

(n-fold wedge product of $\partial\overline{\partial}u$ with itself), where n is the complex dimension of the ambient complex manifold. In this section we will show that if ϕ is a solution of the "Monge-Ampére type" problem posed in theorem 2.2.1, then $u = \phi^{\frac{1}{2}}$ satisfies the complex homogeneous Monge-Ampère equation away from the "center" M. We will also show that u satisfies the nondegeneracy condition

$$(\partial\overline{\partial}u)^{n-1} \neq 0$$

away from M.

If ϕ is a solution of the "Monge-Ampére type" problem posed in theorem 2.2.1, the initial conditions on ϕ imply that it has a strict minimum on M. We may assume, by shrinking Ω if need be, that $\phi \neq 0$ and $u = \phi^{\frac{1}{2}}$ is smooth on $\Omega \backslash M$. If ω, α, and Ξ are as in the statement of theorem 2.2.1, then on $\Omega \backslash M$

$$\alpha = 2u\ \mathrm{Im}\,\overline{\partial}u$$

$$\omega = (-2)(du \wedge \mathrm{Im}\,\overline{\partial}u + u\, d(\mathrm{Im}\,\overline{\partial}u))$$

$$\Xi u = u.$$

The equation $\iota(\Xi)\omega = -\alpha$ can be written

$$du(\Xi)\mathrm{Im}\,\overline{\partial}u - \mathrm{Im}\,\overline{\partial}u(\Xi)du + u\iota(\Xi)d\,\mathrm{Im}\,\overline{\partial}u = u\,\mathrm{Im}\,\overline{\partial}u.$$

It's clear from the definition of Ξ that $\alpha(\Xi) = 0$, so $\mathrm{Im}\,\overline{\partial}u(\Xi) = 0$. Using $du(\Xi) = \Xi u = u$ we have

$$\iota(\Xi)d\,\mathrm{Im}\,\overline{\partial}u = 0. \tag{2.3}$$

Now $\Xi \neq 0$ on $\Omega\backslash M$, so this says that the two-form $\partial\overline{\partial}u$ has (real) rank less than $2n$. Then the $2n$ form $(\partial\overline{\partial}u)^n$ must be zero on $\Omega\backslash M$. Since

$$\omega^n = 2^n(nu^{n-1}\partial u \wedge \overline{\partial}u \wedge (\partial\overline{\partial}u)^{n-1})$$

and ω^n is a volume form, it is clear that the nondegeneracy condition $(\partial\overline{\partial}u)^{n-1} \neq 0$ holds.

Conversely, suppose u is a solution of the homogeneous Monge-Ampère equation. If $\phi = u^2$ is strictly plurisubharmonic, then it is shown in §3 of [25] that ϕ satisfies

$$\phi^{\alpha\overline{\beta}}\phi_\alpha\phi_{\overline{\beta}} = 2\phi.$$

Set $\alpha = \mathrm{Im}\,\overline{\partial}\phi$, $\omega = -d\alpha$, and define Ξ by $\iota(\Xi)\omega = -\alpha$. This definition may be written as

$$\iota(\Xi)\partial\overline{\partial}\phi = \frac{1}{2}(\overline{\partial}\phi - \partial\phi).$$

Write $\Xi = \Xi^\gamma\frac{\partial}{\partial z^\gamma} + \Xi^{\overline{\gamma}}\frac{\partial}{\partial \overline{z}^\gamma}$. Then a short computation shows that

$$\Xi^\gamma = \frac{1}{2}\phi^{\gamma\overline{\beta}}\phi_{\overline{\beta}} \tag{2.4}$$

$$\Xi^{\overline{\gamma}} = \frac{1}{2}\phi^{\alpha\overline{\gamma}}\phi_\alpha,$$

so that $\Xi\phi = \phi^{\alpha\overline{\beta}}\phi_\alpha\phi_{\overline{\beta}} = 2\phi$.

2.4 Monge-Ampère Manifolds and Foliations

Let Ω be a n dimensional complex manifold, $u: \Omega \to [0, \infty)$ a strictly plurisubharmonic function, and let $M = \{u = 0\}$. We say that Ω is a Monge-Ampère manifold if u is continuous on Ω, smooth on $\Omega \backslash M$ and

$$(\partial\overline{\partial}u)^n = 0$$

on $\Omega \backslash M$. We will assume that M is a smooth manifold of real dimension n. Monge-Ampère manifolds have been studied by several authors. See for example Wong [22], [24] and Burns [3], [4]. The results in this section are mostly due to Wong [24]. They are new only in that they can be applied to the cotangent bundle of a compact real analytic Riemannian manifold, assuming theorem 2.2.1. Note that the initial conditions on u in theorem 2.2.1 insure that u is strictly plusrisubharmonic and positive on a neighborhood of M, so by shrinking Ω we may assume this is true on Ω.

Since $(\partial\overline{\partial}u)^n = 0$ and $(\partial\overline{\partial}u)^{n-1} \neq 0$, the distribution F defined pointwise by

$$F_\zeta = \{V \in T_\zeta\Omega: \iota(V)\partial\overline{\partial}u = 0\}$$

is two (real) dimensional. The distribution F is integrable, since if V, W are vector fields with values in F, then

$$\iota([V, W])\partial\overline{\partial}u = L_V\iota(W)(\partial\overline{\partial}u) - \iota(W)L_V(\partial\overline{\partial}u)$$

$$= 0 - \iota(W)(d\iota(V)\partial\overline{\partial}u - \iota(V)d\partial\overline{\partial}u) = 0$$

since $\partial\overline{\partial}u$ is closed. F is also a complex distribution: if $V \in F$ and X is any vector field, then

$$\iota(JV)\partial\overline{\partial}u(X) = \partial\overline{\partial}(JV, X) = -\partial\overline{\partial}(V, JX) = 0.$$

Thus Ω carries a foliation $\mathcal{F}$ whose leaves are complex one dimensional submanifolds, i.e.

Riemann surfaces. $\mathcal{F}$ is called the Monge-Ampère foliation.

Let Ξ be the vector field defined in theorem 2.2.1. By equation 2.3, Ξ and $J\Xi$ span F. The leaves of $\mathcal{F}$ may be thought of as the flows of the type $(1,0)$ vector field

$$Z = \Xi - \sqrt{-1}J\Xi.$$

This vector field is the complex gradient of $\phi = u^2$ with respect to the Hermitian inner product

$$< V, W >= \partial\overline{\partial}\phi(V, \overline{W}).$$

In other words Z is defined by

$$\partial\overline{\partial}\phi(Z, \overline{W}) = \overline{\partial}\phi(\overline{W}).$$

In coordinates it is easy to see that

$$Z = \phi^{\alpha\overline{\beta}}\phi_{\overline{\beta}}\frac{\partial}{\partial z^{\alpha}},$$

so by equation 2.4, $\Xi = \mathrm{Re}(Z)$. Note that the "Monge-Ampère type" equation in theorem 2.2.1 can be written as

$$Z\phi = 2\phi =< Z, Z > .$$

The first interesting result about the Monge-Ampère foliation is that the leaves are flat and totally geodesic. This is proved in Wong [24]. We will give a slightly different and more intrinsic proof. First we compute the covariant differentiation along the leaves, using a few simple observations which are of independent interest.

Lemma 2.4.1 *Let ω, ϕ, and Ξ be as in theorem 2.2.1 and let b denote the Kähler metric $b(X, Y) = \omega(X, JY)$. Then the vector field Ξ is orthogonal to the level sets*

$$\Omega_c = \{\zeta \in \Omega : \phi(\zeta) = c\} \ \ (c > 0),$$

$J\Xi$ is tangent to the level sets Ω_c, and the vector fields $\frac{1}{\sqrt{\phi}}\Xi$, $\frac{1}{\sqrt{\phi}}J\Xi$ form an orthonormal basis for the Monge-Ampère distribution.

Proof. Let V be a vector tangent to one of the level sets Ω_c. Then

$$b(\Xi, V) = \omega(\Xi, JV) = -\alpha(JV)$$

$$= -\mathrm{Im}\,\overline{\partial}\phi(JV) = \frac{1}{2}d\phi(V) = 0.$$

To see that $J\Xi$ is tangent to Ω_c, note that

$$(J\Xi)\phi = d\phi(J\Xi) = 2\mathrm{Im}\,\overline{\partial}\phi(\Xi) = -2\omega(\Xi, \Xi) = 0.$$

The modulus squared of Ξ with respect to the metric b is

$$b(\Xi, \Xi) = \omega(\Xi, J\Xi) = -\mathrm{Im}\,\overline{\partial}\phi(J\Xi)$$

$$= \frac{1}{2}d\phi(\Xi) = \phi.$$

This implies that the modulus squared of $J\Xi$ is also equal to ϕ. □

Lemma 2.4.2 $L_\Xi\omega = \omega$.

Proof. $L_\Xi\omega = \iota(\Xi)d\omega + d\iota(\Xi)\omega = 0 + d(-\alpha) = \omega$. □

Lemma 2.4.3 $[\Xi, J\Xi] = J\Xi$.

Proof. For any vector field V we have $b(J\Xi, V) = \omega(\Xi, V) = -\alpha(V) = -\frac{1}{2}d\phi(JV)$, and $b([\Xi, J\Xi], V) = \iota([\Xi, J\Xi])\omega(JV)$. We compute

$$\iota([\Xi, J\Xi])\omega = L_\Xi\iota(J\Xi)\omega - \iota(J\Xi)L_\Xi\omega$$

$$= L_\Xi\iota(J\Xi)\omega - \iota(J\Xi)\omega.$$

A simple computation shows that $\iota(J\Xi)\omega = -\frac{1}{2}d\phi$. Then

$$\iota([\Xi, J\Xi])\omega = -\frac{1}{2}dL_\Xi\phi + \frac{1}{2}d\phi = -d\phi + \frac{1}{2}d\phi = -\frac{1}{2}d\phi.$$

Thus $b(J\Xi, V) = b([\Xi, J\Xi], V)$. □

Proposition 2.4.4 *Let ∇ be the Levi-Cevita connection of the Kähler metric b, and let Ξ be as above. Then*

$$\begin{aligned} \nabla_\Xi \Xi &= \Xi \\ \nabla_\Xi J\Xi &= J\Xi \\ \nabla_{J\Xi}\Xi &= \nabla_{J\Xi}J\Xi = 0. \end{aligned}$$

Proof. Since Ω is a Kähler manifold, the complex structure is parallel with respect to ∇. Then since ∇ has no torsion, we need only compute $\nabla_\Xi\Xi$. Suppose V is a vector field tangent to the level sets $\Omega_c = \{\phi = c\}$ $(c > 0)$. Then

$$2b(\nabla_\Xi\Xi, V) = 2\Xi b(\Xi, V) - Vb(\Xi, \Xi) - 2b([\Xi, V], \Xi).$$

The first term is zero by lemma 2.4.1. By lemma 2.4.1, the second term is $-V\phi$, which is also zero if V is tangent to the level sets of ϕ. The last term is

$$b([\Xi, V], \Xi) = \omega(\Xi, J[\Xi, V]) = \frac{1}{2}d\phi([\Xi, V])$$

$$= (\Xi V\phi - V\Xi\phi)$$

$$= \frac{1}{2}(0 - V\phi) = 0.$$

Thus $\nabla_\Xi\Xi$ is orthogonal to the level sets of ϕ, so it must be a multiple of Ξ. To find which one, we compute

$$2b(\nabla_\Xi \Xi, \Xi) = \Xi b(\Xi, \Xi) = \Xi\phi = 2\phi.$$

This proves that $\nabla_\Xi \Xi = \Xi$. It follows immediately that $\nabla_\Xi J\Xi = J\Xi$ and $\nabla_{J\Xi}\Xi = J\Xi + [J\Xi, \Xi]$, which in view of lemma 2.4.3 proves the proposition. □

Corollary 2.4.5 *The leaves of the Monge-Ampère foliation are totally geodesic.*

Using this information it is easy to compute that the curvature along leaves is zero.

Proposition 2.4.6 *The leaves of the Monge-Ampère foliation are flat.*

Proof. We will show that the $(1,3)$ curvature tensor

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

vanishes using the basis Ξ, $J\Xi$ of the Monge-Ampère distribution. There are only two potentially nonzero ones that we have to check:

$$R(J\Xi, \Xi, \Xi) = 2\nabla_{J\Xi}\Xi = 0$$

$$R(J\Xi, \Xi, J\Xi) = 2\nabla_{J\Xi} J\Xi - \nabla_\Xi \nabla_{J\Xi} J\Xi = 0.$$

Hence the leaves are flat. □

Remark. It's easy to see that the integral curves of the orthonormal basis $\frac{1}{\sqrt{\phi}}\Xi$, $\frac{1}{\sqrt{\phi}}J\Xi$ are geodesics.

2.5 Some Interesting Results

In this section we collect some interesting results about the Monge-Ampère foliation. Most of them are due to Wong [24]. We also record an interesting connection between the gradient of the exhaustion function ϕ with respect to the Kähler metric b and the Hamiltonian vector field associated to ϕ by the symplectic structure, which seems to be new.

The following was proved in [24], but we will give a much simpler proof.

Proposition 2.5.1 *The base M is a totally geodesic submanifold of Ω.*

Proof. Let σ be the antiholomorphic involution of a neighborhood of M in Ω whose fixed point set is M (see chapter 1). Then σ is an antisymplectic map, since for any vector field V on a neighborhood of M,

$$\sigma^*\alpha(V) = \frac{1}{2}d\phi(Jd\sigma(V))$$

$$= -\frac{1}{2}d(\sigma^*\phi)(JV) = -\alpha(V).$$

It follows easily that σ is an isometry of the Kähler metric b. If γ is a geodesic tangent to M at a point $p \in M$ with tangent V_p at p, then $\sigma \circ \gamma$ is also a geodesic tangent to M at p with tangent vector V_p at p. By uniqueness of geodesics, $\sigma \circ \gamma = \gamma$ (at least for short time). Hence σ fixes γ, so γ lies in M. $\square$

Proposition 2.5.2 *The distance minimizing geodesics between the level sets of ϕ are integral curves of the vector field $\frac{1}{\sqrt{\phi}}\Xi$, and the geodesic distance between level sets Ω_b and Ω_a is $|\phi^{\frac{1}{2}}(b) - \phi^{\frac{1}{2}}(a)|$.*

Proof. See Wong [24]. $\square$

P. M. Wong gives a very explicit description of the leaves of the Monge-Ampère foliation. Let $F: TM \to \Omega$ be the map

$$F(V_p) = \mathrm{Exp}_p(JV)$$

where Exp is the Riemannian exponential map of the Kähler metric associated with $\sqrt{-1}\partial\overline{\partial}\phi$. This map is a global diffeomorphism. If γ is a geodesic on M, then the image of the set

$$\{(\gamma(t), -s\dot{\gamma}(t)) : s, t \in \mathbb{R}\} \subset TM$$

contains a leaf of the Monge-Ampère foliation. This gives the following result.

Proposition 2.5.3 *The Monge-Ampère foliation extends across the center M. The intersection of each leaf with M is a geodesic on M, and through each geodesic on M there passes a unique extended leaf of the Monge-Ampère foliation.*

Proof. See Wong [24], §5, theorem 5.1. □

Finally we show that the complex structure operator takes the gradient vector field of ϕ into (minus one times) the Hamiltonian vector field of ϕ.

Proposition 2.5.4 *Let Ξ be defined as in theorem 2.2.1, and let b denote the Riemannian metric associated with the Kähler form ω. Then*

$$\Xi = \frac{1}{2}\mathrm{grad}_b\phi$$

$$J\Xi = -\frac{1}{2}H_\phi$$

where H_ϕ is the Hamiltonian vector field associated with ϕ by the Kähler form ω.

Proof. For any vector field V on Ω,

$$b(\Xi, V) = -\alpha(JV) = \frac{1}{2}d\phi(V).$$

Hence $\Xi = \frac{1}{2}\mathrm{grad}_b\phi$. On the other hand,

$$\omega(J\Xi, V) = -\omega(\Xi, JV) = \alpha(JV) = -\frac{1}{2}d\phi(V).$$

This shows that $J\Xi = -\frac{1}{2}H_\phi$, where H_ϕ is the Hamiltonian vector field associated with ϕ. □

Corollary 2.5.5 *The Monge-Ampère distribution is spanned by the gradient vector field of ϕ and the Hamiltonian vector field of ϕ.*

This result enables us to interpret Wong's results in terms of Hamiltonian mechanics, using the Kostant-Sternberg-Weinstein identification of Ω with T^*M near M. Motion on a Riemannian manifold is described by the flow of $X_{\frac{1}{2}\phi_o}$, where $\phi_o(\xi) = |\xi|_g^2$. The integral curves of $X_{\frac{1}{2}\phi_o}$, when projected to M by the cotangent projection, are geodesics. This explains why the Monge-Ampère foliation intersects the base in geodesics.

Since the flows of $X_{\frac{1}{2}\phi_o}$ preserve the sphere bundle $\Sigma_c = \{\xi \in T^*M : \phi_o(\xi) = c\}$, it follows that the flow of $X_{\frac{1}{2}\phi}$ on Ω preserves the level sets of ϕ.

Corollary 2.5.6 *The flow of $J\Xi$ preserves the level sets Ω_c.*

Chapter 3

Formal Proof of the Result

3.1 The Formal Power Series Solution

Let (M, g) be a real analytic Riemannian manifold. In section 2.2 we showed that theorem 2.1.1 is equivalent to the following theorem.

Theorem 3.1.1 *There exists a Bruhat-Whitney embedding of M into a complex manifold Ω and a unique real valued, real analytic function ϕ on a neighborhood of M in Ω such that*

1. $\phi = d\phi = 0$ *on* M.

2. *If σ is the complex conjugation about M, then $\sigma^*\phi = \phi$.*

3. *Set $\alpha = \operatorname{Im} \overline{\partial}\phi$ and $\omega = -d\alpha$. If X and Y are vector fields on M and J is the complex structure on Ω, then*

$$\omega(X, JY) = g(X, Y).$$

4. *Define a vector field Ξ by $\iota(\Xi)\omega = -\alpha$. Then*

$$\Xi\phi = 2\phi.$$

In this section we will give a *formal* proof of this theorem. We will show that theorem 3.1.1 is equivalent to a certain local problem. We will write down a power series solution to this problem, and establish the uniqueness of this solution, if it exists. But the proof will be only formal in that we will not establish the convergence of this power series in general. In chapter 4 we will indirectly prove the convergence of this power series for a large class of homogeneous examples, by constructing an explicit complexification of T^*M near the zero section with the properties described in theorem 2.1.1. In many cases this complex structure will exist globally on T^*M (see for example theorem 4.5.1)

3.1.1 Reduction to a Local Problem

We know from the Bruhat-Whitney embedding theorem (see theorem 1.2.4) that it is always possible to embed a compact, real analytic manifold M as a totally real submanifold of a complex manifold Ω. Fix such an embedding. Let $\mathcal{O}$ be a holomorphic coordinate patch on Ω with coordinates $z^1, \ldots, z^n$ such that

$$\mathcal{O} \cap M = \{\operatorname{Im} z^1 = \cdots = \operatorname{Im} z^n = 0\}.$$

In this section we will express the conditions 1 through 4 in theorem 3.1.1 in terms of these coordinates, and reduce the proof of theorem 3.1.1 to a local problem.

Write $z^i = x^i + \sqrt{-1}y^i$, so that M is given (locally) by $y^1 = \cdots = y^n = 0$. The condition that $\phi = d\phi = 0$ on M then means that ϕ vanishes to second order in y at $y = 0$, or in terms of the Taylor series expansion of ϕ,

$$\phi(x, y) = a_{\alpha\beta}(x) y^\alpha y^\beta + O(y^3).$$

We are using the implied summation convention. Here $a_{\alpha\beta}$ are analytic functions of $x^1, \ldots, x^n$ and $O(y^3)$ means $(\phi(x,y) - a_{\alpha\beta}y^\alpha y^\beta)/|y|^3 < \infty$ as $|y| \to 0$. The complex conjugation about M in these coordinates is $\sigma(x + \sqrt{-1}\,y) = x - \sqrt{-1}\,y$, so the condition $\sigma^*\phi = \phi$ means that $\phi(x, -y) = \phi(x, y)$.

To interpret condition 3 in theorem 3.1.1, note that

$$d\,\mathrm{Im}\,\overline{\partial}\phi = \mathrm{Im}\,\partial\overline{\partial}\phi = \frac{1}{\sqrt{-1}}\partial\overline{\partial}\phi,$$

since $\partial\overline{\partial}\phi$ is purely imaginary if ϕ is a real valued function. Then

$$\begin{aligned}\omega &= \sqrt{-1}\partial\overline{\partial}\phi = \frac{1}{2}\frac{\partial^2\phi}{\partial y^i\partial y^j}dx^i\wedge dy^j + O(y)\\ &= a_{ij}(x)dx^i\wedge dy^j + O(y).\end{aligned}$$

So the condition $\omega(X, JY) = g(X, Y)$ for vector fields X, Y on M means that $a_{ij}(x) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \stackrel{def}{=} g_{ij}(x)$. Finally for condition 4 note that in section 2.3, page 26 we have shown that the equation $\Xi\phi = 2\phi$ can be written in coordinates as

$$\phi^{\alpha\overline{\beta}}\phi_\alpha\phi_{\overline{\beta}} = 2\phi.$$

It will be convenient to define ϕ^α by

$$\phi_{\overline{\beta}} = \sqrt{-1}\phi_{\alpha\overline{\beta}}\phi^\alpha.$$

This makes sense near $\{y^1 = \cdots y^n = 0\}$ since condition 3 means that $\phi_{\alpha\overline{\beta}}$ evaluated at $y = 0$ is a nonsingular matrix. Then we can write condition 4 as

$$\sqrt{-1}\phi_\alpha\phi^\alpha = 2\phi.$$

Suppose for any local representation $g_{ij}(x)$ of the metric g on M we can find a unique real valued, real analytic function ρ on a neighborhood in $\mathbb{C}^n$ of an open set in $\mathbb{R}^n$ such that

1. $\rho(x, -y) = \rho(x, y)$
2. $\rho(x, y) = g_{\alpha\beta}(x)y^\alpha y^\beta + O(y^4)$

3. $\sqrt{-1}\rho_\alpha\rho^\alpha = 2\rho$.

We claim that being able to solve this problem implies theorem 3.1.1. Cover M by holomorphic coordinate patches $\mathcal{O}_i$ on Ω with coordinate functions $\psi_i = (z_i^1, \ldots, z_i^n)$ such that each component of $\mathcal{O}_i$ contains only one component of M, a solution to the local problem exists on $\psi(\mathcal{O}_i)$, and such that if $\mathcal{O}_i \cap \mathcal{O}_j$ is not empty, then $\mathcal{O}_i \cap \mathcal{O}_j \cap M$ is not empty. See appendix A.1 for why we may arrange this, given that we can solve the local problem above. Define the function ϕ we seek in theorem 3.1.1 on the open set $\mathcal{O}_i$ to be the local solution $\rho_i \circ \psi_i$, where ρ_i is obtained by solving the local problem on $\psi(\mathcal{O}_i)$ with initial data in condition 2 given by $g_{\alpha\beta} = g(\frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_i^\beta})$. We need to show that if $\mathcal{O}_i \cap \mathcal{O}_j$ is not empty, and ρ_j is the solution to the local problem on $\psi_j(\mathcal{O}_j)$, then $\rho_i \circ \psi_i = \rho_j \circ \psi_j$ on $\mathcal{O}_i \cap \mathcal{O}_j$. Since $\mathcal{O}_i \cap \mathcal{O}_j \cap M$ is not empty, we can pose the local problem on $\psi_i(\mathcal{O}_i \cap \mathcal{O}_j)$, with initial data given by the coordinate representation $g_{\alpha\beta}$ of g in the coordinates $(\mathcal{O}_i, \psi_i)$. We must show that if we set $\tilde{\rho} = \rho_j \circ \psi_j \circ \psi_i^{-1}$ then $\tilde{\rho}$ solves this local problem; then by uniqueness we can conclude that $\rho_i \circ \psi_i = \rho_j \circ \psi_j$. It's obvious that $\tilde{\rho}$ satisfies condition 1, since in both coordinate systems M is given by $\operatorname{Im} z^1 = \cdots \operatorname{Im} z^n = 0$ and so the unique antiholomorphic involution fixing M is given by $z \to \bar{z}$ in both coordinate systems (see lemma 1.2.2). The other two conditions can also be formulated in a coordinate independent way as in theorem 3.1.1, so they are satisfied in any coordinate system where M is given by $\operatorname{Im} z^1 = \cdots = \operatorname{Im} z^n = 0$. Thus $\tilde{\rho}$ solves the local problem on $\psi_i(\mathcal{O}_i \cap \mathcal{O}_j)$. By uniqueness of solutions to the local problem, we conclude that $\tilde{\rho} = \rho_i$ on $\mathcal{O}_i \cap \mathcal{O}_j$. Then the function ϕ we get by patching together the local solutions is well defined, and solves the "Monge-Ampère type" problem posed in theorem 3.1.1. This we have proved that theorem 3.1.1 is equivalent to the following theorem.

Theorem 3.1.2 *Given any positive definite matrix of real analytic functions $g_{ij}(x)$ on an open set $\mathcal{U}$ in $\mathbb{R}^n$, there is a unique real valued, real analytic function ρ on a neighborhood $\mathcal{O}$ of $\mathcal{U}$ in $\mathbb{C}^n$ such that*

1. $\rho(x, -y) = \rho(x, y)$

2. $\rho(x,y) = g_{\alpha\beta}(x)y^\alpha y^\beta + O(y^4)$

3. $\sqrt{-1}\rho_\alpha\rho^\alpha = 2\rho$.

3.1.2 Formal Solution of the Local Problem

In this section we construct a formal power series solution to the local problem in theorem 3.1.2. Let ρ by a real analytic function on some neighborhood of 0 in $\mathbb{C}^n$, with coordinates $z^\alpha = x^\alpha + \sqrt{-1}\, y^\alpha$. We will write $\rho_\alpha = \partial\rho/\partial z^\alpha$, and when it makes sense define ρ^α by

$$\rho_{\overline{\beta}} = \sqrt{-1}\,\rho_{\alpha\overline{\beta}}\rho^\alpha. \tag{3.1}$$

We are of course using the implied summation convention. This will make sense in some neighborhood of zero if $\rho_{\alpha\overline{\beta}}$ evaluated at zero is a nonsingular matrix. Define $\rho^{\overline{\alpha}}$ by taking the complex conjugate of equation 3.1. We will need some preliminary results.

Lemma 3.1.3 *If $\rho(x,-y) = \rho(x,y)$, then the power series expansion of ρ has the form*

$$\rho(x,y) = \sum_{|\alpha|\ even} a_\alpha(x)y^\alpha.$$

Proof. Since ρ is real analytic, we can write

$$\rho(x,y) = \sum_{|\alpha|\ even} a_\alpha(x)y^\alpha + \sum_{|\alpha|\ odd} b_\alpha(x)y^\alpha.$$

The map $(x,y) \to (x,-y)$ fixes the right hand side. $\square$

Lemma 3.1.4 *Suppose $\rho(x,-y) = \rho(x,y)$. Then*

$$\rho_\alpha\rho^\alpha(x,-y) = \rho_\alpha\rho^\alpha(x,y).$$

Proof. If ρ is an even function of y, then $\partial\rho/\partial y^\alpha$ is an odd function of y. Then

$$\rho_\alpha(x,-y) = \rho_{\overline{\alpha}}(x,y),$$

and it's easy to see that

$$\rho_{\alpha\overline{\beta}}(x,-y) = \rho_{\beta\overline{\alpha}}(x,y).$$

Then evaluating the equation 3.1 at $(x,-y)$ gives

$$\rho_\beta(x,y) = \sqrt{-1}\,\rho_{\beta\overline{\alpha}}(x,y)\rho^\alpha(x,-y)$$

which means that $\rho^\alpha = -\rho^{\overline{\alpha}}$. Then

$$\begin{aligned}\rho_\alpha\rho^\alpha(x,-y) &= -\rho_{\overline{\alpha}}\rho^{\overline{\alpha}}(x,y)\\ &= \rho_\alpha\rho^\alpha(x,y),\end{aligned}$$

since $\rho_\alpha\rho^\alpha$ is purely imaginary. □

Lemma 3.1.5 *Suppose $\rho(x,y) = g_{\alpha\beta}(x)y^\alpha y^\beta + O(y^4)$, where $g_{\alpha\beta}(x)$ is a nonsingular matrix of functions. Then*

$$\rho^{\overline{\beta}}(x,y) = 2y^\beta + O(y^2).$$

Proof. Substituting $\rho(x,y) = g_{\alpha\beta}y^\alpha y^\beta + O(y^4)$ into the equation defining $\rho^{\overline{\beta}}$ we see that

$$g_{\alpha i}y^i = \frac{1}{2}(g_{\alpha\beta} + O(y))\rho^{\overline{\beta}} + O(y^2).$$

From this expression it's clear that $\rho^{\overline{\beta}}(x,0) = 0$. Writing $\rho^{\overline{\beta}} = c^{\overline{\beta}}_m(x)y^m + O(y^2)$ and equating terms of first order in y on both sides gives $c^{\overline{\beta}}_m(x)y^m = 2y^\beta$. □

We will need to improve this result considerably.

Lemma 3.1.6 *Let $\rho_{k-1}(x,y)$ be a polynomial in y with coefficients depending on x of degree $2(k-1)$ $(k \geq 2)$ such that $\rho_{k-1}(x,-y) = \rho_{k-1}(x,y)$ and $\rho_{k-1}(x,y) = g_{\alpha\beta}(x)y^\alpha y^\beta +$*

$O(y^4)$, where $g_{\alpha\beta}(x)$ is a nonsingular matrix of functions. Let $\psi_k(x,y)$ be a homogeneous polynomial in y of degree $2k$ (with coefficients depending on x), and suppose that $\rho_k = \rho_{k-1} + \psi_k$. Then

$$(\rho_k)^{\overline{\beta}} = (\rho_{k-1})^{\overline{\beta}} - 2(k-1)g^{\alpha\beta}\frac{\partial\psi_k}{\partial y^\alpha} + O(y^{2k}).$$

Proof. Substitute $\rho_k = \rho_{k-1} + \psi_k$ and $(\rho_k)^{\overline{\beta}} = 2y^\beta + O(y^2)$ into the equation defining $(\rho_k)^{\overline{\beta}}$. This gives

$$\sqrt{-1}(\rho_{k-1})_\alpha + \frac{1}{2}\frac{\partial\psi_k}{\partial y^\alpha} = (\rho_{k-1})_{\alpha\overline{\beta}}(\rho_k)^{\overline{\beta}} + \frac{1}{2}(\frac{\partial^2\psi_k}{\partial y^\alpha\partial y^\beta})y^\beta + O(y^{2k}). \tag{3.2}$$

Recall that Euler's relation says in this context that

$$y^\beta\frac{\partial}{\partial y^\beta}(\frac{\partial\psi_k}{\partial y^\alpha}) = (2k-1)\frac{\partial\psi_k}{\partial y^\alpha}.$$

Using this in equation 3.2 gives

$$\sqrt{-1}(\rho_{k-1})_\alpha + (1-k)\frac{\partial\psi_k}{\partial y^\alpha} = (\rho_{k-1})_{\alpha\overline{\beta}}(\rho_k)^{\overline{\beta}} + O(y^{2k}).$$

Note that $(\rho_{k-1})_{\alpha\overline{\beta}} = \frac{1}{2}g_{\alpha\beta} + O(y)$, so $(\rho_{k-1})^{\alpha\overline{\beta}} = 2g^{\alpha\beta} + O(y)$. Now multiply both sides of the equation above by $(\rho_{k-1})^{\alpha\overline{\beta}}$ and sum on alpha. This gives

$$(\rho_{k-1})^{\overline{\gamma}} - 2(k-1)g^{\alpha\gamma}\frac{\partial\psi_k}{\partial y^\alpha} = (\rho_k)^{\overline{\gamma}} + O(y^{2k})$$

which proves the lemma. □

We are now ready to construct a formal power series solution to the local problem in theorem 3.1.2.

Proposition 3.1.7 *Let $g_{ij}(x)$ be a nonsingular matrix of functions. For all $k = 1, 2, \ldots$ there exists a polynomial $\rho_k(x,y)$ in y of degree $2k$ (with coefficients depending on x) such that*

1. $\rho_k(x,-y) = \rho_k(x,y)$

2. $\rho_k(x,y) = g_{ij}y^iy^j + O(y^4)$

3. $\sqrt{-1}(\rho_k)_\alpha(\rho_k)^\alpha - 2\rho_k = O(y^{2k+2})$.

If we set r_k *equal to the homogeneous part of* $\sqrt{-1}(\rho_{k-1})_\alpha(\rho_{k-1})^\alpha$ *of order* $2k$ *in* y*, then we obtain* ρ_k *from* ρ_{k-1} *by setting*

$$\rho_k = \rho_{k-1} + r_k/(2k-1)(2k-2).$$

Proof. First set $\rho_1 = g_{ij}y^iy^j$. Then an easy calculation shows that

$$\sqrt{-1}(\rho_1)_\alpha(\rho_1)^\alpha - 2\rho_1 = O(y^4).$$

Now suppose we have constructed ρ_{k-1} satisfying 1, 2, and 3 above. We wish to find a homogeneous polynomial ψ_k in y of degree $2k$ such that if we set $\rho_k = \rho_{k-1} + \psi_k$, then 3 holds for ρ_k. Using lemma 3.1.6 we can write

$$\begin{aligned}\sqrt{-1}(\rho_k)_\alpha(\rho_k)^\alpha - 2\rho_k &= \sqrt{-1}(\rho_{k-1})_\alpha(\rho_{k-1})^\alpha - 2\rho_{k-1} \\ &\quad +\sqrt{-1}(\rho_{k-1})_\alpha g^{\alpha\gamma}(2-2k)\frac{\partial\psi_k}{\partial y^\gamma} \\ &\quad +\sqrt{-1}(\psi_k)_\alpha(\rho_{k-1})^\alpha - 2\psi_k + O(y^{2k+1}).\end{aligned} \tag{3.3}$$

Using the inductive hypothesis we define a homogeneous polynomial r_k in y of degree $2k$ by

$$\sqrt{-1}(\rho_{k-1})_\alpha(\rho_{k-1})^\alpha - 2\rho_{k-1} = r_k + O(y^{2k+2}).$$

Notice that since ρ_{k-1} is a polynomial in y of degree $2k-2$, r_k is really the homogeneous part of $\sqrt{-1}(\rho_{k-1})_\alpha(\rho_{k-1})^\alpha$ of order $2k$ in y. Substituting this into 3.3, using lemma 3.1.5 and remembering that $\rho_{k-1} = g_{ij}y^iy^j + O(y^4)$ gives

$$\sqrt{-1}(\rho_k)_\alpha(\rho_k)^\alpha - 2\rho_k \;=\; r_k + (3-2k)\frac{\partial\psi_k}{\partial y^\gamma}y^\gamma - 2\psi_k + O(y^{2k+1})$$

$$= \quad r_k - (2k-1)(2k-2)\psi_k + O(y^{2k+1}).$$

Since the left hand side is even in y we may conclude that the error is in fact $O(y^{2k+2})$. We have shown that if we set $\rho_k = \rho_{k-1} + r_k/(2k-1)(2k-2)$ then ρ_k satisfies 1, 2, and 3 above, which completes the proof. □

An obvious corollary of this is that if a real analytic solution exists to the "Monge-Ampère type" problem in theorem 3.1.1 exists, then it is unique. It may be instructive to write out the first few terms in the formal power series for ϕ and the vector field Ξ in theorem 3.1.1 in terms of the metric g.

Proposition 3.1.8 *Let Ξ, ϕ be as in theorem 3.1.1, and let Γ^i_{jk} be the Christoffel symbols of the metric g. Then*

$$\begin{aligned}
\Xi &= (\Gamma^i_{\alpha\beta} y^\alpha y^\beta + O(y^4))\frac{\partial}{\partial x^i} + \\
&\quad (g^{ik}(\frac{1}{3}\frac{\partial}{\partial x^\gamma}(g_{kr}\Gamma^r_{\alpha\beta}) - \frac{4}{3} g_{rs}\Gamma^r_{i\alpha}\Gamma^s_{\gamma\beta} - (\frac{\partial g_{\beta k}}{\partial x^j} - \frac{\partial g_{\beta j}}{\partial x^k})\Gamma^j_{\alpha\gamma}) y^\alpha y^\beta y^\gamma + O(y^5))\frac{\partial}{\partial y^i} \\
\phi &= g_{\alpha\beta} y^\alpha y^\beta + \frac{1}{3}(g_{rs}\Gamma^r_{i\alpha}\Gamma^s_{j\beta} - \frac{1}{2}\frac{\partial^2 g_{\alpha\beta}}{\partial x^i \partial x^j}) y^i y^j y^\alpha y^\beta + O(y^6).
\end{aligned}$$

Proof. Write $\Xi = f^i \frac{\partial}{\partial x^i} + h^i \frac{\partial}{\partial y^i}$. Substituting $\phi = g_{\alpha\beta} y^\alpha y^\beta$ into the equation defining Ξ,

$$\iota(\Xi) d(\mathrm{Im}\, \overline{\partial}\phi) = \mathrm{Im}\, \overline{\partial}\phi, \tag{3.4}$$

we see that $f^i = O(y^2)$ and $h^i = y^i + O(y^2)$. Applying σ^* to both sides of equation 3.4 we see that

$$\iota(\sigma^*\Xi) d\alpha = \alpha,$$

or $\sigma^*\Xi = \Xi$. This means that $\sigma^* f^i = f^i$ and $\sigma^* h^i = -h^i$, so the power series for f^i has no odd order terms in y, and the power series for h^i has no even order terms in y. Then we can write

$$\begin{aligned}
\Xi &= (f_2^i + O(y^4))\frac{\partial}{\partial x^i} + (y^i + h_3^i + O(y^5))\frac{\partial}{\partial y^i} \\
\phi &= F_2 + F_4 + O(y^6).
\end{aligned}$$

Here f_2^i, h_3^i, and F_4 are homogeneous polynomials in y (with coefficients depending on x), of the degree indicated by the subscripts, and $F_2 = g_{\alpha\beta}y^\alpha y^\beta$. Substituting this in to the equations

$$\iota(\Xi)d\,\mathrm{Im}\,\overline{\partial}\phi = \mathrm{Im}\,\overline{\partial}\phi$$

$$\Xi\phi = 2\phi$$

and equating terms homogeneous of the same degree in y we get the following equations: for $i = 1, \ldots, n$,

$$(\frac{\partial^2 F_2}{\partial x^j \partial y^i} - \frac{\partial^2 F_2}{\partial x^i \partial y^j})y^j - \frac{\partial^2 F_2}{\partial y^j \partial y^i}f_2^j = -\frac{\partial F_2}{\partial x^i}$$

$$(\frac{\partial^2 F_2}{\partial x^i \partial x^j} + \frac{\partial^2 (F_2 + F_4)}{\partial y^i \partial y^j})y^j + (\frac{\partial^2 F_2}{\partial y^i \partial y^j})h_3^j + (\frac{\partial^2 F_2}{\partial x^j \partial y^i} - \frac{\partial^2 F_2}{\partial x^i \partial y^j})f_2^j = \frac{\partial (F_2 + F_4)}{\partial y^i}$$

$$f_2^j\frac{\partial F_2}{\partial x^j} + y^j\frac{\partial (F_2 + F_4)}{\partial y^j} + h_3^j\frac{\partial F_2}{\partial y^j} = 2(F_2 + F_4).$$

Substituting $F_2 = g_{\alpha\beta}y^\alpha y^\beta$ in the first equation gives

$$g_{ij}f_2^j = (\frac{\partial g_{\alpha i}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{\alpha j}}{\partial x^i})y^\alpha y^j.$$

Thus we must take

$$f_2^i = \Gamma^i_{\alpha\beta}y^\alpha y^\beta$$

where $\Gamma^i_{\alpha\beta}$ are the Christoffel symbols of the metric g on M.

To determine h_3^i and F_4 we apply Euler's relation to the second and third equation above. This gives

$$\frac{\partial^2 F_2}{\partial x^i \partial x^j} y^j + \frac{\partial^2 F_2}{\partial y^i \partial y^j} h_3^j + \left(\frac{\partial^2 F_2}{\partial x^j \partial y^i} - \frac{\partial^2 F_2}{\partial x^i \partial y^j}\right) f_2^j = -2 \frac{\partial F_4}{\partial y^i}$$

$$f_2^j \frac{\partial F_2}{\partial x^j} + h_3^j \frac{\partial F_2}{\partial y^j} = -2F_4. \tag{3.5}$$

Now we multiply the first equation by y^i, sum over i, and apply Euler's relation. This allows us to write

$$h_3^j \frac{\partial F_2}{\partial y^j} = -8F_4 - \frac{\partial^2 F_2}{\partial x^i \partial x^j} y^i y^j - \left(\frac{\partial^2 F_2}{\partial x^j \partial y^i} - \frac{\partial^2 F_2}{\partial x^i \partial y^j}\right) y^i f_2^j.$$

Substituting this into equation 3.5 gives

$$F_4 = \frac{1}{6}\left\{f_2^i \frac{\partial F_2}{\partial x^i} - \frac{\partial^2 F_2}{\partial x^i \partial x^j} y^i y^j - \left(\frac{\partial^2 F_2}{\partial x^j \partial y^i} - \frac{\partial^2 F_2}{\partial x^i \partial y^j}\right) y^i f_2^j\right\}.$$

This determines F_4 and hence h_3^i. Substituting $F_2 = g_{\alpha\beta} y^\alpha y^\beta$ we see that

$$F_4 = \frac{1}{3}\left(g_{rs} \Gamma^r_{i\alpha} \Gamma^s_{j\beta} - \frac{1}{2} \frac{\partial^2 g_{\alpha\beta}}{\partial x^i \partial x^j}\right) y^i y^j y^\alpha y^\beta,$$

and after some manipulation,

$$h^i = g^{ik}\left(\frac{1}{3} \frac{\partial}{\partial x^\gamma}(g_{kr} \Gamma^r_{\alpha\beta}) - \frac{4}{3} g_{rs} \Gamma^r_{i\alpha} \Gamma^s_{\gamma\beta} - \left(\frac{\partial g_{\beta k}}{\partial x^j} - \frac{\partial g_{\beta j}}{\partial x^k}\right) \Gamma^j_{\alpha\gamma}\right) y^\alpha y^\beta y^\gamma$$

which proves the lemma. □

3.2 Solution of the One Dimensional Problem

Let $g(x)$ be a real analytic metric defined on an open set $\mathcal{U}$ in $\mathbb{R}$. We want to find an analytic function ϕ on a neighborhood of $\mathcal{U}$ in $\mathbb{C}$ such that, if $z = x + \sqrt{-1}\, y$ is the standard coordinate, then

1. $\phi|_{y=0} = d\phi|_{y=0} = 0$
2. $\frac{\partial^2\phi}{\partial y \partial y}|_{y=0} = g$
3. If $\alpha = \operatorname{Im}\overline{\partial}\phi$ and Ξ is the vector field defined by $\iota(\Xi)d\alpha = \alpha$, then $\Xi\phi = 2\phi$
4. $\phi(x,-y) = \phi(x,y)$.

It is easy to translate the above into a partial differential equation for ϕ. We must find ϕ satisfying

$$\phi_x^2 + \phi_y^2 = 2\phi(\phi_{xx} + \phi_{yy})$$

$$\phi|_{y=0} = d\phi|_{y=0} = 0$$

$$\phi_{yy}|_{y=0} = g$$

(here the subscripts denote partial derivatives). This is a nonlinear, characteristic Cauchy problem. One can't appeal to the standard Cauchy-Kovalevsky theory for existence of solutions. However, a simple change of variables turns this into Laplace's equation. I would like to thank Prof. D. Jerison for pointing this out to me. Set $u = \phi^{1/2}$. Then away from $y = 0$, u is C^∞ and we get the following equation for u:

$$u^3\Delta u = 0.$$

We of course require that u^2 is C^∞ and satisfy the initial conditions posed for ϕ. These ensure that u is real and has a strict minimum at $y = 0$.

It's easy to generate solutions of Laplace's equation; the only difficulty is the slightly unorthodox initial conditions.

Theorem 3.2.1 *The (unique) solution of the one dimensional problem is*

$$\phi(x,y) = (\operatorname{Re}\int_0^y g(x+\sqrt{-1}\,t)^{1/2}dt)^2$$

Proof. In order to see what's going on we give a constructive proof. The distinguishing feature of the one dimensional case is that the boundary conditions do not preclude a C^∞ solution. We seek u such that

$$(u^2)_y|_{y=0} = 2uu_y|_{y=0} = 0$$

$$(u^2)_{yy}|_{y=0} = 2(u_y^2 + uu_{yy})|_{y=0} = 2g$$

We may take for initial data

$$u_y|_{y=0} = g^{1/2}$$

The obvious candidate for a solution of Laplace's equation is the following harmonic function:

$$u(x,y) = \mathrm{Im}(\int_0^{x+\sqrt{-1}\,y} g(\zeta)^{1/2} d\zeta).$$

Here $g(\zeta)$ is the analytic function g continued analytically to a tubular neighborhood of $\mathcal{U}$ in $\mathbb{C}$. The contour integral is independent of path, by Cauchy's theorem. To evaluate it we take the standard path along the coordinate axes. Along the x axis there is no contribution, since the imaginary part is zero. We are left with the integration along the path $\zeta(t) = x + \sqrt{-1}\,t$ from $t = 0$ to $t = y$. This gives

$$\begin{aligned} u(x,y) &= \mathrm{Im}(\sqrt{-1}\int_0^y g(x+\sqrt{-1}\,t)^{1/2}dt) \\ &= \mathrm{Re}(\int_0^y g(x+\sqrt{-1}\,t)^{1/2}dt) \end{aligned}$$

It's now clear that u satisfies the initial condition $u_y|_{y=0} = g^{1/2}$. □

3.3 Some Further Remarks

We have assumed that the manifold M has a real analytic structure and the metric g on M is real analytic. Clearly to have any hope of embedding M as a totally real submanifold of a complex manifold we must assume that M admits a real analytic structure. It is not so clear that the metric g need be real analytic, as the problem posed in theorem 2.1.1 makes sense for any metric (in fact one could consider Finsler metrics). We will show that if we can find a complex structure J_o as in theorem 2.1.1, then g must be real analytic.

We will also show that the construction in theorem 2.1.1 is functorial, in the sense that if g' is equal to G^*g for some real analytic diffeomorphism G, then the complex structure associated with g' by theorem 2.1.1 is the pullback by G of the complex structure associated to g.

3.3.1 The C^∞ Case

Suppose we embed M as a totally real submanifold of a complex manifold, and ask for a defining function ϕ as in theorem 3.1.1 when the metric g is only C^∞. By the formal power series constructions in section 3.1 we have a canonically defined infinite order jet of defining function. But it turns out that there is no hope of finding a function ϕ satisfying the "Monge-Ampère type" problem in theorem 3.1.1 unless g is real analytic. Equivalently, we will show that if a complex structure J_o as in theorem 2.1.1 exists, then g must be real analytic. For this we need the following lemma, which is interesting in its own right.

Lemma 3.3.1 *Let x, ξ be canonical cotangent coordinates on T^*M, and represent the complex structure operator J_o described in theorem 2.1.1 by the matrix*

$$J_o = \begin{pmatrix} J_b^b & J_f^b \\ J_b^f & J_f^f \end{pmatrix}$$

where

$$J_o \frac{\partial}{\partial x^i} = (J_b^b)_{ij} \frac{\partial}{\partial x^j} + (J_b^f)_{ij} \frac{\partial}{\partial \xi_j}$$

$$J_o\frac{\partial}{\partial\xi_i} \quad = \quad (J_f^b)_{ij}\frac{\partial}{\partial x^j} + (J_f^f)_{ij}\frac{\partial}{\partial \xi_j}.$$

Then

$$J_o|_{\xi=0} = \begin{pmatrix} 0 & -g^{ij} \\ g_{ij} & 0 \end{pmatrix}.$$

Proof. Recall that in these coordinates we have

$$\phi_o = g^{ij}\xi_i\xi_j$$

and

$$\alpha_o = \xi_i dx^i.$$

Writing out the condition that $\mathrm{Im}\,\overline{\partial}\phi_o = \alpha_o$ gives

$$\frac{1}{2}\frac{\partial g^{ij}}{\partial x^\alpha}\xi_i\xi_j(J_b^b)_{k\alpha} + g^{i\alpha}\xi_i(J_b^f)_{k\alpha} = \xi_k$$

and

$$\frac{1}{2}\frac{\partial g^{ij}}{\partial x^\alpha}\xi_i\xi_j(J_f^b)_{k\alpha} + g^{i\alpha}\xi_i(J_f^f)_{k\alpha} = 0.$$

Differentiating these equations with respect to ξ and evaluating at $\xi = 0$ gives

$$(J_b^f)_{k\gamma}|_{\xi=0} \quad = \quad g_{k\gamma}$$
$$(J_f^f)_{k\gamma}|_{\xi=0} \quad = \quad 0.$$

Then writing out the condition $J_o^2 = -Id$ shows that

$$(J_b^b)_{k\gamma}|_{\xi=0} \quad = \quad 0$$
$$(J_f^b)_{k\gamma}|_{\xi=0} \quad = \quad -g^{k\gamma}$$

which proves the lemma. □

Corollary 3.3.2 *The metric g must be real analytic in order for any of the equivalent problems given in theorems 2.1.1, 3.1.1, and 3.1.2 to have a solution.*

Proof. The matrix entries of the complex structure operator are real analytic functions. □

3.3.2 Metrics in the Same Isometry Class

Suppose g and g' are real analytic metrics on M in the same isometry class. By this we mean that there is a real analytic diffeomorphism G of M such that $g' = G^*g$. Then G induces a symplectomorphism of T^*M, called the "lift" of G, by the map $\hat{G}$, where

$$\hat{G}(\xi)(V) = \xi(dG^{-1}(V)).$$

The map $\hat{G}$ is not only symplectic; it also preserves the canonical one form α_o. In fact it can be shown that a diffeomorphism of T^*M preserves α_o if and only if it is the lift of a diffeomorphism of M (see Abraham and Marsden [1], theorem 3.2.12 and exercise 3.2F). Note also that $\hat{G}$ commutes with the standard involution σ_o of T^*M.

Proposition 3.3.3 *If J_g is the complex structure on T^*M associated with the metric g on M by theorem 2.1.1, then $\hat{G}^*J_g \overset{def}{=} d\hat{G}^{-1} \circ J_g \circ d\hat{G}$ is the complex structure associated with the metric G^*g.*

Proof. Let ν_g, $\nu_{G^*g}: TM \to T^*M$ be the metric identifications associated with g and G^*g, respectively, and let ϕ_g, ϕ_{G^*g} be the corresponding quadratic functions. Note that $\hat{G}(\nu_{G^*g}(V)) = \nu_g(dG(V))$, so $\hat{G}^*\phi_g = \phi_{G^*g}$. The proposition now follows from the fact that $\hat{G}$ preserves α_o and σ_o. □

Remark. If M is embedded as a totally real submanifold of a complex manifold Ω, then by lemma 1.2.1 G induces a unique biholomorphism $\tilde{G}$ of a tubular neighborhood of M in Ω, preserving M. Furthermore, $\tilde{G}$ preserves the complex conjugation σ about M by lemma 1.2.3. It is easy to see that in the context of theorem 3.1.1, if ϕ_g is the solution to the "Monge-Ampère type" problem for g, then $\tilde{G}^* \phi_g$ is the solution to the problem for $G^* g$.

Chapter 4

Examples and Global Results

In this chapter we will prove theorem 2.1.1 for compact Lie groups, many compact symmetric examples (including those of Helgason's "compact type"), and homogeneous spaces of compact Lie groups. We give all of these examples metrics induced by the bi-invariant metric on a compact Lie group. The proofs will be entirely constructive, and provide a realization of T^*M near the zero section as a complex homogeneous space. We will show that in the case of a compact Lie group or a homogeneous space of a compact, connected, semisimple Lie group, the cotangent bundle has a *globally* defined complex structure with the properties described in theorem 2.1.1.

4.1 Compact Lie Groups with Bi-invariant Metrics

In this section we will show how to construct the complexification of T^*G described in theorem 2.1.1 for a compact Lie group G and the bi-invariant metric on G. Moreover we will see that the complex structure exists globally on T^*G, and identify T^*G as a complex manifold with the complexification of G.

4.1.1 Trivialization of G_c and T^*G

It is well known that a compact Lie group G is a real analytic manifold, and the bi-invariant metric is a real analytic Riemannian metric. A complexification of G is a complex Lie group $G_{\mathbf{C}}$ whose Lie algebra is the complexification of the Lie algebra $\mathbf{g}$ of G as a vector space. In other words we may write as a direct sum decomposition

$$\mathbf{g}_{\mathbf{C}} = \mathbf{g} + \sqrt{-1}\,\mathbf{g}.$$

Often it will be useful to think of $\mathbf{g}_{\mathbf{C}}$ as a $2n$ dimensional real vector space equipped with a complex structure operator J corresponding to multiplication by $\sqrt{-1}$.

Following Ẑelobenko [25] we will say that a complex Lie group $G_{\mathbf{C}}$ is a regular complexification of a real Lie group G if $G_{\mathbf{C}}$ is a complexification of G and every connected component of $G_{\mathbf{C}}$ contains only one component of G. Then we have the following fundamental result.

Theorem 4.1.1 *Every compact Lie group has a regular complexification, which is unique up to isomorphism. The regular complexification is an algebraic subvariety of $Gl(n_o, \mathbf{C})$ for some n_o. The group G is isomorphic to the unitary matrices in the regular complexification.*

Proof. See Ẑelobenko [25], §106. □

From now on we will refer to the regular complexification of G as simply the complexification of G, and write it as $G_{\mathbf{C}}$. If G is semisimple, it is well known that $G_{\mathbf{C}}$ is diffeomorphic to $\mathbf{g} \times G$ by the map

$$(V, g) \to \exp(\sqrt{-1}\,V)g.$$

See Helgason [10], chapter VI, theorem 1. It is perhaps less well known that this is true for any compact Lie group, so we will outline a proof.

Theorem 4.1.2 *Let G be a compact Lie group, $G_{\mathbf{c}}$ the complexification of G. Then $G_{\mathbf{c}}$ is diffeomorphic to $\mathbf{g} \times G$ by the map*

$$(V, g) \to \exp(\sqrt{-1}\, V)g. \tag{4.1}$$

Proof. We may consider $G_{\mathbf{c}}$ to be embedded in $Gl(n_o, V)$ for some n_o. If $\zeta \in G_{\mathbf{c}}$, let $\zeta = \rho u$ be the polar decomposition of ζ in $Gl(n_o, V)$. Here ρ is a positive definite Hermitian matrix, u is a unitary matrix, and the decomposition $\zeta = \rho u$ is unique. Ẑelobenko shows that both ρ and u are in $G_{\mathbf{c}}$; in fact ρ^{λ} is in $G_{\mathbf{c}}$ for any complex λ (see [25], §106). The positive definite Hermitian matrix ρ may be written

$$\rho = \exp X$$

for some $X \in \mathbf{gl}(n_o, \mathbb{C})$. Since ρ is Hermitian and $\rho^{\lambda} \in G_{\mathbf{c}}$ for all real λ, it follows that $X \in \sqrt{-1}\,\mathbf{g}$. We have shown that the map 4.1 is a smooth bijection. To see that the inverse is smooth we need to show that $d\exp_{\sqrt{-1}V}$ is bijective for all $V \in \mathbf{g}$. This is the case if the eigenvalues of $\mathrm{ad}(\sqrt{-1}\, V)$ are real (see Varadarajan [20], theorem 2.14.3). Since $\sqrt{-1}\, V$ is a Hermitian matrix, the eigenvalues of $\sqrt{-1}\, V$ are real. It follows that the eigenvalues of $\mathrm{ad}(\sqrt{-1}\, V)$ are also real. □

For our purposes it will be more convenient to write this identification as

$$(g, V) \to g \exp \sqrt{-1}\, V. \tag{4.2}$$

This is still a diffeomorphism since $\exp(\sqrt{-1}\, V)g = g \exp \sqrt{-1}\, \mathrm{Ad}(g^{-1})V$ and the map

$$(V, g) \to (g, \mathrm{Ad}(g^{-1})V)$$

is a diffeomorphism. Multiplication by $\sqrt{-1}$ commutes with $\mathrm{Ad}(g)$ since left and right multiplication by elements of $G_{\mathbf{c}}$ are holomorphic maps on $G_{\mathbf{c}}$. It is also important to note that the map 4.2 is real analytic. The implicit function theorem is true in the real analytic category, so the inverse is also real analytic. Hence 4.2 is an identification of $G \times \mathbf{g}$ and $G_{\mathbf{C}}$ as real analytic manifolds. We will give $G \times \mathbf{g}$ the complex structure induced by this identification.

Let $\nu : \mathbf{g} \to \mathbf{g}^*$ be the identification via the bi-invariant metric $<,>$ at the identity. We identify T^*G and $G \times \mathbf{g}$ by associating a covector ξ_g with the pair $(\nu^{-1}(dL_g^*\xi_g), g)$. We will write this as

$$\Psi: G \times \mathbf{g} \to T^*G$$

where $\Psi(g, V) = dL_{g^{-1}}^* \nu(V)$. This identification is G-equivariant for the standard left G action on T^*G and the action on $G \times \mathbf{g}$ which is trivial (the identity) on $\mathbf{g}$ and left multiplication on G.

4.1.2 The Complex Structure on $G \times \mathbf{g}$

We have identified $G \times \mathbf{g}$ and $G_{\mathbf{C}}$ by the real analytic diffeomorphism $\Phi: G \times \mathbf{g} \to G_{\mathbf{C}}$ given by

$$\Phi(g, V) = g \exp \sqrt{-1}\, V.$$

In order to prove theorem 2.1.1 we will need an explicit description of the complex structure operator $J_{G\times\mathbf{g}}$ on $G \times \mathbf{g}$. By definition,

$$J_{G\times\mathbf{g}} = d\Phi^{-1} \circ J_{G_{\mathbf{C}}} \circ d\Phi$$

where $J_{G_{\mathbf{C}}}$ is the complex strucure on $G_{\mathbf{C}}$. We will identify the tangent space $T_{(g,V)}(G\times\mathbf{g})$ with $dL_g\mathbf{g} \times \mathbf{g}$ in the obvious manner. To compute $d\Phi$ we need to know how to compute the differential of the exponential map.

Lemma 4.1.3 *Let* $\exp : \mathbf{g}_{\mathbf{C}} \to G_{\mathbf{C}}$ *be the exponential map. Then for* U, $X \in \mathbf{g}_{\mathbf{C}}$,

$$d\exp_U(X) = dL_{\exp U} \circ \left(\frac{1 - e^{-\mathrm{ad}(U)}}{\mathrm{ad}(U)}\right)(X).$$

Here we have identified $T_U(\mathbf{g}_{\mathbf{C}})$ with $\mathbf{g}_{\mathbf{C}}$, and $(1 - e^{-A})/A$ stands for

$$\sum_0^\infty (-A)^m/(m+1)!$$

when A is a linear operator.

Proof. See Helgason [10], theorem 1.7, chapter II. □

We can now compute the differential of Φ.

Lemma 4.1.4 *Let $\Phi: G \times \mathbf{g} \to G_{\mathbf{C}}$ be as above. Then*

$$d\Phi(g,V)(dL_g W, Y) = dL_{g \exp \sqrt{-1}\,V}\Big(\frac{1 - e^{-\mathrm{ad}(\sqrt{-1}\,V)}}{\mathrm{ad}(\sqrt{-1}\,V)}(\sqrt{-1}\,Y) + e^{-\mathrm{ad}(\sqrt{-1}\,V)}W\Big).$$

Proof. It's easy to see that

$$d\Phi_{(g,V)}(0,Y) = dL_g \circ d\exp_{\sqrt{-1}\,V}(\sqrt{-1}\,Y) = dL_{g \exp \sqrt{-1}\,V}\Big(\frac{1 - e^{-\mathrm{ad}(\sqrt{-1}\,V)}}{\mathrm{ad}(\sqrt{-1}\,V)}(\sqrt{-1}\,Y)\Big)$$

and

$$d\Phi_{(g,V)}(dL_g W, 0) = dL_g \circ dR_{\exp \sqrt{-1}\,V}(W) = dL_{g \exp \sqrt{-1}\,V}(e^{-\mathrm{ad}(\sqrt{-1}\,V)}W)$$

which proves the lemma. □

Power series such as $e^{-\mathrm{ad}(\sqrt{-1}\,V)}$ behave very much like their one variable counterparts. For example,

$$e^{-\mathrm{ad}(\sqrt{-1}\,V)} = \cos\mathrm{ad}(V) - \sqrt{-1}\,\sin\mathrm{ad}(V).$$

Note that $\sin x$, $\cos x$, $\sin x/x$ and $(1 - \cos x)/x$ are entire functions. Thus it makes sense to use the operators defined by their power series:

$$\begin{aligned}\sin\mathrm{ad}(V) &= \sum_0^\infty (-1)^n \mathrm{ad}(V)^{2n+1}/(2n+1)! \\ \cos\mathrm{ad}(V) &= \sum_0^\infty (-1)^n \mathrm{ad}(V)^{2n}/(2n)!\end{aligned}$$

$$\frac{\sin \operatorname{ad}(V)}{\operatorname{ad}(V)} = \sum_0^\infty (-1)^n \operatorname{ad}(V)^{2n}/(2n+1)!$$

$$\frac{1-\cos \operatorname{ad}(V)}{\operatorname{ad}(V)} = \sum_0^\infty (-1)^n \operatorname{ad}(V)^{2n+1}/(2n+2)!$$

We interpret $\operatorname{ad}(V)^0$ as the identity operator on $\mathbf{g}$.

Using this remark and the fact that multiplication by $\sqrt{-1}$ commutes with $\operatorname{ad}(\sqrt{-1}\,V)$ we can rewrite the result in the lemma above as $d\Phi_{(g,V)}(dL_g W, Y) =$

$$dL_{g\exp\sqrt{-1}\,V}\Big(\frac{1-\cos \operatorname{ad}(V)}{\operatorname{ad}(V)}(Y) + \cos \operatorname{ad}(V)W + \sqrt{-1}\big(\frac{\sin \operatorname{ad}(V)}{\operatorname{ad}(V)}(Y) - \sin \operatorname{ad}(V)W\big)\Big).$$

Using the notation

$$U + \sqrt{-1}\,S = \begin{pmatrix} U \\ S \end{pmatrix}$$

and identifying $(W, Y) \in \mathbf{g} \times \mathbf{g}$ with the column vector

$$\begin{pmatrix} W \\ Y \end{pmatrix},$$

we may write this as $d\Phi_{(g,V)}(dL_g W, Y) =$

$$dL_{g\exp\sqrt{-1}\,V} \begin{pmatrix} \cos \operatorname{ad}(V) & (1-\cos \operatorname{ad}(V))/\operatorname{ad}(V) \\ -\sin \operatorname{ad}(V) & \sin \operatorname{ad}(V)/\operatorname{ad}(V) \end{pmatrix} \begin{pmatrix} W \\ Y \end{pmatrix}. \tag{4.3}$$

We want to find the inverse of $d\Phi$. Identify the tangent space to $G_{\mathbf{c}}$ at $g\exp\sqrt{-1}\,V$ with $dL_{g\exp\sqrt{-1}\,V}\mathbf{g}_{\mathbf{c}}$ in the obvious way. Note that if $S_1(x)$, $S_2(x)$, $S_3(x)$ are analytic functions near zero in the single indeterminate x which satisfy $S_1(x)S_2(x) = S_3(x)$, then if A is a continuous linear operator of small enough norm we have $S_1(A)S_2(A) = S_3(A)$ (and similarly for sums). The adjoint representation

$$\operatorname{ad}\colon \mathbf{g} \to \operatorname{End}(\mathbf{g})$$

by $V \to \mathrm{ad}(V)$ is a linear map between finite dimensional vector spaces, hence bounded. Then it is easily checked that for small enough V,

$$d\Phi^{-1}(dL_{g\exp\sqrt{-1}V}(U+\sqrt{-1}\,S)) = \\ d(L_g \times Id)\begin{pmatrix} I & (\cos\mathrm{ad}(V)-1)/\sin\mathrm{ad}(V) \\ \mathrm{ad}(V) & \cos\mathrm{ad}(V)(\mathrm{ad}(V)/\sin\mathrm{ad}(V)) \end{pmatrix}\begin{pmatrix} U \\ S \end{pmatrix}. \tag{4.4}$$

We can now give an explicit form for the complex structure on $G \times \mathbf{g}$.

Lemma 4.1.5 *The complex structure on $G \times \mathbf{g}$ induced by the identification Φ is given near $G \times 0$ by $(J_{G\times\mathbf{g}})_{(g,V)} =$*

$$d(L_g \times Id)\begin{pmatrix} (1-\cos\mathrm{ad}(V))/\sin\mathrm{ad}(V) & 2(\cos\mathrm{ad}(V)-1)/\mathrm{ad}(V)\sin\mathrm{ad}(V) \\ \mathrm{ad}(V)/\sin\mathrm{ad}(V) & (\cos\mathrm{ad}(V)-1)/\sin\mathrm{ad}(V) \end{pmatrix} d(L_{g^{-1}} \times Id).$$

Proof. Represent the complex structure on $\mathbf{g}_{\mathbf{C}}$ by the matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Formally multiplying the relevant matrices gives the expression for $J_{G\times\mathbf{g}}$ above. For small enough V, the result will converge. □

Remark. In fact this expression for $J_{G\times\mathbf{g}}$ is valid on all of $G \times \mathbf{g}$. The expression for $d\Phi$ in equation 4.3 is valid for all $V \in \mathbf{g}$. The argument in lemma 4.3.11 shows that $\sin\mathrm{ad}(V)/\mathrm{ad}(V)$ is an invertible linear operator on $\mathbf{g}$ for all $V \in \mathbf{g}$. This shows that the expression for $d\Phi^{-1}$ in equation (4.4) is valid for all $V \in \mathbf{g}$. Hence the expression for $J_{G\times\mathbf{g}}$ is valid on all of $G \times \mathbf{g}$.

4.1.3 Proof of the Result

Let $\phi_o: T^*G \to \mathbb{R}$ be the modulus of a covector squared with respect to the bi-invariant metric on G. If $\nu: \mathbf{g} \to \mathbf{g}^*$ is the identification via the bi-invariant metric at the identity, it is easy to see that

$$\phi_o(\xi_g) =< \nu^{-1}(dL_g^*\xi_g), \nu^{-1}(dL_g^*\xi_g) >$$

where $<,>$ is the Ad(G)-invariant inner product on $\mathbf{g}$. Let $\phi = \Psi^*\phi_o$ be the object on $G \times \mathbf{g}$ corresponding to ϕ_o under the identification Ψ of $G \times \mathbf{g}$ with T^*G given above. Since (g, V) is mapped to the covector $dL_{g^{-1}}^*\nu(V)$ by Ψ, it is clear that

$$\phi(g, V) =< V, V > .$$

From this it's easy to see that

$$d\phi_{(g,V)}(dL_gW, Y) = 2 < V, Y > .$$

Let α_o be the canonical one-form on T^*G. At a point $\xi_g \in T^*G$, α_o applied to a tangent vector $X \in T_{\xi_g}(T^*G)$ is given by

$$\alpha_o(\xi_g)(X) = \xi_g(d\pi_{T^*G}(X))$$

where $\pi_{T^*G}: T^*G \to G$ is the cotangent projection. Let $\alpha = \Psi^*\alpha_o$ be the corresponding object on $G \times \mathbf{g}$. Then for $(g, V) \in G \times \mathbf{g}$ and $(dL_gW, Y) \in dL_g\mathbf{g} \times \mathbf{g}$ (which we have identified with $T_{(g,V)}(G \times \mathbf{g})$), we have

$$\alpha(g, V)(dL_gW, Y) =< V, dL_{g^{-1}} \circ d\pi_{T^*G} \circ d\Psi(dL_gW, Y) > .$$

It is easy to see that $d\pi_{T^*G} \circ d\Psi(dL_gW, Y) = dL_gW$. This shows that

$$\alpha(g, V)(dL_gW, Y) =< V, W > .$$

We can now prove the main result of this section.

Proposition 4.1.6 *Give T^*G the structure of a complex manifold by identifying it with $G_{\mathbb{C}}$ via the map*

$$\xi_g \to g \exp \sqrt{-1}\, \nu^{-1}(dL_g^* \xi_g).$$

Let α_o be the canonical one-form on T^*G and let ϕ_o be the quadratic function on T^*G associated with the bi-invariant metric on G. Then with this complex structure, $\mathrm{Im}\,\overline{\partial}\phi_o = \alpha_o$ and the standard involution of T^*G is an antiholomorphic map.

Proof. To show that $\mathrm{Im}\,\overline{\partial}\phi_o = \alpha_o$ we may as well show that $\mathrm{Im}\,\overline{\partial}\phi = \alpha$ when we give $G \times \mathbf{g}$ the complex structure induced by the identification with $G_{\mathbb{C}}$ above. Denote the complex strucure on $G \times \mathbf{g}$ simply by J. We need to show that for all $(g, V) \in G \times \mathbf{g}$ and all $(dL_g W, Y) \in dL_g \mathbf{g} \times \mathbf{g}$,

$$\frac{1}{2}(d\phi)_{(g,V)}(dL_g W, Y) = -\alpha(g, V)(J_{(g,V)}(dL_g W, Y)).$$

Using the observations above, this means we must show that

$$< V, Y >= - < V, \mathrm{pr}_1 d(L_{g^{-1}} \times Id) J_{(g,V)}(dL_g W, Y) > \tag{4.5}$$

where pr_1 denotes projection onto the first factor in $\mathbf{g} \times \mathbf{g}$. By lemma 4.1.5 we have

$$\mathrm{pr}_1 d(L_{g^{-1}} \times Id) J_{(g,V)}(dL_g W, Y) =$$
$$(1 - \cos \mathrm{ad}(V))/ \sin \mathrm{ad}(V)(W) + 2(\cos \mathrm{ad}(V) - 1)/\mathrm{ad}(V) \sin \mathrm{ad}(V)(Y).$$

This is a power series in $\mathrm{ad}(V)$, and the term of order zero in $\mathrm{ad}(V)$ is $-Y$. Higher order terms in $\mathrm{ad}(V)$ will make no contribution in equation 4.5 since for all $X \in \mathbf{g}$,

$$< V, \mathrm{ad}(V)X >= 0$$

if $<,>$ is $\mathrm{Ad}(G)$-invariant. This shows that $\mathrm{Im}\,\overline{\partial}\phi = \alpha$.

It remains to show that $\sigma^* J = -J$, where σ is the involution on $G \times \mathbf{g}$ sending (g, V) to $(g, -V)$. Explicitly, we need to show that for all $V \in \mathbf{g}$ and all $g \in G$,

$$d\sigma_{(g,-V)} J_{(g,-V)} d\sigma_{(g,V)} = -J_{(g,V)}.$$

Examining the matrix of J we see that evaluating J at $(g, -V)$ instead of (g, V) has the effect of multiplying the diagonal entries by -1. It is easily checked that this operation followed by conjugation by $d\sigma$ is the same as multiplying J by -1 and evaluating at (g, V). □

We have proved theorem 2.1.1 for a compact Lie group and the bi-invariant metric. We will have another proof of this when we prove theorem 2.1.1 for a compact Riemannian symmetric space, since a compact Lie group with the bi-invariant metric is also a Riemannian symmetric space.

We close this section with a remark about stability of the Kähler form $\omega_{\mathbf{C}}$ on $G_{\mathbf{C}}$ induced by the symplectic structure of T^*G. We say $\omega_{\mathbf{C}}$ is stable if zero is a regular value of the associated moment map (see [7], §2). It is easy to see that $\omega_{\mathbf{C}}$ is a G-invariant Kähler form on $G_{\mathbf{C}}$. If we set $\phi_{\mathbf{C}} = (\Phi^{-1})^*\phi$ then it is easily checked that $\omega_{\mathbf{C}} = \sqrt{-1}\partial\bar{\partial}\phi_{\mathbf{C}}$. In [7] it is shown that either $\phi_{\mathbf{C}}$ has no critical points, or the set of critical points of $\phi_{\mathbf{C}}$ consists of a single G orbit on which $\phi_{\mathbf{C}}$ takes its unique minimum value; furthermore, it is shown that $\omega_{\mathbf{C}}$ is stable if and only if the second of these alternatives is true.

Since $d\phi_{\mathbf{C}} = d\phi \circ d\Phi^{-1}$, it is clear that the set of critical points of $\phi_{\mathbf{C}}$ is exactly G, where $\phi_{\mathbf{C}}$ is zero. Hence $\omega_{\mathbf{C}}$ is stable.

4.2 Rank One Compact Riemannian Symmetric Spaces

In this section we will review the work of P. M. Wong [25], who essentially proved theorem 2.1.1 for compact Riemannian symmetric spaces of rank one. We will interpret his results in terms of theorem 2.2.1. The compact Riemannian symmetric spaces of rank one are:

1. S^n, the standard n-sphere ($n \geq 2$)
2. $\mathbb{R}P^n$, real projective n-space ($n \geq 2$)
3. $\mathbb{C}P^n$, complex projective n-space ($n \geq 1$)
4. $\mathbb{H}P^{n-1}$, the quaternionic projective n-space ($n \geq 2$)
5. $\mathbb{K}P^2$, the Cayley projective plane.

These, with the exception of the Cayley projective plane, have standard models as "center" manifolds of complex affine algebraic submanifolds of $\mathbb{C}^N$ for some N. The Cayley projective plane is difficult to describe geometrically and will not be dealt with here. A center manifold of a complex manifold carrying a strictly plurisubharmonic exhaustion is the minimum set of that exhaustion. Center manifolds are also totally real submanifolds (see Harvey and Wells [9]). These complex algebraic manifolds are natural candidates for the complex manifolds Ω in theorem 2.2.1, in which the compact symmetric space M is embedded as a totally real submanifold. We will describe these manifolds below, following closely the exposition and notation in Wong [24]. Superscripts indicate the real dimension of M (= complex dimension of Ω) and subscripts refer to the symmetric spaces on the list above.

1. $\Omega_1^n = \{z_1^2 + \cdots + z_{n+1}^2 = 1\} \subset \mathbb{C}^{n+1}$, the complex affine hyperquadric. The condition $z_1^2 + \cdots + z_{n+1}^2 = 1$ means that if $z = x + \sqrt{-1}\, y$, then

$$|x|^2 - |y|^2 = 1 \quad \text{and} \quad x \cdot y = 0.$$

   It's clear that S^n is given by $|y|^2 = 0$. This is the minimum set of the exhaustion $\rho = |z|^2$, since on Ω_1^n we have

$$|z|^2 = |x|^2 + |y|^2 = 1 + 2|y|^2.$$

Note also that S^n is the fixed point set of the complex conjugation of Ω_1^n induced by the complex conjugation of $\mathbb{C}^{n+1}$.

2. $\Omega_2^n = \mathbb{C}P^n \backslash \bar{Q}^{n-1}$. Here $\bar{Q}^{n-1}$ is the compact hyperquadric in $\mathbb{C}P^n$,

$$\bar{Q}^{n-1} = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n : z_o^2 + \cdots + z_{n+1}^2 = 0\}.$$

Note that $\mathbb{R}P^n$ is embedded in $\mathbb{C}P^n$ as the fixed point set of the complex conjugation of $\mathbb{C}P^n$, $[z] \to [\bar{z}]$. It's clear that $\mathbb{R}P^n \subset \mathbb{C}P^n \backslash \bar{Q}^{n-1}$. We obtain a two-fold unramified cover of Ω_2^n by Ω_1^n by the map β sending a point in Q^n to the complex line determined by it. When restricted to the real points S^n in Q^n, the map β is the standard two-fold unramified cover of $\mathbb{R}P^n$ by S^n. The exhaustion $\rho = |z|^2$ is constant on preimages of β, so gives an exhaustion $\tilde{\rho} = \rho \circ \beta^{-1}$ on Ω_2^n. Then $\mathbb{R}P^n$ is given by the minimum set, $\tilde{\rho} = 1$.

3. $\Omega_3^{2n} = (\mathbb{C}P^n \times \mathbb{C}P^n) \backslash P_\infty(\mathbb{C}^{N-1})$, $N = (n+1)^2 - 1$. We consider $\mathbb{C}P^n \times \mathbb{C}P^n$ to be embedded in $\mathbb{C}P^N$ by the Segre embedding,

$$([z],[w]) \to [\zeta],$$

where $\zeta_{\alpha\beta} = z_\alpha w_\beta$, $0 \le \alpha, \beta \le n$. $P_\infty(\mathbb{C}^{N-1})$ is the hyperplane at infinity in $\mathbb{C}P^N$ given by

$$\sum_{\alpha=0}^{n} \zeta_{\alpha\alpha} = 0.$$

The underlying real manifold of $\mathbb{C}P^n$ is embedded in $\mathbb{C}P^n \times \mathbb{C}P^n$ by the map

$$[z] \to ([z],[\overline{z}]).$$

This is the fixed point set of the antiholomorphic involution

$$([z],[w]) \to ([\overline{w}],[\overline{z}]).$$

The image of $\mathbb{C}P^n$ under the Segre embedding is given by

$$[\zeta_{\alpha\beta}] = [z_\alpha \overline{z}_\beta],$$

which is disjoint from $P_\infty(\mathbb{C}^{N-1})$. The image of $\mathbb{C}P^n$ is the minimum set of the strictly plurisubharmonic exhaustion $\aleph$ of $\mathbb{C}P^N \backslash P_\infty(\mathbb{C}^{N-1})$ given by

$$\aleph([\zeta]) = \frac{\sum_{0 \le \alpha,\beta \le n} |\zeta_{\alpha\beta}|^2}{|\sum_{0 \le \alpha \le n} \zeta_{\alpha\alpha}|^2} \ge 1.$$

The metric on $\mathbb{C}P^n$ induced by the Fubini-Study metric on $\mathbb{C}P^N$ is the metric on $\mathbb{C}P^n$ as a Riemannian symmetric space.

4. $\Omega_4^{4(n-1)} = \mathrm{Gr}(2, 2n, \mathbb{C}) \backslash P_\infty(\mathbb{C}^{N-1})$, $N = n(2n-1) - 1$. $\mathrm{Gr}(2, 2n, \mathbb{C})$ is the Grassmannian manifold of complex two-planes through the origin in $\mathbb{C}^{2n}$. We regard $\mathrm{Gr}(2, 2n, \mathbb{C})$ as embedded in $\mathbb{C}P^N$ by the Plücker embedding: if a plane $p \in \mathrm{Gr}(2, 2n, \mathbb{C})$ is spanned by two linearly independent vectors $z = (z_1, \ldots, z_{2n})$ and $w = (w_1, \ldots, w_{2n})$, then

$$p \to [\zeta_{\alpha\beta}] = [z_\alpha w_\beta - z_\beta w_\alpha]$$

where $1 \le \alpha < \beta \le 2n$. Here $P_\infty(\mathbb{C}^{N-1})$ is the hyperplane at infinity

$$\sum_{k=1}^{n} \zeta_{2k-1,2k} = 0.$$

The quaternions $\mathbb{H}$ are identified with $\mathbb{C}^2$ by $q = (a + b\sqrt{-1}) + (c + d\sqrt{-1})j$, and so $\mathbb{H}^n$ is identified with $\mathbb{C}^{2n}$. This identification induces a "multiplication by j" operation on $\mathbb{C}^{2n}$ which takes a complex subspace to a complex subspace, and so induces a map on $\mathrm{Gr}(2, 2n, \mathbb{C})$. This map is in fact an antiholomorphic involution of $\mathrm{Gr}(2, 2n, \mathbb{C})$, and the quaternionic projective space $\mathbb{H}P^{n-1}$ is naturally identified with the fixed point set of this involution. Hence $\mathbb{H}P^{n-1}$ is embedded as a totally real submanifold of $\mathrm{Gr}(2,2n,\mathbb{C})$. See Huckleberry-Snow [12] for more details.

With respect to the basis $\{e_1, je_1, \ldots, e_n, je_n\}$ of $\mathbb{C}^{2n}$ (where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$), the map "multiplication by j" is given by

$$((z_1, w_1), \ldots, (z_n, w_n)) \to ((-\overline{w_1}, \overline{z_1}), \ldots, (-\overline{w_1}, \overline{z_1})).$$

If p is a quaternionic line in $\mathrm{Gr}(2,2n,\mathbb{C})$ and z is a point on p, then z and jz span p. If p is a quaternionic line and $[\zeta]$ is the image of p under the Plücker embedding then

$$[\zeta_{2k-1,2k}] = [|z_{2k-1}|^2 + |z_{2k}|^2],$$

so the image of $\mathbb{H}P^{n-1}$ does not intersect the hyperplane $P_\infty(\mathbb{C}^{N-1})$. $\mathbb{H}P^{n-1}$ is the minimum set of the strictly plurisubharmonic exhaustion

$$\tilde{\aleph}(\zeta) = \frac{\sum_{1 \le \alpha < \beta \le 2n} |\zeta_{\alpha\beta}|^2}{|\sum_{1 \le \alpha \le n} \zeta_{2\alpha-1,2\alpha}|^2} \ge 1.$$

5. Ω_5^{16} is a 16 dimensional Stein manifold. It is difficult to describe geometrically. See Huckleberry-Snow [12] for a description in terms of quotients of Lie groups.

Let ρ_i be the strictly plurisubharmonic exhaustions of Ω_i given above ($i = 1, \ldots, 4$):

$$\rho_1(z) = |z|^2$$

$$\rho_2([z]) = |\beta^{-1}([z])|^2$$

$$\rho_3([\zeta]) = 2\aleph([\zeta]) - 1$$

$$\rho_4([\zeta]) = 2\tilde{\aleph}([\zeta]) - 1.$$

It can be shown that ρ_i is a smooth (in fact real analytic), strictly plurisubharmonic exhaustion of Ω_i such that the center M_i is given by $\{\rho_i = 1\}$. The function

$$u_i = \cosh^{-1} \rho_i$$

is strictly plurisubharmonic on $\Omega_i \backslash M_i$, and on $\Omega_i \backslash M_i$, u_i satisfies the homogeneous Monge-Ampère equation. See §2 of Wong [24] for proof. Thus the $(u_i)^2$ are obvious candidates for the function ϕ in theorem 2.2.1. It is not immediately obvious that the $(u_i)^2$ are even C^∞, since the function $\cosh^{-1} t$ is not differentiable at $t = 1$. However the singularity is of the correct type, as the following lemma shows.

Lemma 4.2.1 *Suppose f is a real analytic function on a manifold Ω such that $f \geq 1$. Then $(\cosh^{-1} f)^2$ is also a real analytic function on Ω.*

Proof. It suffices to show that $(\cosh^{-1} s)^2$ is analytic at $s = 1$. Let $y = \cosh^{-1} s$. Then

$$s = \cosh y = \sum_0^\infty \frac{y^{2n}}{(2n)!}.$$

Let $\nu = y^2$. Then

$$s = \sum_0^\infty \frac{\nu^n}{(2n)!} \stackrel{def}{=} F(\nu).$$

Note $F'(0) \neq 0$, so the inverse exists (and is analytic) near $\nu = 0$. This shows that $\nu = (\cosh^{-1} s)^2$ is an analytic function of s near $s = 1$. □

We can now state the main result of this section.

Proposition 4.2.2 *The Monge-Ampère manifolds Ω_i and their exhaustions ρ_i constructed above prove theorem 2.1.1 for the compact rank one Riemannian symmetric spaces with their standard metrics, with the exception of the Cayley projective plane.*

Proof. We need only show that if we set $\phi_i = (\cosh^{-1} \rho_i)^2$ then ϕ_i solves the "Monge-Ampère type" problem posed in part 2 of theorem 2.2.1. We have shown that ϕ_i is real analytic, and it's clear that $\phi_i = d\phi_i = 0$ on the center M_i. By inspection it is clear that the functions ρ_i are invariant under the antiholomorphic involutions of Ω_i fixing M_i, so the ϕ_i are also. Since $u_i = \cosh^{-1}\rho_i$ satisfies the homogeneous Monge-Ampère equation, the remarks in section 2.3 show that ϕ_i satisfies the "Monge-Ampère type" equation $\Xi\phi_i = 2\phi_i$. It can be shown that ϕ_i is strictly plurisubharmonic, since the ρ_i are. Hence we get a metric on M_i by

$$g_i(X,Y) = -d(\mathrm{Im}\,\overline{\partial}\phi_i)(X,JY).$$

Thus we have proved theorem 2.1.1 for some metric g_i on M_i. To see which metric, note that $d\rho_i = 0$ on the center M_i (see theorem 4.1 in Wong [25] for the Taylor series expansion of ρ_i). Hence

$$\frac{\partial^2\phi_i}{\partial z^\alpha \partial \overline{z}^\beta}|_{M_i} = G'(1)(\frac{\partial^2\rho_i}{\partial z^\alpha \partial \overline{z}^\beta})|_{M_i}$$

where G is the inverse of the function F in lemma 4.2.1 (in fact $G'(1) = 2$). The Kähler metrics on Ω_i associated with the Kähler form $\sqrt{-1}\partial\overline{\partial}\rho_i$ induce the standard metrics on M_i as Riemannian symmetric spaces. This shows we have proved theorem 2.1.1 for the compact rank one Riemannian symmetric spaces. □

The Cayley projective plane is not treated explicitly by Wong in [25]. In particular he does not construct a Monge-Ampère exhaustion for Ω_5^{16}. The proof of theorem 2.1.1 for this case follows from our treatment of compact Riemannian symmetric spaces as quotients of Lie groups in section 4.3. In fact we will prove theorem 2.1.1 for compact Riemannian symmetric spaces of arbitrary rank.

4.3 Compact Riemannian Symmetric Spaces

We now consider the case where M is a Riemannian locally symmetric space of the compact type. By this we mean that $M = G/K$ where G is a compact, connected, semisimple Lie group, K is a closed subgroup of G, and there exists an involutive automorphism θ of the Lie algebra g of G such that the Lie algebra k of K is the $+1$ eigenspace of θ. The -1 eigenspace of θ is denoted by p, and $\mathsf{g} = \mathsf{k} + \mathsf{p}$ is a direct decomposition of g. The eigenspaces are orthogonal with respect to the $\mathrm{Ad}(G)$-invariant nondegenerate inner product on g, since in this case the invariant inner product is the Killing form, and any automorphism of g leaves the Killing form invariant. If $\pi_{G/K}$ denotes the projection of G onto G/K, then $(d\pi_{G/K})|_{\mathsf{p}}$ is a bijection of p onto $T_{eK}G/K$. This identification gives $T_{eK}G/K$ a nondegenerate, positive definite inner product corresponding to minus one times the Killing form on g. G acts on G/K as a transitive group of diffeomorphisms, so we get a metric on G/K which is well-defined due to the $\mathrm{Ad}(G)$-invariance of the Killing form. This metric turns G/K into a Riemannian locally symmetric space. See Helgason [10] for details, for example proposition 1.1 of chapter VII. We show that there is a global identification of T^*G/K with a complex homogeneous manifold, and that this identification induces a complex structure on T^*G/K with the properties described in theorem 2.1.1.

4.3.1 Complexification of G/K

In this section we will embed G/K as a totally real submanifold of a canonically determined non-compact complex homogeneous space. It is important to note that this construction does not require any assumptions on G and K other than that both are compact. We will refer to this section when we prove theorem 2.1.1 for other homogeneous examples (see also theorems 4.1.1 and 4.1.2).

Since every compact Lie group has a faithful unitary representation, we may assume that G is a subgroup of the unitary group $U(n_o)$ for some n_o. Then we have the following result concerning the existence of complexifications of G and K.

Theorem 4.3.1 *i) There exists an algebraic variety $\mathcal{A}(G) \subset GL(n_o, \mathbb{C})$ such that $\mathcal{A}(G)$ is a Lie group, $G = \mathcal{A}(G) \cap U(n_o)$, and $\mathcal{A}(G)$ is a regular complexification of G (i.e., $\mathcal{A}(G)$ is a complexification of G and each component of $\mathcal{A}(G)$ contains only one component of G).*

ii) If $K \subset G$ is a compact subgroup, then we can find an algebraic variety $\mathcal{A}(K) \subset \mathcal{A}(G)$ such that $\mathcal{A}(K)$ is a Lie group, $K = \mathcal{A}(K) \cap U(n_o)$, and $\mathcal{A}(K)$ is a regular complexification of K.

Proof. i) See Ẑelobenko [26], §106, lemmas 3-5. ii) Clear from the construction of $\mathcal{A}(G)$ in [26], §106.□

Let $G_{\mathbb{C}} = \mathcal{A}(G)$, $K_{\mathbb{C}} = \mathcal{A}(K)$. Then $G_{\mathbb{C}}$, $K_{\mathbb{C}}$ are the complexifications of G, K respectively as Lie groups. Since $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ are algebraic varieties in $Gl(n_o, \mathbb{C})$, $K_{\mathbb{C}}$ is a closed subgroup of $G_{\mathbb{C}}$. We can now form the homogeneous manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$. The following standard result gives the basic information about $G_{\mathbb{C}}/K_{\mathbb{C}}$ that will be needed.

Theorem 4.3.2 *$G_{\mathbb{C}}/K_{\mathbb{C}}$ is a complex manifold. The $G_{\mathbb{C}}$ action on $G_{\mathbb{C}}/K_{\mathbb{C}}$ consists of holomorphic maps, and the projection of $G_{\mathbb{C}}$ onto $G_{\mathbb{C}}/K_{\mathbb{C}}$ is a holomorphic map.*

Proof. See [19], page 227 (3), or [17], chapter X, §6. □

The complex manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ is an obvious candidate for the complexification of M. We will now show that M is naturally embedded as a totally real submanifold of $G_{\mathbb{C}}/K_{\mathbb{C}}$.

Lemma 4.3.3 *The map $\imath: M \to G_{\mathbb{C}}/K_{\mathbb{C}}$ given by $\imath(gK) = gK_{\mathbb{C}}$ is well-defined and embeds M as a totally real submanifold of $G_{\mathbb{C}}/K_{\mathbb{C}}$.*

Proof. To see that $\imath$ is well-defined, suppose $gK = \overline{g}K$. Then $g\overline{g}^{-1} \in K \subset K_{\mathbb{C}}$, so $gK_{\mathbb{C}} = \overline{g}K_{\mathbb{C}}$. To see that $\imath$ is globally one-to-one, suppose $gK_{\mathbb{C}} = \overline{g}K_{\mathbb{C}}$. Then $g\overline{g}^{-1} \in G \cap K_{\mathbb{C}}$. But by theorem 4.3.1, $G = G_{\mathbb{C}} \cap U(n_o)$ so $G \cap K_{\mathbb{C}} = U(n_o) \cap K_{\mathbb{C}} = K$. Thus $gK = \overline{g}K$.

To show that $\imath$ is an embedding we go to local coordinates on G/K. Let g be the Lie algebra of G, let k be the Lie algebra of K, and let $\mathsf{k}^{\perp}$ be the orthogonal complement of k with respect to the Ad(G)-invariant inner product on g. In the symmetric case $\mathsf{k}^{\perp}$ is usually denoted as p, and

$$\mathsf{g} = \mathsf{k} + \mathsf{p}$$

is a decomposition of $\mathfrak{g}$ into eigenspaces of an involutive automorphism of $\mathfrak{g}$. In any case, local coordinates near gK in G/K are given by, for a suitable neighborhood $\mathfrak{u}$ of zero in $\mathfrak{k}^{\perp}$, the map

$$\mathfrak{u} \ni X \longmapsto \pi_{G/K} \circ L_g \circ \exp X \in G/K$$

where $\pi_{G/K}$ is the projection of G onto G/K. In these coordinates $\imath$ is given by the map

$$\mathfrak{u} \ni X \longmapsto \pi_{G_{\mathbb{C}}/K_{\mathbb{C}}} \circ L_g \circ \exp X$$

(here $\pi_{G_{\mathbb{C}}/K_{\mathbb{C}}}$ is the projection of $G_{\mathbb{C}}$ onto $G_{\mathbb{C}}/K_{\mathbb{C}}$, and we are considering G as a subset of $G_{\mathbb{C}}$; the exponential map is the same for G and $G_{\mathbb{C}}$). From this expression it is clear that the kernel of $(d\imath)_{gK}$ is zero and that $\imath$ is an open map into $\imath(G/K)$ with the relative topology, so that $\imath$ is an embedding.

To show that $\imath$ embeds G/K as a totally real submanifold of $G_{\mathbb{C}}/K_{\mathbb{C}}$ we use complex local coordinates on $G_{\mathbb{C}}/K_{\mathbb{C}}$. These are given near $gK_{\mathbb{C}}$ by, for a suitable neighborhood $\mathfrak{u}_{\mathbb{C}}$ of zero in $\mathfrak{k}_{\mathbb{C}}^{\perp}$, the map

$$\mathfrak{u}_{\mathbb{C}} \ni X + \sqrt{-1}\,Y \longmapsto \pi_{G_{\mathbb{C}}/K_{\mathbb{C}}} \circ L_g \circ \exp(X + \sqrt{-1}\,Y).$$

Clearly G/K is defined near $gK_{\mathbb{C}}$ by $Y = 0$.□

This construction produces essentially the same complex manifolds as the ones described in section 4.2, as the following examples show.

1. $M = S^n$ is the homogeneous space $SO(n+1,\mathbb{R})/SO(n,\mathbb{R})$. $SO(n,\mathbb{R})$ is embedded in $SO(n+1,\mathbb{R})$ as matrices of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}.$$

The natural complexification of M is $SO(n+1,\mathbb{C})/SO(n,\mathbb{C})$. $SO(n+1,\mathbb{C})$ consists of those complex matrices of determinant one which preserve the standard quadratic form on $\mathbb{C}^n$,

$$z \to z_1^2 + \cdots + z_{n+1}^2.$$

We will show that $SO(n+1,\mathbb{C})/SO(n,\mathbb{C})$ is biholomorphically equivalent to the standard hyperquadric in $\mathbb{C}^n$, Ω_1^n. It's clear that $SO(n+1,\mathbb{C})$ acts holomorhpically on $\mathbb{C}^{n+1}$ and preserves the hyperquadric Ω_1^n. An arbitrary non-zero vector in $\mathbb{C}^{n+1}$ can be made, by the ordinary Gram-Schmidt orthogonalization process, to be the first column in an orthogonal matrix with determinant one, so $SO(n+1,\mathbb{C})$ acts transitively. This shows that Ω_1^n is biholomorphically equivalent to $SO(n+1,\mathbb{C})$ modulo the stabilizer of the point

$$z_o = (1,0,\ldots,0) \in \Omega_1^n.$$

A matrix A_o fixing z_o must have the form

$$A_o = \begin{pmatrix} 1 & & \\ 0 & & * \\ \vdots & & \\ 0 & & \end{pmatrix}.$$

It follows from the identity $A_o^t A_o = I$ that $A_o \in SO(n,\mathbb{R})$. This shows that Ω_1^n is biholomorphically equivalent to $SO(n+1,\mathbb{C})/SO(n,\mathbb{C})$.

2. $M = \mathbb{R}P^n$ is the homogeneous space $SO(n+1,\mathbb{R})/O(n,\mathbb{R})$. We consider $O(n,\mathbb{R})$ to

be the closed subgroup of $SO(n+1,\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} 1/\det A & \cdots & 0 \\ 0 & & \\ \vdots & A & \\ 0 & & \end{pmatrix}$$

with $A \in O(n,\mathbb{R})$. The natural complexification of M is $SO(n+1,\mathbb{C})/O(n,\mathbb{C})$. The action of $SO(n+1,\mathbb{C})$ on Ω_1^n descends to a transitive holomorphic action on Ω_2^n, by the map β sending a point z in Ω_1^n to the complex line $[z]$ in $\mathbb{C}P^n$ determined by z. If A_o fixes the point

$$[z_o] = [(1,0,\ldots,0)] \in \Omega_2^n$$

then A_o must have the form

$$\begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}.$$

The stabilizer of $[z_o]$ is $O(n,\mathbb{C})$. This shows that Ω_2^n is biholomorphically equivalent to $SO(n+1,\mathbb{C})/O(n,\mathbb{C})$.

3. $M = \mathbb{C}P^n$ is the homogeneous space $SU(n+1)/U(n)$ (here $U(n)$ is embedded in $SU(n+1)$ in the same way that $O(n)$ is in $SO(n+1,\mathbb{R})$). The natural complexification of M is $SL(n+1,\mathbb{C})/GL(n,\mathbb{C})$. We will show that $SL(n+1,\mathbb{C})/GL(n,\mathbb{C})$ is biholomorphically equivalent to $\Omega_3^{2n} = \mathbb{C}P^n \times \mathbb{C}P^n \backslash P_\infty(\mathbb{C}^{N-1})$, $N = (n+1)^2 - 1$. Recall that $\mathbb{C}P^n \times \mathbb{C}P^n$ is embedded in $\mathbb{C}P^N$ by the Segre embedding

$$([z],[w]) \to [\zeta_{\alpha\beta}] = [z_\alpha w_\beta].$$

Notice that Ω_3^{2n} is the projective image of the complex $n+1$ by $n+1$ matrices with complex rank one and non-zero trace. Consider the action of $SL(n+1,\mathbb{C})$ on $\mathbb{C}P^N$ by matrix conjugation:

$$[\zeta] \to [A\zeta A^{-1}].$$

This is clearly a holomorphic action of $SL(n+1,\mathbb{C})$ on Ω_3^{2n}, since matrix conjugation preserves the rank and the trace. If ζ is a rank one matrix with nonzero trace, then the Jordan canonical form of ζ is diagonal[1]. Thus there is an A in $SL(n+1,\mathbb{C})$ such that

$$A\zeta A^{-1} = \operatorname{tr}\zeta \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

This shows that $SL(n+1,\mathbb{C})$ acts transitively on Ω_3^{2n}. It is easy to see that if A fixes the projective image of the point

$$\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

then A must be in $GL(n,\mathbb{C})$. This shows that Ω_3^{2n} is biholomorphically equivalent to $SL(n+1,\mathbb{C})/GL(n,\mathbb{C})$.

4.3.2 Identification of $G_{\mathbb{C}}/K_{\mathbb{C}}$ and T^*G/K.

Let G be a compact Lie group, and K a closed subgroup of G. Let $\mathbf{k}$ be the Lie algebra of K, and let $\mathbf{k}^\perp$ be the orthogonal complement of $\mathbf{k}$ with respect to the Ad(G)-invariant inner product on $\mathbf{g}$. Define the vector bundle $G \times_K \mathbf{k}^\perp$ to be $G \times \mathbf{k}^\perp$ modulo the proper,

[1]See Strang [20], p. 81-82.

free action of K given by

$$(g, X) \to (gk, \mathrm{Ad}(k^{-1})X).$$

This makes sense since $\mathrm{Ad}(K)$ preserves the orthogonal complement of $\mathbf{k}$. In the case of a Riemannian symmetric space, this inner product is (minus one times) the Killing form, and $\mathbf{k}^\perp$, denoted by $\mathbf{p}$, is the -1 eigenspace of an involutive automorphism of $\mathbf{g}$.

Identify TG/K with T^*G/K by a Riemannian metric on G/K. We will show that TG/K is diffeomorphic to $G \times_K \mathbf{k}^\perp$ as a real analytic manifold. If G is a compact, connected, semisimple Lie group and $G_{\mathbf{C}}$, $K_{\mathbf{C}}$ are the complexifications of G and K described above, then we will show that $G_{\mathbf{C}}/K_{\mathbf{C}}$ is diffeomorphic to $G \times_K \mathbf{k}^\perp$ as a real analytic manifold. We are very grateful to Professor David Vogan for his help with this problem.

Proposition 4.3.4 *Let G be a compact Lie group, K a closed subgroup. Then TG/K is diffeomorphic to $G \times_K \mathbf{k}^\perp$ as a real analytic manifold.*

Proof. Let $\tau(g)$ denote the natural action of G on G/K. Consider the following diagram:

$$\begin{array}{ccc} & b & \\ G \times \mathbf{k}^\perp & \longrightarrow & G \times_K \mathbf{k}^\perp \\ a \quad \downarrow & & \\ TG/K & & \end{array}$$

where b is the natural projection sending (g, X) onto its equivalence class in $G \times_K \mathbf{k}^\perp$, and a is the equally natural map

$$(g, X) \to d\tau(g) d\pi_{G/K}(V).$$

We want to show that there is a real analytic map c,

$$c: G \times_K \mathbf{k}^\perp \to TG/K,$$

which makes the diagram above commute. To see this it suffices to show that a and b are surjective submersions, and a is constant on the fibers of b (this follows from the implicit function theorem; see Loos [13], lemma 1.5, chapter I). The differentiable structure on $G \times_K \mathbf{k}^\perp$ is defined so that b is a submersion, and b is clearly surjective. If $b(g, X) = b(g', X')$, then there exists a $k \in K$ such that $(g, X) = (g'k, \mathrm{Ad}(k^{-1})X')$. Then

$$a(g, X) = d\tau(g')d\tau(k)d\pi_{G/K}(\mathrm{Ad}(k^{-1})X').$$

It is easy to verify that $d\tau(k)d\pi_{G/K}(\mathrm{Ad}(k^{-1})X') = d\pi_{G/K}(X')$, since for all $X \in \mathbf{k}^\perp$,

$$\tau(k)\pi_{G/K}(\exp \mathrm{Ad}(k^{-1})X) = \tau(k)\pi_{G/K}(k^{-1} \exp Xk) = \pi_{G/K}(\exp X).$$

This shows that

$$a(g, X) = d\tau(g')d\pi_{G/K}(X') = a(g', X'),$$

so a is constant on the fibers of b. It is clear that a is surjective, since G acts transitively on G/K and $T_{eK}G/K$ is identified with $\mathbf{k}^\perp$ via $d\pi_{G/K}$. The map

$$\alpha: G \times \mathbf{g} \to TG/K$$

given by $\alpha(g, X) = d\tau(g)d\pi_{G/K}(X)$ is clearly a submersion, and a is just α restriced to $G \times \mathbf{k}^\perp$. To show a is a submersion we need to show that for all $(g, V) \in G \times \mathbf{k}^\perp$, $d\alpha_{(g,V)}(T_{(g,V)}(G \times \mathbf{k}^\perp))$ is all of $T_{\alpha(g,V)}(TG/K)$. This is clear since $T_{(g,V)}(G \times \mathbf{g}) = T_gG \times \mathbf{g}$, and $d\alpha_{(g,V)}(0, \mathbf{k}) = 0$. Hence there is a unique map

$$c: G \times_K \mathbf{k}^\perp \to TG/K$$

such that $a = c \circ b$. This map is a surjective submersion, and is real analytic by the implicit function theorem for real analytic functions.

To show that c is in fact a real analytic diffeomorphism we will construct an inverse by reversing the roles of a and b above. We need only check that b is constant on the fibers of a. If $a(g,X) = a(g',X')$, then

$$d\pi_{G/K}(X) = d\tau(g^{-1}g')d\pi_{G/K}(X').$$

Then there is a $k \in K$ such that $g^{-1}g' = k$, and

$$d\pi_{G/K}(X) = d\pi_{G/K}(\mathrm{Ad}(k)X').$$

This shows that $g' = gk$ and $X' = \mathrm{Ad}(k^{-1})X$, so $b(g,X) = b(g',X')$. Then there is a unique real analytic map c',

$$c' : TG/K \to G \times_K \mathsf{k}^{\perp},$$

such that $b = c' \circ a$. Then $b = c' \circ c \circ b$ and $a = c \circ c' \circ a$, which shows that c is invertible and $c' = c^{-1}$. □

Remark. Note that the action of G on $G \times \mathsf{k}^{\perp}$ which is the standard action on G and the trivial action on $\mathsf{k}^{\perp}$ preserves the fibers of b, inducing a G action on $G \times_K \mathsf{k}^{\perp}$. It is easy to see that the map c is equivariant with respect to this action and the $d\tau(G)$ action on TG/K.

We will need the following results to show that if G is a compact, connected, semisimple Lie group, then $G_{\mathbf{C}}/K_{\mathbf{C}}$ is diffeomorphic to $G \times_K \mathsf{k}^{\perp}$ as a real analytic manifold.

Theorem 4.3.5 (Mostow, 1955) *Let $\tilde{G}$ be a connected, semisimple Lie group and let $\mathsf{G} = \mathsf{K} + \mathsf{E}$ be a Cartan decomposition of its Lie algebra with K compact. Let E' be a linear subspace of E such that $[X,[X,Y]] \in \mathsf{E}'$ for all $X, Y \in \mathsf{E}'$. Let $\mathsf{F} = \{X \in \mathsf{E} : \mathrm{B}(X, \mathsf{E}') = 0\}$, where B is the Killing form of $\tilde{G}$. Then $\tilde{G}$ decomposes topologically into $\tilde{K} \cdot F \cdot E'$, where $\tilde{K}$ is the analytic subgroup determined by K, $F = \exp \mathsf{F}$, $E' = \exp \mathsf{E}'$.*

Proof. See Mostow [16], or Helgason [10], theorem 1.4, §1, chapter VI. In Helgason the order of the factors is reversed, but that is easily remedied by taking inverses. □

To apply this we take $\tilde{G}$ to be $G_{\mathbf{C}}$. $G_{\mathbf{C}}$ is semisimple and connected if G is, and $\mathbf{g}_{\mathbf{C}} = \mathbf{g} + \sqrt{-1}\,\mathbf{g}$ is a Cartan decomposition with $\mathbf{g}$ compact. Take for E' the linear subspace $\sqrt{-1}\,\mathsf{k}$ of $\sqrt{-1}\,\mathbf{g}$. It is clear that for $X, Y \in \mathsf{k}$, $[\sqrt{-1}\,X, [\sqrt{-1}\,X, \sqrt{-1}\,Y]] \in \sqrt{-1}\,\mathsf{k}$. Note that the orthogonal complement of $\sqrt{-1}\,\mathsf{k}$ in $\sqrt{-1}\,\mathbf{g}$ with respect to the Killing form $\mathrm{B}_{\mathbf{C}}$ of $G_{\mathbf{C}}$ is the same as $\sqrt{-1}$ times the orthogonal complement of k in $\mathbf{g}$, since $\mathrm{B}_{\mathbf{C}}(\sqrt{-1}\,X, \sqrt{-1}\,Y) = -\mathrm{B}(X, Y)$ and the bi-invariant metric at the identity is just (minus one times) the Killing form of G. Thus we take for F the subspace $\sqrt{-1}\,\mathsf{k}^{\perp}$ of $\sqrt{-1}\,\mathbf{g}$. The analytic subgroup determined by $\mathbf{g}$ is just G, since G is connected. Then $G_{\mathbf{C}}$ decomposes topologically as

$$G_{\mathbf{C}} = G \cdot \exp \sqrt{-1}\,\mathsf{k}^{\perp} \cdot \exp \sqrt{-1}\,\mathsf{k}.$$

We will need the following sharpening of theorem 4.3.5.

Theorem 4.3.6 *Let G be a compact, connected, semisimple Lie group, and let k, $\mathsf{k}^{\perp}$ be as above. Then the map $\Gamma\colon \sqrt{-1}\,\mathsf{k} \times \sqrt{-1}\,\mathsf{k}^{\perp} \times G \to G_{\mathbf{C}}$ given by*

$$\Gamma(\sqrt{-1}\,X, \sqrt{-1}\,V, g) = \exp \sqrt{-1}\,X \exp \sqrt{-1}\,V g$$

is a real analytic diffeomorphism.

Proof. We follow the proof in Loos [13], page 160-161. The first step is to prove that $G_{\mathbf{C}}/G$ is diffeomorphic to $\sqrt{-1}\,\mathsf{k} \times \sqrt{-1}\,\mathsf{k}^{\perp}$ by the map γ,

$$\gamma\colon \sqrt{-1}\,\mathsf{k} \times \sqrt{-1}\,\mathsf{k}^{\perp} \to G_{\mathbf{C}}/G,$$

given by

$$\gamma(\sqrt{-1}\,X, \sqrt{-1}\,Y) = \tau(\exp \sqrt{-1}\,X)\mathrm{Exp}\,\sqrt{-1}\,Y$$

where Exp is the Riemannian exponential map from $\mathbf{g}$ to $G_{\mathbf{C}}/G$. Since $G_{\mathbf{C}}/G$ is a Riemannian globally symmetric space of the noncompact type, the map Exp is a diffeomorphism[2]. Explicitly, Exp is given by[3]

$$\operatorname{Exp}\sqrt{-1}\,X = \pi_{G_{\mathbb{C}}/G}(\exp\sqrt{-1}\,X)$$

where $\pi_{G_{\mathbb{C}}/G}$ is the coset projection of $G_{\mathbb{C}}$ onto $G_{\mathbb{C}}/G$. Let $S = \operatorname{Exp}\sqrt{-1}\,\mathsf{k}$. The normal bundle of S is naturally identified with $\sqrt{-1}\,\mathsf{k}\times\sqrt{-1}\,\mathsf{k}^{\perp}$, and the map γ is the exponential map from the normal bundle of S to $G_{\mathbb{C}}/G$. It is clearly smooth, in fact it is real analytic. To see that it is a bijection, note that $G_{\mathbb{C}}/G$ is a complete, simply connected Riemannian manifold of negative sectional curvature[4]. S is a closed, totally geodesic submanifold of $G_{\mathbb{C}}/G$, since $\sqrt{-1}\,\mathsf{k}$ is a "Lie triple system"[5], i.e., $[\sqrt{-1}\,\mathsf{k},[\sqrt{-1}\,\mathsf{k},[\sqrt{-1}\,\mathsf{k}]]] \subset \sqrt{-1}\,\mathsf{k}$. This implies[6] that at each point $\zeta \in S$, the geodesics perpendicular to S through ζ are a submanifold $S(\zeta)^{\perp}$ of $G_{\mathbb{C}}/G$, and $G_{\mathbb{C}}/G$ is the disjoint union of the geodesics through S perpendicular to S,

$$G_{\mathbb{C}}/G = \bigcup_{\zeta\in S} S(\zeta)^{\perp}.$$

This shows that the exponential map from the normal bundle of S, γ, is a real analytic bijection. Since $G_{\mathbb{C}}/G$ is a complete Riemannian manifold of negative curvature, the exponential map is everywhere regular[7]. Hence γ is a real analytic diffeomorphism.

Now consider the map Γ,

$$\Gamma\colon \sqrt{-1}\,\mathsf{k}\times\sqrt{-1}\,\mathsf{k}^{\perp}\times G \to G_{\mathbb{C}},$$

given by

$$\Gamma(\sqrt{-1}\,X,\sqrt{-1}\,V,g) = \exp\sqrt{-1}\,X\exp\sqrt{-1}\,Vg.$$

This is a real analytic bijection, by theorem 4.3.5. We need to see that the inverse

[2]See Helgason [10], theorem 1.1, Chapter VI.
[3]Ibid, §3 and §4 of chapter IV.
[4]Ibid, theorem 3.1, chapter V.
[5]Ibid, theorem 7.2, chapter IV .
[6]Ibid, theorem 14.6, chapter I
[7]Ibid, theorem 13.3(i), chapter I.

is real analytic. The map $\gamma^{-1} \circ \pi_{G_{\mathbb{C}}/G}$ is real analytic by the above discussion. Let $\gamma^{-1} \circ \pi_{G_{\mathbb{C}}/G}(\zeta) = (\sqrt{-1}\,X(\zeta), \sqrt{-1}\,Y(\zeta))$. By theorem 4.3.5,

$$(\exp -\sqrt{-1}\,X(\zeta) \exp -\sqrt{-1}\,Y(\zeta))\zeta$$

is an element of G, and the map

$$\zeta \to g(\zeta) = (\exp -\sqrt{-1}\,X(\zeta) \exp -\sqrt{-1}\,Y(\zeta))\zeta$$

is analytic. Then the map Γ^{-1} is given by

$$\zeta \to (\sqrt{-1}\,X(\zeta), \sqrt{-1}\,V(\zeta), g(\zeta)),$$

so Γ is a real analytic diffeomorphism. □

Corollary 4.3.7 *The map* $\Upsilon: G \times \mathsf{k}^{\perp} \times \mathsf{k} \to G_{\mathbb{C}}$ *given by*

$$\Upsilon(g, V, X) = g \exp \sqrt{-1}\,V \exp \sqrt{-1}\,X$$

is a real analytic diffeomorphism.

Proof. Υ can be expressed as the composition of real analytic diffeomorphisms

$$\Upsilon(g, V, X) = (\exp \sqrt{-1}\,X \exp \sqrt{-1}\,V g^{-1})^{-1}.$$

□

We are now ready to prove the main result of this section.

Proposition 4.3.8 *Let G be a compact, connected, semisimple Lie group, let K be a closed subgroup of G, and let $G_{\mathbb{C}}$, $K_{\mathbb{C}}$ be the complexifications of G and K described in section 4.3.1. Then $G_{\mathbb{C}}/K_{\mathbb{C}}$ is diffeomorphic to $G \times_K \mathsf{k}^{\perp}$ as a real analytic manifold.*

Proof. As in proposition 4.3.4 we consider the following natural maps:

$$\begin{array}{ccc} & b & \\ G \times \mathbf{k}^{\perp} & \longrightarrow & G \times_K \mathbf{k}^{\perp} \\ a_{\mathbb{C}} \downarrow & & \\ G_{\mathbb{C}}/K_{\mathbb{C}} & & \end{array}$$

where b is the coset projection and $a_{\mathbb{C}}$ is the map

$$a_{\mathbb{C}}(g, V) = \pi_{G_{\mathbb{C}}/K_{\mathbb{C}}}(g \exp \sqrt{-1}\, V)$$

($\pi_{G_{\mathbb{C}}/K_{\mathbb{C}}}$ is the natural projection of $G_{\mathbb{C}}$ onto $G_{\mathbb{C}}/K_{\mathbb{C}}$). As in proposition 4.3.4, we want to construct a real analytic map $c_{\mathbb{C}}$,

$$c_{\mathbb{C}}: G_K \times \mathbf{k}^{\perp} \to G_{\mathbb{C}}/K_{\mathbb{C}},$$

such that $a_{\mathbb{C}} = c_{\mathbb{C}} \circ b$. All the maps involved are real analytic, and b is a submersion. It is easy to see that $a_{\mathbb{C}}$ is constant on the fibers of b, since $g \exp \sqrt{-1}\, V \cdot K_{\mathbb{C}} = gk \exp \sqrt{-1}\, \mathrm{Ad}(k^{-1})V \cdot K_{\mathbb{C}}$. This shows that there is a unique real analytic map $c_{\mathbb{C}}$ such that $a_{\mathbb{C}} = c_{\mathbb{C}} \circ b$.

To show that $c_{\mathbb{C}}$ is a real analytic diffeomorphism we need to be able to reverse the role of $a_{\mathbb{C}}$ and b above. Suppose we can show that

1. b is constant on the fibers of $a_{\mathbb{C}}$
2. $a_{\mathbb{C}}$ is a submersion.

Then there exists a unique real analytic map $c'_{\mathbb{C}}$,

$$c'_{\mathbb{C}}: G_{\mathbb{C}}/K_{\mathbb{C}} \to G \times_K \mathbf{k}^{\perp},$$

such that $b = c'_{\mathbb{C}} \circ a_{\mathbb{C}}$. By the same reasoning as before we can conclude that $c'_{\mathbb{C}} = c_{\mathbb{C}}^{-1}$

and $c_{\mathbf{c}}$ is a real analytic diffeomorphism.

Proof of 1. Suppose $\pi_{G_{\mathbf{c}}/K_{\mathbf{c}}}(g \exp \sqrt{-1}\, V) = \pi_{G_{\mathbf{c}}/K_{\mathbf{c}}}(g' \exp \sqrt{-1}\, V')$. Then there is a $k_{\mathbf{c}} \in K_{\mathbf{c}}$ such that

$$g \exp \sqrt{-1}\, V = g' \exp \sqrt{-1}\, V' k_{\mathbf{c}}.$$

By theorem 4.1.2 and the remarks following we have the polar decomposition

$$k_{\mathbf{c}} = k \exp \sqrt{-1}\, X$$

with $k \in K$ and $X \in \mathsf{k}$. Thus

$$\begin{aligned} g \exp \sqrt{-1}\, V &= g' \exp \sqrt{-1}\, V' k \exp \sqrt{-1}\, X \\ &= g' k \exp \sqrt{-1}\, \mathrm{Ad}(k^{-1}) V' \exp X. \end{aligned}$$

Note that V, $\mathrm{Ad}(k^{-1})V' \in \mathsf{k}^{\perp}$, $X \in \mathsf{k}$, and g, $g' \in G$. By the uniqueness of the decomposition

$$G_{\mathbf{c}} = G \cdot \exp \sqrt{-1}\, \mathsf{k}^{\perp} \cdot \exp \sqrt{-1}\, \mathsf{k}$$

in corollary 4.3.7, we conclude that $X = 0$, $g = g'k$, and $V = \mathrm{Ad}(k^{-1})V'$. This shows that $b(g, V) = b(g', V')$, and proves 1.

Proof of 2. Let Υ be as in corollary 4.3.7. To show that $a_{\mathbf{c}}$ is a submersion, note that $a_{\mathbf{c}} \circ \mathrm{pr}_{G \times \mathsf{k}^{\perp}} = \pi_{G_{\mathbf{c}}/K_{\mathbf{c}}} \circ \Upsilon$. Since Υ is a diffeomorphism, $\pi_{G_{\mathbf{c}}/K_{\mathbf{c}}} \circ \Upsilon$ is a submersion. This shows that $a_{\mathbf{c}}$ is a submersion, and completes the proof of 2 (and the proposition). □

Remark. Note that G acts on $G_{\mathbf{c}}/K_{\mathbf{c}}$ in the obvious way, as a subgroup of $G_{\mathbf{c}}$. It's easy to see that the map $c_{\mathbf{c}}$ is equivariant with respect to this action on $G_{\mathbf{c}}/K_{\mathbf{c}}$ and the G action on $G \times_K \mathsf{k}^{\perp}$ described above.

Combining propositions 4.3.4 and 4.3.8 we have proved that, under suitable hypotheses on G, there is a real analytic diffeomorphism of TG/K and $G_{\mathbf{c}}/K_{\mathbf{c}}$,

$$c_{\mathbf{c}} \circ c^{-1} \colon TG/K \to G_{\mathbf{c}}/K_{\mathbf{c}}.$$

We can easily write down this map for the record. If $V_{gK} \in TG/K$ can be written as

$$V_{gK} = d\tau(g) d\pi_{G/K}(V)$$

for some $(g, V) \in G \times \mathsf{k}^{\perp}$, then

$$c_{\mathbf{C}} \circ c^{-1}(V_{gK}) = g \exp \sqrt{-1}\, V \cdot K_{\mathbf{C}}.$$

It may be instructive to check directly that this procedure for computing $c_{\mathbf{C}} \circ c^{-1}$ is well-defined. If $V_{gK} = d\tau(g') d\pi_{G/K}(V')$ for another $(g', V') \in G \times \mathsf{k}^{\perp}$, then $g^{-1}g' \in K$ and $V = \mathrm{Ad}(g^{-1}g')V$. Then

$$g \exp \sqrt{-1}\, V \cdot K_{\mathbf{C}} = g \exp \sqrt{-1}\, \mathrm{Ad}(g^{-1}g')V' \cdot K_{\mathbf{C}} = g' \exp \sqrt{-1}\, V' \cdot K_{\mathbf{C}}.$$

It's also a trivial matter to check directly that $c_{\mathbf{C}} \circ c^{-1}$ is equivariant with respect to the $d\tau(G)$ action on TG/K and the obvious G action on G/K.

It may be useful to keep in mind the following picture.

$$\begin{array}{ccccc} & & G \times \mathsf{k}^{\perp} & & \\ & \mathrm{a} \swarrow & & \searrow a_{\mathbf{C}} & \\ TG/K & & \downarrow b & & G_{\mathbf{C}}/K_{\mathbf{C}} \\ & \mathrm{c} \nwarrow & & \nearrow c_{\mathbf{C}} & \\ & & G \times_K \mathsf{k}^{\perp} & & \end{array}$$

We summarize the results of this section in the following theorem.

Theorem 4.3.9 *Let G be a compact, connected, semisimple Lie group and let K be a closed subgroup of G. Then the tangent bundle of G/K is canonically identified with a complex homogeneous manifold. A real analytic Riemannian metric on G/K induces a complex structure on the cotangent bundle of G/K by the metric identification of TG/K and T^*G/K.*

4.3.3 The Complex Structure on $G \times_K \mathsf{p}$

We now have global identifications of T^*G/K, TG/K, $G \times_K \mathsf{k}^\perp$, and $G_{\mathbf{C}}/K_{\mathbf{C}}$ as real analytic manifolds. Each of these inherits a symplectic structure from T^*G/K, and a complex structure from $G_{\mathbf{C}}/K_{\mathbf{C}}$. In this section we will study the complex structure so obtained. In order to write down the complex structure operator explicitly, we need to know how to compute the projection onto $\mathsf{k}^\perp$ of $\mathrm{ad}(V)(X)$ for $V \in \mathsf{k}^\perp$ and $X \in \mathbf{g}$. In the non-symmetric case there is no reason to expect to know how to do this, so it is not possible in general to write down the complex structure operator on $G \times_K \mathsf{k}^\perp$ explicitly. In the symmetric case the situation is more satisfactory. We have a decomposition of $\mathbf{g}$ into eigenspaces of an involutory automorphism θ of $\mathbf{g}$:

$$\mathsf{g} = \mathsf{k} + \mathsf{p}.$$

Here k is the $+1$ eigenspace of θ (since k is a subalgebra), and p is the -1 eigenspace. The eigenspaces are orthogonal with respect to the $\mathrm{Ad}(G)$-invariant inner product on $\mathbf{g}$. Since this is an eigenspace decomposition, it is easy to see how the action of $\mathrm{ad}(\mathbf{g})$ permutes the subspaces k and p.

Lemma 4.3.10 $[\mathsf{p},\mathsf{k}] \subset \mathsf{p}$, $[\mathsf{p},\mathsf{p}] \subset \mathsf{k}$, *and* $[\mathsf{k},\mathsf{k}] \subset \mathsf{k}$.

Proof. Since θ is an automorphism of $\mathbf{g}$, we have

$$\theta[X,Y] = [\theta X, \theta Y].$$

If $X \in \mathsf{p}$ and $Y \in \mathsf{k}$, then $\theta[X,Y] = -[X,Y]$, so $[X,Y] \in \mathsf{p}$. The others follow similarly. □

Since $\sin x/x$ and $\cos x$ are entire functions, it makes sense to speak of the operators

$$\sin \mathrm{ad}(V)/\mathrm{ad}(V) \;=\; \sum_0^\infty (-1)^n \mathrm{ad}(V)^{2n}/(2n+1)!$$

$$\cos \mathrm{ad}(V) \quad = \quad \sum_0^\infty (-1)^n \mathrm{ad}(V)^{2n}/(2n)!$$

with $V \in \mathbf{g}$. We interpret $\mathrm{ad}(V)^0$ as the identity operator on $\mathbf{g}$.

Lemma 4.3.11 *The operators* $\sin \mathrm{ad}(V)/\mathrm{ad}(V)$ *and* $\cos \mathrm{ad}(V)$ *are symmetric, invertible linear operators on* $\mathbf{g}$. *If* $V \in \mathbf{p}$ *or* $V \in \mathbf{k}$, *then these operators preserve the subspaces* $\mathbf{k}$ *and* $\mathbf{p}$.

Proof. If $<,>$ is any $\mathrm{Ad}(G)$-invariant nondegenerate inner product on $\mathbf{g}$, then $\mathrm{ad}(\mathbf{g})$ consists of skew-symmetric endomorphisms. Furthermore, $-\mathrm{ad}(V)^2$ is a non-negative operator, since

$$-<\mathrm{ad}(V)^2 X, X> = <\mathrm{ad}(V)X, \mathrm{ad}(V)X> \geq 0.$$

Since $\mathrm{ad}(V)^0$ is a strictly positive operator, it follows that $\sin \mathrm{ad}(V)/\mathrm{ad}(V)$ and $\cos \mathrm{ad}(V)$ are strictly positive, symmetric operators. This proves the invertibility. It's clear from lemma 4.3.10 that if $V \in \mathbf{p}$ then $\mathrm{ad}(V)^2$ preserves $\mathbf{k}$ and $\mathbf{p}$. If $V \in \mathbf{k}$ then $\mathrm{ad}(V)$ preserves the subspaces $\mathbf{k}$ and $\mathbf{p}$. $\square$

To describe the complex structure on TG/K we will need a convenient method of representing the tangent bundles of these manifolds. We will do so in terms of the image of vector fields on $G \times \mathbf{p}$ under the differentials of the maps a and $a_{\mathbf{C}}$. Note that $\mathbf{p}$, being a vector space, is an Abelian Lie group and $G \times \mathbf{p}$ has the product group structure. If $(W, Y) \in \mathbf{g} \times \mathbf{p}$, we denote the corresponding left invariant vector field on $G \times \mathbf{p}$ by $(W, Y)^\sim$. Using the natural identification of $T_{(g,V)}(G \times \mathbf{p})$ with $T_g G \times \mathbf{p}$, the vector field $(W, Y)^\sim$ is simply

$$(g, V) \rightarrow (dL_g W, Y).$$

Such vector fields form a basis for the tangent space of $G \times \mathbf{p}$ at each point. Since $a_{\mathbf{C}}$ is a surjective submersion, every vector in $TG_{\mathbf{C}}/K_{\mathbf{C}}$ can be represented as $da_{\mathbf{C}}(W, Y)^\sim_{(g,V)}$ for some (not necessarily unique) $(g, V) \in G \times \mathbf{p}$ and $(W, Y) \in \mathbf{g} \times \mathbf{p}$. Similarly, each vector in $T(TG/K)$ can be represented as $da(W, Y)^\sim_{(g,V)}$, and each vector in $TG \times_K \mathbf{p}$ can be represented as $db(W,Y)^{\sim}_{(g,V)}$. Then the following gives an explicit description of the complex structure on TG/K, $G\times_K \mathbf{p}$, and $G_{\mathbf{C}}/K_{\mathbf{C}}$ in terms of these vector fields.

Proposition 4.3.12 *Let $J_{G_{\mathbf{C}}/K_{\mathbf{C}}}$ denote the operator "multiplication by $\sqrt{-1}$" on the tangent bundle of $G_{\mathbf{C}}/K_{\mathbf{C}}$, and let $J_{G\times_K\mathbf{p}}$, $J_{TG/K}$ denote the corresponding operators on the tangent bundles of $G\times_K \mathbf{p}$ and TG/K under the identifications given in section 4.3.2. Let $(g,V)\in G\times\mathbf{p}$ and let $T(V)$ denote the operator*

$$(\sin \mathrm{ad}(V)/\mathrm{ad}(V))^{-1}\circ\cos\mathrm{ad}(V),$$

which preserves the subspaces $\mathbf{k}$ and $\mathbf{p}$ by lemma 4.3.11. Then for all $(W,Y)\in\mathbf{g}\times\mathbf{p}$,

$$\begin{aligned} J_{G_{\mathbf{C}}/K_{\mathbf{C}}}da_{\mathbf{C}}(W,Y)^{\sim}_{(g,V)} &= da_{\mathbf{C}}(-T(V)^{-1}Y+\tan\mathrm{ad}(V)W_{\mathbf{k}},\ T(V)W_{\mathbf{p}})^{\sim}_{(g,V)}\\ J_{TG/K}da(W,Y)^{\sim}_{(g,V)} &= da(-T(V)^{-1}Y+\tan\mathrm{ad}(V)W_{\mathbf{k}},\ T(V)W_{\mathbf{p}})^{\sim}_{(g,V)}\\ J_{G\times_K\mathbf{p}}db(W,Y)^{\sim}_{(g,V)} &= db(-T(V)^{-1}Y+\tan\mathrm{ad}(V)W_{\mathbf{k}},\ T(V)W_{\mathbf{p}})^{\sim}_{(g,V)}\end{aligned}$$

Here $\tan\mathrm{ad}(V)$ means $(\cos\mathrm{ad}(V))^{-1}\circ\sin\mathrm{ad}(V)$, and the subscripts $\mathbf{k}$ and $\mathbf{p}$ denote projection onto those spaces relative to the decomposition $\mathbf{g}=\mathbf{k}+\mathbf{p}$.

Proof. Recall that $a_{\mathbf{C}}: G\times\mathbf{p}\to G_{\mathbf{C}}/K_{\mathbf{C}}$ is the map

$$a_{\mathbf{C}}(g,V)=\pi_{G_{\mathbf{C}}/K_{\mathbf{C}}}(g\exp\sqrt{-1}\,V).$$

We need to compute the differential of $a_{\mathbf{C}}$. Let $M_{\sqrt{-1}}$ denote multiplication by $\sqrt{-1}$ in $\mathbf{g}_{\mathbf{C}}$, and let $\mathcal{M}: G_{\mathbf{C}}\times G_{\mathbf{C}}\to G_{\mathbf{C}}$ denote the group multiplication. Then we can write $a_{\mathbf{C}}$ as

$$a_{\mathbf{C}}=\pi_{G_{\mathbf{C}}/K_{\mathbf{C}}}\circ\mathcal{M}\circ(Id_G\times(\exp\circ M_{\sqrt{-1}}))$$

and so for $(W,Y)\in\mathbf{g}\times\mathbf{p}$,

$$da_{\mathbf{C}}(W,Y)^{\sim}_{(g,V)}=$$

$$d(\pi_{G_{\mathbf{C}}/K_{\mathbf{C}}})_{g\exp\sqrt{-1}V}\circ d(\mathcal{M})_{(g,\exp\sqrt{-1}V)}\circ d(Id_G\times(\exp\circ M_{\sqrt{-1}}))(W,Y)^{\sim}_{(g,V)}.$$

To compute this we write $(W,Y)^{\sim}_{(g,V)}=(dL_gW,Y)$. Then

$$d(Id_G\times(\exp\circ M_{\sqrt{-1}}))(W,Y)^{\sim}_{G,V)}=(dL_gW,d(\exp)_{\sqrt{-1}V}(\sqrt{-1}Y)).$$

The differential of the exponential map is given by, for $U\in\mathbf{g_C}$,

$$d(\exp)_U=dL_{\exp U}\circ\frac{1-e^{-\mathrm{ad}(U)}}{\mathrm{ad}(U)}$$

(see lemma 4.1.3). Then

$$\begin{aligned}&d(Id_G\times(\exp\circ M_{\sqrt{-1}}))(W,Y)^{\sim}_{(g,V)}=\\&\quad(dL_gW,\ dL_{\exp\sqrt{-1}V}\circ\frac{1-e^{-\mathrm{ad}(\sqrt{-1}V)}}{\mathrm{ad}(\sqrt{-1}V)}(\sqrt{-1}Y)).\end{aligned}$$

The differential of the group multiplication is given by, for ζ, $\eta\in G_{\mathbf{C}}$ and $U_\zeta\in T_\zeta G_{\mathbf{C}}$, $V_\eta\in T_\eta G_{\mathbf{C}}$,

$$d(\mathcal{M})_{(\zeta,\eta)}(U_\zeta,V_\eta)=d(L_\zeta)_\eta(V_\eta)+d(R_\eta)_\zeta(U_\zeta).\tag{4.6}$$

Then

$$\begin{aligned}&d(\mathcal{M})_{(g,\exp\sqrt{-1}V)}\circ d(Id_G\times(\exp\circ M_{\sqrt{-1}}))(W,Y)^{\sim}_{(g,V)}=\\&\quad d(L_{g\exp\sqrt{-1}V})(\frac{1-e^{-\mathrm{ad}(\sqrt{-1}V)}}{\mathrm{ad}(\sqrt{-1}V)}(\sqrt{-1}Y))+d(R_{\exp\sqrt{-1}V}\circ L_g)(W).\end{aligned}$$

Using the fact that left and right multiplication commutes we can write this as

$$\begin{aligned}&d(\mathcal{M})_{(g,\exp\sqrt{-1}V)}\circ d(Id_G\times(\exp\circ M_{\sqrt{-1}}))(W,Y)^{\sim}_{(g,V)}\\&=\ d(L_{g\exp\sqrt{-1}V})(\frac{1-e^{-\mathrm{ad}(\sqrt{-1}V)}}{\mathrm{ad}(\sqrt{-1}V)}(\sqrt{-1}Y)+\mathrm{Ad}(\exp-\sqrt{-1}V)(W))\\&=\ d(L_{g\exp\sqrt{-1}V})(\frac{1-e^{-\mathrm{ad}(\sqrt{-1}V)}}{\mathrm{ad}(\sqrt{-1}V)}(\sqrt{-1}Y)+e^{-\mathrm{ad}(\sqrt{-1}V)}(W)).\end{aligned}$$

Note that $\pi_{G_\mathbb{C}/K_\mathbb{C}} \circ L_\zeta = \tau(\zeta) \circ \pi_{G_\mathbb{C}/K_\mathbb{C}}$. Thus we have computed

$$da_\mathbb{C}(W,Y)^\sim_{(g,V)} =$$
$$d\tau(g \exp \sqrt{-1}\, V) \circ d\pi_{G_\mathbb{C}/K_\mathbb{C}}\Big(\frac{1-e^{-\mathrm{ad}(\sqrt{-1}\,V)}}{\mathrm{ad}(\sqrt{-1}\,V)}(\sqrt{-1}\,Y) + e^{-\mathrm{ad}(\sqrt{-1}\,V)}(W)\Big).$$

Recall that the kernel of $d\pi_{G_\mathbb{C}/K_\mathbb{C}}$ is $\mathfrak{k}_\mathbb{C}$, and

$$(d\pi_{G_\mathbb{C}/K_\mathbb{C}})|_{\mathfrak{p}_\mathbb{C}} \colon \mathfrak{p}_\mathbb{C} \to T_{eK_\mathbb{C}} G_\mathbb{C}/K_\mathbb{C}$$

is a bijection. So it will be useful to find the projection onto $\mathfrak{p}_\mathbb{C}$ of the expression

$$\frac{1-e^{-\mathrm{ad}(\sqrt{-1}\,V)}}{\mathrm{ad}(\sqrt{-1}\,V)}(\sqrt{-1}\,Y) + e^{-\mathrm{ad}(\sqrt{-1}\,V)}(W).$$

Since $\mathrm{ad}(\mathfrak{g}_\mathbb{C})$ consists of complex linear maps, it is easy to see that

$$\begin{aligned}\frac{1-e^{-\mathrm{ad}(\sqrt{-1}\,V)}}{\mathrm{ad}(\sqrt{-1}\,V)}(\sqrt{-1}\,Y) &= \frac{1-e^{\mathrm{ad}(\sqrt{-1}\,V)}}{\mathrm{ad}(V)}(Y)\\ &= \frac{1-\cos \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y) + \sqrt{-1}\,\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y)\end{aligned}$$

and

$$e^{-\mathrm{ad}(\sqrt{-1}\,V)}(W) = \cos \mathrm{ad}(V)(W) - \sqrt{-1}\,\sin \mathrm{ad}(V)(W).$$

Examining the power series for $(1-\cos \mathrm{ad}(V))/\mathrm{ad}(V)$, $\sin \mathrm{ad}(V)/\mathrm{ad}(V)$ and using lemma 4.3.10, we see that since $V, Y \in \mathfrak{p}$,

$$\frac{1-\cos \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y) \in \mathfrak{k}$$

$$\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y) \in \mathfrak{p}.$$

Given $W \in \mathbf{g}$, write $W_{\mathbf{k}}$ and $W_{\mathbf{p}}$ to denote its projections onto $\mathbf{k}$ and $\mathbf{p}$ respectively. Then recalling that $V \in \mathbf{p}$ we can see that

$$\begin{aligned}
\cos \mathrm{ad}(V)(W) &= \underbrace{\cos \mathrm{ad}(V)(W_{\mathbf{k}})}_{\in \mathbf{k}} + \underbrace{\cos \mathrm{ad}(V)(W_{\mathbf{p}})}_{\in \mathbf{p}} \\
\sin \mathrm{ad}(V)(W) &= \underbrace{\sin \mathrm{ad}(V)(W_{\mathbf{k}})}_{\in \mathbf{p}} + \underbrace{\sin \mathrm{ad}(V)(W_{\mathbf{p}})}_{\in \mathbf{k}}.
\end{aligned}$$

Combining these we see that

$$\begin{aligned}
&\mathrm{pr}_{\mathbf{p}_{\mathbb{C}}}\left(\frac{1 - e^{-\mathrm{ad}(\sqrt{-1}\,V)}}{\mathrm{ad}(\sqrt{-1}\,V)}(\sqrt{-1}\,Y) + e^{-\mathrm{ad}(\sqrt{-1}\,V)}(W)\right) \\
&= \cos \mathrm{ad}(V)(W_{\mathbf{p}}) + \sqrt{-1}\left(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y) - \sin \mathrm{ad}(V)(W_{\mathbf{k}})\right).
\end{aligned}$$

Thus we have shown that

$$\begin{aligned}
&da_{\mathbb{C}}(W, Y)^{\sim}_{(g,V)} = \qquad (4.7) \\
&\quad d\tau(g \exp \sqrt{-1}\,V) \circ (d\pi_{G_{\mathbb{C}}/K_{\mathbb{C}}})|_{\mathbf{p}_{\mathbb{C}}}\left(\cos \mathrm{ad}(V)(W_{\mathbf{p}}) + \sqrt{-1}\left(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y) - \sin \mathrm{ad}(V)(W_{\mathbf{k}})\right)\right).
\end{aligned}$$

Conversely, suppose we are given a vector $U + \sqrt{-1}\,X$ with U, $X \in \mathbf{p}$. Then this expression shows that

$$d\tau(g \exp \sqrt{-1}\,V) \circ (d\pi_{G_{\mathbb{C}}/K_{\mathbb{C}}})|_{\mathbf{p}_{\mathbb{C}}}(U + \sqrt{-1}\,X) = da_{\mathbb{C}}\left((\cos \mathrm{ad}(V))^{-1}(U), \left(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}\right)^{-1}(X)\right)^{\sim}_{(g,V)}.$$

Now it is easy to see that

$$\begin{aligned}
&\sqrt{-1}\, da_{\mathbb{C}}(W, Y)^{\sim}_{(g,V)} \\
&= d\tau(g \exp \sqrt{-1}\,V) \circ (d\pi_{G_{\mathbb{C}}/K_{\mathbb{C}}})|_{\mathbf{p}_{\mathbb{C}}}\left(\sin \mathrm{ad}(V)(W_{\mathbf{k}}) - \frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y) + \sqrt{-1}\, \cos \mathrm{ad}(V)(W_{\mathbf{p}})\right) \\
&= da_{\mathbb{C}}\left((\cos \mathrm{ad}(V))^{-1}\left(\sin \mathrm{ad}(V)(W_{\mathbf{k}}) - \frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y)\right), \left(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}\right)^{-1} \cos \mathrm{ad}(V)(W_{\mathbf{p}})\right)
\end{aligned}$$

$$= \quad da_{\mathbb{C}}(-T(V)^{-1}(Y) + \tan \mathrm{ad}(V)(W_{\mathsf{k}}),\ T(V)(W_{\mathsf{p}})).$$

This proves the first equality in the proposition. The other equalities follow more or less functorially. Recall

$$J_{TG/K} = d(c \circ c_{\mathbb{C}}^{-1}) \circ J_{G_{\mathbb{C}}/K_{\mathbb{C}}} \circ d(c_{\mathbb{C}} \circ c^{-1}).$$

Since $b = c^{-1} \circ a$, we have

$$\begin{aligned} J_{TG/K} \circ da &= d(c \circ c_{\mathbb{C}}^{-1}) J_{G_{\mathbb{C}}/K_{\mathbb{C}}} d(c_{\mathbb{C}} \circ b) \\ &= d(c \circ c_{\mathbb{C}}^{-1}) J_{G_{\mathbb{C}}/K_{\mathbb{C}}} da_{\mathbb{C}}. \end{aligned}$$

Now using the expression for $J_{G_{\mathbb{C}}/K_{\mathbb{C}}} \circ da_{\mathbb{C}}$ we get the corresponding expression for $J_{TG/K} \circ da$. Since

$$J_{G \times_K \mathsf{p}} = dc_{\mathbb{C}}^{-1} \circ J_{G_{\mathbb{C}}/K_{\mathbb{C}}} \circ dc_{\mathbb{C}},$$

the expression for $J_{G \times_K \mathsf{p}} \circ db$ follows similarly. □

Corollary 4.3.13 *At each point $(g, V) \in G \times \mathsf{p}$, da maps $(\mathsf{p}, \mathsf{p})^{\sim}_{(g,V)}$ bijectively onto the tangent space $T_{a(g,V)}(TG/K)$. If $(W, Y) \in \mathsf{p} \times \mathsf{p}$, then*

$$\begin{aligned} J_{G_{\mathbb{C}}/K_{\mathbb{C}}} da_{\mathbb{C}}(W, Y)^{\sim}_{(g,V)} &= da_{\mathbb{C}}(-T(V)^{-1}Y, T(V)W)^{\sim}_{(g,V)} \\ J_{TG/K} da(W, Y)^{\sim}_{(g,V)} &= da(-T(V)^{-1}Y, T(V)W)^{\sim}_{(g,V)} \\ J_{G \times_K \mathsf{p}} db(W, Y)^{\sim}_{(g,V)} &= db(-T(V)^{-1}Y, T(V)W)^{\sim}_{(g,V)}. \end{aligned}$$

Proof. If $(W, Y) \in \mathsf{p} \times \mathsf{p}$ and $da_{\mathbb{C}}(W, Y)^{\sim}_{(g,V)} = 0$, then from equation 4.7 we must have

$$\cos \mathrm{ad}(V)(W) = \frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)}(Y) = 0,$$

which implies that $W = Y = 0$. Now bijectivity follows from dimensional arguments. The expressions for the complex structure are obvious, since $W \in \mathfrak{p}$. $\square$

4.3.4 Proof of the Result

In this section we will complete the proof of theorem 2.1.1 for the compact locally symmetric space G/K. Let Q denote the metric on G/K corresponding to the bi-invariant metric on G as described in the beginning of this section, and let $\nu: TG/K \to T^*G/K$ be the metric identification,

$$\nu(V_{gK})(W_{gK}) = Q(V_{gK}, W_{gK}).$$

Let ϕ_o be the quadratic function on T^*G/K associated with the metric Q on G/K, and let $\phi = \nu^*\phi_o$ be the corresponding function on TG/K under the metric identification. Similarly let α_o be the canonical one form on T^*G/K, and let $\alpha = \nu^*\alpha_o$ be the corresponding one form on TG/K.

We have turned TG/K into a complex manifold by identifying it with $G_{\mathbf{C}}/K_{\mathbf{C}}$, and we give T^*G/K the complex structure induced by the metric identification ν. To show that $\operatorname{Im} \overline{\partial}\phi_o = \alpha_o$ it suffices to show that $\operatorname{Im} \overline{\partial}\phi = \alpha$, since $\alpha_o = (\nu^{-1})^*\alpha$ and

$$\begin{aligned} \operatorname{Im} \overline{\partial}\phi_o(U) &= \frac{1}{2} d\phi_o(J_{T^*G/K}U) \\ &= \frac{1}{2} d\phi_o(d\nu \circ J_{TG/K} \circ d\nu^{-1}(U)) \\ &= (\nu^{-1})^* \operatorname{Im} \overline{\partial}\phi(U). \end{aligned}$$

To show that $\operatorname{Im} \overline{\partial}\phi = \alpha$ it suffices to show that $a^*\operatorname{Im} \overline{\partial}\phi = a^*\alpha$, since a is a surjective submersion. This is the course we will take to prove theorem 2.1.1 in this case. The following lemma shows how to compute these. Since we don't need to assume that G/K is locally symmetric for this lemma, we will use the notation $\mathfrak{k}^\perp$ instead of $\mathfrak{p}$. In fact we don't even need to assume that G is semisimple, only that G is compact.

Lemma 4.3.14 *Let* $(m,E) \in G \times \mathbf{k}^{\perp}$ *and* $(A,B) \in \mathbf{g} \times \mathbf{k}^{\perp}$. *Let* $<,>$ *denote the* Ad(G)-*invariant inner product on* $\mathbf{g}$. *Then*

1. $a^*\alpha((A,B)^{\sim}_{(m,E)}) =< E, A >$

2. $a^*\phi((m,E)) =< E, E >$

3. $d(a^*\phi)((A,B)^{\sim}_{(m,E)}) = 2 < E, B >$.

Proof. Let $\xi_{gK} \in T^*G/K$, $U \in T_{\xi_{gK}}(T^*G/K)$. Then the canonical one form on T^*G/K is given by

$$\alpha_o(\xi_{gK})(U) = \xi_{gK}(d\pi_{T^*G/K}U)$$

where $\pi_{T^*G/K}$ is the cotangent projection. For $V_{gK} \in TG/K$, $X \in T_{V_{gK}}(TG/K)$, α is given by

$$\begin{aligned} \alpha(V_{gK})(X) &= \alpha_o(\nu(V_{gK}))(d\nu(X)) \\ &= \nu(V_{gK})(d(\pi_{T^*G/K} \circ \nu)(X)) \\ &= Q(V_{gK}, d(\pi_{T^*G/K} \circ \nu)(X)) \end{aligned}$$

where Q is the Riemannian metric on G/K. Now let $(m,E) \in G \times \mathbf{k}^{\perp}$ and $(A,B) \in \mathbf{g} \times \mathbf{k}^{\perp}$. Then

$$\begin{aligned} (a^*\alpha)((A,B)^{\sim}_{(m,E)}) &= a^*\alpha(m,E)((dL_mA,B)) \\ &= \alpha(a(m,E))((da)_{(m,E)}(dL_mA,B)) \\ &= Q(a(m,E), d(\pi_{T^*G/K} \circ \nu \circ a)_{(m,E)}(dL_mA,B)). \end{aligned}$$

It's clear that ν is a fiber preserving map, so $\pi_{T^*G/K} \circ \nu = \pi_{TG/K}$, where $\pi_{TG/K}: TG/K \to G/K$ is the tangent projection. Recall that

$$a(m,E) = d\tau(m) d\pi_{G/K}(E)$$

where $\pi_{G/K}: G \to G/K$ is the coset projection. Then $\pi_{TG/K} \circ a$ is just $\pi_{G/K} \circ \mathrm{pr}_1$, where pr_1 denotes projection onto the first factor in $G \times \mathbf{k}^\perp$. Then

$$\begin{aligned}(a^*\alpha)((A,B)^{\sim}_{(m,E)}) &= Q(d\tau(m)d\pi_{G/K}(E), d\pi_{G/K} \circ dL_m(A)) \\ &= Q(d\pi_{G/K}(E), d\pi_{G/K}(A))\end{aligned}$$

since $\pi_{G/K} \circ L_m = \tau(m) \circ \pi_{G/K}$, and the metric Q is invariant under the action of G. Recall that the metric Q at the identity coset is given by the $\mathrm{Ad}(G)$-invariant inner product $<,>$ on $\mathbf{g}$ (minus one times the Killing form if G is semisimple) under the identification of $T_{eK}G/K$ and $\mathbf{k}^\perp$ by $d\pi_{G/K}$ restricted to $\mathbf{k}^\perp$. This means that

$$(a^*\alpha)((A,B)^{\sim}_{(m,E)}) = < \mathrm{pr}_{\mathbf{k}^\perp}(E), \mathrm{pr}_{\mathbf{k}^\perp}(A) > .$$

But we assumed that $A \in \mathbf{k}^\perp$, and so

$$< \mathrm{pr}_{\mathbf{k}^\perp}(E), \mathrm{pr}_{\mathbf{k}^\perp}(A) > = < \mathrm{pr}_{\mathbf{k}^\perp}(E), A > = < E, A > .$$

This proves 1. To prove 2, just compute

$$\begin{aligned}a^*\phi(m,E) &= \phi(d\tau(m) \circ d\pi_{G/K}(E)) \\ &= Q(d\tau(m) \circ d\pi_{G/K}(E), d\tau(m) \circ d\pi_{G/K}(E)) \\ &= Q(d\pi_{G/K}(E), d\pi_{G/K}(E)) \\ &= < \mathrm{pr}_{\mathbf{k}^\perp}(E), \mathrm{pr}_{\mathbf{k}^\perp}(E) > = < E, E >\end{aligned}$$

since $E \in \mathbf{k}^\perp$ by assumption. Now 3 follows immediately. □

Remark. When G is semisimple we have a global identification of TG/K and $G_{\mathbf{C}}/K_{\mathbf{C}}$. If we let $\phi_{\mathbf{C}}$, $\alpha_{\mathbf{C}}$ be the objects on $G_{\mathbf{C}}/K_{\mathbf{C}}$ corresponding to ϕ and α on TG/K, then it's easy to see that $a^*\phi = a^*_{\mathbf{C}}\phi_{\mathbf{C}}$ and $a^*\alpha = a^*_{\mathbf{C}}\alpha_{\mathbf{C}}$.

Returning to the symmetric case, we are now ready to show that $a^*\mathrm{Im}\,\overline{\partial}\phi = a^*\alpha$. Let

$(g,V) \in G \times \mathbf{p}$, $(W,Y) \in \mathbf{g} \times \mathbf{p}$. Then

$$a^*\mathrm{Im}\,\overline{\partial}\phi((W,Y)^{\sim}_{(g,V)}) = \frac{1}{2}d\phi(J_{TG/K}\,da(W,Y)^{\sim}_{(g,V)}).$$

By proposition 4.3.12 this is

$$a^*\mathrm{Im}\,\overline{\partial}\phi((W,Y)^{\sim}_{(g,V)}) = \frac{1}{2}d(a^*\phi)((-T(V)^{-1}Y + \tan\mathrm{ad}(V)W_{\mathbf{k}}, T(V)W_{\mathbf{p}})^{\sim}_{(g,V)})$$

which by lemma 4.3.14 becomes

$$a^*\mathrm{Im}\,\overline{\partial}\phi((W,Y)^{\sim}_{(g,V)}) = < V, T(V)W_{\mathbf{p}} > .$$

Note that $T(V) = (\cos\mathrm{ad}(V))^{-1} \circ (\sin\mathrm{ad}(V)/\mathrm{ad}(V))$ is a symmetric operator by lemma 4.3.11, so

$$a^*\mathrm{Im}\,\overline{\partial}\phi((W,Y)^{\sim}_{(g,V)}) = < T(V)V, W_{\mathbf{p}} > .$$

Now note that V is an eigenvector of $\cos\mathrm{ad}(V)$ and $\sin\mathrm{ad}(V)/\mathrm{ad}(V)$, with eigenvalue 1. Hence $T(V)V = V$, and

$$\begin{aligned} a^*\mathrm{Im}\,\overline{\partial}\phi((W,Y)^{\sim}_{(g,V)}) &= < V, W_{\mathbf{p}} > \\ &= < V, W > \end{aligned}$$

since $V \in \mathbf{p}$. On the other hand, by lemma 4.3.14,

$$a^*\alpha((W,Y)^{\sim}_{(g,V)}) = < V, W > .$$

This proves that $\mathrm{Im}\,\overline{\partial}\phi = \alpha$.

To complete the proof of theorem 2.1.1 in the symmetric case, we need to show that the standard involution σ_o of T^*G/K is an antiholomorphic map. Equivalently, we must show that the involution σ of TG/K that takes a tangent vector to its negative is an antiholomorphic map. Let $U \in T_{a(g,V)}(TG/K)$, and write

$$U = da(W,Y)^{\sim}_{(g,V)}$$

for some $(g,V) \in G \times \mathbf{p}$, $(W,Y) \in \mathbf{g} \times \mathbf{p}$. Note that $\sigma \circ a$ is equal to a composed with multiplication by -1 on the second factor. Thus

$$d\sigma \circ da(W,Y)^{\sim}_{(g,V)} = da(W,-Y)^{\sim}_{(g,-V)}.$$

Then

$$J_{TG/K} \circ d\sigma(U) = J_{TG/K} da(W,-Y)^{\sim}_{(g,-V)}$$

$$= da(-T(-V)^{-1}(-Y) + \tan \operatorname{ad}(-V)W_{\mathbf{k}}, T(-V)W_{\mathbf{p}})^{\sim}_{(g,-V)}.$$

Note that $T(-V) = T(V)$ and $\tan \operatorname{ad}(-V) = -\tan \operatorname{ad}(V)$. Then

$$\begin{aligned}
J_{TG/K} \circ d\sigma(U) &= da(T(V)^{-1}Y - \tan \operatorname{ad}(V)W_{\mathbf{k}}, T(V)W_{\mathbf{p}})^{\sim}_{(g,-V)} \\
&= -d\sigma \circ da(-T(V)^{-1}Y + \tan \operatorname{ad}(V)W_{\mathbf{k}}, T(V)W_{\mathbf{p}})^{\sim}_{(g,V)} \\
&= -d\sigma \circ J_{TG/K} da(W,Y)^{\sim}_{(g,V)} \\
&= -d\sigma \circ J_{TG/K}(U).
\end{aligned}$$

This completes the proof of theorem 2.1.1 in the symmetric case. Moreover, we have shown that the complex structure described in theorem 2.1.1 exists globally on T^*G/K in this case.

4.4 Homogeneous Spaces of Compact Lie Groups

We now consider the case of a homogeneous space of an arbitrary compact Lie group G. Let $M = G/H$ with G a compact Lie group and H a closed subgroup. We will use H instead of K to distinguish this case from the symmetric case. The bi-invariant metric on

G induces an Ad(G)-invariant inner product $<,>$ on g, and an orthogonal decomposition of the Lie algebra g of G,

$$\mathsf{g} = \mathsf{h} + \mathsf{h}^{\perp}.$$

Let $\pi_{G/K}: G \to G/H$ be the coset projection. Then $d\pi_{G/K}$ restricted to $\mathsf{h}^{\perp}$ gives an identification of $\mathsf{h}^{\perp}$ and $T_{eH}G/H$, and hence a positive definite inner product on $T_{eH}G/H$. Due to the Ad(G)-invariance of the inner product, this gives rise to a metric Q on G/H. In this section we will prove theorem 2.1.1 for $(M, g) = (G/H, Q)$.

If G is a compact, connected, semisimple Lie group, then TG/H is globally diffeomorphic to $G_{\mathbf{C}}/H_{\mathbf{C}}$. If G is not semisimple, we have only been able to obtain an identification of a tubular neighborhood of G/H in TG/H with a neighborhood of G/H in $G_{\mathbf{C}}/H_{\mathbf{C}}$. All of our results become local in this situation, and the proofs consist mainly of computations in local coordinate systems.

4.4.1 Complexification of T^*G/H

Using the results of section 4.3.1, we can embed G/H as a totally real submanifold of $G_{\mathbf{C}}/H_{\mathbf{C}}$. We want to construct a real analytic identification of a neighborhood of G/H in T^*G/H with a neighborhood of G/H in $G_{\mathbf{C}}/H_{\mathbf{C}}$. As before we will identify T^*G/H with TG/H by the Riemannian metric on G/H, and then complexify TG/H in a tubular neighborhood of G/H. Let $\pi_{G_{\mathbf{C}}/H_{\mathbf{C}}}: G_{\mathbf{C}} \to G_{\mathbf{C}}/H_{\mathbf{C}}$ denote the coset projection, and let τ denote the natural action of G on G/H.

Proposition 4.4.1 *Let* $\Phi: TG/H \to G_{\mathbf{C}}/H_{\mathbf{C}}$ *be the map*

$$\Phi(V_{gH}) = \pi_{G_{\mathbf{C}}/H_{\mathbf{C}}}(g \exp \sqrt{-1}(d\pi_{G/H}|_{\mathsf{h}^{\perp}})^{-1} d\tau(g^{-1})V_{gH}).$$

This map is well defined, and is a diffeomorphism of a neighborhood of G/H *in* TG/H *with a neighborhood of* G/H *in* $G_{\mathbf{C}}/H_{\mathbf{C}}$.

Before proving the proposition we will explain what the map Φ is doing. If $V_{gH} \in T_{gH}G/H$, then $d\tau(g^{-1})V_{gH}$ is in the tangent space to the identity coset. We are using $(d\pi_{G/H})|_{\mathfrak{h}^\perp}$ to identify $T_{eH}G/H$ with $\mathfrak{h}^\perp$. Then $\sqrt{-1}(d\pi_{G/H}|_{\mathfrak{h}^\perp})^{-1}d\tau(g^{-1})V_{gH}$ is in $\mathfrak{g}_{\mathbf{C}}$, and we can exponentiate onto $G_{\mathbf{C}}$ and project onto $G_{\mathbf{C}}/H_{\mathbf{C}}$.

Proof. First we show that Φ is well-defined. If $V_{gH} = V'_{g'H}$ then there is an $h \in H$ such that $g' = gh^{-1}$. Then

$$\begin{aligned}\Phi(V_{gH}) &= (g'h \exp \sqrt{-1}(d\pi_{G/H}|_{\mathfrak{h}^\perp})^{-1} d\tau((g'h)^{-1})V'_{g'H}) \cdot H_{\mathbf{C}} \\ &= (g' \exp \sqrt{-1}\, \mathrm{Ad}(h)(d\pi_{G/H}|_{\mathfrak{h}^\perp})^{-1} d\tau(h^{-1}) d\tau(g'^{-1})V'_{g'H}) \cdot H_{\mathbf{C}}.\end{aligned}$$

We know that for $X \in \mathfrak{h}^\perp$, $h \in H$,

$$d\tau(h) d\pi_{G/H}(X) = d\pi_{G/H} \circ \mathrm{Ad}(h)(X).$$

Since $\mathrm{Ad}(h)$ preserves $\mathfrak{h}^\perp$, it follows that

$$(d\pi_{G/H}|_{\mathfrak{h}^\perp})^{-1} d\tau(h) = \mathrm{Ad}(h)(d\pi_{G/H}|_{\mathfrak{h}^\perp})^{-1}.$$

This shows that $\Phi(V_{gH}) = \Phi(V'_{g'H})$.

Next we show that Φ is real analytic. This may seem obvious, but we will give a proof using real analytic local coordinates because we will need these later. Let κ_g be the real analytic local coordinate near $gH \in G/H$ given by, for X in a neighborhood of zero in $\mathfrak{h}^\perp$,

$$\kappa_g(X) = \pi_{G/H} \circ L_g \circ \exp X.$$

Then coordinates on a neighborhood of 0_{gH} in TG/H are given by

$$\kappa_g^\sharp(X, V) = d(\pi_{G/H} \circ L_g \circ \exp)_X(V) = d\tau(g \exp X)(d\pi_{G/H})(\frac{1 - e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(V))$$

(here X is as above and $V \in \mathfrak{h}^{\perp}$; see Helgason [10], theorem 1.7, chapter II for the differential of the exponential map). Thus it's easy to see that

$$\Phi \circ \kappa_g^{\sharp}(X, V) = (g \exp X) \exp \sqrt{-1}\, \mathrm{pr}_{\mathfrak{h}^{\perp}}\left(\frac{1 - e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(V)\right) \cdot H_{\mathbf{C}} \tag{4.8}$$

where $\mathrm{pr}_{\mathfrak{h}^{\perp}}$ denotes projection onto the subspace $\mathfrak{h}^{\perp}$ with respect to the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^{\perp}$. From the expression 4.8 it is clear that Φ is real analytic.

To show that Φ is a local diffeomorphism near the zero section it suffices to show that $d(\Phi \circ \kappa_g^{\sharp})_{(0,0)}$ is nonsingular for every $g \in G$. Note for W, $Y \in \mathfrak{h}^{\perp}$,

$$d(\Phi \circ \kappa_g^{\sharp})_{(0,0)}(W, 0) = \frac{d}{dt}|_{t=0}\, g \exp tW \cdot H_{\mathbf{C}} = d\tau(g) \circ d\pi_{G_{\mathbf{C}}/H_{\mathbf{C}}}(W),$$

$$d(\Phi \circ \kappa_g^{\sharp})_{(0,0)}(0, Y) = \frac{d}{dt}|_{t=0}\, g \exp \sqrt{-1}(tY) \cdot H_{\mathbf{C}} = d\tau(g) \circ d\pi_{G_{\mathbf{C}}/H_{\mathbf{C}}}(\sqrt{-1}\, Y)$$

This shows that

$$d(\Phi \circ \kappa_g^{\sharp})_{(0,0)}(W, Y) = d\tau(g) \circ d\pi_{G_{\mathbf{C}}/H_{\mathbf{C}}}(W + \sqrt{-1}\, Y).$$

Since $W + \sqrt{-1}\, Y \in \mathfrak{h}_{\mathbf{C}}^{\perp}$ and $(d\pi_{G_{\mathbf{C}}/H_{\mathbf{C}}})|_{\mathfrak{h}_{\mathbf{C}}^{\perp}}$ is an isomorphism, it follows that $d(\Phi \circ \kappa_g^{\sharp})_{(0,0)}$ is nonsingular.

To show that Φ is a diffeomorphism of a tubular neighborhood of G/H in TG/H to a neighborhood of G/H in $G_{\mathbf{C}}/H_{\mathbf{C}}$, note that Φ restricted to G/H is globally one-to-one, since it is just the embedding of G/H into $G_{\mathbf{C}}/H_{\mathbf{C}}$ given in lemma 4.3.3. Then by a standard argument (see for example Lang [17], page 97-98), Φ is a diffeomorphism when restricted to a sufficiently small neighborhood of the zero section in TG/H. □

From now on we will give a neighborhood of G/H in TG/H the complex structure induced by this identification.

4.4.2 Proof of the Result

In this section we will prove theorem 2.1.1 for homogeneous spaces of compact Lie groups, with the Riemannian metric Q described above. If ϕ_o is the quadratic function on T^*G/H associated with the Riemannian metric Q and α_o is the canonical one form on T^*M, we must show that $\operatorname{Im} \overline{\partial}\phi_o = \alpha_o$, and that the standard involution σ_o of T^*G/H is an antiholomorphic map with respect to the complex structure given by proposition 4.4.1.

Let α, ϕ be the objects on TG/H corresponding to α_o and ϕ_o under the metric identification $\nu: TG/H \to T^*G/H$. Then it suffices to show that $\operatorname{Im} \overline{\partial}\phi = \alpha$, and that the corresponding involution of TG/H is an antiholomorphic map with respect to the complex structure on TG/H.

We first show that it suffices to verify that the equation

$$\operatorname{Im} \overline{\partial}\phi = \alpha \tag{4.9}$$

holds on a neighborhood of zero in the fiber over the identity coset, $T_{eH}G/H$. The group G permutes the fibers transitively, and it's easy to see that ϕ and α are invariant under the action of G on TG/H. The identification of TG/H with $G_{\mathbf{C}}/H_{\mathbf{C}}$ is equivariant with respect to the natural G actions on TG/H and $G_{\mathbf{C}}/H_{\mathbf{C}}$. It follows that the complex structure on TG/H induced by this identification is G-invariant, so equation 4.9 is G-invariant. If we can verify that equation 4.9 holds on a neighborhood of eH in $T_{eH}G/H$, we will have shown it holds locally near each point of the zero section G/H in TG/H, and hence on a neighborhood of G/H in TG/H.

Let $J_{TG/H}$ denote the complex structure operator on TG/H near the zero section given by the identification in proposition 4.4.1. Let o denote the identity coset eH. We must show that for all V_o in a neighborhood of zero in T_oG/H and all $U_o \in \mathrm{T}_{V_o}(TG/H)$,

$$\frac{1}{2}(d\phi)_{V_o}(U_o) = -\alpha(V_o)(J_{TG/H}U_o).$$

The proof is a detailed computation in local coordinates, which will be broken up into several steps. Throughout we will use the following local coordinates. Let $\mathfrak{u}$ be a neighborhood of zero in $\mathfrak{h}^{\perp}$. Then local coordinates on G/H near zero are given by $\kappa: \mathfrak{u} \to G/H$, where

$$\kappa(X) = \pi_{G/H}(\exp X).$$

Vector bundle coordinates on TG/H near 0_o are given by $\kappa^{\sharp}: \mathfrak{u} \times \mathfrak{h}^{\perp} \to TG/H$, where

$$\kappa^{\sharp}(X, V) = (d\kappa)_X(V).$$

It is easily seen, using the formula for $d\exp$ in Helgason [10], theorem 1.7, chapter II, that

$$\kappa^{\sharp}(X, V) = d\tau(\exp X) \circ d\pi_{G/H} \circ (\frac{1 - e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)})(V).$$

The fiber over the identity coset is the image of the set $\{0\} \times \mathfrak{h}^{\perp}$ under the map $\kappa^{\sharp}$. Let $\phi^{\sharp}$, $\alpha^{\sharp}$, $J^{\sharp}_{TG/H}$ denote the objects on $\mathfrak{u} \times \mathfrak{h}^{\perp}$ corresponding to ϕ, α, $J_{TG/H}$ on TG/H under the identification $\kappa^{\sharp}$. Then it suffices to show that for all $(0, V)$ in the domain of $J^{\sharp}_{TG/H}$ and all $(W, Y) \in T_{(0,V)}(\mathfrak{u} \times \mathfrak{h}^{\perp}) \cong \mathfrak{h}^{\perp} \times \mathfrak{h}^{\perp}$,

$$\frac{1}{2}(d\phi^{\sharp})_{(0,V)}(W, Y) = -\alpha^{\sharp}(0, V)((J^{\sharp}_{TG/H})_{(0,V)}(W, Y)). \tag{4.10}$$

To prove theorem 2.1.1 we must get reasonably good expressions for the complex structure operator $J^{\sharp}_{TG/H}$ and the forms $d\phi^{\sharp}$, $\alpha^{\sharp}$ over $\{0\} \times \mathfrak{h}^{\perp}$. The difficult part is obtaining an expression for $J^{\sharp}_{TG/H}$, so we will tackle that problem first. By definition,

$$(J^{\sharp}_{TG/H})_{(0,V)} = d(\Phi \circ \kappa^{\sharp})^{-1} M_{\sqrt{-1}}\, d(\Phi \circ \kappa^{\sharp})_{(0,V)}$$

where $M_{\sqrt{-1}}$ is multiplication by $\sqrt{-1}$ on the tangent space $T_{\Phi\circ\kappa^{\sharp}(0,V)} G_{\mathbb{C}}/H_{\mathbb{C}}$. The first step is to compute $d(\Phi \circ \kappa^{\sharp})_{(0,V)}$.

Lemma 4.4.2 *Let* $(0, V) \in \{0\} \times \mathfrak{h}^{\perp}$, *and let* $(W, Y) \in \mathfrak{h}^{\perp} \times \mathfrak{h}^{\perp}$, *which we identify with* *the tangent space* $T_{(0,V)}(\mathfrak{u} \times \mathfrak{h}^{\perp})$. *Then*

$$d(\Phi \circ \kappa^{\sharp})_{(0,V)}(W,Y) = \\ d\tau(\exp \sqrt{-1}\, V) d\pi_{G_{\mathbb{C}}/H_{\mathbb{C}}}((\frac{1-e^{-\mathrm{ad}(\sqrt{-1}\,V)}}{\mathrm{ad}(V)}(Y - \frac{1}{2}\mathrm{pr}_{\mathfrak{h}^{\perp}}\mathrm{ad}(W)V)) + e^{-\mathrm{ad}(\sqrt{-1}\,V)}(W)).$$

Proof. We first compute $d(\Phi \circ \kappa^{\sharp})_{(0,V)}(W,0)$. From equation 4.8 we have

$$\begin{aligned} d(\Phi \circ \kappa^{\sharp})_{(0,V)}(W,0) &= \frac{d}{dt}|_{t=0}\, (\Phi \circ \kappa^{\sharp})(tW,V) \\ &= \frac{d}{dt}|_{t=0}\, \pi_{G_{\mathbb{C}}/H_{\mathbb{C}}}(\exp tW \exp \sqrt{-1}\,\mathrm{pr}_{\mathfrak{h}^{\perp}}(\frac{1-e^{-\mathrm{ad}(tW)}}{\mathrm{ad}(tW)}(V))) \\ &= d\pi_{G_{\mathbb{C}}/H_{\mathbb{C}}}(\frac{d}{dt}|_{t=0}\, \exp tW \exp \sqrt{-1}\,\mathrm{pr}_{\mathfrak{h}^{\perp}}(\frac{1-e^{-\mathrm{ad}(tW)}}{\mathrm{ad}(tW)}(V))). \end{aligned}$$

Let $\mathcal{M}: G_{\mathbb{C}} \times G_{\mathbb{C}} \to G_{\mathbb{C}}$ denote the group multiplication. We can write this as

$$d(\Phi \circ \kappa^{\sharp})_{(0,V)}(W,0) = \\ d\pi_{G_{\mathbb{C}}/H_{\mathbb{C}}} \circ d\mathcal{M}_{(e,\exp\sqrt{-1}\,V)}(W, d(\exp)_{\sqrt{-1}\,V}\frac{d}{dt}|_{t=0}\, \sqrt{-1}\,\mathrm{pr}_{\mathfrak{h}^{\perp}}(\frac{1-e^{-\mathrm{ad}(tW)}}{\mathrm{ad}(tW)}(V))).$$

By examining the power series for $(1-e^{-\mathrm{ad}(tW)})/\mathrm{ad}(tW)$ we see that

$$\frac{d}{dt}|_{t=0}\, \mathrm{pr}_{\mathfrak{h}^{\perp}}(\frac{1-e^{-\mathrm{ad}(tW)}}{\mathrm{ad}(tW)}(V)) = \frac{-1}{2}\mathrm{pr}_{\mathfrak{h}^{\perp}}\mathrm{ad}(W)V.$$

Recalling the formula for the differential of group multiplication given in equation 4.6, we see that

$$d(\mathcal{M})_{(e,\exp\sqrt{-1}\,V)}(W, d(\exp)_{\sqrt{-1}\,V}(\frac{-\sqrt{-1}}{2})\mathrm{pr}_{\mathfrak{h}^{\perp}}\mathrm{ad}(W)V) = \\ d(\exp)_{\sqrt{-1}\,V}(\frac{-\sqrt{-1}}{2}\mathrm{pr}_{\mathfrak{h}^{\perp}}\mathrm{ad}(W)V) + dR_{\exp\sqrt{-1}\,V}(W).$$

Using Helgason's formula for $d\exp$ in lemma 4.1.3, we have

$$d(\mathcal{M})_{(e,\exp\sqrt{-1}V)}(W, d(\exp)_{\sqrt{-1}V}(\frac{-\sqrt{-1}}{2})\mathrm{pr}_{\mathfrak{h}^\perp}\mathrm{ad}(W)V) =$$
$$dL_{\exp\sqrt{-1}V}(\sqrt{-1}(\frac{1-e^{-\mathrm{ad}(\sqrt{-1}V)}}{\mathrm{ad}(\sqrt{-1}V)}(\frac{-1}{2}\mathrm{pr}_{\mathfrak{h}^\perp}\mathrm{ad}(W)V)) + e^{-\mathrm{ad}(\sqrt{-1}V)}(W)).$$

Putting all this together we find that

$$d(\Phi\circ\kappa^\sharp)_{(0,V)}(W,0) =$$
$$d\tau(\exp\sqrt{-1}V)\circ d\pi_{G_\mathbb{C}/H_\mathbb{C}}(\sqrt{-1}(\frac{1-e^{\sqrt{-1}V}}{\mathrm{ad}(\sqrt{-1}V)}(\frac{-1}{2}\mathrm{pr}_{\mathfrak{h}^\perp}\mathrm{ad}(W)V)) + e^{-\mathrm{ad}(\sqrt{-1}V)}(W)).$$

Next we compute $d(\Phi\circ\kappa^\sharp)_{(0,V)}(0,Y)$; this is somewhat easier. Using equation 4.8 we have

$$\begin{aligned}
d(\Phi\circ\kappa^\sharp)_{(0,V)}(0,Y) &= \frac{d}{dt}|_{t=0}\,(\Phi\circ\kappa^\sharp)(0,V+tY)\\
&= \frac{d}{dt}|_{t=0}\,\pi_{G_\mathbb{C}/H_\mathbb{C}}(\exp\sqrt{-1}(V+tY))\\
&= d\pi_{G_\mathbb{C}/H_\mathbb{C}}\circ d(\exp)_{\sqrt{-1}V}(\sqrt{-1}Y)\\
&= d\tau(\exp\sqrt{-1}V)\circ d\pi_{G_\mathbb{C}/H_\mathbb{C}}(\sqrt{-1}(\frac{1-e^{-\mathrm{ad}(\sqrt{-1}V)}}{\mathrm{ad}(\sqrt{-1}V)}(Y)))
\end{aligned}$$

This shows that

$$d(\Phi\circ\kappa^\sharp)_{(0,V)}(W,Y) =$$
$$d\tau(\exp\sqrt{-1}V)\circ d\pi_{G_\mathbb{C}/H_\mathbb{C}}(\sqrt{-1}(\frac{1-e^{-\mathrm{ad}(\sqrt{-1}V)}}{\mathrm{ad}(\sqrt{-1}V)}(Y-\frac{1}{2}\mathrm{pr}_{\mathfrak{h}^\perp}\mathrm{ad}(W)V)) + e^{-\mathrm{ad}(\sqrt{-1}V)}(W)).$$

This proves the lemma, since

$$\sqrt{-1}(1-e^{-\mathrm{ad}(\sqrt{-1}V)})/\mathrm{ad}(\sqrt{-1}V) = (1-e^{-\mathrm{ad}(\sqrt{-1}V)})/\mathrm{ad}(V).$$

□

The next step is to understand how to invert $d(\Phi \circ \kappa^\sharp)_{(0,V)}$. Our strategy will be to write $d(\Phi \circ \kappa^\sharp)_{(0,V)}$ as a composition of maps which are invertible for small enough V. Note that for any W, $X \in \mathfrak{h}^\perp$,

$$\frac{1 - e^{-\sqrt{-1}\,\mathrm{ad}(V)}}{\mathrm{ad}(V)}(X) + e^{-\mathrm{ad}(\sqrt{-1}\,V)}(W) =$$
$$\frac{1 - \cos\mathrm{ad}(V)}{\mathrm{ad}(V)}(X) + \cos\mathrm{ad}(V)(W) + \sqrt{-1}\Big(\frac{\sin\mathrm{ad}(V)}{\mathrm{ad}(V)}(X) - \sin\mathrm{ad}(V)(W)\Big).$$

We are thus led to define the following maps:

1. $P_V\colon \mathfrak{h}^\perp \times \mathfrak{h}^\perp \to \mathfrak{h}^\perp \times \mathfrak{h}^\perp$ given by

$$P_V(W,X) = (W, X + \frac{1}{2}\mathrm{pr}_{\mathfrak{h}^\perp}\mathrm{ad}(V)W)$$

2. $R_V\colon \mathfrak{h}^\perp \times \mathfrak{h}^\perp \to \mathfrak{h}^\perp \times \mathfrak{h}^\perp$ given by

$$R_V(W,X) = \mathrm{pr}_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp}\Big(\frac{1 - \cos\mathrm{ad}(V)}{\mathrm{ad}(V)}(X) + \cos\mathrm{ad}(V)(W), \frac{\sin\mathrm{ad}(V)}{\mathrm{ad}(V)}(X) - \sin\mathrm{ad}(V)(W)\Big)$$

3. $S\colon \mathfrak{h}^\perp \times \mathfrak{h}^\perp \to \mathfrak{h}^\perp_{\mathbb{C}}$ is the standard identification, $S(W,X) = W + \sqrt{-1}\,X$.

Then it is clear that we can write

$$d(\Phi \circ \kappa^\sharp)_{(0,V)}(W,Y) = d\tau(\exp\sqrt{-1}\,V) \circ (d\pi_{G_{\mathbb{C}}/H_{\mathbb{C}}})|_{\mathfrak{h}^\perp_{\mathbb{C}}} \circ S \circ R_V \circ P_V(W,Y). \qquad (4.11)$$

Obviously S, P_V, and $(d\pi_{G_{\mathbb{C}}/H_{\mathbb{C}}})|_{\mathfrak{h}^\perp_{\mathbb{C}}}$ are bijections. For R_V we have the following lemma.

Lemma 4.4.3 *For small enough V, R_V is a bijection.*

Proof. We can write R_V as

$$R_V(W,X) = (W,X) +$$
$$\mathrm{pr}_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp}\Big(\frac{1 - \cos\mathrm{ad}(V)}{\mathrm{ad}(V)}(X) + (\cos\mathrm{ad}(V) - 1)(W), (\frac{\sin\mathrm{ad}(V)}{\mathrm{ad}(V)} - 1)(X) - \sin\mathrm{ad}(V)(W)\Big).$$

Define the linear operator $H_V: \mathfrak{h}^\perp \times \mathfrak{h}^\perp \to \mathfrak{h}^\perp \times \mathfrak{h}^\perp$ by

$$H_V(W, X) =$$
$$\mathrm{pr}_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp}\left(\frac{1 - \cos \mathrm{ad}(V)}{\mathrm{ad}(V)}(X) + (\cos \mathrm{ad}(V) - 1)(W), \left(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)} - 1\right)(X) - \sin \mathrm{ad}(V)(W)\right).$$

Then R_V can be written as

$$R_V = Id_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} + H_V.$$

Suppose we put a norm $\|\cdot\|$ on $\mathfrak{h}^\perp \times \mathfrak{h}^\perp$ (all norms on a finite dimensional vector space are equivalent). Then it suffices to show that for all V sufficiently small,

$$\|H_V\| < 1.$$

If this is true, then we can invert R_V by a Neumann series and conclude that R_V is a bijection. The most convenient norm to use on $\mathfrak{h}^\perp \times \mathfrak{h}^\perp$ is the "box" norm:

$$\|(W, X)\| = \|W\| + \|X\|$$

where $\|\cdot\|$ is the norm on $\mathfrak{h}^\perp$ induced by the $\mathrm{Ad}(G)$-invariant inner product. Then we have

$$\begin{aligned} \|H_V(W, X)\| \leq\ & \|\mathrm{pr}_{\mathfrak{h}^\perp}\left(\frac{1 - \cos \mathrm{ad}(V)}{\mathrm{ad}(V)}\right)(X)\| + \|\mathrm{pr}_{\mathfrak{h}^\perp}(\cos \mathrm{ad}(V) - 1)(W)\| \\ & + \|\mathrm{pr}_{\mathfrak{h}^\perp}\left(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)} - 1\right)(X)\| + \|\mathrm{pr}_{\mathfrak{h}^\perp}(\sin \mathrm{ad}(V))(W)\|. \end{aligned}$$

It's clear that $\|\mathrm{pr}_{\mathfrak{h}^\perp}\| \leq 1$, so this becomes

$$\begin{aligned} \|H_V(W, X)\| \leq\ & \left(\left\|\frac{1 - \cos \mathrm{ad}(V)}{\mathrm{ad}(V)}\right\| + \left\|\left(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)} - 1\right)\right\|\right)\|X\| \\ & + (\|(\cos \mathrm{ad}(V) - 1)\| + \|\sin \mathrm{ad}(V)\|)\|W\|. \end{aligned} \tag{4.12}$$

The adjoint representation

$$\mathrm{ad}\colon \mathbf{g} \rightarrow \mathrm{End}(\mathbf{g})$$

by $V \rightarrow \mathrm{ad}(V)$ is a linear map between finite dimensional vector spaces, hence bounded. So there exists a constant C, independent of V, such that

$$\|\mathrm{ad}(V)\| \leq C\|V\|.$$

The power series for the analytic functions appearing in the estimate 4.12 have no constant term. By examining these power series one sees that

$$\max(\|\frac{1-\cos \mathrm{ad}(V)}{\mathrm{ad}(V)}\|, \|(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)} - 1)\|, \|(\cos \mathrm{ad}(V) - 1)\|, \|\sin \mathrm{ad}(V)\|)$$

$$\leq \sum_{n=1}^{\infty} C^n \|V\|^n.$$

So by taking V sufficiently small we can arrange that $\|H_V\| < 1$. □

We can now write down an expression for the complex structure $J^{\sharp}_{TG/H}$. Recall that by definition,

$$J^{\sharp}_{TG/H} = d(\Phi \circ \kappa^{\sharp})^{-1} \circ M_{\sqrt{-1}} \circ d(\Phi \circ \kappa^{\sharp}).$$

Since $\pi_{G_{\mathbb{C}}/H_{\mathbb{C}}}$ and the $G_{\mathbb{C}}$ action are holomorphic maps, we can write using equation 4.11

$$(J^{\sharp}_{TG/H})_{(0,V)} = P_V^{-1} \circ R_V^{-1} \circ J_{\mathbf{h}^{\perp}\times\mathbf{h}^{\perp}} \circ R_V \circ P_V \tag{4.13}$$

where $J_{\mathbf{h}^{\perp}\times\mathbf{h}^{\perp}}$ is the standard complex structure on $\mathbf{h}^{\perp} \times \mathbf{h}^{\perp}$, $J_{\mathbf{h}^{\perp}\times\mathbf{h}^{\perp}}(W,Y) = (-Y,W)$. Using the Neumann series to invert R_V, for small V, we have

$$\begin{aligned} R_V^{-1} &= Id_{\mathbf{h}^{\perp}\times\mathbf{h}^{\perp}} - H_V + H_V^2 - \cdots \\ &= Id_{\mathbf{h}^{\perp}\times\mathbf{h}^{\perp}} - H_V \circ R_V^{-1}. \end{aligned}$$

Substituting this into 4.13 gives us the expression that will be useful for proving theorem 2.1.1:

$$\begin{aligned}(J^{\sharp}_{TG/H})_{(0,V)} &= P_V^{-1} \circ J_{\mathfrak{h}^{\perp}\times\mathfrak{h}^{\perp}} \circ P_V + P_V^{-1} \circ J_{\mathfrak{h}^{\perp}\times\mathfrak{h}^{\perp}} \circ H_V \circ P_V \\ &\quad - P_V^{-1} \circ H_V \circ R_V^{-1} \circ J_{\mathfrak{h}^{\perp}\times\mathfrak{h}^{\perp}} \circ R_V \circ P_V.\end{aligned} \tag{4.14}$$

To see why this is useful we must compute the other things that enter into the expression for $\operatorname{Im}\overline{\partial}\phi = \alpha$ in these coordinates, which is given by equation 4.10.

Lemma 4.4.4 *Let $(X,V) \in \mathfrak{u} \times \mathfrak{h}^{\perp}$, let $(W,Y) \in T_{(0,V)}(\mathfrak{u} \times \mathfrak{h}^{\perp})$ (which we identify with $\mathfrak{h}^{\perp} \times \mathfrak{h}^{\perp}$), and let $<,>$ denote the* Ad(G)*-invariant inner product on $\mathfrak{g}$. Then*

1. $\alpha^{\sharp}(0,V)(W,Y) =< V, W >$
2. $\phi^{\sharp}(X,V) =< \frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(V), \mathrm{pr}_{\mathfrak{h}^{\perp}}(\frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(V)) >$
3. $(d\phi^{\sharp})_{(0,V)}(W,Y) = 2 < V, Y >$.

Proof. Since $\alpha^{\sharp} = (\nu \circ \kappa^{\sharp})^{*}\alpha_o$ where $\nu: TG/H \to T^{*}G/H$ is the metric identification, we have

$$\alpha^{\sharp}(0,V)(W,Y) = \nu \circ \kappa^{\sharp}(0,V)(d(\pi_{T^{*}G/H} \circ \nu \circ \kappa^{\sharp})_{(0,V)}(W,Y)).$$

It's easy to see that $\pi_{T^{*}G/H} \circ \nu = \pi_{TG/H}$, and $\pi_{TG/H} \circ \kappa^{\sharp} = \kappa \circ \mathrm{pr}_1$ where pr_1 is projection onto the first component. By the definitions we see that

$$d(\kappa \circ \mathrm{pr}_1)_{(0,V)}(W,Y) = \kappa^{\sharp}(0,W) = d\pi_{G/H}(W).$$

Thus $\alpha^{\sharp}(0,V)(W,Y) = Q_o(d\pi_{G/H}(V), d\pi_{G/H}(W)) =< V, W >$ since $V, W \in \mathfrak{h}^{\perp}$, which proves 1.

To prove 2, we just unravel the definitions:

$$\begin{aligned}\phi^{\sharp}(X,V) &= \phi_o \circ \nu \circ \kappa^{\sharp}(X,V) \\ &= Q(\kappa^{\sharp}(X,V), \kappa^{\sharp}(X,V))\end{aligned}$$

$$
\begin{aligned}
&= Q(d\pi_{G/H}(\frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(V)), d\pi_{G/H}(\frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(V))) \\
&= <\mathrm{pr}_{\mathfrak{h}^\perp}(\frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(V)), \mathrm{pr}_{\mathfrak{h}^\perp}(\frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}(V)) > .
\end{aligned}
$$

This proves 2 since the projection onto $\mathfrak{h}^\perp$ is a symmetric operator.

It's clear that $(d\phi^\sharp)_{(0,V)}(0,Y) = 2 < V, Y >$. To find $(d\phi^\sharp)_{(0,V)}(W,0)$, we note that

$$
\frac{1-e^{-\mathrm{ad}(tW)}}{\mathrm{ad}(tW)} = I - \frac{t}{2}\mathrm{ad}(W) + O(t^2).
$$

Then

$$
\begin{aligned}
(d\phi^\sharp)_{(0,V)}(W,0) &= \frac{d}{dt}|_{t=0} < V - \frac{t}{2}[W,V], \mathrm{pr}_{\mathfrak{h}^\perp}(V - \frac{t}{2}[W,V]) > \\
&= -\frac{1}{2}(< V, \mathrm{pr}_{\mathfrak{h}^\perp}[W,V] > + < [W,V], V >).
\end{aligned}
$$

Since $\mathrm{pr}_{\mathfrak{h}^\perp}$ is a symmetric operator, V is in $\mathfrak{h}^\perp$, and $\mathrm{ad}(V)$ is a skew symmetric operator, this quantity is zero. This proves 3. □

We are now ready to verify the equation

$$
\frac{1}{2}(d\phi^\sharp)_{(0,V)}(W,Y) = -\alpha^\sharp(0,V)((J^\sharp_{TG/H})_{(0,V)}(W,Y)). \tag{4.15}
$$

By the lemma above, we must show that

$$
< V, Y >= - < V, \mathrm{pr}_1((J^\sharp_{TG/H})_{(0,V)}(W,Y)) >
$$

where pr_1 means projection onto the first factor in $\mathfrak{h}^\perp \times \mathfrak{h}^\perp$. First we note that since P_V is only a translation in the second factor, $\mathrm{pr}_1 \circ P_V^{-1} = \mathrm{pr}_1$. So we may forget about the P_V^{-1} factors in the expression for $(J^\sharp_{TG/H})_{(0,V)}$ in equation 4.14. Then recalling the definition of P_V we can write

$$\begin{aligned} -<V, \mathrm{pr}_1(J^{\sharp}_{TG/H})_{(0,V)}(W,Y)> &= <V, Y + \frac{1}{2}\mathrm{pr}_{\mathfrak{h}^\perp}\mathrm{ad}(V)W> \\ &\quad - <V, \mathrm{pr}_1(J_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} \circ H_V \circ P_V(W,Y))> \\ &\quad + <V, \mathrm{pr}_1(H_V \circ R_V^{-1} \circ J_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} \circ R_V \circ P_V(W,Y))> . \end{aligned}$$

We now come to the crucial point that makes this whole thing work: *the range of* $\mathrm{ad}(V)$ *is orthogonal to* V. This means in particular that

$$<V, Y + \frac{1}{2}\mathrm{pr}_{\mathfrak{h}^\perp}\mathrm{ad}(V)W> = <V, Y> .$$

Since $(\sin \mathrm{ad}(V)/\mathrm{ad}(V)) - 1$, $\sin \mathrm{ad}(V)$, $(1 - \cos \mathrm{ad}(V))/\mathrm{ad}(V)$, and $\cos \mathrm{ad}(V) - 1$ are analytic functions of $\mathrm{ad}(V)$ without constant term, the range of these operators is orthogonal to V also. This means that

$$\begin{aligned} 0 &= <V, \mathrm{pr}_1(J_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} \circ H_V \circ P_V(W,Y))> \\ &= <V, \mathrm{pr}_1(H_V \circ R_V^{-1} \circ J_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} \circ R_V \circ P_V(W,Y))> . \end{aligned}$$

So we have shown that

$$-<V, \mathrm{pr}_1((J^{\sharp}_{TG/H})_{(0,V)}(W,Y))> = <V, Y>,$$

verifying equation 4.15.

To complete the proof of theorem 2.1.1 we need to show that the standard involution σ_o of T^*G/H is an antiholomorphic map. We need only show that the corresponding map on TG/H, $\sigma: V_{gH} \to -V_{gH}$, is an antiholomorphic map. Since σ commutes with the G action on TG/H, it suffices to show that for all $V \in \mathfrak{h}^\perp$ sufficiently small,

$$(J_{TG/H})_{(0,-V)} \circ d\sigma = -d\sigma \circ (J_{TG/H})_{(0,V)}.$$

To prove this we use the coordinates introduced above and consider the corresponding map on $\mathfrak{u} \times \mathfrak{h}^{\perp}$, $\sigma^{\sharp}: (X, V) \to (X, -V)$. We need to show that

$$(J^{\sharp}_{TG/H})_{(0,-V)} \circ d\sigma^{\sharp} = -d\sigma^{\sharp}_{(0,V)} \circ (J^{\sharp}_{TG/H})_{(0,V)}.$$

We will use the decomposition of $(J^{\sharp}_{TG/H})_{(0,V)}$ given by equation 4.13, and the following trivial observations.

Lemma 4.4.5 $P_{-V} \circ d\sigma^{\sharp}_{(0,V)} = (Id \times -Id) \circ P_V$.

Proof. Just compute from the definitions:

$$\begin{aligned} P_{-V} \circ d\sigma^{\sharp}_{(0,V)}(W,Y) &= P_{-V}(W,-Y) = (W, -Y - \frac{1}{2}\mathrm{ad}(V)W) \\ &= (Id \times -Id) \circ P_V(W,Y). \end{aligned}$$

□

Lemma 4.4.6 $R_{-V} \circ (Id \times -Id) = (Id \times -Id) \circ R_V$.

Proof. Recall we can write $R_V = Id_{\mathfrak{h}^{\perp}\times\mathfrak{h}^{\perp}} + H_V$, where H_V is given by

$$H_V(W,X) =$$
$$\mathrm{pr}_{\mathfrak{h}^{\perp}\times\mathfrak{h}^{\perp}}\Big(\frac{1-\cos \mathrm{ad}(V)}{\mathrm{ad}(V)}(X) + (\cos \mathrm{ad}(V) - 1)(W), \Big(\frac{\sin \mathrm{ad}(V)}{\mathrm{ad}(V)} - 1\Big)(X) - \sin \mathrm{ad}(V)(W)\Big).$$

Note that $(1-\cos x)/x$, $\sin x$ are odd functions, and $\cos x$, $(\sin x/x)-1$ are even functions. Then it's easy to see that

$$H_{-V} \circ (Id \times -Id)(W,X) = (Id \times -Id) \circ H_V(W,X).$$

□

To complete the proof of theorem 2.1.1, we simply observe that

$$\begin{aligned}
(J^\sharp_{TG/H})_{(0,-V)} \circ d\sigma^\sharp_{(0,V)} &= P_{-V}^{-1} \circ R_{-V}^{-1} \circ J_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} \circ R_{-V} \circ P_{-V} \circ d\sigma^\sharp_{(0,V)} \\
&= P_{-V}^{-1} \circ R_{-V}^{-1} \circ J_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} \circ (Id \times -Id) \circ R_V \circ P_V \\
&= -P_{-V}^{-1} \circ R_{-V}^{-1} \circ (Id \times -Id) \circ J_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} \circ R_V \circ P_V \\
&= -d\sigma^\sharp_{(0,V)} \circ P_V^{-1} \circ R_V^{-1} \circ J_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp} \circ R_V \circ P_V \\
&= -d\sigma^\sharp_{(0,V)} \circ (J^\sharp_{TG/H})_{(0,V)}.
\end{aligned}$$

This shows that the standard involution σ_o of T^*G/H is antiholomorphic. The uniqueness of this complex structure now follows from the formal power series construction in section 3.1 (see the remark on page 43). Thus we have completed the proof of theorem 2.1.1 in the homogeneous case.

4.5 Homogeneous Spaces of Compact Semisimple Lie Groups

Suppose G is a compact, connected, semisimple Lie group, and H is a closed subgroup. Then the map Φ in proposition 4.4.1 is the same as the map $c_{\mathbb{C}} \circ c^{-1}$ in the proof of proposition 4.3.8. So by theorem 4.3.9, Φ extends to a *global* diffeomorphism of TG/H and $G_{\mathbb{C}}/H_{\mathbb{C}}$. The identification of T^*G/H and TG/H by the Riemannian metric is also global, so we get a globally defined complex structure on T^*G/H. We can hope for a global version of theorem 2.1.1 in this case. The following theorem shows that in fact we do have such a global theorem.

Theorem 4.5.1 *Let G be a compact, connected, semisimple Lie group and let H be a closed subgroup. Give G/H the Riemannian metric corresponding to the bi-invariant metric on G. Then there is a globally defined complex structure on T^*G/H having the properties described in theorem 2.1.1.*

Proof. The one form $\mathrm{Im}\,\overline{\partial}\phi_o - \alpha_o$ exists globally, and when paired with an analytic vector field gives an analytic function. By the results of section 4.4.2, this analytic function vanishes on a neighborhood of G/H, so by the uniqueness of analytic continuation it must be identically zero. We conclude that $\mathrm{Im}\,\overline{\partial}\phi_o = \alpha_o$ on all of T^*G/H. Similarly, the analytic $(1,1)$-tensor $\sigma_o^* J_o + J_o$ vanishes in a neighborhood of G/H in T^*G/H, so it must be identically zero. □

Chapter 5

Toeplitz Operators

In this final chapter we will apply the ideas we have worked out in the preceding sections to a problem in analysis. We will construct a pseudodifferential operator on M, a totally real submanifold of a complex manifold, that models the process of analytic continuation of Hardy functions from the boundary of a tubular neighborhood of M in Ω. We will find that this process is very much like solving a certain "heat type" equation on M, and make explicit computations in the case of spheres and tori.

5.1 Analytic Continuation as Heat Flow

Suppose M is embedded as a totally real submanifold of a complex manifold Ω, and let ρ be a strictly plurisubharmonic exhaustion of a neighborhood of M in Ω such that $\rho \geq 0$ and $\rho = d\rho = 0$ on M only. Let

$$\Omega_t = \{\zeta \in \Omega : \rho(\zeta) < e^{-t}\}.$$

Note that as t goes to infinity, Ω_t shrinks to M. Due to the strict plurisubharmonicity of ρ, Ω_t is a strictly pseudoconvex domain for t sufficiently large. The two form $-d\,\mathrm{Im}\overline{\partial}\rho$ will be nondegenerate for t sufficiently large, so by the Kostant-Sternberg-Weinstein theorem we have a symplectic identification of a neighborhood of M in Ω and a neighborhood of M in T^*M. By scaling ρ we may assume that Ω_0 is compactly contained in such a neighborhood, still denoted by Ω.

In this way the boundary of Ω_t ($t \geq 0$) inherits the cosphere fibration, which we will denote by

$$p_t : \partial\Omega_t \to M.$$

The fibration of course depends on the choice of a symplectic structure on Ω, and therefore the choice of exhaustion function ρ. This fibration is important because it allows us to associate functions on $\partial\Omega_t$ with functions on M by "fiber integration". Let $m \in M$. The fibers $p_t^{-1}(m) \subset \partial\Omega_t$ are submanifolds of Ω, and so inherit a measure $\mu_{t,m}$ from the Riemannian structure of Ω, which varies smoothly with m. If h is a smooth function on $\partial\Omega_t$, we can transform h into a smooth function $p_{t*}h$ on M by integrating over the fibers of p_t:

$$(p_{t*}h)(m) = \int_{p_t^{-1}(m)} h \, d\mu_{t,m}.$$

This makes sense since the fibers $p_t^{-1}(m)$ are compact. We could also use some other measure besides the Riemannian measure on the fibers; we will suppress from the notation the dependence of p_{t*} on the choice of measure. The notation p_{t*} is used because $p_{t*}h$ is in a sense the push-forward of h by the map p_t. See Guillemin [8], §5 for more details.

Let $\mathcal{H}^\infty$ denote the smooth Hardy space,

$$\mathcal{H}^\infty(\partial\Omega_t) = \{h \in C^\infty(\partial\Omega_t) \colon \exists \text{ a holomorphic function } \tilde{h} \text{ on } \Omega_t \text{ such that } h = \tilde{h}|_{\partial\Omega_t}\}.$$

We could also characterize this space as the kernel of a certain differential operator on $\partial\Omega_t$, the $\overline{\partial}_b$ operator, and define Hardy spaces of L^2 functions or distributions. For simplicity, we will consider only smooth functions. L. Boutet de Monvel and V. Guillemin have obtained the following remarkable result.

Theorem 5.1.1 *The map p_{t*} when restricted to $\mathcal{H}^\infty$ has finite dimensional kernel and cokernel.*

Proof. See V. Guillemin [8], theorem 5.1. □

Recently Melrose and Epstein [14] have proven that this correspondence is actually *bijective*, for all t sufficiently large (as t becomes large, Ω_t becomes small). By scaling the exhaustion ρ we may assume this is true for all $t \geq 0$. This allows us to identify $C^\infty(M)$ with $\mathcal{H}^\infty(\partial\Omega_t)$, $(t \geq 0)$, and operators on $\mathcal{H}^\infty(\partial\Omega_t)$ with operators on $C^\infty(M)$.

One of the most natural operators on Hardy functions is analytic continuation to the interior of $\partial\Omega_t$, since these are precisely the functions on the boundary which have such extensions. There is a natural map from $\mathcal{H}^\infty(\partial\Omega_0)$ to $\mathcal{H}^\infty(\partial\Omega_t)$, given by analytic continuation followed by restriction: if $h \in \mathcal{H}^\infty(\partial\Omega_0)$ and $\tilde{h}$ is the holomorphic function on Ω_0 whose boundary value is h, then we send

$$\mathcal{H}^\infty(\partial\Omega_0) \ni h \longrightarrow \tilde{h}|_{\partial\Omega_t} \stackrel{def}{=} \tilde{h}_t \in \mathcal{H}^\infty(\partial\Omega_t).$$

This gives us an operator S_t on M, which is the bottom arrow of the following diagram:

$$\begin{array}{cccccc} & \mathcal{H}^\infty(\partial\Omega_0) & \longrightarrow & \mathcal{H}^\infty(\partial\Omega_t) & \\ (p_{0*}|_{\mathcal{H}^\infty(\partial\Omega_0)})^{-1} & \uparrow & & \downarrow & p_{t*}|_{\mathcal{H}^\infty(\partial\Omega_t)} \\ & C^\infty(M) & \xrightarrow{S_t} & C^\infty(M) & \end{array}$$

It is easy to see that microlocally this operator is not very exciting; it can be extended to a map which sends distributions into smooth functions. Its interest is in that it is the solution operator to a certain type of heat equation, which we will study in the rest of this chapter.

5.1.1 Heat Flow From the Boundary of a Complex Tube

The ideas we will describe can be formulated in terms of a Bruhat-Whitney embedding of M into a complex manifold and any strictly plurisubharmonic exhaustion ρ. In what follows we will use the complex structure on T^*M near the zero section described in theorem 2.1.1, and the quadratic exhaustion function ϕ_o induced by a Riemannian metric g on M. Then the fibration p_t is just the cotangent fibration restricted to the cosphere bundle (measured with respect to the dual metric on the fibers of T^*M), and the vector field Ξ in theorem 2.1.1 is the radial vector field $\Xi_o = \sum \xi_i \partial/\partial\xi_i$.

We can use the flow of Ξ_o to identify $\overline{\Omega}_0 \backslash M$ with $[0,\infty) \times \partial\Omega_0$ by

$$F: [0,\infty) \times \partial\Omega_0 \ni (t, \xi_m) \longrightarrow e^{-t/2}\xi_m \in \Omega_t \backslash M.$$

We have chosen to use the flow of $-\frac{1}{2}\Xi_o$ so that $F_t \stackrel{def}{=} F(t,\cdot)$ identifies $\partial\Omega_0$ with $\partial\Omega_t$. Let H_{ϕ_o} be the Hamiltonian vector field associated with ϕ_o, defined by

$$\omega_o(H_{\phi_o}, X) = d\phi_o(X).$$

Then H_{ϕ_o} is tangent to $\partial\Omega_0$, since

$$H_{\phi_o}(\phi_o) = d\phi_o(H_{\phi_o}) = \omega_o(H_{\phi_o}, H_{\phi_o}) = 0.$$

We can consider H_{ϕ_o} as a differential operator on $[0,\infty)\times\partial\Omega_0$. This allows us to formulate a simple yet elegant result relating the analytic continuation of Hardy functions on $\partial\Omega_0$ to a Cauchy initial value problem on $[0,\infty) \times \partial\Omega_0$.

Proposition 5.1.2 *Let $h \in \mathcal{H}^\infty(\partial\Omega_0)$. Then the unique solution to the Cauchy problem*

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\sqrt{-1}}{4} H_{\phi_o} u \\ u|_{t=0} &= h \end{aligned}$$

is given by $u(t,\xi_m) = \tilde{h}(e^{-t/2}\xi_m)$, where $\tilde{h}$ is the holomorphic extension of h to Ω_0.

Proof. By proposition 2.5.4 we know that $H_{\phi_o} = -2J_o\Xi_o$. Since this is a real analytic differential operator, by classical Cauchy-Kovalevsky theory we know that there exists a unique solution u to the initial value problem above (see for example Courant and

Hilbert [5]). Let $\tilde{h}$ be the holomorphic extension of h to Ω_0. If $\overline{Z}$ is any type $(0,1)$ vector field (i.e., $\overline{Z}$ can be written locally as $\overline{Z} = \sum a_\alpha(x,y)\frac{\partial}{\partial \overline{z}^\alpha}$), then $\overline{Z}\tilde{h} = 0$ on Ω_0. Now let $\overline{Z} = \Xi_o + \sqrt{-1}J\Xi_o$. The identification of $\overline{\Omega}_0 \backslash M$ with $[0,\infty) \times \partial\Omega_0$ has been constructed so that if we set $u(t,\xi_m) = \tilde{h}(e^{-t/2}\xi_m)$, then

$$\Xi_o \tilde{h} = -2\frac{\partial u}{\partial t}.$$

The equation $\overline{Z}\tilde{h} = 0$ becomes

$$\frac{\partial u}{\partial t} = \frac{\sqrt{-1}}{2}(J\Xi_o)u$$

which proves the proposition. □

The analytic continuation of Hardy functions is analogous to a sort of heat flow on $[0,\infty) \times \partial\Omega_0$, governed by the equation

$$\frac{\partial u}{\partial t} = -\frac{\sqrt{-1}}{4}H_{\phi_o}u. \tag{5.1}$$

Under the Boutet de Monvel–Guillemin identification of $\mathcal{H}^\infty(\partial\Omega_t)$ with $C^\infty(M)$, the differential operator H_{ϕ_o} corresponds to a pseudodifferential operator on M, and the partial differential equation 5.1 corresponds to a pseudodifferential equation on M. In the remainder of this chapter we will study this operator, and work out some examples.

5.1.2 Analytic Continuation as a Pseudodifferential Equation on M

In order to convert equation 5.1 into a pseudodifferential equation on M by fiber integration, we will need to specify the measures $\mu_{t,m}$ to be used in computing the fiber integral more precisely. We want to choose these measure so that "$\partial/\partial t$" commutes with p_{t*}. If $\tilde{h}$ is a smooth function on $\overline{\Omega}_0$ and $\tilde{h}_t$ denotes its restriction to $\partial\Omega_t$, then we would like to have

$$\frac{d}{dt}|_{t_o}(p_{t*}\tilde{h}_t) = p_{t_o*}((-\frac{1}{2}\Xi_o\tilde{h})_{t_o}).$$

The simplest way to arrange this is to fix a smooth measure $\mu_{0,m}$ on the fibers $p_0^{-1}(m)$ and set

$$\mu_{t,m} = (F_t)^*\mu_{0,m}$$

where $F_t\colon \partial\Omega_0 \to \partial\Omega_t$ is the map $F_t(\xi_m) = e^{-t/2}\xi_m$. Then

$$\begin{aligned}
\frac{d}{dt}|_{t_o}(p_{t*}\tilde{h}_t)(m) &= \frac{d}{dt}|_{t_o}\int_{p_0^{-1}(m)} \tilde{h}(e^{-t/2}\xi_m)\,d\mu_{0,m}(\xi_m) \\
&= \int_{p_0^{-1}(m)} -\frac{1}{2}(\Xi_o\tilde{h})(e^{-t_o/2}\xi_m)\,d\mu_{0,m}(\xi_m) \\
&= p_{t_o*}((-\frac{1}{2}\Xi_o\tilde{h})_{t_o})(m).
\end{aligned}$$

Using the measures $\mu_{t,m}$ we can translate the differential equation 5.1 into a pseudodifferential heat equation on M.

Proposition 5.1.3 *There exists a first order pseudodifferential operator P_t on M such that S_t is the solution operator for the associated heat equation,*

$$\frac{\partial g}{\partial t} = P_t g. \tag{5.2}$$

Proof. Let $f \in C^\infty(M)$, and let h be the Hardy function $(p_{0*})^{-1}f$ (we write $(p_{t*})^{-1}$ to mean the inverse of p_{t*} restricted to $\mathcal{H}^\infty(\partial\Omega_t)$). Let $u(t,\xi_m)$ be the solution to the Cauchy problem

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{\sqrt{-1}}{4}H_{\phi_o}u \\
u|_{t=0} &= h
\end{aligned} \tag{5.3}$$

which by proposition 5.1.2 is given by $u(t,\xi_m) = \tilde{h}(e^{-t/2}\xi_m)$, where $\tilde{h}$ is the analytic continuation of h to Ω_0. Now transfer this equation to M by integrating over the fibers of p_t. This gives for the left hand side of 5.3

$$p_{t*}(\frac{\partial u}{\partial t}) = p_{t*}(-\frac{1}{2}\Xi_o\tilde{h}) = \frac{d}{dt}(p_{t*}\tilde{h}_t),$$

where $\tilde{h}_t$ is the restriction of $\tilde{h}$ to $\partial\Omega_t$. Recalling the construction of the operator S_t on page 113, we see that after integrating over the fibers of p_{t*}, the left hand side of 5.3 becomes

$$p_{t*}(\frac{\partial u}{\partial t}) = \frac{d}{dt}(S_t f).$$

For the right hand side of 5.3, note that

$$u(t,\xi_m) = \tilde{h}(e^{-t/2}\xi_m) = ((p_{t*})^{-1}S_t f)(e^{-t/2}\xi_m).$$

So after integrating over the fibers, equation 5.3 becomes

$$\frac{d}{dt}(S_t f) = -\frac{\sqrt{-1}}{4}p_{t*}H_{\phi_o}(p_{t*})^{-1}(S_t f).$$

Here we are considering H_{ϕ_o} as an operator on $\partial\Omega_t$. Thus we are led to the operator

$$P_t \stackrel{def}{=} -\frac{\sqrt{-1}}{4}p_{t*}H_{\phi_o}(p_{t*})^{-1}.$$

The operators p_{t*} and $(p_{t*})^{-1}$ are Fourier integral operators, which always compose well with differential operators. So P_t is also a Fourier integral operator. In fact it is a pseudodifferential operator, since the canonical relation associated with the vector field H_{ϕ_o} is the identity canonical relation, so the canonical relation associated with P_t is also the identity relation (i.e., the conormal bundle of the diagonal in $T^*(M \times M)$). It's clear that P_t is a first order pseudodifferential operator. □

In fact P_t is an elliptic pseudodifferential operator. Rather than prove this, we will refer to the examples to follow. These examples will show that P_t looks very much like the Laplace operator on M. From these examples we make the following conjecture.

Conjecture. $P_t = c_1(t)\sqrt{\Delta_g} + c_2(t)Id$ modulo a compact operator.

5.2 The Torus

Let $\mathbb{T}^n$ be the standard n torus, $\mathbb{R}^n/2\pi\mathbb{Z}^n$, with the metric induced by the standard metric on $\mathbb{R}^n$. This is exactly the bi-invariant metric on $\mathbb{T}^n$ as a Lie group. The tangent bundle of $\mathbb{T}^n$ is naturally identified with $\mathbb{T}^n \times \mathbb{R}^n$, which is then identified with $T^*\mathbb{T}^n$ by the metric. The complex structure on $\mathbb{T}^n \times \mathbb{R}^n$ described by theorem 2.1.1 is given in lemma 4.1.5. Since $\mathbb{T}^n$ is commutative, $\mathrm{ad}(V)$ is identically zero and so the complex structure is the one obtained by identifying the tangent space $T_{(x,V)}(\mathbb{T}^n \times \mathbb{R}^n)$ with $\mathbb{C}^n$ in the usual way. It is easy to see that $\mathbb{C}^n/2\pi\mathbb{Z}^n$ is the universal complexification of $\mathbb{T}^n$ given in theorem 4.1.1. Here $\mathbb{C}^n/2\pi\mathbb{Z}^n$ is $\mathbb{C}^n$ modulo the $2\pi\mathbb{Z}^n$ action

$$x + \sqrt{-1}\, y \to (x + 2\pi q) + \sqrt{-1}\, y.$$

It is easy to check that the obvious global identification of $T^*\mathbb{T}^n$ with $\mathbb{C}^n/2\pi\mathbb{Z}^n$ is the complex structure described in theorem 2.1.1. The exhaustion function ϕ is $\phi(x,\xi) = |\xi|^2$.

The functions $e_k(x) = e^{\sqrt{-1}\,k\cdot x}$, $k \in \mathbb{Z}^n$, form a Hilbert space basis for $L^2(\mathbb{T}^n)$, and every smooth function on $\mathbb{T}^n$ can be expanded in a Fourier series in this basis. It can be shown that the functions

$$e_{k,t}(x,\xi) = e^{\sqrt{-1}k\cdot(x+\sqrt{-1}\xi)}|_{\partial\Omega_t}$$

form a Hilbert space basis for the L^2 closure of the smooth Hardy space on $\partial\Omega_t \subset T^*\mathbb{T}^n$. Due to the invariance under the obvious group actions of our operators, it will be enough to compute with these elementary functions.

Lemma 5.2.1 $(p_{t*}e_{k,t})(x) = J_n(e^{-t/2}|k|)e_k(x)$, *where J_n is the zeroth order Bessel function*

$$J_n(\nu) = \Gamma_{n-2}\int_{-1}^{1} e^{-\nu s}(1-s^2)^{\frac{n-3}{2}}\, ds.$$

Here Γ_{n-2} is the volume of the $n-2$ sphere ($n \geq 3$).

Proof. Let $\epsilon = e^{-t/2}$. Then

$$\begin{aligned} (p_{t*}e_{k,t})(x) &= \int_{|\xi|^2=1} e^{\sqrt{-1}k\cdot(x+\sqrt{-1}\epsilon\xi)}\, d\xi \\ &= e_k(x)\int_{S^{n-1}} e^{-\epsilon k\cdot\omega}\, d\omega \end{aligned}$$

where $d\omega$ is the standard measure on S^{n-1}. The integral is rotationally invariant in k, so it is a function J_n of $\epsilon|k|$. To compute it we set $\epsilon k = \nu(0,\dots,0,1)$. Then

$$J_n(\nu) = \int_{S^{n-1}} e^{-\nu\omega_n}\, d\omega.$$

Using cylindrical polar coordinates on S^{n-1},

$$\begin{aligned} \omega' &= (\omega_1,\dots,\omega_{n-1}) \\ r &= |\omega'| = \cos\theta \\ \omega_n &= \sin\theta \\ d\omega &= \Gamma_{n-2}(\cos\theta)^{n-2}d\theta \\ &\quad (\Gamma_{n-2} = \text{volume of } S^{n-2}), \end{aligned}$$

we see that

$$J_n(\nu) = \Gamma_{n-2}\int_{-\pi/2}^{\pi/2} e^{-\nu\sin\theta}(\cos\theta)^{n-2}\, d\theta.$$

Setting $s = \sin\theta$ this becomes

$$J_n(\nu) = \Gamma_{n-2}\int_{-1}^{1} e^{-\nu s}(1-s^2)^{\frac{n-3}{2}}\, ds$$

which proves the lemma. □

Remark. Note that $J_n(\nu) > 0$, so in this case we have a direct proof that p_{t*} is bijective when restricted to the Hardy space.

Corollary 5.2.2 *The operator S_t is given by*

$$S_t e_k = \frac{J_n(e^{-t/2}|k|)}{J_n(|k|)} e_k.$$

We can now describe the operator P_t on $\mathbf{T}^n$.

Proposition 5.2.3 *The operator P_t is given by*

$$P_t e_k = \frac{(e^{-t/2}|k|)^2}{2(1-n)} \frac{\Gamma_{n-2}}{\Gamma_n} \frac{J_{n+2}(e^{-t/2}|k|)}{J_n(e^{-t/2}|k|)} e_k$$

where Γ_n is the volume of the n sphere.

Proof. The Hamiltonian vector field H_{ϕ_o} is

$$H_{\phi_o} = 2 \sum \xi_i \frac{\partial}{\partial x_i}.$$

Then

$$H_{\phi_o}(p_{t*})^{-1} e_k(x,\xi) = \frac{2}{J_n(e^{-t/2}|k|)} \sqrt{-1}(k \cdot \xi) e_{k,t}(x,\xi).$$

The operator P_t is then

$$P_t e_k(x) = \frac{1}{2J_n(e^{-t/2}|k|)} p_{t*}((k \cdot \xi) e_{k,t}).$$

Setting $\epsilon = e^{-t/2}$ we have

$$p_{t*}((k \cdot \xi) e_{k,t}) = \int_{|\xi|=1} (k \cdot \epsilon\xi) e^{-\epsilon k \cdot \xi} \, d\xi \; e_k(x).$$

As before the integral depends only on $|k|$. We can write it as

$$\begin{aligned} \int_{|\xi|=1} (k \cdot \epsilon\xi) e^{-\epsilon k \cdot \xi} \, d\xi &= \epsilon|k| \int_{S^{n-1}} \omega_n e^{\epsilon|k|\omega_n} \, d\omega \\ &= \epsilon|k|\Gamma_{n-2} \int_{-1}^{1} e^{-\epsilon|k|s} s(1-s^2)^{\frac{n-3}{2}} \, ds. \end{aligned}$$

After integrating by parts this becomes

$$\int_{|\xi|=1}(k \cdot \epsilon\xi)e^{-\epsilon k\cdot\xi}\,d\xi = \frac{(\epsilon|k|)^2}{1-n}\Gamma_{n-2}\int_{-1}^{1} e^{-\epsilon|k|s}(1-s^2)^{\frac{n-1}{2}}\,ds$$
$$= \frac{(\epsilon|k|)^2}{1-n}\frac{\Gamma_{n-2}}{\Gamma_n}J_{n+2}(\epsilon|k|).$$

This shows that

$$p_{t*}((k\cdot\xi)e_{k,t}) = \frac{(\epsilon|k|)^2}{1-n}\frac{\Gamma_{n-2}}{\Gamma_n}J_{n+2}(\epsilon|k|)e_k$$

which proves the proposition. □

To understand this operator we need the following asymptotic expansion of J_n.

Lemma 5.2.4 *There exist constants α_i, depending only on the dimension n, such that*

$$e^{-\nu}\nu^{\frac{n-1}{2}}J_n(\nu) \sim \sum_{i=0}^{\infty}\alpha_i\nu^{-i}$$

with $\alpha_0 > 0$.

Proof. See appendix A.2 □

Let Δ be the Laplacian on $\mathbb{T}^n$, $\Delta = -\sum(\frac{\partial}{\partial x_i})^2$. Then it follows immediately from the lemma that there exist constants a_i, depending only on the dimension n, such that

$$P_t \sim a_1 e^{-t/2}\sqrt{\Delta} + a_o + \sum_{i=1}^{\infty} a_{-i}(e^{-t/2}\sqrt{\Delta})^{-i}$$

with $a_1 < 0$. The meaning of the symbol "$\sim$" is that the difference between P_t and the first k terms on the right maps $L^2(\mathbb{T}^n)$ into the Sobolov space $W^{k-1}(\mathbb{T}^n)$.

This verifies conjecture given at the end of section 5.1.2 for the n torus with its standard metric. It also provides further justification for thinking of analytic continuation as a sort of heat flow.

When $n = 3$ it is easy to compute the integrals defining $J_n(\nu)$. The result of this computation (which we will omit) is that for $n = 3$,

$$\begin{aligned} J_n(\nu) &= \Gamma_1(\frac{1}{\nu})(e^{\nu} - e^{-\nu}) \\ J_{n+2}(\nu) &= \Gamma_3(\frac{2}{\nu^2})(e^{-\nu} + e^{\nu} + \frac{1}{\nu}(e^{-\nu} - e^{\nu})). \end{aligned}$$

Then if we set $\epsilon = e^{-t/2}$ we have

$$P_t e_k = \frac{-\epsilon|k|}{2}(\frac{e^{-2\epsilon|k|} + 1 + \frac{1}{\epsilon|k|}(e^{-2\epsilon|k|} - 1)}{1 - e^{-2\epsilon|k|}})e_k.$$

This shows that in the ring of pseudodifferential operators we have for $n = 3$,

$$P_t = \frac{1}{2}(-e^{-t/2}\sqrt{\Delta} + Id).$$

5.3 The Sphere

In this section we will study the operator P_t on the n sphere, for n odd, and write down an exact expression for P_t as a pseudodifferential operator when $n = 3$. The results will be very similar to the case of the torus.

To do the computations it will be most convenient to work in the context of theorem 3.1.1, thinking of S^n as the "center" of a complex Monge-Ampère manifold. We have given a very explicit realization of this in the case of the sphere in section 4.2, due to P. M. Wong [25]. Let

$$\Omega = \{z = x + \sqrt{-1}\,y \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1\}$$

$$\phi = (\cosh^{-1}\rho)^2$$

where $\rho(z) = |z|^2$. We have shown that ϕ is the solution to the "Monge-Ampère type" problem posed in theorem 3.1.1 for S^n with its standard metric (see proposition 4.2.2). This identifies Ω with T^*S^n, and we have shown in theorem 4.5.1 that this identification is global. We need to describe the cotangent foliation on Ω, the radial vector field and its integral curves. P. M. Wong shows in [25] that the Monge-Ampère distribution, as a complex one dimensional subbundle of the holomorphic tangent bundle of Ω, is spanned by the vector field

$$V = \sum_{i=1}^{n+1} (\rho z^i - \bar{z}^i) \frac{\partial}{\partial z^i}|_\Omega.$$

Recalling the discussion in section 2.4, we see that the vector field Ξ must be a multiple of the real part of V. Using the condition that $\Xi\phi = 2\phi$ we can determine Ξ. The result is that

$$\Xi = \frac{\cosh^{-1}\rho}{2|x||y|} \sum_{i=1}^{n+1} x^i |y|^2 \frac{\partial}{\partial x^i} + y^i |x|^2 \frac{\partial}{\partial y^i}. \tag{5.4}$$

Let

$$\Omega_t = \Omega \cap \{\phi(z) < e^{-t}\}.$$

Since Ω is defined by $|x|^2 - |y|^2 = 1$ and $x \cdot y = 0$, we see that this can be written as

$$\Omega_t = \Omega \cap \{|y| < \frac{1}{\sqrt{2}}(-1 + \cosh e^{-t/2})^{\frac{1}{2}}\}.$$

One can check that if $x + \sqrt{-1}\,y$ is in $\partial\Omega_0$, then the integral curve of $-\frac{1}{2}\Xi$ through $x + \sqrt{-1}\,y$ is given by

$$F(t, x, y) = \frac{1}{\sqrt{2}}(1 + \cosh e^{-t/2})^{\frac{1}{2}} \frac{x}{|x|} + \frac{\sqrt{-1}}{\sqrt{2}}(-1 + \cosh e^{-t/2})^{\frac{1}{2}} \frac{y}{|y|}.$$

The cotangent fibration is given by following the integral curves of $-\frac{1}{2}\Xi$ through $x+\sqrt{-1}\,y$ to where they intersect the center S^n. This shows that the cotangent fibration of $\partial\Omega_t$ over S^n is just the obvious one,

$$p_t(x + \sqrt{-1}\,y) = \frac{x}{|x|}.$$

The map $F\colon \partial\Omega_0 \to \overline{\Omega}\backslash M$ above gives the identification of $\partial\Omega_0 \times [0, \infty)$ with $\Omega\backslash M$ needed to construct the one parameter family of operators P_t.

To compute the "averaging" operator p_{t*} we will have to rely heavily on the group invariance present. The orthogonal group $O(n+1)$ acts on S^n by rotations. Under this action $L^2(S^n)$ decomposes into a Hilbert space direct sum of inequivalent, irreducible representations of $O(n+1)$,

$$L^2(S^n) = \sum_{k=0}^{\infty} V_k,$$

where the finite dimensional subspaces V_k are the "spherical harmonics of degree k",

$$V_k = \text{span of } \{f_{a,k}(m) = (a_1 m_1 + \cdots + a_{n+1} m_{n+1})^k \colon a_1^2 + \cdots + a_{n+1}^2 = 0,\ a_i \in \mathbb{C}\}.$$

See for example Helgason [11], theorem 3.1, introduction. The orthogonal group $O(n+1)$ acts on Ω as well by

$$A \cdot (x + \sqrt{-1}\, y) = Ax + \sqrt{-1}\, Ay.$$

This action preserves the complex structure, so it induces an action on the L^2 closure of the smooth Hardy space, $\mathcal{H}^2(\partial\Omega_t)$. We will show that we get a similar decomposition of $\mathcal{H}^2(\partial\Omega_t)$.

Lemma 5.3.1 *Under this $O(n+1)$ action, $\mathcal{H}^2(\partial\Omega_t)$ breaks up into a Hilbert space direct sum of inequivalent, irreducible representations of $O(n+1)$,*

$$\mathcal{H}^2(\partial\Omega_t) = \sum_{0}^{\infty} V_{k,t}^{\#},$$

where

$$V_{k,t}^{\#} = \{\text{polynomials } h(z)|_{\partial\Omega_t} : h(z)|_{S^n} \in V_k\}.$$

Proof. The $V_{k,t}^{\#}$ are clearly inequivalent, irreducible representations of $O(n+1)$, since the V_k are. Since polynomials in z restricted to $\partial\Omega_t$ are dense in $\mathcal{H}^2(\partial\Omega_t)$, we need only show that if $P(z)$ is a polynomial in z homogeneous of degree k, then P restricted to $\partial\Omega_t$ is in the algebraic direct sum of the $V_{k,t}^{\#}$. Since P restricted to $\mathbb{R}^n$ is a homogeneous polynomial on $\mathbb{R}^n$ of degree k, there exist harmonic polynomials $h_k, h_{k-2}, \ldots, h_{k-2p}$ ($p =$

integer part of $k/2$) such that

$$P|_{\mathbb{R}^n} = h_k(x) + |x|^2 h_{k-2}(x) + \cdots + |x|^{2p} h_{k-2p}(x)$$

(see for example Helgason [11], proof of theorem 3.1, introduction). This shows that $P(z)$ can be written as

$$P(z) = h_k(z) + (z_1^2 + \cdots + z_{n+1}^2) h_{k-2} + \cdots + (z_1^2 + \cdots + z_{n+1}^2)^p h_{k-2p}(z).$$

Then when restricted to $\partial\Omega_t$ we have, since $z_1^2 + \cdots + z_{n+1}^2 = 1$ on Ω,

$$P|_{\partial\Omega_t} = h_k|_{\partial\Omega_t} + \cdots + h_{k-2p}|_{\partial\Omega_t}.$$

This proves the lemma, since the h_{k-2j} restricted to S^n are in V_{k-2j}. □

The fibers p_t^{-1} are $n-1$ spheres, so if $h_t \in \mathcal{H}^\infty(\partial\Omega_t)$ we see that

$$(p_{t*}h_t)(m) = \int_{\{y \in S^n : y \cdot m = 0\}} h_t\left(\frac{1}{\sqrt{2}}(1 + \cosh e^{-t/2})^{\frac{1}{2}} m + \frac{\sqrt{-1}}{\sqrt{2}}(-1 + \cosh e^{-t/2})^{\frac{1}{2}} y\right) d\mu_{0,m}.$$

We take for the measures $\mu_{0,m}$ the standard measure on S^{n-1}. It is clear from this expression that p_{t*} commutes with the $O(n+1)$ actions. It follows that $p_{t*}(V_{k,t}^{\#})$ is an irreducible representation of $O(n+1)$. Schur's lemma says that either $p_{t*}(V_{k,t}^{\#})$ is inequivalent as a representation to $V_{k,t}^{\#}$ and $p_{t*}(V_{k,t}^{\#})$ is zero, or it is an equivalent representation, in which case there is a unique map from $V_{k,t}^{\#}$ to $p_{t*}(V_{k,t}^{\#})$ commuting with the $O(n+1)$ actions, up to a constant multiple. See Želobenko [25], §20. We conclude that there is a function $\lambda_t(k)$ such that for each $h_{k,t} \in V_{k,t}^{\#}$,

$$p_{t*}(h_{k,t}) = \lambda_t(k) h_{k,t}|_{S^n}.$$

To compute $\lambda_t(k)$ we set $h_{k,t}(z) = (z_1 + \sqrt{-1}\, z_2)^k$ and evaluate both sides of the above equation at $m = (1, 0, \ldots, 0)$. Set $a = \frac{1}{\sqrt{2}}(1 + \cosh e^{-t/2})^{\frac{1}{2}}$, $b = \frac{1}{\sqrt{2}}(-1 + \cosh e^{-t/2})^{\frac{1}{2}}$, and let $\omega = (y_2, \ldots, y_{n+1})$. Then

$$\lambda_t(k) = \int_{\omega \in S^{n-1}} (a - b\omega_1)^k \, d\omega$$

where $d\omega$ is the standard measure on S^{n-1}. Using cylindrical polar coordinates as in section 5.2, we can write this as

$$\lambda_t(k) = \Gamma_{n-2} \int_{-1}^{1} (a + bs)^k (1 - s^2)^{\frac{n-3}{2}} \, ds.$$

This is clearly nonzero for all $t \in [0, \infty)$, $k = 0, 1, \ldots$. Thus we have shown the following.

Proposition 5.3.2 *The map p_{t*} is a bijection from $\mathcal{H}^2(\partial\Omega_t)$ to $L^2(S^n)$, and is given by, for $h_{k,t} \in V^{\#}_{k,t}$,*

$$p_{t*}(h_{k,t}) = \Gamma_{n-2} \int_{-1}^{1} (a + bs)^k (1 - s^2)^{\frac{n-3}{2}} \, ds \, h_{k,t}|_{S^n}.$$

where Γ_{n-2} is the volume of the $n-2$ sphere.

We can now write down an expression for P_t. Using the expression for Ξ in equation 5.4 we have

$$H_\phi = -2J\Xi = \frac{\cosh^{-1} \rho}{|x||y|} \sum_{i=1}^{n+1} y^i |x|^2 \frac{\partial}{\partial x^i} - x^i |y|^2 \frac{\partial}{\partial y^i}.$$

A quick computation shows that H_ϕ commutes with the $O(n+1)$ action on $\partial\Omega_t$, so P_t commutes with the $O(n+1)$ action on S^n. By Schur's lemma there is a function $\beta_t(k)$ such that

$$P_t|_{V_k} = \beta_t(k) Id. \tag{5.5}$$

Set $f_k(m) = (m_1 + \sqrt{-1}\, m_2)^k$ and evaluate both sides of the above equation at $m = (1, 0, \ldots, 0)$. Then

$$\beta_t(k) = \frac{-\sqrt{-1}}{4} \frac{1}{\lambda_t(k)} p_{t*}(H_\phi f^{\#}_{k,t})(1, 0, \ldots, 0)$$

where $f^{\#}_{k,t}$ is the function $(z_1 + \sqrt{-1}\, z_2)^k$ restricted to $\partial\Omega_t$. We need to evaluate

$$\begin{aligned} p_{t*}(H_\phi f^{\#}_{k,t})(1,0,\ldots,0) &= \int_{\{y\in S^n:\, y_1=0\}} H_\phi f^{\#}_{k,t}(am + \sqrt{-1}\, by)\, d\mu \\ &= -k\sqrt{-1}\, \cosh^{-1}(1+2b^2)\int_{\omega\in S^{n-1}} (b - a\omega_1)(a - b\omega_1)^{k-1}\, d\omega \end{aligned}$$

where $a(t)$, $b(t)$ are as above. We are assuming that $k > 0$; if $k = 0$, then $P_t f_k = 0$. Using cylindrical polar coordinates, this becomes

$$\begin{aligned} & p_{t*}(H_\phi) f^{\#}_{k,t}(1,0,\ldots,0) = \\ & \quad -k\sqrt{-1}\, \cosh^{-1}(1-2b^2)\Gamma_{n-2}\int_{-1}^{1} (b+as)(a+bs)^{k-1}(1-s^2)^{\frac{n-3}{2}}\, ds \end{aligned}$$

where Γ_{n-2} is the volume of the $n-2$ sphere. Recalling the expression for $\lambda_t(k)$ above and the definition of b, we have proved the following.

Proposition 5.3.3 *The operator P_t on S^n is given by, for $f_k \in V_k$,*

$$P_t f_k = \frac{-k}{4} e^{-t/2} \frac{\int_{-1}^{1}(b+as)(a+bs)^{k-1}(1-s^2)^{\frac{n-3}{2}}\, ds}{\int_{-1}^{1}(a+bs)^k(1-s^2)^{\frac{n-3}{2}}\, ds} f_k.$$

To understand this operator we need to get asymptotic expansions for these integrals as $k \to \infty$. We will assume that $\frac{n-3}{2} \stackrel{def}{=} q$ is an integer, i.e. that n is odd.

Lemma 5.3.4 *Let $I_t(k)$ denote the integral in the denominator in proposition 5.3.3. Then*

$$I_t(k) \sim k^{-(q+1)}(a+b)^k \sum_{l=0}^{\infty} k^{-l} c_{n,t,l}$$

where if $r(s) = \ln\left(\frac{a+bs}{a+b}\right)$, $\psi = (1-s^2)^q$, then

$$c_{n,t,l} = \frac{(-1)^{l+q}}{r'(1)}\left(\frac{d}{ds}\frac{1}{r'}\right)^{l+q}\psi|_{s=1}.$$

Proof. We can write the integral as

$$I_t(k) = (a+b)^k \int_{-1}^{1} e^{kr(s)} \psi(s)\, ds.$$

Integrating by parts N times gives

$$\begin{aligned} I_t(k) = k^{-1}(a+b)^k \Big(\sum_{l=0}^{N} k^{-l} \frac{(-1)^l}{r'(1)} \Big(\frac{d}{ds}\frac{1}{r'}\Big)^l \psi|_{s=1} \\ -(-k)^{-N} \int_{-1}^{1} e^{kr(s)} \Big(\frac{d}{ds}\frac{1}{r'}\Big)^{N+1} \psi(s)\, ds\Big) + O(k^{-\infty}). \end{aligned}$$

Since $r(s) < 0$ for $s \in [-1, 1)$, the remainder term is of order k^{-N}. Since ψ vanishes to qth order at $s = 1$, this proves the lemma. □

Lemma 5.3.5 *Let $\tilde{I}_t(k)$ denote the integral in the denominator in proposition 5.3.3. Then*

$$\tilde{I}_t(k) = k^{-(q+1)}(a+b)^k \sum_{l=0}^{\infty} k^{-l} d_{n,t,l}$$

where if $r(s)$ is as above and $\tilde{\psi} = (\frac{b+as}{a+bs})(1-s^2)^{\frac{n-3}{2}}$, then

$$d_{n,t,l} = \frac{(-1)^{l+q}}{r'(1)} \Big(\frac{d}{ds}\frac{1}{r'}\Big)^{l+q} \tilde{\psi}|_{s=1}.$$

Proof. Note that for $s \in [-1, 1]$, $a + bs > a - b = 1/(a+b) > 0$, since $a^2 - b^2 = 1$. This shows that $\tilde{\psi}$ is a smooth function. The proof of the asymptotic expansion is exactly the same as in the previous lemma. □

We can now obtain an asymptotic expansion for the Fourier coefficients $\beta_t(k)$ in equation 5.5. Using the asymptotic expansion of $I_t(k)$ and $\tilde{I}_t(k)$ above and the expression for $P_t f_k$ in proposition 5.3.3, we obtain

$$\beta_t(k) = \frac{-e^{-t/2}}{4}\Big(k\frac{d_{n,t,0}}{c_{n,t,0}} + \frac{1}{c_{n,t,0}}\Big(d_{n,t,1} - \frac{d_{n,t,0}}{c_{n,t,0}} c_{n,t,1}\Big)\Big) + O(k^{-1}).$$

One can easily compute the first term in this expansion:

$$c_{n,t,0} = d_{n,t,0} = (\frac{a+b}{b})^{(q+1)} q! 2^q.$$

Then

$$\beta_t(k) = \frac{-e^{-t/2}}{4} k + \alpha_{t,n} + O(k^{-1})$$

where

$$\alpha_{t,n} = \frac{e^{-t/2}}{4} (\frac{b}{a+b})^{q+1} (\frac{1}{q! 2^q})(d_{n,t,1} - c_{n,t,1}).$$

We want to compare this with $\sqrt{\Delta}$, where Δ is the (positive) Laplace operator on S^n associated with the standard metric. The Fourier coefficients of $\sqrt{\Delta}$ are

$$\begin{aligned} \sqrt{\Delta} f_k &= \sqrt{k(k+n-1)}\, f_k \\ &= (k + \frac{n-1}{2} + O(k^{-1})) f_k. \end{aligned}$$

This shows that

$$P_t = \frac{-e^{-t/2}}{4} \sqrt{\Delta} + \frac{2}{n-1} \alpha_{t,n} Id + K$$

where K is a compact operator on $L^2(S^n)$.

When $n = 3$ it is possible to evaluate the integrals directly. The result is

$$\begin{aligned} I_t(k) &= \frac{a+b}{b(k+1)}((a+b)^k - (a+b)^{-k}) \\ \tilde{I}_k(t) &= \frac{a+b}{bk}((a+b)^k - (a+b)^{-k} - \frac{a}{b(k+1)}((a+b)^k - (a+b)^{-k})). \end{aligned}$$

Note that $(a+b)^{-k}$ is rapidly decreasing[1] as $k \to \infty$, since $a + b > 1$ for $t \in [0, \infty)$. Then we see that when $n = 3$, the Fourier coefficients of P_t are

$$\beta_t(k) = \frac{-e^{-t/2}}{4}(k + 1 - \frac{a}{b}) + O(k^{-\infty}).$$

[1]By this we mean that $k^N (a+b)^{-k} \to 0$ as $k \to \infty$ for all N.

Then

$$P_t = \frac{e^{-t/2}}{4}(-\sqrt{\Delta} + \frac{a}{b} Id) + K$$

where K is a compact operator.

Appendix A

Appendix

A.1 A Covering Lemma

We will show in this appendix that given any open cover V_j of M by open sets in Ω we can cover M by holomorphic coordinate patches $\mathcal{O}_i$ with coordinate functions $\psi_i = (z_i^1, \ldots, z_i^n)$ such that

1. Each $\mathcal{O}_i$ is contained in some V_j
2. $M \cap \mathcal{O}_i$ is given by $\{\zeta \in \mathcal{O}_i : \operatorname{Im} z_i^1(\zeta) = \cdots = \operatorname{Im} z_i^n(\zeta) = 0\}$
3. Each $\mathcal{O}_i$ contains ony one component of M
4. If $\mathcal{O}_i \cap \mathcal{O}_j$ is not empty, then $\mathcal{O}_i \cap \mathcal{O}_j \cap M$ is not empty.

It is clear that we can arrange for conditions 1 through 3 above to be satisfied, and that if $\tilde{\mathcal{O}}_i$ is a covering of M with open sets with $\tilde{\mathcal{O}}_i \subset \mathcal{O}_i$, then conditions 1 through 3 above continue to hold for the coordinate system $(\tilde{\mathcal{O}}_i, \psi_i|_{\tilde{\mathcal{O}}_i})$.

To arrange for condition 4 to be satisfied we use a standard tubular neighborhood theorem. We may assume that there is a neighborhood $\mathcal{M}$ of M in Ω, a neighborhood $\mathcal{M}_N$ of M in the normal bundle of M, and a diffeomorphism F from $\mathcal{M}_N$ to $\mathcal{M}$ (see for example [1], theorem 2.7.5). We may assume that $\mathcal{O}_i \subset \mathcal{M}$, and that $N^{-1}(\mathcal{O}_i)$ contains a set diffeomorphic to $\mathcal{O}_i \cap M \times B_\epsilon(0)$ for some ϵ sufficiently small, where $B_\epsilon(0)$ is a ball of radius ϵ about 0 in $\mathbb{R}^n$. So we may assume by shrinking the $\mathcal{O}_i$ that $\mathcal{O}_i$ is diffeomorphic to $\mathcal{O}_i \cap M \times B_\epsilon(0)$. Then if $\mathcal{O}_i \cap \mathcal{O}_j$ is not empty, neither is

$$(\mathcal{O}_i \cap M \times B_{\epsilon_i}(0)) \cap (\mathcal{O}_j \cap M \times B_{\epsilon_j}(0))$$

$$= \mathcal{O}_i \cap \mathcal{O}_j \cap M \times B_{min(\epsilon_i,\epsilon_j)}(0).$$

This shows that $\mathcal{O}_i \cap \mathcal{O}_j \cap M$ is not empty.

A.2 Asymptotic Expansion of J_n

The results in this appendix are from a course on hyperfunctions given by Prof. V. Guillemin. Write

$$J_n(\nu) = \Gamma_{n-2} \int_0^1 e^{\nu s}(1-s^2)^{\frac{n-3}{2}}\, ds + O(\nu^0).$$

Set $r = 1 - s$. Then

$$J_n(\nu) = \Gamma_{n-2} e^{\nu} \int_0^1 e^{-\nu r} r^{\frac{n-3}{2}} (2-r)^{\frac{n-3}{2}}\, dr + O(\nu^0).$$

Now let $u = \nu r$. Then

$$J_n(\nu) = \Gamma_{n-2} e^{\nu} \nu^{\frac{-(n-1)}{2}} \int_0^{\nu} e^{-u} u^{\frac{n-3}{2}} (2 - \frac{u}{\nu})^{\frac{n-3}{2}}\, du + O(\nu^0).$$

We can write for some constants a_j, depending only on the dimension n,

$$(2 - \frac{u}{\nu})^{\frac{n-3}{2}} \sim \sum_{j=0}^{\infty} a_j (\frac{u}{\nu})^j.$$

Then

$$e^{-\nu} \nu^{\frac{(n-1)}{2}} J_n(\nu) = \Gamma_{n-2} \sum_{j=0}^{\infty} \nu^{-j} a_j \int_0^{\nu} e^{-u} u^{\frac{n-3}{2}+j}\, du + O(\nu^0).$$

Note that

$$\begin{aligned}\int_0^{\nu} e^{-u} u^{\frac{n-3}{2}+j}\, du &= \int_0^{\infty} e^{-u} u^{\frac{n-3}{2}+j}\, du + O(\nu^{-\infty}) \\ &= \Gamma(\frac{n-3}{2}+j) + O(\nu^{-\infty}).\end{aligned}$$

This shows that

$$e^{-\nu}\nu^{\frac{n-1}{2}} J_n(\nu) \sim \Gamma_{n-2} \sum_{j=0}^{\infty} a_j \Gamma(\frac{n-3}{2}+j) + O(\nu^{-\infty})$$

which proves lemma 5.2.4.

A.3 Notational Conventions

We will denote the interior product of a vector V with a form ω by $\iota(V)\omega$. The Lie derivative of ω by the vector field V will be denoted by $L_V\omega$.

The subscript o will usually refer to objects on the cotangent bundle of a manifold M. For example, α_o denotes the canonical one form, ω_o denotes the symplectic form on T^*M.

Lie algebras will be written with sans serif letters. For example, if G is a Lie group, then the Lie algebra of G is written as g.

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