Social Networks in Industrial Organization

by

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Abstract

Chapter 1 studies the optimal strategies of a monopolist selling a good to consumers who engage in word of mouth communication. The monopolist uses the price it charges to influence both the proportion of the population that is willing to purchase the good and the pattern of communication that takes place within the social network. I find a number of results: (i) demand is more elastic in the presence of word of mouth; (ii) the monopolist reduces the price to induce additional word of mouth for regular goods, however for goods whose valuation is greater for well connected individuals the price may, in fact, be greater; (iii) the optimal pattern of diffusion involves introductory prices which vary up and down; and (iv) exclusive (high priced) products will optimally target advertising towards individuals with many friends whereas common (low priced) products will target individuals with fewer friends. Chapter 2 presents a model of friendship formation in a social network. During each period a new player enters the social network, this player searches for and forms friendships with the existing population and all individuals play a prisoner's dilemma game with each of their friends. The set of friendships a player forms reveals some information to a friend about how likely she is to subsequently cooperate. Cooperative types are able to separate themselves from uncooperative types by becoming friends with people who know one another. The threat of communication amongst people who know one another prevents an uncooperative type mimicking a cooperative type. Chapter 3 analyzes the effects of policies which support electricity generation from intermittent technologies (wind, solar). I find that intermittent generation is a substitute for base-load technologies but may be complementary or substitutable for peaking/intermediate technologies. I characterize the long run implications of this for carbon emissions.

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Contents

Introduction

The thesis contains three chapters. The first chapter considers a setting where consumers engage in word of mouth communication about a product, it characterizes product demand and a firm's optimal pricing and advertising behavior in this environment. The second chapter presents a model of friendship formation where individuals signal their willingness to cooperate through their choice of friends. The final chapter characterizes the long-run implications of support for intermittent electricity generating technologies on the mix of conventional technologies and carbon emissions.

Chapter 1 studies the optimal strategies of a monopolist selling a good to consumers who engage in word of mouth communication. In the model consumers spread news about the monopolist's good to uninformed consumers through a social network. The monopolist uses the price it charges to influence both the proportion of the population that is willing to purchase the good and the pattern of communication that takes place within the social network. I find a number of results: (i) demand is more elastic in the presence of word of mouth and this induces a downward bias in estimates of consumers' valuation for the good which ignore word of mouth; (ii) the monopolist reduces the price to induce additional word of mouth for regular goods however for goods whose valuation is greater for well connected individuals the price may, in fact, be greater; (iii) the optimal pattern of diffusion involves introductory prices which vary up and down; and (iv) exclusive (high priced) products will optimally target advertising towards individuals with many friends whereas common (low priced) products will target individuals with fewer friends.

Chapter 2 presents a model of friendship formation in a social network. During each period a new player enters the social network, this player searches and forms friendships with the existing population and all individuals play a prisoner's dilemma game with each of their friends. The way in which new players search for and then establish friendships reveals some information to a friend about how likely they are to be willing to cooperate. This willingness is modeled through the discount factor of an individual which is private information. Types who have high discount factors, and are hence cooperative, can signal their type to new friends by also becoming friends with some of their friends. It is the threat of communication amongst friends who know one another which deters an uncooperative type from copying the cooperative type. I relate the sequential equilibrium in this paper to a model of network formation by Jackson and Rogers (2007) which describes the relative frequency of random and network based meetings and can explain many of the stylized facts associated with social networks. The model of network formation in this paper is used to predict when how the parameters describing random and network based meeting might very across environments and the welfare implications in these cases.

Chapter 3 analyzes the effects of policies which support electricity generation from intermittent technologies (wind, solar) on the long-run incentives for investment and generation from dispatchable electricity generation technologies (gas, coal, nuclear, oil) and the implications for carbon emissions. The nature of electricity markets, instantaneous matching of supply and demand, means that intermittent technologies are not perfect substitutes for any one dispatchable technology. The variability of load usually determines the long run mix of generating technologies in a competitive electricity market. When there is a significant amount of intermittent production the mix of other generating technologies is determined by the variability of net load (load net of intermittent output). Net load may be more variable than load itself if the intermittent output is not too positively correlated with load. This increase in variability results in a substitution away from baseload generating technologies towards peaking and intermediate technologies. If peaking and intermediate technologies are more carbon intensive than non-renewable "baseload" technologies, this substitution can more than offset the emission benefits derived from the output of the renewable technology.

Chapter 1

Tell Your Friends! Word of Mouth and Percolation in Social Networks

1.1 Introduction

A widely recognized phenomenon is the diffusion of products and innovations within populations. A key conduit for this diffusion is often word of mouth (WOM hereafter) between members of the population. A large number of studies have found that WOM is an important source of information for consumers' purchase decisions.[1] The significant influence of WOM on purchasing decisions raises a number of questions pertaining to an environment where consumers share their experience of a firm's good or service with each other. How does WOM affect demand for a product? What strategies does a firm employ in the presence of WOM? Do these strategies differ across different product categories? How are traditional advertising strategies affected by WOM? This paper characterizes product demand, and the firm's optimal pricing and advertising behavior by considering how a firm may strategically affect the probability consumers engage in WOM through the price and the subsequent pattern of communication which takes place. It combines a model of a monopolist and a percolation model of WOM in a social network. The

[1] See Bass (1969), Sheth (1979), Arndt (1967), Day (1971) and Richins (1983), Mobius et al (2006), Chevalier and Mayzlin (2006) Godes and Mayzlin (2003,2004), and Reichheld (2003).

percolation process describes the pattern of WOM that takes place in the social network as a function of a firm's pricing and advertising strategies, and consumer's valuations.

The paper studies a monopolist selling a good to an initially uninformed population with heterogeneous valuations for the good. Consumers are connected within the population by a social network which is modelled as a random graph with an arbitrary degree distribution. Consumers may communicate with their friends in the social network. The content of communication is to inform the receiver that the good exists. In order to purchase the good, consumers must first find out about the good and second be prepared to purchase it at the price charged by the monopolist. The analysis assumes that an infinitessimal fraction of the population become informed and the remainder of the population may only find out about the good via WOM diffusing through the social network. Later, I also consider the case where consumers may become informed from costly advertising undertaken by the monopolist.

WOM is modeled as a percolation process on the social network. Representing the social network by a random graph with arbitrary degree distributions makes the analysis of the percolation process particularly tractable and maintains a great deal of freedom in the distribution of friendships across the population. The percolation process assumes individuals are prepared to engage in WOM with a certain probability, which is a function of the individual's valuation for the good and the price charged by the monopolist. This probability is modelled by a step function, whereby the consumer only engages in WOM if she is prepared to purchase the product. When the price is zero everyone is willing to engage in WOM and the potential pathways for communication correspond to the social network, however as the price increases fewer people are prepared to engage in WOM and there are fewer pathways for communication. The connectedness of the network over which communication may take place becomes increasingly disconnected at higher prices, mitigating the effect of WOM. The analysis proceeds in two steps: firstly the formulation of WOM as a percolation process allows one to map the primitive of the model, the social network, and firm's strategy, price, to the network describing the potential communication pathways between individuals; second the demand and profit of the monopolist may then be derived as a function of this communication network, the price and level advertising.

The model is able to provide insights into how the social network affects the nature and shape of demand for a good, what pricing strategies a monopolist may use in a static and dynamic setting, the individuals to target advertising towards and how an owner of the rights to advertise to individuals within a social network can benefit from utilizing information about a consumer's relative connectivity. Demand has two regions, one at high prices, where very few people hear about the good, and another at lower prices, where there is a significant fraction of the population communicating about the good through WOM. At these lower prices demand is more elastic than demand when the population is fully informed. Estimates of consumers' valuation for the good are biased downwards and estimated counterfactual responses to price increases are overstated when WOM is ignored. Regular goods are priced below what a monopolist would charge absent WOM; however for goods whose valuation is greater for people with many friends, the price can in fact be greater. Introductory prices may include periods of sales as prices fluctuate up and down to facilitate a more effective spread of the good through the population. Increasing advertising costs may benefit consumers. Exclusive (high priced) products will optimally target advertising towards individuals with many friends, whereas common (low priced) products will target individuals with fewer friends.

1.1.1 Related literature

This paper is related to a recent economics literature which considers the optimal strategy for an outside party trying to maximize an objective which is a function of agents' actions in a social network (see for instance Goyal and Galleotti (2007), Ballester et al. (2006), Banerji and Dutta (2006) and Galleotti and Mattozzi (2008)). The most related of these is Goyal and Galleotti (2007) which considers the optimal advertising decisions of a monopolist in the presence of local information sharing and local adoption externalities. In contrast to these papers the present paper is the first to use a model of percolation to capture the pattern of communication and endogenize the probability that individuals engage in WOM as a function of their valuation for the good and monopolist's strategies. The model addresses new questions concerning the optimal strategies the monopolist employs when it can affect the diffusion rate of information, and gains fresh insights into the shape and nature of demand and the effects of diffusion of information via WOM on the pricing and advertising behavior of the monopolist.

There are also a number of papers which consider diffusion of an action or adoption decision of agents interacting in social networks. In these papers an agent's payoff is a function of the actions of agents connected to them in the social network. Some of these papers, like this paper, find that there is some critical threshold which determines whether an action or behavior will successfully propagate through a population (for instance Ellison (1993), Morris (2000), Jackson and Yariv (2007), Lopez-Pintado (2007)). In these papers the probability an agent is prepared to propagate/pass on the action/information is a function of the decisions of other agents, in this paper the focus is different, it is on the strategic decision making of an outside party, the monopolist, when it can influence this probability (also known as the percolation probability) and hence the rate and distance of diffusion.

Within the broad literature that considers percolation processes, some other papers, as this paper does, consider the spread of phenomenon on social networks which are modelled by random graphs with arbitrary degree distributions, for epidemic diseases (Newman (2002), Sander et al. (2002)) and fads/innovations (Watts (2002)). In contrast to these papers, the innovation of this paper is to endogenize the percolation probability itself by making it a function of the strategy (price) chosen by the monopolist. In doing so I am able to relate the strategy of the monopolist to the characteristics of the network and diffusion process.

1.2 Model

There is a monopolist selling a good to a population of consumers $N = \{1, .., n\}$ who have heterogeneous preferences for this good and are initially unaware that it exists. A fraction $\varepsilon \approx 0$ of these people will find out about the good exogenously, everyone else must find out about it either through WOM from one of their friends or from informative advertising undertaken by the monopolist. Consumers have a uniform valuation for the good $\theta_i \sim U[0,1]$ and they derive utility $\theta_i - P$ if they purchase the good and 0 otherwise. The individuals who desire the product will be those for whom $\theta_i \geq P$. Hence the demand for the good if the population is fully informed is $1 - P$.

The population is connected by a social network described by a graph (N, Ξ) with n nodes and a set of edges $\Xi \subseteq \{(i,j) | i \neq j \in N\}$ where an element $(i,j) \in \Xi$ indicates there is a friendship between individuals i and j. The social network considered here is an undirected network so if $(i,j) \in \Xi$ then $(j,i) \in \Xi$. Each person may engage in WOM with their friends. I assume that the probability an individual i passes on information about the good to her friends is a function $\nu(\theta, P)$ of the individual's valuation for the good and the price charged by the monopolist. Specifically:

$$\begin{aligned} \nu(\theta, P) &= 1 \text{ if } \theta_i - P \geq 0 \\ &= 0 \text{ if } \theta_i - P < 0 \end{aligned} \tag{1.1}$$

The key characteristic of this function is that it is increasing in the individual's willingness to purchase the good $\theta - P$. It is the relationship between the probability and the price, that allows the monopolist to affect the rate, and distance which WOM about the good spreads within the social network.

All consumers are initially unaware of the good, so the fraction of the population that eventually buy it is in part determined by how many people find out about it. The timing of the model is as follows:

1. Each person in the population becomes informed with independent probability $\varepsilon \approx 0$

2. Monopolist chooses a fraction of the population ω to inform directly through a costly advertising technology

3. Informed consumers purchase the product if $\theta_i \geq P$ and tell all their friends about the product through WOM according to $\nu(\theta, P)$

4. Step 3 is repeated for newly informed consumers until there are no more consumers being informed

An important part of the analysis will be to describe the social network and how this network affects the number of people who become informed about the product. The study of random graphs goes back to the influential work of Erdös and Renyí (1959, 1960, 1961). One of the key insights of Erdös and Renyí is to consider the properties of a "typical" graph in a probability space consisting of graphs of a particular type. I assume the social network is described by random graphs with an arbitrary defined degree distribution $\{p_k\}$ (as per Newman, Strogatz and Watts (2001)) where p_k represents the fraction of individuals in the population with k friends and $\sum p_k = 1$. There are several different algorithms for constructing random graphs of this type, one is the "configuration model". Consider the following formation process for the configuration model. For a given N consider forming a sequence of n numbers which are i.i.d. draws from p_k. This is known as the "degree sequence" where the ith number k_i is the number of friends of individual i. One can think of individual i as having k_i stubs of friendships to be. Stubs are then chosen at random and connected together until there are no stubs left.[2] It has been shown that this produces every possible graph with the given degree sequence with equal probability (Molloy and Reed (1995)). The configuration model is the ensemble of graphs $\Omega_{N,\{p_k\}}$ produced via this procedure and the properties derived in the analysis are for the average over this ensemble of graphs in the limit as $n \to \infty$.

I now define a number of characteristics of networks.

Definition 1 *A path exists between two individuals i and j if there exists a sequence of individuals where i is the first member of the sequence and j is the last member of the sequence such that for the $(t+1)$th member of the sequence $(t, t+1) \in \Xi$*

Using this definition of a path I define a component.

Definition 2 *A component $C(i)$ of individual i is the set $\{j | \exists$ path from i to $j\}$*

[2]This process assumes there is an even number of stubs to begin with and does not rule out two stubs from the same individual connecting to one another or multiple links existing between two individuals. Under some regularity conditions on $\{p_k\}$ the instances of own or multiple links become small in a variety of senses as the size of the network $n \to \infty$. For an excellent discussion of these issues see Jackson (2008).

The size of a component $|C|$ is the number of individuals in it. In undirected networks components are connected subsets of the population, who may all reach one another by following friendships in the network, such that $j \in C(i) \Leftrightarrow i \in C(j)$. The set of components in a network represents a partition of the set N. Denote this partition of N induced by Ξ as $\Pi(N, \Xi)$. An important part of the analysis will be the distribution of component sizes in the partition $\Pi(N, \Xi)$. Define the size of the largest component $\bar{s}$ in a graph (N, Ξ) by

$$\bar{s} = \max_{C \in \Pi(N, \Xi)} |C|$$

Definition 3 *A giant component is said to exist in a random graph with degree distribution* $\{p_k\}$ *if* $E_{\Omega_{N,\{p_k\}}}[\bar{s}] = \Theta(n)$.

In subsequent sections the question of the existence and size of a giant component in a network will be central to the analysis. In the next section I explain how to represent a social network with a probability generating function and some of the characteristics of generating functions which will become useful for deriving the distribution of component sizes.

1.3 Representing social networks with random graphs with arbitrary degree distributions

A social network with an arbitrary degree distribution given by $\{p_k\}$ can be described by a probability generating function. The probability generating function $G_0(x)$ for the social network is written as:

$$G_0(x) = \sum_{k=0}^{\infty} p_k x^k$$

This is a polynomial in the generating function argument x where the coefficient on the kth power is the probability p_k that a randomly chosen individual has k friends. Generating functions have a number of useful properties that can allow one to calculate a variety of local and global properties of the social network. A good exposition of these and the formalism for calculating various properties can also be found in Newman, Strogatz and Watts (2001). I will briefly reproduce some of them here for clarity.

Derivatives The probability p_k is given by the kth derivative of G_0 according to:

$$p_k = \frac{1}{k!} \frac{d^k G_0}{dx^k}\bigg|_{x=0}$$

Moments Moments of the probability distribution can be calculated from the derivative of the generating function. The mth moment equals:

$$\sum_k k^m p_k = \left[\left(x\frac{d}{dx}\right)^m G_0(x)\right]_{x=1}$$

Where the average degree, which I denote by z_1, is given by $z_1 = G_0'(1) = \sum_k p_k k$ and the terminology $\left(x\frac{d}{dx}\right)^m$ means repeating m times the operation: differentiate with respect to x and then multiply by x.

Powers The distribution of the sum of m independent draws from the probability distribution $\{p_k\}$ is generated by the mth power of the generating function $G_0(x)$. For example, if I choose two individuals at random from the population and sum together the number of friends each person has then the distribution of this sum is generated by the function $[G_0(x)]^2$. To see this, consider the expansion of $[G_0(x)]^2$:

$$\begin{aligned}
[G_0(x)]^2 &= \left[\sum_k p_k x^k\right]^2 \\
&= \sum_{j,k} p_j p_k x^{j+k} \\
&= p_0 p_0 x^0 + (p_0 p_1 + p_1 p_0) x^1 \\
&\quad + (p_0 p_2 + p_1 p_1 + p_2 p_0) x^2 \\
&\quad + (p_0 p_3 + p_1 p_2 + p_2 p_1 + p_3 p_0) x^3 \ldots
\end{aligned}$$

In this expression the coefficient of the power of x^l is the sum of all products $p_k p_j$ such that $k+j = l$ and is thus the probability that the sum of the degrees of the two individuals will be l. This property can be extended to any power m of the generating function.

The distribution of the number of friends of a person found by following a randomly chosen friendship will be important in the analysis to come. This is not the same as the distribution of the number of friends of a person chosen at random because people with many friends are more likely to be found, when selected in this way, since they have more friendships. A person with k friends is k times more likely to be found than a person with 1 friend. Therefore the probability of finding a person with k friends is proportional to kp_k. After the correct normalization the generating function for this distribution is:

$$\frac{\sum_k kp_k x^k}{\sum_k kp_k} = x\frac{G_0'(x)}{z_1}$$

Now consider choosing a person randomly and looking at each of her friends. Then for each friend, the distribution of the number of friendships these people have, which do not lead back to the originally chosen person (this is $k-1$ if the friend has k friends themselves since one must lead back to original individual chosen), is generated by the function $G_1(x)$:

$$G_1(x) = \frac{G_0'(x)}{z_1}$$

The assumption that friendships between individuals are independent of one another means that as the network becomes large, $(n \to \infty)$, then the probability that any of the neighbors also know one another goes as n^{-1} and can be ignored in the limit of large n. Making use of the powers property of generating functions the probability distribution of second neighbors of the individual is given by:

$$\sum_k p_k [G_1(x)]^k = G_0(G_1(x))$$

In the analysis I utilize results regarding the robustness of random graphs with arbitrary degree distributions. Part of the analysis will consider the resultant network of individuals when the individuals who do engage in WOM are removed according to

$\nu(\theta, P)$ (those people with $\theta_i - P < 0$)[3]. This is equivalent to a percolation problem on a random graph. A depiction of this process is given in Figure 1-1. Consider the original social network shown in Figure 1-1 where the individuals who desire the product ($\theta_i \geq P$) are represented as the black nodes. The process of percolation takes one from this network to the network of WOM on the right where only the friendships between individuals who are willing to engage in WOM is shown.

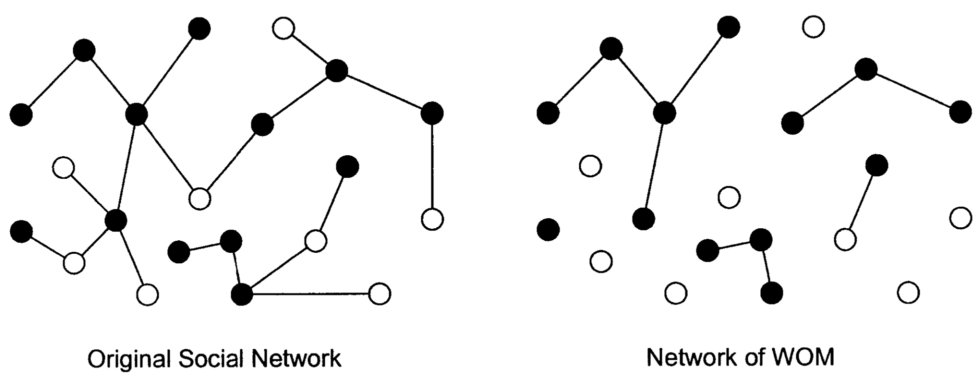


Figure 1-1: Percolation process

In general a giant component does not always exist in either network. If it exists in the network of WOM then it will in the original social network but not vice versa. One of the most important and well studied topics in the random graph literature is characterizing the conditions under which the giant component does or does not exist when a parameter(s) describing the network is varied. Typically the cases, also known as phases, where the giant component does and does not exist, are separated by a critical threshold. In the model developed here this quantity will be the price which affects the probability individuals will engage in WOM. When the giant component does exist in the network of WOM every individual who knows someone in the giant component will become informed about the good. This will occur as a result of the tiny fraction ε of the population who find out about the good independently. By the law of large numbers in the limit as $n \to \infty$ at least one of these individuals will be in the giant component, and thus WOM will spread out from this person to all the people in the giant component.

[3]This process of percolation is a variant of the Reed-Frost model in the epidemiology literature

The network of WOM is also a random graph I denote the probability space of these graphs by $\Omega_{N,\{p_k\}P}$. The methodology for describing the network of WOM that I use in this paper was developed in Callaway et al. (2000). For expositional purposes I reproduce part of their analysis here to derive the probability generating function for this second network in terms of the first network and the probability that a person engages in WOM. To this end let q_k be the probability that an individual with k friends is willing to engage in WOM. Note this allows for some correlation between θ_i and the number of friends of individual i. The product $p_k q_k$ is the probability that a randomly chosen individual has k friends and is willing to engage in WOM. The probability generating function $F_0(x)$ for this distribution of people is given by:

$$F_0(x) = \sum_{k=0}^{\infty} p_k q_k x^k \tag{1.2}$$

where $\frac{\partial^k}{\partial x} F_0(0) = p_k q_k$ and when $\nu(\theta, P)$ is given by equation 1.1 $F_0(1) = 1 - P$ which is the fraction of the population with valuations $\theta_i \geq P$. If we again consider following a randomly chosen friendship from the social network, the individual we reach has degree distribution proportional to kp_k rather than just p_k. Hence the probability generating function that an individual has k friends and also desires the good, when she is chosen by randomly following a friendship is:

$$F_1(x) = \frac{\sum_k k p_k q_k x^{k-1}}{\sum_k k p_k} = \frac{F_0'(x)}{z} \tag{1.3}$$

1.3.1 Distribution of component sizes

Now let $H_1(x)$ be the generating function for the probability that one end of a randomly chosen friendship from the original social network in Figure 1-1 leads to a component of a given size in the network of WOM. Denote the probability that this component is of size s by h_s'. The component may in fact be empty if the individual at the end of the friendship has $\theta_i < P$ which occurs with probability $1 - F_1(1)$ or the individual may purchase the good and have k friends (distributed according to $F_1(x)$) any of whom may also purchase the good. Note that the giant component, if there is one, is *excluded* from $H_1(x)$. When component sizes are finite the chances of a finite component containing a closed loop goes as n^{-1} which becomes negligible as n becomes large. This means that the distribution of components can be represented as in Figure 1-2 where each component is represented as a tree like structure consisting of the single individual reached at the end of the randomly chosen friendship plus any number, including zero, of other tree-like structures.

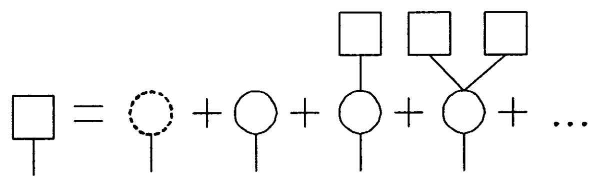


Figure 1-2: Schematic representation of the sum rule for components found by following a randomly chosen friendship

Therefore $H_1(x)$ must satisfy the following self consistency condition:

$$\begin{aligned} H_1(x) &= 1 - F_1(1) + xq_0p_0 + xq_1p_1[H_1(x)] + xq_2p_2[H_1(x)]^2 + \ldots \\ H_1(x) &= 1 - F_1(1) + xF_1(H_1(x)) \end{aligned} \tag{1.4}$$

If an individual is chosen randomly then there is one such component at the end of each friendship of that person. Therefore the generating function for the size of the connected components in the network of WOM that a randomly chosen individual belongs, is similarly generated by a function $H_0(x)$ which satisfies:

$$H_0(x) = 1 - F_0(1) + xF_0(H_1(x)) = \sum_{s=0}^{\infty} h_s x^s \tag{1.5}$$

where h_s is probability a randomly chosen individual from the population belongs to a component of size s. These four relationships, equations 1.2, 1.3, 1.4 and 1.5, determine the distribution of the sizes of the connected groups of individuals who can communicate to one another about the good. The size of the giant component, if it exists, is given by the number of people not in components that are of finite size, $1 - H_0(1)$.

The analysis proceeds in the next section by making assumptions about the correlation between the probability a person has k friends p_k and the individual's valuation θ, and how the probability q_k that individuals pass on information about the good is related to the individual's valuation and the price charged by the monopolist. Denoting the joint distribution of θ and k in the population by $\Phi(\theta, k)$ and the conditional distribution of θ given k as $\phi(\theta|k)$ then p_k and $q_k(P)$ can be calculated as

$$\begin{aligned} p_k &= \int_\theta \Phi(\theta, k)\, d\theta \\ q_k(P) &= \int_\theta \nu(\theta, P)\, \phi(\theta|k)\, d\theta \end{aligned}$$

and substituted into the above relationships to derive $F_0(x,P)$, $F_1(x,P)$, $H_0(x,P)$, $H_1(x,P)$ in terms of the price P to describe the network of WOM. One then relates this network to the subsequent demand for the good and hence the profits of the monopolist. In principle the methodology described here may accommodate a variety of functional forms for ν relating valuations, price and even advertising effort to an individuals' probability of talking about the good. Indeed any two functions $\check{\nu}, \hat{\nu}$ such that

$$\int_\theta \check{\nu}(\theta, P)\, \phi(\theta|k)\, d\theta = \int_\theta \hat{\nu}(\theta, P)\, \phi(\theta|k)\, d\theta$$

holds will result in the same level of demand. For example when θ and k are uncorrelated an equivalent specification for $\nu(\theta, P)$ would be $\nu(\theta, P) = \theta^{\frac{P}{1-P}}$.

Also the methodology can easily incorporate a probability less than 1 that an individual passes on news about the good along any individual friendship. Denoting this probability by q_b the above expressions would be unchanged except that $F_1(x) = q_b \times \frac{\sum_k k p_k q_k x^{k-1}}{\sum_k k p_k}$. For the remainder of the paper I will focus on the case where $q_b = 1$.

1.4 Demand

In this section I bring the insights of percolation processes on random graphs to bear on the characterization of demand for a good in the presence of WOM. One of the central insights from the random graph literature is that there is a critical threshold which determines whether a giant component does or does not exist. I find that absent any advertising by the monopolist, demand for the good, as measured by the fraction of the population who purchase the good, exhibits two distinct regions: one where demand is zero and the giant component does not exist, and another where demand is non-zero and the giant component exists. These two regions are separated by a critical price P^{crit} below which the giant component exists and above which it does not. Provided a giant component exists in the social network, then as prices rise demand shrinks from a positive fraction of the population continuously to a 0 fraction of the population at the critical price, where it will in general have a strictly negatively slope. In comparison to the fully informed demand curve the demand curve under WOM is more elastic. Ignoring the effect of WOM introduces a downward bias in welfare calculations and an upward bias of consumers' response to price increases after the population has become informed.

1.4.1 Critical price

The following theorem characterizes the conditions when a giant component exists in terms of a critical price below which it exists and above which it does not.

Theorem 1 *Suppose an individual's valuation is independent of the number of friends, $q_k = q = 1 - P$ for all k and $\{p_k\}$ is such that $\frac{E[k]}{E[k^2]-E[k]} < 1$, then, there exists a critical price P^{crit} such that*

$$E_{\Omega_{n,P,\{p_k\}}}[\bar{s}] = \Theta(n), \text{ and } H_1(1,P) < 1 \text{ if } 0 \leq P < P^{crit}$$

and

$$E_{\Omega_{n,P,\{p_k\}}}[\bar{s}] = O\left(n^{1/8}\log n\right), \text{ and } H_1(1,P) = 1 \text{ if } P^{crit} < P < 1$$

Moreover the critical price satisfies $1 - P^{crit} = \frac{E[k]}{E[k^2]-E[k]}$.

Proof. The result follows immediately from results on percolation thresholds in the statistical physics literature cited in the appendix. ∎

The intuition behind the result is best illustrated by considering the number of people who subsequently buy the product after an individual who is prepared to engage in WOM hears about the good from a friend. If a person has k friends then the expected number of first neighbors who are informed by this person and then purchase the good themselves will be $(1-P)(k-1)$, where it is $k-1$ because the individual hears about the good from one of her friends. Now taking the expectation over the expected number of friends of a person found by following a randomly chosen friendship is $(1-P)\frac{\sum_k p_k k(k-1)}{E[k]} = (1-P)\frac{E[k^2]-E[k]}{E[k]}$. When this quantity is greater than 1 the component will initially grow exponentially, while for values less than 1 the component will decay and die out. This is known as the reproduction rate. The critical price is the price at which this reproduction rate equals 1. Subsequently when $P < P^{crit}$ a giant component exists and when $P > P^{crit}$ it does not.

As alluded to earlier, for the monopolist to sell to a non-zero fraction of the population there needs to be a giant component. If this is not the case all of the εn individuals will belong to components whose average size is finite so total demand will be approximately a fraction ε of the population and therefore negligible as $\varepsilon \to 0$. If the giant component is of size $\Theta(n)$ then almost surely at least one of the εn individuals will belong to the giant component and thus the fraction of the population which becomes informed about the good is the fraction of people who know someone in the giant component.[4] This reasoning implies that demand exhibits two distinct regions one where the giant component is of size $\Theta(n)$ and the other where it is not, which depend on the price chosen by the monopolist. An important quantity is the probability that a friend does *not* belong to the giant component and thus does *not* become informed via WOM. If there exists a giant component of size $\Theta(n)$ then $H_1(1,P) < 1$ and the probability a person with $k > 0$ friends is informed is $1 - H_1(1,P)^k > 0$. The fraction of the population informed is therefore $\sum_k p_k \left(1 - H_1(1,P)^k\right)$.

[4]When taking the limit of limits this requires that $\varepsilon \to 0$ slower than $\frac{1}{n}$.

1.4.2 Level of demand

The first step of the analysis is to determine the fraction of the population who become informed about the product. This is the fraction of people who know someone in the giant component. The probability that the person at the end of a randomly chosen friendship does *not* belong to the giant component $H_1(1, P)$ is the smallest non-negative solution to the self consistency condition equation 1.4:

$$H_1(1, P) = 1 - F_1(1, P) + F_1(H_1(1, P), P)$$

where F_1 is now written as a function of the price P when the percolation probability is written in terms of price, $q_k(P) = \int \nu(\theta, P)\, \phi(\theta|k)\, d\theta$. The probability a person with k friends becomes informed is hence $1 - (H_1(1, P))^k$ and the total fraction of the population which is informed is:

$$\sum_k p_k \left(1 - (H_1(1, P))^k\right)$$

The second step of the analysis is to determine how many of these people purchase the product $S(P)$, this is given by:

$$S(P) = \sum_k p_k \left(1 - (H_1(1, P))^k\right) \int_P^1 \phi(\theta|k)\, d\theta$$

which given the functional form chosen for $\nu(\theta, P)$ is the size of the giant component[5]

$$S(P) = F_0(1, P) - F_0(H_1(1, P), P)$$

where again price is now an argument of F_0. Suppose there is no correlation between θ and k then $\int_P^1 \phi(\theta|k)\, d\theta = 1 - P$ for all k and $S(P)$ can be written in terms of the price P as:

$$S(P) = (1 - P) \sum_k p_k \left(1 - (H_1(1, P))^k\right) \tag{1.6}$$

[5] Callaway et al. (2000) derive this expression using the generating function approach. Molloy and Reed (1995) derive an equivalent expression for the size of the giant component using a different methodology.

It should be obvious from this expression that the difference between demand as generated here and the standard fully informed demand comes through the $H_1(1,P)$ term in equation 1.6. The distribution of valuations within the fraction of the population who find out about the product is $U[0,1]$ because the probability a person finds out about the product is independent of her own valuation as she hears about it from a neighbor. Demand is the product of the probability that an individual finds out about the good $\sum_k p_k \left(1-(H_1(1,P))^k\right)$ and the probability a person is prepared to purchase the good $(1-P)$, given the price and distribution of valuations amongst the informed individuals. When the monopolist chooses a price it influences both the fraction of the population who find out about the product $\sum_k p_k \left(1-(H_1(1,P))^k\right)$ and the proportion of these people $(1-P)$ who are prepared to purchase it.

The following example considers a Homogeneous and a Hub social network to illustrate some of the characteristics of demand with WOM. I will then formalize these for a more general class of networks later in this section. The mean degree is 3 for both networks, in the first homogeneous social network (triangles) every individual has exactly 3 friends so the generating function is $G_0 = x^3$ and in the second Hub network (asterisks) 98% of the population have 2 friends and 2% have 52 friends so $G_0(x) = 0.98x^2 + 0.02x^{52}$. The inverse demand curves are shown below along with the fully informed inverse demand $P = 1 - Q$.Decreasing the price from $P = 1$, a giant component appears first in the Hub network, where there is greater variance in the distribution of friendships, at a price $P \approx 0.94$. Demand grows relatively slowly because it is unlikely that the individuals with $\theta_i \geq P$ and 2 friends become informed when the giant component is very small. As the price falls further the giant component grows faster and the inverse demand curve appears convex in this region. When the price reaches $P = 0.5$ a giant component appears in the Homogenous network. Initially the giant component in the Homogeneous network grows very quickly compared to the Hub network because everyone has the same number of friends. In fact the giant component in the homogeneous network becomes larger than in the Hub network at $P \approx 0.45$ and at a price of 0.3 it contains approximately 40% more individuals. This difference is driven by the relative likelihood of a person with 3 friends versus 2 friends becoming informed in this range of prices. Eventually the giant

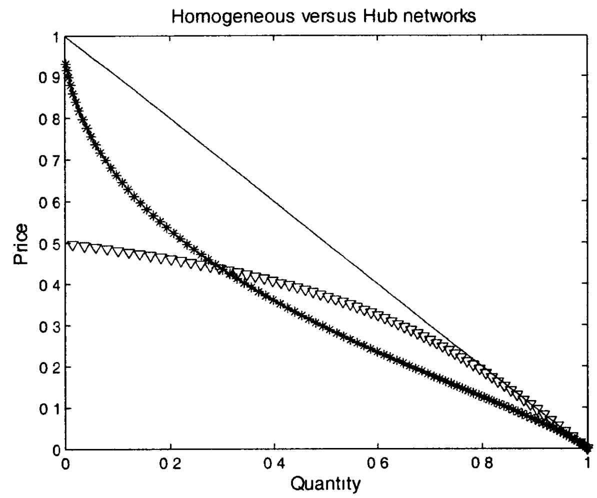


Figure 1-3: Homogeneous versus Hub networks

component in the Homogenous network consists of almost all individuals for whom $\theta_i > P$ so $S \approx 1-P$ and it can only grow at the rate at which new people are willing to purchase the product for a given price change. Since both networks are fully connected eventually the giant component in the Hub network also approaches $1-P$ and for both networks $S=1$ at $P=0$.

The following theorem characterizes demand as price varies absent any direct advertising by the monopolist.

Theorem 2 *Suppose θ and k are uncorrelated then demand for the good $S(P)$ is*

1. *Continuous*

2. $S(P) = 0$ *for* $P \geq P^{crit}$
   $S(P) > 0$ *for* $P < P^{crit}$

3. $\frac{dS}{dP} = 0$ *for* $P \geq P^{crit}$
   $\frac{dS}{dP} < 0$ *for* $P < P^{crit}$

4. $\lim_{P \to P^{crit-}} \frac{dS}{dP} = -\left(1 - P^{crit}\right) \frac{G_0'''(1)}{\left(G_0''(1)\right)^2} < 0$

5. $\left|\frac{P}{S}\frac{dS}{dP}\right| > \left|\frac{P}{1-P}\right|$ *for* $P < P^{crit}$

Proof. See appendix. ∎

This theorem establishes that demand is continuous in price and that at the critical price the slope of demand makes a discontinuous change from zero in the region $P > P^{crit}$ to a strictly negative amount at P^{crit}. This change in the growth rate distinguishes the two regions of demand. This change in the behavior of demand does not come as a result of the fully informed demand having a negative slope at $P = 1$. Indeed provided that a fraction $1 - P^{crit}$ of the population have valuations θ greater than P^{crit} and valuations are locally distributed uniform with density 1 around P^{crit} the above theorem will continue to be true. This means that the fully informed demand may in fact asymptote to 0 as P increases such that for the inverse demand curve $\lim_{Q \to 0} \frac{dP}{dQ} = 0$ and the theorem will be unchanged. The elasticity of demand when there is WOM $\frac{P}{S}\frac{dS}{dP}$ is:

$$\frac{P}{S}\frac{dS}{dP} = \frac{-P}{1-P}\left[1 + \frac{(1-P)}{1 - \sum_k p_k H_1(1,P)^k}\frac{dH_1(1,P)}{dP}\sum_k p_k k H_1(1,P)^{k-1}\right]$$

which is the fully informed elasticity adjusted by a factor $1 + \frac{(1-P)}{1-\sum_k p_k u^k}\frac{du}{dP}\sum_k p_k k u^{k-1}$ where the second term comes from the increase in connectivity of the network from lowering the price. This is new customers, with $k \geq 2$, forming a bridge to the giant component to connect previously disjoint components of individuals.

1.4.3 Biases in estimates which ignore WOM

One can imagine using cross-sectional data to non-parametrically identify the relationship for $S(P)$. In this section I find that failing to recognize the effects of WOM in generating this demand may lead to several of biases. The first is a downward bias in welfare calculations of consumer surplus of the form $\int_{\tilde{P}}^{\infty} S(P)\,dP$.

Corollary 1 *Suppose θ and k are uncorrelated and the price of the good is $\tilde{P}$ then an estimate of consumer surplus $\widehat{CS}\left(\tilde{P}\right) = \int_{\tilde{P}}^{\infty} S(P)\,dP$ is biased downwards.*

Proof. See Appendix. ∎

An estimate of the valuations of consumers who purchase the product based on the demand curve $S(P)$ understate the valuations of the *purchasing consumers*. It is obvious that the population being uninformed leads to fewer consumers purchasing the product than if they were fully informed, however this corollary implies that even amongst the consumers who do find out about the product and purchase it, an estimate based on $S(P)$ of *their* valuations will be biased downwards. The reason is that the marginal consumers at a price of P are a combination of individuals who know about the product and have valuations $\theta \approx P$, and consumers in previously disjoint components with valuations $\theta \sim U[P,1]$ who become informed via one of the consumers with $\theta \approx P$. Failing to recognize that demand changes through this second channel induces a downward bias in estimates of consumer valuations because it attributes a valuation of $\theta \approx P$ to a group of consumers with valuations $\theta \sim U[P,1]$. Thus welfare calculations such as evaluating the introduction of a new good will understate the consumer surplus.

A second bias may occur when considering how consumers will respond to an increase in price once WOM has diffused.

Corollary 2 *Suppose θ and k are uncorrelated and the price of the good is $\tilde{P}$ then an estimate $\Delta\hat{S}$*

$$\Delta\hat{S} = S\left(\tilde{P}\right) - S\left(\tilde{P} + \Delta P\right)$$

of the consumer response to a price increase ΔP overstates the actual response ΔQ

$$\Delta\hat{S} < \Delta Q$$

Proof. See Appendix ■

The distribution of valuations are distributed $U[0,1]$ across those people who are informed about the product. An increase in the price by ΔP will change demand by $\frac{\Delta P}{1-P}\%$ however an estimate based on $S(P)$ overstates the elasticity with respect to price of the consumer's preferences for the product and will predict a greater response. A monopolist choosing to increase its price or a policy maker introducing a tax will estimate a larger change in demand than what would actual take place.

1.5 Static pricing

In this section I study the optimal static pricing decision of the monopolist. For regular goods, where valuations and number of friends are uncorrelated, I show that the monopolist will set a lower price when there is WOM compared to when consumers are fully informed. However, for goods where there is significant positive correlation between valuation for the good and an individual's number of friends then the monopolist may in fact price above the fully informed level. When the monopolist can price discriminate between consumers based on numbers of friends, then better connected individuals are charged lower prices.

1.5.1 Regular goods

The first result in this section is that the monopolist will set a lower price when there is WOM compared to when consumers are fully informed.

Theorem 3 *Suppose valuations and number of friends are uncorrelated and marginal costs* $c < 1$*, then a monopolist facing demand given by* $S(P)$ *charges a lower price* P^*_{WOM} *than a monopolist facing a fully informed population* P^*_{FI}*, where demand is given by* $Q(P) = 1 - P$*.*

Proof. See appendix. ■

This theorem comes as an immediate consequence of demand being more elastic under WOM in Theorem 2. The WOM monopolist has an additional incentive to stimulate demand through the word of mouth channel and will lower prices below the price that would be charged by the monopolist facing a fully informed population. The effect can be so large that consumers may in fact be better off being uninformed than fully informed.

Corollary 3 *Consumer surplus may be greater when consumers are uninformed and the monopolist charges* P^*_{WOM} *than if consumers are fully informed and the monopolist charges* P^*_{FI}*.*

Proof. See appendix ■

This proposition illustrates that consumers may in fact be better off when they are uninformed because the monopolist lowers the price below P^*_{FI} to stimulate word of mouth in the population. The gains to consumers may in fact be quite significant, for instance in a social network where everyone has 3 friends the consumer surplus is 65% larger in the WOM setting than the fully informed setting. Of course a social networks which does not have a significant fraction of the population in the giant component at any price it will no longer be true.

1.5.2 Correlation between valuations and number of friends

For goods where there is significant correlation between the connectivity of individuals and their valuation for the product, in contrast to Theorem 3, it can be the case that the monopolist will charge a price higher than it would if everyone is informed. When there is significant positive correlation, the network of WOM is much better connected at higher prices than a network with no correlation. The following proposition illustrates a case where significant positive correlation leads to prices above the fully informed monopoly price $\frac{1+c}{2}$.

Theorem 4 *If $P^{crit} > \underline{\theta}$ and all consumers with $\theta \in [c, \underline{\theta}]$ have $k = 1$ then the monopoly price will be greater than the fully informed monopoly price $\frac{1+c}{2}$.*

Proof. See Appendix ■

The intuition for this result is that when the mix of marginal consumers has a large fraction of individuals with low connectivity then demand will be relatively inelastic. In this theorem the mix contains only individuals with 1 friend. These consumers can not provide a bridge to connect components which are disjoint from the giant component for $c \leq P \leq \underline{\theta}$, thus demand is relatively inelastic compared to the fully informed demand over the range of prices $P \in [c, \underline{\theta}]$ and the monopolist will not price at or below the fully informed monopoly price $\frac{1+c}{2}$.

The types of goods which would naturally have some correlation between valuation and the number of friends are fashion and status products, where the value is, at least in part, increasing in the consumer's ability to display them to others. The example given here suggests that these types of goods will receive a higher mark up than other types of goods all else equal.

1.5.3 Price discrimination

When the monopolist can discriminatingly price to consumers with different numbers of friends, the optimal set of prices will be decreasing in the number of friends each person has. For the monopolist there is a greater incentive to offer a lower price to individuals with more friends because these individuals are the most effective at informing others. When the monopolist decreases the price to one of the groups it can increase the number of people informed of all groups through WOM.

Monopolist's maximization problem when it can discriminate between consumers with different numbers of friends is:

$$\pi\left(\{P_k\}\right) = \max_{\{P_k\}\in[0,1]^n} \sum p_k q_k \left(1 - H_1\left(1, \{P_k\}\right)^k\right)\left(P_k - c\right) \tag{1.7}$$

where the value of $H_1\left(1, \{P_k\}\right)$ is now a function of the set of prices $\{P_k\}$.

Theorem 5 *If valuations and number of friends are uncorrelated and* $\exists\{P_k\}$ *such that* $\pi\left(\{P_k\}\right) > 0$ *then the optimal set of prices are* $P_1 = \frac{1+c}{2}$ *and* $\exists \underline{k} : \{P_k\}$ *is decreasing for* $2 \leq k \leq \underline{k}$ *and* $P_k = 0$ *for* $k \geq \underline{k}$.

Proof. See appendix. ■

The proof considers the complementarity of demand from the different groups of consumers. In fact the problem is equivalent to a multiproduct monopolist's problem where $p_k q_k\left(1 - u^k\right)$ in equation 1.7 is the demand for good k and the demands for each good are complementary through the value of $u\left(\{P_k\}\right)$. When marginally adjusting a price P_k the monopolist faces the usual pricing incentives over the informed population $(1 + c - 2P_k)$ plus the impact of changing the price on the size of the informed population through $u\left(\{P_k\}\right)$. The relative trade-off between these two effects is proportional to $\frac{k\left(1-u^{k-1}\right)}{1-u^k}$ which is increasing in the number of friends k. Hence P_k is decreasing in the number of friends. In fact it can be profitable to give the good away for free to individuals with sufficiently many friends because of the size of their influence on demand from individuals with fewer friends.

This is a very intuitive result that offering discounts to the individuals who are best able to spread news about the good increases the profits of the monopolist. As discussed earlier the individuals with a large number of friends are very influential because these individuals are both more likely to hear about the good and able to inform more people. There have been a number of authors who have emphasized the importance of market mavens for spreading information about products (for instance Feick and Price (1987) and Gladwell (2000)). Interpreting market mavens as people who are able to influence many people within the social network then this theorem underlines the importance of providing a discount to these types of consumers because of the significant complementarity between their choice to buy the product and the total number of people who hear about it.

1.6 Introductory pricing

In this section I find that introductory pricing involves periods of sales. The monopolist increases and decreases the price of the good to optimally diffuse news of the good in the population. The trade off facing the monopolist is to sacrifice immediate profits to facilitate greater WOM today and a larger population of informed consumers in the future. The natural intuition in this situation, is that the dynamic sequence of prices will be increasing because as more and more people become informed there is less incentive for the monopolist to keep the price below the monopoly level. I show that this not necessarily the case for prices during the early stages of diffusion of WOM.

I will assume the good is non-durable to avoid the added complexity of strategic purchasing decisions by consumers. In this section I also assume for tractability that the marginal cost is 0 and that valuations and number of friends are uncorrelated. In each period consumers who know about the good will purchase it if $\theta_i \geq P_t$. In the first period, $t = 0$, a small number of people M_0, hear about the good and decide whether to purchase it at the price P_0. Those that purchase the good tell their friends, who are then added to the total population of informed consumers in the next period denoted M_1. In this way M_t grows over time. The current period payoff can be written as $M_t P_t (1 - P_t)$ where $1 - P_t$ represents the distribution of valuations ($\theta \sim U[0,1]$) across this population. The distribution of valuations within M_t does not change because becoming informed via WOM from a friend is independent of an individual's own valuation. Hence it is a random draw from the distribution of valuations within the population. The change from one period to the next $M_{t+1} - M_t$ comes through the number of people who purchase the good for the first time during period t and then tell their friends about it. The number of people who know about the good, but have never purchased it, are the conduit for this change. I will denote this population of people by R_t and the distribution of valuations in it by $F_t(\theta)$. Unlike the distribution of valuations across M_t, $F_t(\theta)$ may change as M_t grows. When a person in R_t purchases the good that person will not be in R_{t+1}, since they have now purchased the good, however all of their friends, who are now informed via WOM, will be in R_{t+1}, since they are now informed about the good but are yet to have purchased it. If a person is in R_t but does not purchase the good during period t, $\theta < P_t$, then that person will also be in R_{t+1}. Thus after a sequence of prices $P > 0$ a stock of people with low valuations can build up in R_t. Depending on the sequence of prices the distribution of valuations within R_t changes.

In general the number of friends a person tells, when they purchase the good for the first time, is a function of the time since the good was introduced and the size of the informed population by that point in time. Individuals with many friends will find out about the good earlier than those with few so over time the next individual informed will have fewer friends. When a large fraction of the total population knows about the product there is a probability that more than one of their friends have already found out about the product from someone else in the past or the current period. The transition M_t to M_{t+1} is a stochastic process and depends on the distribution of both valuations and number of friends of individuals within R_t. Characterizing how this distribution and R_t evolve over time is a complicated problem. To illustrate why a monopolist may increase and decrease the price over time I will consider a simplified problem to avoid a number of the complexities that occur in the more general setting.

I will focus on a branching problem which assumes that the market is a mass of people M_t which can grow without bound such that it never consists of a significant fraction of the population. This is of course unrealistic over long time horizons since, if the market continues to grow, at some point it will be bound by the size of the population. Notwithstanding this, it does allow a much more tractable characterization of the problem, which I argue is a reasonable approximation of behavior close to when the product is first introduced and characterizes the incentives the monopoly faces for introductory prices. This setting allows one to characterize how the change of valuations within R_t can lead the monopolist to increase and decrease the price over time.

1.6.1 Infinite horizon branching problem

At the start of period 0 a unit mass $M_0 = 1$ of individuals find out about the good. During each period the monopolist chooses prices $\{P_0, P_1, P_2...\}$ and in each period the mass of informed individuals M_t chooses whether or not to purchase the good. The monopolist faces a trade off between making profits over the existing population of informed individuals and lowering the price to sell to a greater number of individuals in R_t, thereby increasing the mass of informed individuals tomorrow. The expected number of individuals who become informed when a member of R_t purchases the good for the first time is the reproduction rate $G_1'(1) = \frac{z_2}{z_1}$. The growth rate conditional on price is deterministic because I have assumed M_0 is a unit mass of consumers.

In this problem there are three state variables and one control variable. The state variables are the number of people informed of the good M_t, the number of people who are both informed about the good but are yet to purchase it R_t and the distribution of valuations within these people F_t. The control variable is the price in each period P_t.

Reducing the number of state variables

In this section I reduce the number of state variables from three to two by considering the ratio of individuals who are informed but have never purchased to those that are informed, this is $\frac{R_t}{M_t}$. I assume that the set of individuals in M_0 are found in such a way that $\frac{R_0}{M_0} = \frac{z_2 - z_1}{z_2}$. Consider how $\frac{R_0}{M_0}$ changes when someone in R_0 purchases the good, the change of the state variables M_0 and R_0 are $\Delta M_0 = \frac{z_2}{z_1}$ and $\Delta R_0 = \frac{z_2}{z_1} - 1$. The reproduction rate $\frac{z_2}{z_1}$ is the expected number of additional people who become informed ΔM_t when a person in R_t purchases the good, and $\frac{z_2}{z_1} - 1$ is the number of additional people in R_t when this happens ΔR_t (the -1 comes from the purchasing individual no longer being in R_t after purchasing). Therefore the new ratio is

$$\begin{aligned} \frac{R_t + \Delta R_t}{M_t + \Delta M_t} &= \frac{\frac{z_2 - z_1}{z_2} M_0 + \frac{z_2}{z_1} - 1}{M_0 + \frac{z_2}{z_1}} \\ &= \frac{z_2 - z_1}{z_2} \end{aligned}$$

thus as more and more individuals purchase, the ratio $\frac{R_t}{M_t}$ remains constant. Using this relationship I can eliminate one state variable which I choose to be R_t.

Characterizing the transition functions

In this section I characterize the transition functions for both M_t and F_t which I denote Γ_M and Γ_F respectively. The population of informed individuals next period M_{t+1} is the population last period M_t plus the number of people who hear about the good through WOM from the consumers in R_t. This relationship is:

$$M_{t+1} = M_t + R_t \left(1 - F_t \left(P_t\right)\right) \frac{z_2}{z_1}$$

Using the relationship $\frac{R_t}{M_t} = \frac{z_2 - z_1}{z_2}$and substituting this into the transition function for M_t :

$$\begin{aligned} M_{t+1} &= \left(\left(1 - F_t \left(P_t\right)\right) \frac{z_2}{z_1} + F_t \left(P_t\right) \right) M_t \\ &= \Gamma_M \left(M, F, P\right) \end{aligned}$$

The distribution of valuations across the set of people yet to purchase R_t will depend on the distribution the previous period and the price in the previous period. The cumulative distribution function this period F_t (with associated pdf f_t) will be a weighted combination of the distribution last period f_{t-1} truncated at P which is the set of people in R_{t-1} who didn't buy last period $(F_{t-1}(P_{t-1})\,R_{t-1})$ and a uniform distribution over the newly informed people $(1-F_{t-1}(P_{t-1}))\left(\frac{z_2}{z_1}-1\right)R_{t-1}$. The relative weights for each are

$$\frac{1}{1+(1-F_{t-1}(P_{t-1}))\left(\frac{z_2-z_1}{z_1}\right)}$$

for f_{t-1} and

$$\frac{\frac{z_2}{z_1}(1-F_{t-1}(P_{t-1}))}{1+\left(\frac{z_2}{z_1}-1\right)(1-F_{t-1}(P_{t-1}))}$$

on the uniform. Thus the transition function for F_t is

$$\begin{aligned} F_{t+1}(\theta) &= \frac{\min\left[F_t(\theta), F_t(P_t)\right] + \frac{z_2}{z_1}(1-F_t(P_t))\,\theta}{1+\left(\frac{z_2}{z_1}-1\right)(1-F_t(P_t))} \\ &= \Gamma_F(F, P) \end{aligned}$$

Define $\mathcal{F}$ as the set of continuous cdfs on $[0,1]$ which satisfy $\frac{F(x)-F(x-\delta)}{\delta} \leq \frac{z_2}{z_2-z_1}$.

Lemma 1 *If $F \in \mathcal{F}$ then $\Gamma_F(F,P) \in \mathcal{F}$.*

Proof. See appendix ■

This lemma bounds the density of valuations in R_t above and is used to establish the continuity of the mapping Γ_F.

Lemma 2 *$\Gamma_M : [1,\infty) \times \mathcal{F} \times [0,1] \rightarrow [1,\infty)$ and $\Gamma_F : \mathcal{F} \times [0,1] \rightarrow \mathcal{F}$ are continuous mappings*

Proof. See appendix ■

The transition functions are single valued mappings and their continuity helps ensure the problem is well behaved. The following lemma derives the limiting distribution of F_t for a constant price P^*

Lemma 3 *If $P_t = P^* < P^{crit}$ for all t and $F_t \in \mathcal{F}$ then the limiting distribution $f^*(\theta) = \lim_{t\to\infty} f_t(\theta)$ will be*

$$\begin{aligned} f^*(\theta) &= \frac{z_2}{z_2 - z_1} \text{ if } \theta < P^* \\ &= \frac{z_2 - \frac{z_1}{1-P^*}}{z_2 - z_1} \text{ if } \theta \geq P^* \end{aligned}$$

Proof. See appendix ■

Given a distribution f_t and price P_t then $\frac{df_t(\theta)}{dt} > 0$ if $f_t^*(\theta) > f_t(\theta)$ and $\frac{df_t(\theta)}{dt} < 0$ if $f_t^*(\theta) < f_t(\theta)$ and $\frac{df_t(\theta)}{dt} = 0$ if $f_t(\theta) = f^*(\theta)$ for all θ. The key characteristic of this problem is that there is a discontinuity in the incentives between marginally increasing vs marginally decreasing the price above and below P^*. When the monopolist charges a price greater than zero there is a stock of people who know about the good but are yet to purchase it. This stock is the difference between the density $f^*(\theta)$ at $\theta < P$ compared to $\theta \geq P$. I show in the following section that it is this characteristic which leads the monopolist to increase and decrease the price over time.

1.6.2 Introductory pricing problem

The monopolist's problem is the following:

$$\begin{aligned} J(M_0, F_0) &= \max_{\{P_t\}} \sum_{t=0}^{\infty} \beta^{t-1} P_t (1 - P_t) M_t \\ &\quad st \\ M_{t+1} &= \left((1 - F_t(P_t)) \frac{z_2}{z_1} + F_t(P_t) \right) M_t \\ F_{t+1}(\theta) &= \frac{\min[F_t(\theta), F_t(P_t)] + \frac{z_2}{z_1}(1 - F_t(P_t))\theta}{1 + \left(\frac{z_2}{z_1} - 1\right)(1 - F_t(P_t))} \\ M_0 &= 1 \\ F_0 &= \theta \end{aligned}$$

I make the following assumption about the network structure and discount factor $\beta < \frac{z_1}{z_2} < \frac{1}{2}$ so that the problem is well posed.

The problem is an optimal control problem where the state is an element of $(M, F) \in [1, \infty) \times \mathcal{F}$ and the control is the price $P \in [0, 1]$. Writing it recursively:

$$V(M, F) = \max_{P \in [0,1]} P(1 - P) M + \beta V(M', F')$$

subject to

$$\begin{aligned} M' &= \Gamma_M(M, F, P) \\ F' &= \Gamma_F(F, P) \end{aligned}$$

Theorem 6 *The monopolist's problem has a unique solution, the value function is continuous and homogeneous of degree 1 in M and the policy function $P(F)$ is upper hemicontinuous and only a function of the state F.*

Proof. See appendix ■

A brief outline of the argument is as follows. The proof proceeds by defining a contraction mapping T :

$$(TV)(M, F) = \max_{\substack{P \in [0,1] \\ M' = \Gamma_M(M,F,P) \\ F' = \Gamma_F(F,P)}} P(1 - P) M + \beta V(M', F')$$

and looking for a solution in the space of continuous functions $V : [1, \infty) \times \mathcal{F} \rightarrow \mathbb{R}$ which are bounded in the norm

$$\|V\| = \max_{\substack{F \in \mathcal{F} \\ M=1}} V(M, F)$$

Letting $H(M, F)$ be the space of these functions. Then the maximization is for a continuous function over a compact set $P \in [0, 1]$ so the maximum exists. Then from the Theorem of the Maximum (Berge 1963) the maximum is continuous and from the homogeneity of the problem with respect to M the contraction T maps $H(M, F) \rightarrow H(M, F)$. Using the contraction mapping theorem the contraction has a unique fixed point which satisfies the recursive relationship. The properties of the policy function then follow immediately from the theorem of the maximum and homogeneity of the value function with respect to its first argument.

The value function is linear in M and the policy function is only a function of the distribution of valuations in the set of people who are informed but yet to purchase the good. I am able to further characterize the dynamic set of prices in the following theorem which highlights the incentives of the monopolist to increase and decrease the price over time.

Theorem 7 *$\nexists T$ such that for all $t > T$ the optimal price sequence $\{P_t^*\}$ is weakly increasing or decreasing.*

Proof. See appendix ■

The argument is a proof by contradiction. I first show that the optimal prices $P_t^* \in [0, \frac{1}{2}]$ and that if $\{P_t^*\}$ is weakly increasing or decreasing then the sequence will converge to a price $P^* \in [0, \frac{1}{2}]$. In this case F_t will converge to the distribution F^* given in lemma 3. This distribution is kinked at the price P^* where the density is discontinuous. The contradiction comes from considering deviations $P_t + \delta$ and $P_t - \delta$. The growth rate $\frac{M_{t+1}}{M_t}$ is $\left((1 - F_t(P_t)) \frac{z_2}{z_1} + F_t(P_t)\right)$ thus the marginal change in growth rate is proportional to $\lim_{P \to P_t^+} f^*(P_t)$ for $P_t + \delta$ and $\lim_{P \to P_t^-} f^*(P_t)$ for $P_t - \delta$. The kink in F^* means that $\lim_{P \to P_t^+} f^*(P_t) < \lim_{P \to P_t^-} f^*(P_t)$. The contradiction then comes from showing that for a small enough δ one of the two deviations is profitable.

This theorem shows that the monopolist will increase and decrease the price over time. One can gain an intuition for the result from the proof. The proof by contradiction assumes the price remains approximately constant, when this occurs for a period of time there is a stock of individuals with valuations slightly below the price who know about the good but are yet to purchase it. At some point in time it becomes worthwhile for the monopolist to drop the price to allow these consumers to purchase the product and subsequently inform their friends. If this is not the case then the monopolist can profit from increasing the price. This provides an intuitive explanation of sales whereby the benefit of the sale is reaped in future periods from the increased WOM it induces. This theory of sales is a rather natural one, the sale generates greater future demand through the additional WOM from people who wouldn't normally purchase the good.

1.7 Advertising

In this section I study the advertising decision of the monopolist by allowing it to engage in informative advertising. Advertising allows the monopolist to spread news of the good to individuals in components outside the giant component. I find that in the presence of WOM, marginal returns to advertising exhibit a peak at the critical price, a monopolist selling an exclusive (high price) good will target advertising at individuals with many friends whereas a monopolist selling a common (low price) good will target advertising at individuals with relatively fewer friends, and an owner of the rights to advertise to people within the social network will optimally allocate advertising for exclusive products to well connected individuals and advertising for common products to less well connected individuals.

Throughout this section I will talk about the returns to advertising not in terms of profit or revenue but rather in terms of how many consumers a specified level of advertising attracts to the product. The effects of direct advertising can be thought of as striking entire components of individuals within the network of WOM represented by F_0 and H_0. Whenever anyone within a component of individuals finds out about the product, the entire component becomes informed via WOM as members of the component pass on news about it. The marginal returns from increasing the level of advertising are the number of additional consumers found by advertising to another individual chosen at random from the population of people not already advertised to. This can be thought of as a traditional advertisement where ω (fraction of the population) represents the level of exposure it gets in the population. For a given level of advertising, the marginal returns from advertising can be written as a function of the distribution of component sizes, where $h_s(P)$ is the probability an individual chosen at random belongs to a component of size s, for a given price P. When the level of advertising is ω the probability that the next person advertised to belongs to a component of size s, which has not already been found via advertising (none of the other members of the component have been advertised to) is $h_s(P) \times (1-\omega)^{s-1}$ where $(1-\omega)^{s-1}$ is the probability that no one else in the component has been advertised to as well. The marginal return is therefore:

$$\sum_s sh_s(P)\,(1-\omega)^{s-1} = H_0'\,(1-\omega,P)$$

and the aggregate return is:

$$\begin{aligned} & \int_0^w H_0'\,(1-\omega,P)\,d\omega \\ = \;& H_0\,(1,P) - H_0\,(1-\omega,P) \\ = \;& 1 - S\,(P) - H_0\,(1-\omega,P) \end{aligned}$$

Assuming a constant cost per unit of advertising α and marginal cost of production c, the monopolist's profit is defined by

$$\pi\,(P,\omega) = (P-c)\,(1 - H_0\,(1-\omega,P)) - \alpha\omega$$

Theorem 8 *For all* $(\omega,P) \in [0,1]^2 \setminus (0,P^{crit})$, $\pi\,(P,\omega)$ *is continuous and differentiable with respect to both price and advertising, and* $\lim_{(\omega,P)\to(0,P^{crit})} \pi\,(\omega,P) = 0.$

Proof. See appendix ■

Corollary 4 *If* $\pi\,(\omega',P') > 0$ *for some* (ω',P') *then* $\exists\varepsilon > 0$ *such that for all* $(\omega,P) \in B_\varepsilon\,(0,P^{crit})$ *where* B_ε *is an open ball* $\pi\,(\omega,P) < \pi\,(\omega',P')$.

Proof. See appendix ■

This theorem and corollary mean that if we find $(\omega^*,P^*) \in [0,1]^2 \setminus B_\varepsilon\,(0,P^{crit})$ which maximizes $\pi\,(\omega,P)$ then this is the optimal strategy for the monopolist. The set $[0,1]^2 \setminus B_\varepsilon\,(0,P^{crit})$ is compact and $\pi\,(\omega,P)$ is continuous so the optimal strategy exists, and we can apply the theorem of the maximum to the problem hence $\pi\,(\alpha)$ is continuous and $(\omega\,(\alpha),P\,(\alpha))$ is upper hemicontinuous. Necessary conditions for the optimal price and level of advertising are

$$(1 - H_0\,(1-\omega^*,P^*)) - (P^*-c)\left.\frac{\partial H_0}{\partial P}\right|_{(1-w^*,P^*)} \leq 0$$

and

$$(P^{*} - c)\ H_0'(x,P)|_{(1-w^{*},P^{*})} - \alpha \leq 0$$

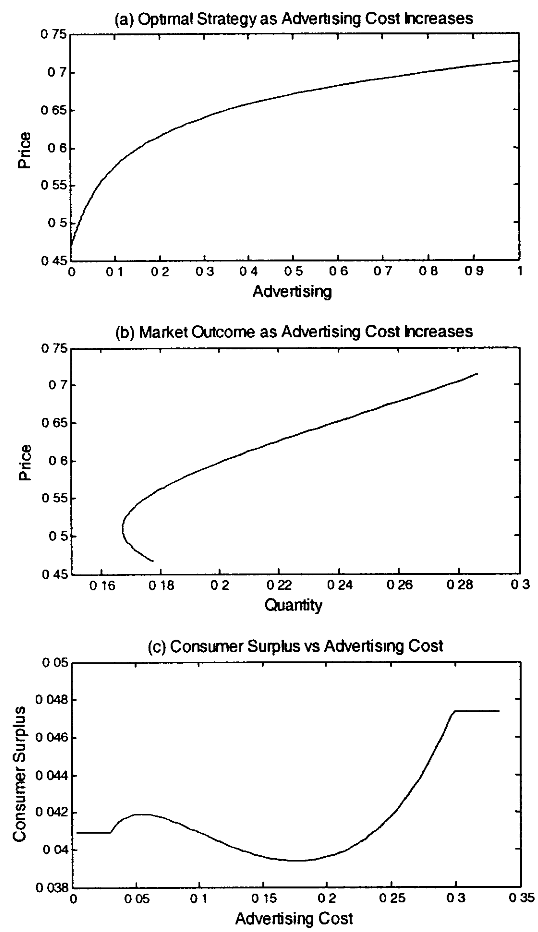


Figure 1-4: Effects of increasing advertising costs

An implication of Corollary 3 is that increasing the advertising cost α, can in fact increase consumer surplus as the monopolist reduces advertising and relies more on the price to stimulate WOM amongst consumers. If advertising is free $\alpha = 0$ then the monopolist chooses $\omega = 1$ and a price $P = \frac{1+c}{2}$, as $\alpha \to \infty$ the monopolist will choose $\omega = 0$ and $P = P^{*}_{WOM}$ for α high enough. Corollary 3 shows that consumers can in fact be better off in the latter case. Indeed Figure 1-4 illustrates how consumer surplus can increase or decrease when the advertising cost increases for a social network $G_0 = x^3$ and marginal cost $c = 0.42$. In Figure 1-4 (a) and (b) increasing the advertising cost corresponds to moving along the curves shown starting from the upper right. As the advertising cost increases the monopolist cuts back on the level of advertising and compensates by decreasing the price to stimulate word of mouth. When advertising costs exceed 0.18 the monopolist starts to dramatically decrease the price which increases the equilibrium quantity and consumer surplus despite the lower levels of advertising taking place.

There is a wide range of potential equilibrium price and advertising pairs depending on the marginal costs of production and advertising. In the following sections I focus on characterizing the marginal returns to advertising.

1.7.1 Marginal returns to advertising

In this section I find that the marginal returns to advertising exhibit a peak as $P \rightarrow P^{crit}$ $\omega \rightarrow 0$, and are decreasing and convex with respect to advertising. The marginal return to the first unit of advertising is the average size of components containing uninformed individuals. I find that the average size of these components (marginal returns of the first unit of advertising) exhibit a very distinctive feature around the critical price. In particular the average component size asymptotes to infinity as the price advertising strategy pair approaches the critical price with zero advertising. This implies that for low levels of advertising there are regions where marginal returns are sharply increasing and decreasing at prices close to the critical price. I provide examples of how the marginal returns vary across a number of networks.

The following theorem characterizes the marginal returns to advertising close to the critical price and zero advertising.

Theorem 9 *If* $0 < P^{crit} < 1$, *then* $\lim_{(\omega,P)\rightarrow(0,P^{crit})} H_0'(1,P) = \infty$.

Proof. See appendix ■

This theorem implies that around the critical price the marginal returns to the first units of direct advertising increase and decrease very sharply. The sharp increase is caused

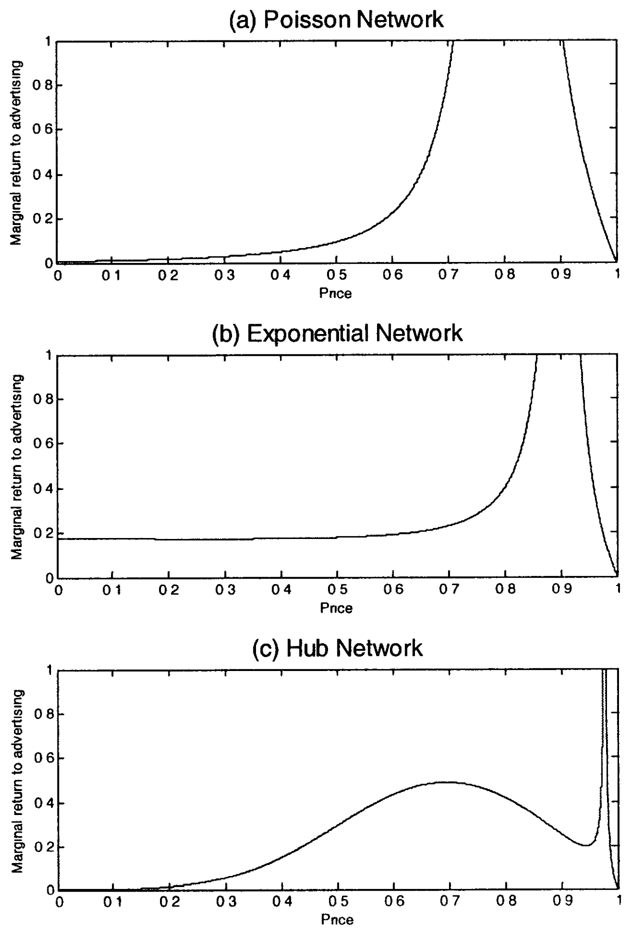


Figure 1-5: Marginal returns to advertising for different social networks

by the phase transition where the giant component appears. The distribution of component sizes contains more and more very large components as the price approaches the critical price. I conclude that for low levels of advertising marginal returns to advertising will exhibit a peak at a price close to the critical price. I contrast this result to an identical model of advertising without WOM. In such a model if the monopolist advertises to $w\%$ of the population $(1 - P)\,w\%$ of the people will end up buying the product if the price is P. The demand in this model is linear in the level of advertising and in contrast to the WOM case the marginal returns are constant. Figure 1-5 illustrates how the marginal returns to the first unit of advertising vary across three networks, each with a mean number of friendships per individual of 5, Poisson, Exponential and Hub (2% have 103 friends and 98% have 3 friends). Each has the distinctive spike at the critical price, however for prices below the critical price the networks are very different. In the Poisson network the marginal returns are strictly increasing, in the Exponential network they are approximately constant until $P = 0.6$ before they start to increase, and in the Hub network the marginal returns are non-monotonic.

The following theorem characterizes marginal returns as the level of advertising changes.

Theorem 10 *Advertising exhibits decreasing and convex marginal returns.*

Proof. See appendix ■

As advertising increases the largest components are relatively more likely to be struck first by the advertising because of their size. Thus as the level of advertising increases the marginal returns fall away sharply at first and then flatten out at higher levels of advertising as the mix of unadvertised components contains a greater fraction of small sized components. This can be seen for the Poisson network in Figure 1-6 where the distinctive spike in marginal returns is evident close to the critical price and zero advertising but as advertising increases the marginal returns fall away sharply and are much flatter at higher levels of advertising.

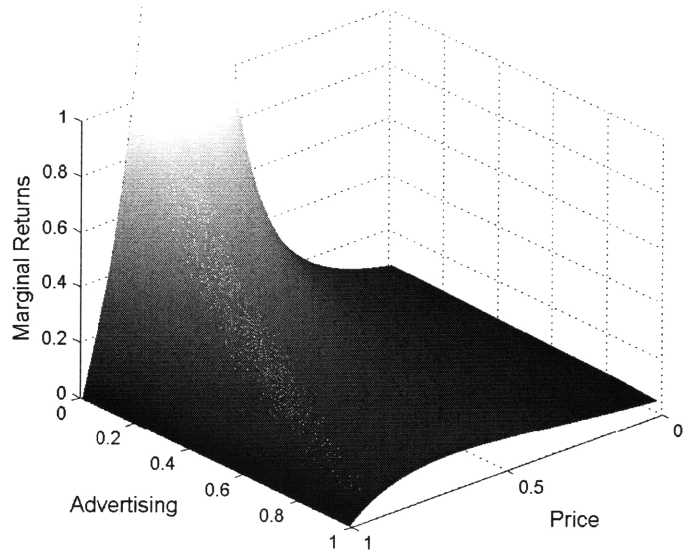


Figure 1-6: Marginal returns vs Advertising and Price

1.7.2 Targeted marketing

If the monopolist can target its advertising at individuals with a certain number of friends then how should it do so? In this section I find that when prices are high and the giant component doesn't exist or is small then individuals with many friends should be targeted, if on the other hand the giant component is large then it is more effective to target advertising at those people with few friends who are least likely to be in the giant component.

Consider the question of which individual the monopolist should advertise to first? I fix the price at a level P and assume the monopolist can observe the number of friends of an individual. The return from advertising to an individual with k friends is the size of the component the individual belongs to, $(1-P)\left(1+k\frac{H_1'(1,P)}{(H_1(1,P))}\right)$, multiplied by the probability the individual is not in the giant component, $(H_1(1,P))^k$. The optimal target individual is the individual which maximizes the return for a given value of P, this person is:

$$k^* = \arg\max_k (1-P)(H_1(1,P))^k \left(1+k\frac{H_1'(1,P)}{(H_1(1,P))}\right)$$

where k is constrained to be an integer.

Theorem 11 *Suppose $p_k > 0$ for all k then the highest return type of individual k^* is:*

$$k^* \in \{\lfloor k^{**} \rfloor, \lceil k^{**} \rceil\} \text{ for } P < P^{crit}$$

where

$$k^{**} = \max \left\{ 0, -\left(\frac{1}{\ln H_1(1,P)} + \frac{u(P)}{H_1'(1,P)} \right) \right\}$$

Proof. See appendix ■

This theorem allows one to characterize the optimal target individual for a monopolist charging P. Note that the floor and ceiling functions $(\lfloor \cdot \rfloor, \lceil \cdot \rceil)$ are necessary because k is an integer. The following corollary illustrates how the optimal target, ignoring integer constraints, k^{**} changes as price changes.

Corollary 5 *The optimal target k^{**} is continuous in P for $P < P^{crit}$, $\lim_{P \to P^{crit}} k^{**} = \infty$, $k^{**} \leq \left\lceil \frac{-1}{\ln H_1(1,P)} \right\rceil$ for $P < P^{crit}$.*

Proof. See appendix ■

The optimal target individual depends on the price. When there is no giant component at high prices $P > P^{crit}$ $(H_1(1,P) = 1)$ then the individuals with the most friends should be targeted. However when the giant component exists $P < P^{crit}$ $(H_1(1,P) < 1)$ then individuals with fewer friends should be targeted. The intuition is that as a greater proportion of the population become informed those people with many friends are very likely find out about the good via WOM.

A firm selling an exclusive product, which is sold at a high price such that only a small fraction of the population is prepared to purchase it, should target its marketing at individuals who can pass on information about the product to as many people as possible. On the other hand if the firm is selling a common product, which a larger fraction of the population is prepared to purchase, then the optimal targets for advertising are individuals with few friends. In this case these are the people most likely to be on the "fringe" of the network, or in other words the least well connected parts of the network. This means that they are unlikely to hear about the good via WOM and in expectation will provide the highest return to direct advertising. As the level of advertising increases then the targeted consumer for the next unit of advertising should be a person with fewer friends than the previous consumer. In other words the targeted individual moves towards the fringes of the network. Again the reason for targeting individuals with fewer and fewer friends is that these are the people least likely to be already informed when a greater proportion of the population are already informed.

1.7.3 Application to social networking sites: Matching consumers to advertising

In this section I assume there is an owner of the rights to advertise to people within the social network. An example of this entity is an online social networking site such as Myspace or Facebook which can sell the rights to advertise to the individuals on their websites. Part of the value from the rights to advertise on these websites is the additional information that the websites have about each consumer. I find that when the website utilizes the information it has about the number of friends each person has, it will optimally allocate advertising for high priced products to individuals with many friends and low priced products to consumers with relatively few.

I assume that there are m different monopolists each selling m different goods. The problem for the owner of the advertising rights is to allocate advertising rights across the m monopolists to maximize the profits from advertising. Each Consumer's valuation for the goods is represented by a vector $\theta_i = \left(\theta_i^1 .... \theta_i^m\right)$ where each θ_i^j is an independent draw from $U\left[0,1\right]$. Thus the demand for each good is independent of the other $m-1$ goods. All monopolists have different prices P^j. Initially a vanishingly small fraction $\varepsilon \to 0$ of the population find out about each good exogenously and WOM is sufficient for the people in the giant component for each good to become informed. Note there are now different giant components for each good. The opportunity to advertise is scarce and only a small fraction δ of individuals may be advertised to.

To simplify the analysis I will assume that the owner of the advertising rights maximizes its own profits by allocating the rights across the monopolists to maximize the total returns to advertising aggregated across all goods, valuing each good equally. The problem is therefore to allocate the δ consumers to the monopolists where they are expected to be in the largest component outside the giant component. I assume that δ is small relative to the giant component thus if multiple individuals are allocated to the same monopolist the probability that they are in the same component outside the giant component is ≈ 0.

Benefit from knowing an individual's number of friends

The return to advertising to a person chosen at random from the population for a monopolist charging P^j is:

$$H_0'\left(1, P^j\right) = \sum_k p_k \left(1 - P^j\right)\left(H_1\left(1, P^j\right)\right)^k \left[1 + k\frac{H_1'\left(1, P^j\right)}{H_1\left(1, P^j\right)}\right]$$

where $H_1\left(1, P^j\right)$ is the probability that a randomly chosen link does not lead to an individual in the giant component when the price is P^j. This is the weighted sum over all people with k friends where the return from each person is the product of the probability they will purchase the good $\left(1 - P^j\right)$, the probability they are not in the giant component $\left(H_1\left(1, P^j\right)\right)^k$, and the expected size of the component they belong to $\left[1 + k\frac{H_1'(1,P^j)}{H_1(1,P^j)}\right]$.

When advertising cannot be targeted consumers are optimally shown an advertisement for product j^{**} defined as:

$$j^{**} = \arg\max_j H_0'\left(1, P^j\right)$$

which is the good with the largest component size outside the giant component. If there is a broad range of prices covered by the m products then for the networks presented in Figure 1-5, (a) and (b) suggest that for the Poisson and Exponential networks j^{**} is going to be the product with the price closest to the critical price. Unless there is a product with a price almost precisely at the critical price for a very heterogeneous networks such as the Hub network in Figure 1-5 (c) j^{**} will be a product with a price significantly less than the critical price with a price close to 0.7.

When the advertising is targeted at an individual with k friends, the optimal market

j^{*k} for the advertising is the market where

$$j^{*k} = \arg\max_j \left(1 - P^j\right) \left(H_1\left(1, P^j\right)\right)^k \left(1 + k\frac{H_1'\left(1, P^j\right)}{\left(H_1\left(1, P^j\right)\right)}\right)$$

This is very similar to the targeted marketing case in the previous section, except instead of choosing an individual with k friends for a given P, we are choosing a product with price P^j for a person with k friends. The optimal target follows the intuition of Corollary 5, the optimal product is a common (lower priced) product for individuals with few friends whereas the optimal product for an individual with many friends is a relatively more exclusive (higher priced) product.

The difference between the return to advertising for product j^{*k} and j^{**} is the allocative benefit from knowing the connectivity information of an individual. For a person with k friends this benefit is

$$\begin{aligned}
&\left(1 - P^{j^{*k}}\right) \left(H_1\left(1, P^{j^{*k}}\right)\right)^k \left(1 + k\frac{H_1'\left(1, P^{j^{*k}}\right)}{\left(H_1\left(1, P^{j^{*k}}\right)\right)}\right) \\
&- \left(1 - P^{j^{**}}\right) \left(H_1\left(1, P^{j^{**}}\right)\right)^k \left(1 + k\frac{H_1'\left(1, P^{j^{**}}\right)}{\left(H_1\left(1, P^{j^{**}}\right)\right)}\right)
\end{aligned}$$

The benefit from knowing a person's connectivity is that it provides an indication of the probability that person is in the giant component and would find out about the good otherwise. For networks such as the Poisson and Exponential networks j^{**} is likely to be a product close to the critical price where $H_1\left(1, P^{j^{**}}\right)$ is close to 1.

Theorem 12 *If $P^{j^{**}} > P^{crit}$ then $\exists\widehat{k}$ such that for $k \geq \widehat{k}$ $j^{**} = j^{*k}$.*

Proof. See Appendix ■

When P^{j**} is above the critical price then this benefit is zero for all individuals with a connectivity above a threshold $\widehat{k}$, and all the benefit comes from allocating the rights to advertise to individuals with few friends, to products with a price below the critical price. This is more likely to be the case in the Poisson and Exponential networks compared to the Hub Network shown earlier.

1.8 Conclusion

Word of mouth is one of the most influential sources of information for consumers when making purchasing decisions. This paper considers informative WOM and how a monopolist can affect the pattern of WOM when the probability an individual engages in WOM is related to her willingness to purchase the product. A key innovation of the paper is to allow the monopolist to strategically affect the probability an individual is willing to engage in WOM. A model of percolation on a random graph with an arbitrary degree distribution is used and enables me to relate the pricing strategy of the monopolist to the pattern of communication which takes place in the social network. It allows me to study a number of new questions concerning the effect of WOM on demand, pricing and advertising when a firm can affect the pattern of communication which takes place for its own benefit. The setting is very tractable and I am able to introduce correlation between valuations and friendships, price discrimination, regular and targeted advertising and in an application I extend the model to consider how the owner of the rights to advertise on a social network can optimally allocate advertising for specific individuals to different products.

I find a range of interesting results: (i) demand has two distinct regions separated by a critical price related to the first and second moments of the distribution of friendships in the social network; (ii) estimates of consumers valuations are biased downwards and estimates of consumer responses to counterfactual policy/strategy changes are biased upwards if WOM is ignored; (iii) prices are below the fully informed monopoly level for goods where there is no correlation between an individual's valuation of the good and their number of friends, however the opposite may be true if there is significant positive correlation; (iv) introductory prices may have periods of sales to optimally diffuse news of the good through the population; (v) increasing advertising costs can benefit consumers; (vi) marginal returns to advertising are peaked close to the critical price; and (vii) targeted advertising should be directed towards individuals with many friends for "exclusive" high priced products and towards people with relatively fewer friends for "common" low priced products.

The tractability of the model suggests a number of avenues for future research. One is to incorporate communication structures which include good and bad quality information about the good. This would lead to different inference problems for agents in components of different sizes. In the case of negative WOM the greater connectivity can be a double-edged sword, on one hand it facilitates a greater diffusion of any negative information but on the other, may also permit better statistical inference by aggregating information in larger components. An aspect of the percolation process explored in sections 1.5.3 and 1.7.2 is the targeting of strategies at individuals depending on their degree. The resilience of a network to the targeted removal of individuals has been studied in the context of immunization and computer networks and may offer further insights in economic applications. More broadly percolation processes can provide a great deal of information about the structure and pattern of communication that takes place through the distribution of component sizes and how this changes in response to endogenously chosen variables. There may be other applications in economics where this information is important and a similar approach may be fruitful. This paper highlights its application to the pricing and advertising strategies of a monopolist facing a population which engages in WOM about its good.

1.9 Appendix A: Proofs

Proof of Theorem 1

Suppose an individual's valuation is independent of the number of friends, $q_k = q = 1 - P$ *for all* k *and* $\{p_k\}$ *is such that* $\frac{E[k]}{E[k^2]-E[k]} < 1$*, then, there exists a critical price* P^{crit} *such that*

$$E_{\Omega_{N,P,\{p_k\}}}[\bar{s}] = \Theta(n), \ H_0'(1,P) < \infty \text{ and } H_1(1,P) < 1 \text{ if } P < P^{crit}$$

and

$$E_{\Omega_{N,P,\{p_k\}}}[\bar{s}] = O\left(n^{1/8}\log n\right), \ H_0'(1,P) < \infty \text{ and } H_1(1,P) = 1 \text{ if } P > P^{crit}$$

Moreover the critical price satisfies $1 - P^{crit} = \frac{E[k]}{E[k^2]-E[k]}$.

Proof. Molloy and Reed (1995) show that the critical percolation threshold is $q_c = \frac{\sum_k p_k k}{\sum_k p_k k(k-1)}$. The result follows immediately by substituting $1 - P^{crit} = q_c$:

$$1 - P^{crit} = \frac{E[k]}{E[k^2] - E[k]}$$

$H_1(1, P)$ is the smallest non-negative solution to

$$\begin{aligned} H_1(1,P) &= 1 - F_1(1,P) + F_1(H_1(1,P), P) \\ &= P + (1-P)\sum_k \frac{kp_k}{z_1}(H_1(1,P))^{k-1} \end{aligned}$$

Note $H_1(1, P) = 1$ is always a solution to this equation and the right hand side is a convex function (polynomial with only positive coefficients) in $H_1(1, P)$. Thus $H_1(1, P) < 1$ iff the derivative evaluated at $H_1(1, P) = 1$ is > 1. This condition is

$$(1-P)\frac{E[k^2] - E[k]}{E[k]} > 1$$

which is true provided $P < 1 - \frac{E[k]}{E[k^2]-E[k]} = P^{crit}$ ■

Proof of Theorem 2

Suppose θ and k are uncorrelated then demand for the good $S(P)$ is

1. *Continuous*

2. $S(P) = 0$ for $P \geq P^{crit}$
   $S(P) > 0$ for $P < P^{crit}$

3. $\frac{dS}{dP} = 0$ for $P \geq P^{crit}$
   $\frac{dS}{dP} < 0$ for $P < P^{crit}$

4. $\lim_{P \to P^{crit-}} \frac{dS}{dP} = -(1 - P^{crit})\frac{G_0'''(1)}{(G_0''(1))^2} < 0$

5. $\left|\frac{P}{S}\frac{dS}{dP}\right| > \left|\frac{P}{1-P}\right|$ for $P < P^{crit}$

Proof. Let $u = H_1(1, P)$. Demand is given by

$$S = F_0(1) - F_0(u)$$

where u is the smallest non-negative solution to the self consistency condition:

$$u = 1 - F_1(1) + F_1(u) \tag{1.8}$$

■

The following lemma illustrates some properties of u with respect to the price which I will subsequently use to prove the above theorem.

Lemma 4 *Suppose $H_1(1, P)$ is given by equation 1.8 then*

1. $H_1(1, P) = 1$ and $\frac{du}{dP} = 0$ for $P^{crit} \leq P \leq 1$
2. $u < 1$ and $\frac{du}{dP} > 0$ for $0 \leq P < P^{crit}$
3. $H_1(1, P)$ is continuous in P

Proof. $H_1(1, P)$ is the smallest non-negative solution to:

$$u = P + (1 - P) \frac{\sum_k k p_k u^{k-1}}{z_1}$$

Now consider the function $f(u, P) = P + (1 - P) \frac{\sum_k k p_k u^{k-1}}{z_1}$ first note $f(1, P) = 1$ and so $u = 1$ always satisfies the above relationship, second $f(u, P)$ is a polynomial in u with positive coefficients so it is continuous, increasing and convex in the region $0 \leq u \leq 1$ and thus combined with $f(0, P) = P$ there is at most one other solution $0 \leq u < 1$.

When $f'(1, P) \leq 1$ there is no solution for $0 \leq u \leq 1$ and $u = 1$ is the only solution. When $f'(1) > 1$ there is a solution for $0 \leq u < 1$. The condition $f'(1) \leq 1$ is equivalent to $P \geq P^{crit}$:

$$f'(1) = (1 - P) \frac{\sum_k k(k-1) p_k}{z_1} \leq 1$$

$$\begin{aligned} 1-\frac{z_1}{\sum_k k\left(k-1\right)p_k} &\leq P \\ P^{crit} &\leq P \end{aligned}$$

Therefore $u = 1$ for $P \geq P^{crit}$ and $0 \leq u < 1$ for $P < P^{crit}$. $u = 1$ for $P \geq P^{crit}$ immediately implies $\frac{du}{dP} = 0$ for $1 \geq P \geq P^{crit}$.

To show that $\frac{du}{dP} > 0$ in $P < P^{crit}$ I look at the derivative for $\frac{du}{dP}$:

$$\frac{du}{dP} = \frac{\left(1-G_1\left(u\right)\right)^2}{1-G_1\left(u\right)-\left(1-u\right)G_1'\left(u\right)}$$

The numerator is positive for $u < 1$ and the denominator

$$1-G_1\left(u\right)-\left(1-u\right)G_1'\left(u\right) = 1-\sum_{k=0}^{\infty} p_k' u^{k-2}\left[k-1-u\left(k-2\right)\right]$$

(where $p_k' = \frac{kp_k}{z_1}$) is continuous and equal to 1 at $u = 0$, equal to 0 at $u = 1$ and is decreasing in u for $0 \leq u \leq 1$ provided $G_1''\left(1\right) > 0$ which is a necessary condition for $P^{crit} > 0$. Therefore in the range $P \in \left[0, P^{crit}\right)$ $H_1\left(1, P\right)$ is continuous and $\frac{du}{dP} > 0$. ■

Returning to the theorem. Using this lemma I conclude that for $P \geq P^{crit}$ $S\left(P\right) = 0$ and for $P \in \left[0, P^{crit}\right)$ $S\left(P\right) = \left(1-P\right)\left(1-\sum_k p_k u^k\right)$ is a continuous function since u is continuous in P. I now prove the continuity of $S\left(P\right)$ by showing that as the price approaches the critical price from below $S \rightarrow 0$:

If there exists a critical price, $0 < P^{crit} < 1$ *then as price approaches the critical price from below* $\lim_{P \rightarrow P^{crit-}} S = 0$

Proof. I rewrite the relationship between P and u

$$P\left(u\right) = \frac{u-G_1\left(u\right)}{1-G_1\left(u\right)}$$

such that $P\left(u\right)$ is a continuous, monotonically increasing (one to one) function $\left[0,1\right) \rightarrow \left[-1,1\right]$. I will now show that $\lim_{u \rightarrow 1^-} P\left(u\right) = P^{crit}$.

$P(1) = \frac{0}{0}$ so applying L'Hopital's rule

$$\begin{aligned}
\lim_{u \to 1^-} P(u) &= \lim_{u \to 1^-} P'(u) \\
&= \frac{1 - G_1'(1)}{G_1'(1)} \\
&= 1 - \frac{E[k]}{E[k^2] - E[k]} \\
&= P^{crit}
\end{aligned}$$

Now $P(u)$ is a one to one function and $0 < P^{crit} < 1$ this implies that $\lim_{P \to P^{crit-}} u = 1$ and hence $\lim_{P \to P^{crit-}} S = 0$ ■

This completes the argument for the continuity of S. So far we have shown $P \in [P^{crit}, 1]$ $S(P) = 0$, for $P \in [0, P^{crit})$ $S(P)$ is continuous, and finally $\lim_{P \to P^{crit-}} S = 0$. The next part of the theorem is:

If there exists a critical price, $0 < P^{crit} < 1$ then as price approaches the critical price from below $\lim_{P \to P^{crit-}} \frac{dS}{dP} = -(1 - P^{crit}) \frac{G_0'''(1)}{(G_0''(1))^2}$

Proof.

$$S = (1 - P)\left[1 - \sum_k p_k u^k\right]$$

$$\lim_{u \to 1^-} \frac{dS}{dP} = \lim_{u \to 1^-} -\left[1 - \sum_k p_k u^k\right] - \frac{du}{dP}(1 - P)\left(-\sum_k k p_k u^{k-1}\right)$$

$$\left.\frac{dS}{dP}\right|_{u=1} = z\left(1 - P^{Crit}\right) \lim_{u \to 1^-} \left.\frac{du}{dP}\right|$$

$$\lim_{u \to 1^-} \left.\frac{du}{dP}\right| = \lim_{u \to 1^-} \left.\frac{1 - G_1(u) - (1 - u) G_1'(u)}{(1 - G_1(u))^2}\right| = \frac{0}{0}$$

using L'Hopitals rule,

$$\begin{aligned}
&\lim_{u \to 1^-} \left.\frac{1 - G_1(u) - (1 - u) G_1'(u)}{(1 - G_1(u))^2}\right| \\
&= \lim_{u \to 1^-} \left.\frac{(1 - u) G_1''(u)}{2G_1'(u)(1 - G_1(u))}\right| = \frac{0}{0}
\end{aligned}$$

and again

$$= \lim_{u \to 1^-} \frac{G_1'''(u)(1-u) - G_1''(u)}{2G_1''(u)(1-G_1(u)) - 2(G_1'(u))^2}$$
$$= \frac{G_1''(1)}{2G_1'(1)^2}$$

Furthermore provided that $G_1''(1)$ is non zero (which also implies $G_0'(1)$ is non zero) then the demand curve will exhibit a non-zero slope ($\frac{dS}{dP} < 0$) as the price approaches the critical price from below. $G_1''(1) > 0$ also implies that there are some people with 3 or more friends, which is also necessary for $P^{crit} > 0$ so that for any network where $P^{crit} > 0$ then demand will exhibit a kink at P^{crit} separating the two regions of demand. ■

At $P < P^{crit}$ $\frac{dS}{dP} < 0$

Proof. Consider the expression for $\frac{P}{S}\frac{dS}{dP}$:

$$\frac{P}{S}\frac{dS}{dP} = \frac{-P}{1-P}\left[1 + \frac{(1-P)}{1-\sum_k p_k u^k}\frac{du}{dP}\sum_k p_k k u^{k-1}\right]$$

the result follows immediately from $u < 1$ and $\frac{du}{dP} > 0$ for $P < P^{crit}$. ■

The final element of the proof is

For $P < P^{crit}$ $\left|\frac{P}{S}\frac{dS}{dP}\right| > \left|\frac{P}{1-P}\right|$

Proof. From above

$$\frac{P}{S}\frac{dS}{dP} = \frac{-P}{1-P}\left[1 + \frac{(1-P)}{1-\sum_k p_k u^k}\frac{du}{dP}\sum_k p_k k u^{k-1}\right]$$

where the second term inside the brackets is strictly positive from lemma 4 and the result follows immediately. ■

Proof of Corollary 1

Suppose valuations and number of friends are uncorrelated and the price of the good is $\tilde{P}$ *then an estimate of consumer surplus* $CS\left(\tilde{P}\right) = \int_{\tilde{P}}^{\infty} S(P)\,dP$ *is biased downwards*

Proof. I show that the estimate of the distribution of valuations implied by $S(P)$ is first order stochastically dominated by the actual distribution of valuations of the consumers purchasing the product. Denote the actual distribution of valuations for the consumers who purchase the good by $G(\theta)$ and the estimate by $\tilde{G}(\theta)$. Preferences are distributed uniformly across informed consumers when θ and k are uncorrelated thus the actual distribution of valuations is

$$\begin{aligned} G(\theta) &= \frac{\theta - \tilde{P}}{1 - \tilde{P}} \text{ for } \tilde{P} \leq \theta \leq 1 \\ &= 0 \text{ for } \theta < \tilde{P} \end{aligned}$$

The estimate $\tilde{G}(\theta)$ from $S(P)$ is

$$\begin{aligned} \tilde{G}(\theta) &= 1 - \frac{S(\theta)}{S\left(\tilde{P}\right)} \text{ for } \tilde{P} \leq \theta \leq 1 \\ &= 0 \text{ for } \theta < \tilde{P} \end{aligned}$$

For any $\theta \in \left[\tilde{P}, 1\right]$

$$\tilde{G}(\theta) - G(\theta) = 1 - \frac{S(\theta)}{S\left(\tilde{P}\right)} - \frac{\theta - \tilde{P}}{1 - \tilde{P}}$$

substituting in for $S\left(\tilde{P}\right)$, $S(\theta)$ and rearranging

$$\frac{1-\theta}{S\left(\tilde{P}\right)} \sum p_k \left(H_1 (1, \theta)^k - H_1 \left(1, \tilde{P}\right)^k \right) > 0 \text{ for } \theta > \tilde{P}$$

because $\theta \geq \tilde{P}$ and $u(\cdot)$ is an increasing function. First Order Stochastic Dominance implies that estimates of consumer welfare using the distribution of valuations implied by $S(P)$ are going to be too small. ■

Proof of Corollary 2

Suppose valuations and number of friends are uncorrelated and the price of the good is $\tilde{P}$ *then an estimate of the consumer response* $\Delta\hat{S} = S\left(\tilde{P}\right) - S\left(\tilde{P} + \Delta P\right)$ *to an increase in the price by* ΔP *overstates the actual response* ΔS

$$\Delta\hat{S} < \Delta Q$$

Proof. Denote the actual distribution of valuations for the consumers who purchase the good at $\tilde{P}$ by $G(\theta)$ and the estimate by $\tilde{G}(\theta)$. Then $\Delta Q = -G\left(\tilde{P} + \Delta P\right)$ $\Delta \hat{S} = -\tilde{G}\left(\tilde{P} + \Delta P\right)$. The result follows immediately from Corollary 1 where $\tilde{G}(\theta) - G(\theta) \geq 0$ for any $\theta \geq \tilde{P}$. ■

Proof of Theorem 3

Suppose valuations and number of friends are uncorrelated and marginal costs $c < 1$, then a monopolist facing demand given by $S(P)$ charges a lower price P^*_{WOM} than a monopolist facing a fully informed population P^*_{FI}, where demand is given by $Q(P) = 1 - P$.

Proof. Define the fully informed monopoly price as P^*_{FI} and the WOM monopoly price as P^*_{WOM}. A monopolist facing a fully informed population has a strictly concave profit maximization problem and charges the unique monopoly price $P^*_{FI} = \frac{1+c}{2}$ provided $c < 1$. If $c \geq 1$ then there is clearly no price where the monopolist can make positive profits. It is also true that

$$\frac{P - c}{P} \geq \frac{1}{\varepsilon_{FI}} \text{ for any } P \geq P^*_{FI}$$

it was shown in Theorem 2 that $|\varepsilon_{WOM}| > |\varepsilon_{FI}|$ which implies that:

$$\frac{P - c}{P} > \frac{1}{\varepsilon_{WOM}} \text{ for any } P \geq P^*_{FI}$$

when demand is positive in the range of prices $P^{crit} > P \geq P^*_{FI}$. The WOM monopolists profit function $(P - c) S(P)$ is continuous and differentiable for $P < P^{crit}$. Therefore the first order conditions for the monopolist are necessary and hence $\frac{P-c}{P} > \frac{1}{\varepsilon_{WOM}}$ for all $P \geq P^*_{FI}$ implies $P^{Mon} \ngeq P^*_{FI}$. ■

Proof of Corollary 3

Consumer surplus may be greater when consumers are uninformed and the monopolist charges P^*_{WOM} than if consumers are fully informed and the monopolist charges P^*_{FI}

Proof. Consider the social networks where everyone has 3 friends $G_0(x) = x^3$. If the marginal cost of the monopolist is 0 then the profit maximizing price when the population is fully informed is 0.5 and consumer surplus is 0.125. On the other hand if the population is uninformed the WOM monopoly price $P^*_{WOM} = 0.3215$ and consumer surplus is 0.2057. If marginal cost is higher, > 0.5, then there is no price above the monopolist's marginal cost where the giant component exists thus consumer surplus is 0. ■

Proof of Theorem 4

If all consumers with $\theta \in [c, \underline{\theta}]$ have $k = 1$ where $\underline{\theta} > \frac{1+c}{2}$ then provided the giant component exists at $P = \underline{\theta}$ the monopoly price will be greater than the fully informed monopoly price $\frac{1+c}{2}$

Proof. I first show that demand will be linear in the region $P \in [c, \underline{\theta}]$. Consider

$$\begin{aligned} S &= 1 - H_0(1, P) \\ \frac{dS}{dP} &= -\frac{dH_0(1, P)}{dP} = -\frac{d\left(1 - \sum p_k q_k \left(1 - u^k\right)\right)}{dP} \\ &= -(1-u) + \frac{du}{dP}\sum k p_k q_k u^{k-1} \end{aligned}$$

where $u = H_1(1, P)$. In the range of prices $P \in [c, \underline{\theta}]$, $\frac{dq_k}{dP} = 0$ for $k \neq 1$ and $\frac{dq_1}{dP} = -\frac{1}{p_1}$ for $k = 1$ because all consumers $\theta \in [c, \underline{\theta}]$ have $k = 1$. Now consider the self consistency relationship for $H_1(1, P)$:

$$u = 1 - \frac{1}{z_1}\sum_{k=2}^{\infty} k p_k q_k \left(1 - u^{k-1}\right)$$

This is independent of q_1, thus for $P \in [c, \underline{\theta}]$ $H_1(1, P)$ is constant, $\frac{dS}{dP} = -(1-u)$ and S is linear. Denote $\underline{u} = H_1(1, P)$ for $P \in [c, \underline{\theta}]$.

Consider the first order condition of the monopolist in the range $P \in [c, \underline{\theta}]$

$$\frac{d\pi}{dP} = S - (P - c)(1 - \underline{u})$$

this is decreasing in P and positive if $\frac{S(P)}{1-\underline{u}} > P - c$. Therefore the optimal price cannot be less than or equal to $\frac{1+c}{2}$ if $\frac{S\left(\frac{1+c}{2}\right)}{1-\underline{u}} > \frac{1-c}{2}$ which is equivalent to

$$\underline{\theta} + \frac{S(\underline{\theta})}{1 - \underline{u}} > 1$$

provided $P^{crit} > \theta$ and hence $\underline{u} < 1$, this can be rewritten

$$\begin{aligned} \sum p_k q_k \left(1 - \underline{u}^k\right) - (1 - \underline{\theta})(1 - \underline{u}) &> 0 \\ \sum p_k q_k \left(\underline{u} - \underline{u}^k\right) &> 0 \end{aligned}$$

which is true for $\underline{u} < 1$ hence the monopoly price is greater than $\frac{1+c}{2}$. ■

Proof of Theorem 5

If valuations and number of friends are uncorrelated then the optimal set of prices $P_0 = P_1 = \frac{1+c}{2}$ and $\exists \underline{k} : \{P_k\}$ is decreasing for $2 \leq k \leq \underline{k}$ and $P_k = 0$ for $k \geq \underline{k}$

Proof. Monopolist's maximization

$$\pi = \max_{\{P_k\}} \sum p_k \left(1 - P_k\right) \left(1 - u^k\right) \left(P_k - c\right)$$

Where $u = H_1(1, P)$. Assuming $P_k > 0$ for all k. First order condition for price P_k :

$$p_k (1 - P_k) \left(1 - u^k\right) - p_k \left(1 - u^k\right) (P_k - c) - \frac{\partial u}{\partial P_k} \sum_j p_j (1 - P_j)(P_j - c) j u^{j-1} = 0 \text{ for } P_k \in (0, 1)$$

Equation defining the probability that a randomly chosen link is outside the giant component:

$$D(u) = u - 1 + \frac{\sum k p_k (1 - P_k) \left(1 - u^{k-1}\right)}{z} = 0$$

Implicit function theorem

$$\begin{aligned} \frac{\partial D}{\partial u} &= 1 - \frac{\sum k(k-1) p_k (1 - P_k) u^{k-2}}{z} \\ \frac{\partial D}{\partial P_k} &= -\frac{k p_k \left(1 - u^{k-1}\right)}{z} \end{aligned}$$

$$\frac{du}{dP_k} = \frac{k p_k \left(1 - u^{k-1}\right)}{z - \sum k(k-1) p_k (1 - P_k) u^{k-2}}$$

where $\frac{du}{dP_1} = 0$ so $P_1 = \frac{1+c}{2}$. Now defining $\alpha = \frac{\sum_j p_j (1-P_j)(P_k - c) j u^{j-1}}{z - \sum k(k-1) p_k (1-P_k) u^{k-2}}$ which is the same for all k and going back to the first order condition for P_k

$$\begin{aligned} p_k q_k \left(1-u^k\right) - p_k \left(1-u^k\right)\left(P_k - c\right) - \alpha k p_k \left(1-u^{k-1}\right) &= 0 \text{ for } P_k \in (0,1) \\ 1 - 2P_k + c - \alpha k \frac{\left(1-u^{k-1}\right)}{1-u^k} &= 0 \text{ for } P_k \in (0,1) \end{aligned}$$

$$\frac{d\left(k\frac{\left(1-u^{k-1}\right)}{1-u^k}\right)}{dk} > 0$$

and thus P_k is decreasing in k. If $1 + c - \alpha k\frac{\left(1-u^{k-1}\right)}{1-u^k} < 0$ then $P_k = 0$ so defining $\underline{k} = \inf\left\{k|1 + c - \alpha k\frac{\left(1-u^{k-1}\right)}{1-u^k} < 0\right\}$ then for all $k \geq \underline{k}$ $P_k = 0$. ■

Proof of Lemma 1

If $F \in \mathcal{F}$ *then* $\Gamma_F(F,P) \in \mathcal{F}$

Proof. Let $F_t = F$ and $F_{t+1} = \Gamma_F(F,P)$. Consider the value of f_{t+1} for $\theta < P_t$ as a function of P_t and f_t, this may be written as:

$$\begin{aligned} f_{t+1}(\theta) &= \frac{f_t(\theta) + \frac{z_2}{z_1}\left(1 - F_t(P_t)\right)}{1 + \left(\frac{z_2}{z_1} - 1\right)\left(1 - F_t(P_t)\right)} \\ &= \frac{z_2}{z_2 - z_1} \frac{\frac{f_t(\theta)}{\frac{z_2}{z_2 - z_1}} + \left(\frac{z_2}{z_1} - 1\right)\left(1 - F_t(P_t)\right)}{1 + \left(\frac{z_2}{z_1} - 1\right)\left(1 - F_t(P_t)\right)} \end{aligned}$$

Hence if $f_t(\theta) < \frac{z_2}{z_2 - z_1}$ then the second term is < 1 and $f_{t+1}(\theta) < \frac{z_2}{z_2 - z_1}$. For $\theta \geq P_t$ $f_{t+1}(\theta) < 1 < \frac{z_2}{z_2 - z_1}$. There are no mass points in F_t so the cdf $\Gamma_F(F_t, P)$ is also continuous. Thus $\Gamma_F(F,P) \in \mathcal{F}$. ■

Proof of Lemma 2

$\Gamma_M : \mathcal{F} \times [0,1] \times [1,\infty) \to [1,\infty)$ *and* $\Gamma_F : \mathcal{F} \times [0,1] \to \mathcal{F}$ *are continuous mappings*

Proof. Use the sup norm on the space of continuos cdfs on $[0,1]$. Γ_M and Γ_F are single valued mappings so I will proceed with an ε η proof of continuity. That is for a give $\varepsilon > 0$ there exists $\eta > 0$ such that if $|(F_0, P_0), (F,P)| < \eta$ then $|\Gamma_F(F_0,P_0), \Gamma_F(F,P)| < \varepsilon$ in the case of Γ_F and similarly in the case of Γ_M.

First I prove the continuity of Γ_F. For any $F_0 \in \mathcal{F}$ and $P_0 \in [0,1]$

$$F'(\theta) = \Gamma_F(F_0, P_0) = \frac{\min[F_0(\theta), F_0(P_0)] + \frac{z_2}{z_1}(1 - F_0(P_0))\theta}{1 + \left(\frac{z_2}{z_1} - 1\right)(1 - F_0(P_0))}$$

For any ε choose $\eta = \frac{1}{2}\sqrt{\frac{\varepsilon}{(\alpha+1)^2 \frac{z_2}{z_1}\left(\frac{z_2}{z_1}+1\right)}}$ where $\alpha = \frac{z_2}{z_2 - z_1}$.

For any (F, P) where $\|(F_0, P_0), (F\ P)\| < \eta$ we have $|F(\theta) - F_0(\theta)| < \eta$ and $|P - P_0| < \eta$. Hence

$$\begin{aligned}
\left|\frac{z_2}{z_1}(1 - F_0(P_0))\theta - \frac{z_2}{z_1}(1 - F(P))\theta\right| &= \frac{z_2}{z_1}\theta\,|F_0(P_0) - F(P)| \\
&< \frac{z_2}{z_1}\theta\,(|F_0(P_0) - F_0(P)| + |F_0(P) - F(P)|) \\
&< \eta\frac{z_2}{z_1}\theta(\alpha + 1)
\end{aligned}$$

,

$$\begin{aligned}
&\left|\frac{1}{1 + \left(\frac{z_2}{z_1} - 1\right)(1 - F_0(P_0))} - \frac{1}{1 + \left(\frac{z_2}{z_1} - 1\right)(1 - F(P))}\right| \\
<\ & \frac{\frac{z_2}{z_1}\theta\,|F_0(P_0) - F(P)|}{1 + \left(\frac{z_2}{z_1} - 1\right)(1 - \min[F_0(P_0), F(P)])} \\
<\ & \eta\frac{z_2}{z_1}\theta(\alpha + 1)
\end{aligned}$$

$$|\min[F_0(\theta), F_0(P_0)] - \min[F(\theta), F(P)]|$$

wlog say $P_0 \geq P$ now if $\theta < P$ then

$$\begin{aligned}
|\min[F_0(\theta), F_0(P_0)] - \min[F(\theta), F(P)]| &= |F_0(\theta) - F(\theta)| \\
&< \eta
\end{aligned}$$

if $\theta > P_0$

$$\begin{aligned} \left|\min\left[F_0(\theta), F_0(P_0)\right] - \min\left[F(\theta), F(P)\right]\right| &= \left|F_0(P_0) - F(P)\right| \\ &< \eta(\alpha+1) \end{aligned}$$

if $P \leq \theta \leq P_0$

$$\begin{aligned} \left|\min\left[F_0(\theta), F_0(P_0)\right] - \min\left[F(\theta), F(P)\right]\right| &= \left|F_0(\theta) - F(P)\right| \\ &< \left|F_0(P_0) - F(P)\right| \\ &< \eta(\alpha+1) \end{aligned}$$

hence

$$\left|\min\left[F_0(\theta), F_0(P_0)\right] - \min\left[F(\theta), F(P)\right]\right| < \eta(\alpha+1)$$

Now

$$\begin{aligned} \left|\Gamma_F(F_0, P_0) - \Gamma_F(F, P)\right| &< \eta\frac{z_2}{z_1}\theta(\alpha+1)\left(\eta(\alpha+1) + \eta\frac{z_2}{z_1}\theta(\alpha+1)\right) \\ &< \eta^2(\alpha+1)^2\frac{z_2}{z_1}\left(\frac{z_2}{z_1}+1\right) \end{aligned}$$

And therefore

$$\left|\Gamma_F(F_0, P_0) - \Gamma_F(F, P)\right| < \frac{\varepsilon}{2}$$

and $\Gamma_F(F, P)$ is a continuous mapping.

For $M' = \Gamma_M(M_0, F_0, P_0) = \left((1 - F_0(P_0))\frac{z_2}{z_1} + F_0(P_0)\right)M_0$. For any ε choose $\eta = \frac{\varepsilon/2}{M_0\left(\frac{z_2}{z_1}+1\right)(\alpha+1)+\left(\frac{z_2}{z_1}+1\right)}$. Any (M, F, P) for where:

$$\left\|(M_0, F_0, P_0), (M, F, P)\right\| < \eta$$

$$\Rightarrow \quad |M_0 - M| < \eta$$
$$\Rightarrow \quad |F(\theta) - F_0(\theta)| < \eta$$
$$\Rightarrow \quad |P - P_0| < \eta$$

and from earlier

$$|F_0(P_0) - F(P)| < \eta(\alpha+1)$$

Now

$$\begin{aligned} |\Gamma_M(M_0,F_0,P_0) - \Gamma_M(M,F,P)| &< |\Gamma_M(M_0,F_0,P_0) - \Gamma_M(M_0,F,P)| \\ &\quad + |\Gamma_M(M_0,F,P) - \Gamma_M(M,F,P)| \\ &< M_0\left(\frac{z_2}{z_1}+1\right)\eta(\alpha+1) + \left(\frac{z_2}{z_1}+1\right)\eta \\ &< \frac{\varepsilon}{2} \end{aligned}$$

■

Proof of Lemma 3

If $P_t = P^* < P^{crit}$ *for all* t *and* $F_t \in \mathcal{F}$ *then the limiting distribution* $f_t^*(\theta) = \lim_{t\to\infty} f_t(\theta)$ *will be*

$$\begin{aligned} f_t^*(\theta) &= \frac{z_2}{z_2 - z_1} \text{ if } \theta < P \\ &= \frac{z_2 - \frac{z_1}{1-P}}{z_2 - z_1} \text{ if } \theta \geq P \end{aligned}$$

Proof. When P_t remains constant each period $1 - F_t(P)$ fraction of people purchase and inform $\frac{z_2}{z_1}$ others. For $\theta < P$ we have the following expression for $F_t(\theta)$:

$$F_t(\theta) - F_{t-1}(\theta) = \frac{\frac{z_2}{z_1}(1 - F_{t-1}(P_{t-1}))\theta - F_{t-1}(\theta)\left(\frac{z_2}{z_1} - 1\right)(1 - F_{t-1}(P_{t-1}))}{1 + \left(\frac{z_2}{z_1} - 1\right)(1 - F_{t-1}(P_{t-1}))}$$

$F_t(\theta) < \frac{z_2}{z_2 - z_1}\theta$ and $F_t(\theta) > F_{t-1}(\theta)$ when $\frac{z_2}{z_2-z_1}\theta > F_{t-1}(\theta)$ and $F_t(\theta) - F_{t-1}(\theta) \to 0$ as $F_t(\theta) \to \frac{z_2}{z_2-z_1}\theta$. Thus $\lim_{t\to\infty} F_t(\theta) = \frac{z_2}{z_2-z_1}\theta$ for $\theta < P$

For $\theta \geq P$ the only people with $\theta \geq P$ are those that have been newly informed from the period before, so the distribution is uniform for $\theta \geq P$ hence $F_t(\theta)$ can be written as $1 - \alpha_t (1 - \theta)$. Substituting this into the transition function Γ_F :

$$\begin{aligned} 1 - \alpha_t (1 - \theta) &= 1 - \frac{\frac{z_2}{z_1} (\alpha_{t-1} (1 - P)) (1 - \theta)}{1 + \left(\frac{z_2}{z_1} - 1\right) (\alpha_{t-1} (1 - P))} \quad \text{for } \theta \geq P \\ \alpha_t &= \frac{\frac{z_2}{z_1} (\alpha_{t-1} (1 - P))}{1 + \left(\frac{z_2}{z_1} - 1\right) (\alpha_{t-1} (1 - P))} \end{aligned}$$

Hence $\frac{z_2 - \frac{z_1}{1-P}}{z_2 - z_1} < \alpha_t < \alpha_{t-1}$ for any $\alpha_{t-1} > \frac{z_2 - \frac{z_1}{1-P}}{z_2 - z_1}$. Thus $\lim_{t \to \infty} f_t(\theta) = \frac{z_2 - \frac{z_1}{1-P}}{z_2 - z_1}$. ■

Proof of Theorem 6

The monopolist's problem has a unique solution, the value function is homogeneous of degree 1 in M and the policy function $P(F)$ is u.h.c and only a function of the state F.

Proof. The proof involves defining a contraction mapping on the recursive problem and using this to show that there is a unique solution to it. The continuity of the value function and u.h.c of the policy function come from the theorem of the maximum.

I first prove the homogeneity of the problem ■

Lemma 5 *$J(\cdot, F)$ is homogeneous of degree one in its first argument*

Proof. Note that the state variable M does not appear in the transition equation Γ_F thus for a given sequence of prices the states F_t will be unaffected by changing M_0 to λM_0. Also note that $\frac{M_{t+1}}{M_t} = \left(\left(1 - F(P) \frac{z_2}{z_1} + F(P)\right)\right)$ is also unchanged. The objective function can therefore be rewritten

$$J(M_0, F_0) = M_0 \times \max_{\{P_t\}} \sum_{t=0}^{\infty} \beta^{t-1} P_t (1 - P_t) \left(\Pi_{i=0}^{t} \frac{M_i}{M_{i-1}}\right)$$

Thus $J(\lambda M_0, F_0) = \lambda J(M_0, F_0)$ ■

Now define the set of continuous cdfs on $[0, 1]$ which satisfy

$$\frac{F(x) - F(x - \delta)}{\delta} \leq \alpha$$

for some finite $\alpha > 0$ by $\mathcal{F}$. From Lemma 1any cdf $\Gamma_F(F,P)$ satisfies this property provided F does. Also note the space $\mathcal{F}$ with the sup norm is complete.

Let $H(M,F)$ be the space of functions $V : [1,\infty) \times \mathcal{F} \to R$ which are continuous, homogeneous of degree one with respect to their first argument and bounded in the norm $\max_{F\in\mathcal{F}} \frac{V(M,F)}{M}$. Define an operator T on $H(M,F)$ by

$$(TV)(M,F) = \max_{\substack{P\in[0,1] \\ M'=\Gamma_M(M,F,P) \\ F'=\Gamma_F(F,P)}} P(1-P)M + \beta V(M',F')$$

where $F \in \mathcal{F}$ and $M \in [1,\infty)$. Note that the objective and transition functions are continuous and the maximization is over a compact set so the maximum is achieved and by the theorem of the maximum (Berge 1963) TV is also continuous. Also note that M' is a linear function of M so TV will be homogenous of degree 1 in M. Thus TV maps $H(M,F) \to H(M,F)$.

Define the function $(V+a)(M,F) = V(M,F) + aM$

Lemma 6 *Let $(M,F) \subseteq [1,\infty) \times \mathcal{F}$ and let $H(M,F)$ be as above, with the associated norm. Let $T : H(M,F) \to H(M,F)$ satisfy*

(monotonicity) $V, W \in H$ and $V \leq W$ implies $TV \leq TW$

(discounting) there exists $\gamma \in (0,1)$ such that for all $V \in H$ and all $a \geq 0$, $T(V+a) \leq TV + \gamma a$

Then T is a contraction with modulus γ

Proof. By homogeneity of degree 1,

$$V(M,F) = MV(1,F) \text{ for all } V \in H$$

Choose any $V, W \in H(M,F)$. Then

$$\begin{aligned} V(M,F) &= W(M,F) - [V(M,F) - W(M,F)] \\ &= W(M,F) - M[V(1,F) - W(1,F)] \\ &\leq W(M,F) - M\|V - W\| \end{aligned}$$

Hence monotonicity and discounting imply

$$TV \leq TW + \gamma \|V - W\|$$

Reversing the roles of V and W and combining the two results we get

$$\|TV - TW\| \leq \gamma \|V - W\|$$

■

I can now prove the following:

The operator T as defined above has a unique fixed point $V \in H(M,F)$ in addition

$$\|T^n V_0 - V\| \leq (\alpha\beta)^n \|V_0 - V\|, \quad n = 0, 1, 2, ..., \text{ all } V_0 \in H(M,F)$$

and the associated policy correspondence $G : (M,F) \rightarrow P$ is compact valued and u.h.c. Moreover, G is homogeneous of degree one in its first argument

$$P \in G(M,F) \text{ implies } P \in G(\lambda M, F), \text{ all } \lambda > 0$$

Proof. $H(M,F)$ is a complete normed vector space and $T : H(M,F) \rightarrow H(M,F)$. Clearly T satisfies the monotonicity property of 6. Choose $V(M,F) \in H(M,F)$ and

$a > 0$. Then

$$\begin{aligned} T(V+a)(M,F) &= \sup_{\substack{P\in[0,1] \\ M'=\Gamma_M(M) \\ F'=\Gamma_F(F)}} P(1-P)M + \beta(V+a)(M',F') \\ &= \sup_{\substack{P\in[0,1] \\ M'=\Gamma_M(M) \\ F'=\Gamma_F(F)}} P(1-P)M + \beta V(M',F') + \beta a M' \\ &\leq \sup_{\substack{P\in[0,1] \\ M'=\Gamma_M(M) \\ F'=\Gamma_F(F)}} P(1-P)M + \beta V(M',F') + \beta a \frac{z_2}{z_1} M \\ &= (TV)(M,F) + \beta \frac{z_2}{z_1} a M \end{aligned}$$

where the third line uses $M' \leq \frac{z_2}{z_1} M$. Since the V was chosen arbitrarily, if follows that $T(V+a) \leq TV + \beta \frac{z_2}{z_1} a$. Hence given the assumption that $\beta \frac{z_2}{z_1} < 1$ T satisfies the discounting condition in 6 and is a contraction of modulus $\beta \frac{z_2}{z_1}$. It then follows from the Contraction Mapping Theorem that T has a unique fixed point in $H(M,F)$ and that

$$\|T^n V_0 - V\| \leq (\alpha\beta)^n \|V_0 - V\|, \quad n = 0,1,2,..., \text{ all } V_0 \in H(M,F)$$

holds.

That the policy function G is compact valued and u.h.c. follows from the Theorem of the Maximum (Berge 1963). Finally if $P \in G(M,F)$ then $P \in G(\lambda M, F)$ otherwise $\lambda V(M,F) < V(\lambda M, F)$ which by the homogeneity of degree 1 must hold with equality. ■

Proof of Theorem 7

$\nexists T$ such that for all $t > T$ the optimal price sequence $\{P_t^*\}$ is weakly increasing or decreasing

Proof. It is useful to have the following two lemmas before proceeding ■

Lemma 7 *If F_0 FOSD F_0' then $V(M_0,F_0) \geq V(M_0,F_0')$*

Proof. For any sequence of prices $\{P_t\}$, it suffices to show that $M_t \geq M_t'$. Γ_F preserves FOSD so if F_0 FOSD F_0' then for a set of prices $\{P_t\}$ F_t FOSD F_t'. The growth rate each period $\left((1-F_t(P_t))\frac{z_2}{z_1}+F_t(P_t)\right)\geq\left((1-F_t'(P_t))\frac{z_2}{z_1}+F_t'(P_t)\right)$ hence $M_t\geq M_t'$. ■

Lemma 8 *The optimal price each period* $P_t^*\in\left[0,\frac{1}{2}\right]$

Proof. Consider a price sequence $\{P'\}$ where $P_t'>\frac{1}{2}$. A price sequence $\left\{P_0'...P_{t-1}',\frac{1}{2},P_{t+1}'...\right\}$ will result in higher profits. In period t the one period profits are strictly greater because $P_t=\frac{1}{2}$ is the one period monopoly price and for all periods $M_{t+i}>M_{t+i}'$ $i=1,2,....$ ■

Now returning to the proof of the theorem. The proof is by contradiction. Suppose there is a weakly increasing or decreasing price sequence $\{P_t^*\}$. Every P_t^* will be an element of a compact set $\left[0,\frac{1}{2}\right]$ and any sequence which is weakly increasing or decreasing will converge to an element of this set. Call this price $P^*=\lim_{t\to\infty}P_t^*$.

The value function is linear in M_t so I will write it as the product of M_t and a function of $F_t: V(M_t,F_t)=M_tV(F_t)$.

I first rule out that $P^*=0$. A constant $P_t=0$ is not optimal because any deviation to a price above 0 gives a positive payoff. Now take a decreasing price sequence for which $\lim_{t\to\infty}P_t=0$ then an upper bound for $V(F_t^*)$ is $\frac{P_t(1-P_t)}{1-\beta\frac{z_2}{z_1}}$. Therefore $\lim_{t\to\infty}V(F_t^*)=0$ because $\lim_{t\to\infty}P_t=0$. However $V(F)$ is also bound from below by $\frac{1}{4}$ from charging the one period monopoly price $P=\frac{1}{2}$. $\lim_{t\to\infty}V(F_t^*)=0$ is therefore a contradiction and $P^*=0$ is never the case.

$\Gamma_F(F_t,P_t)$ is continuous in P_t which implies that

$$\begin{aligned}\lim_{t\to\infty}f_t^*(\theta) &= f^*(\theta)=\frac{z_2}{z_2-z_1}\text{ for }\theta<P^*\\ \lim_{t\to\infty}f_t^*(\theta) &= f^*(\theta)=\frac{z_2-\frac{z_1}{1-P^*}}{z_2-z_1}\text{ for }\theta\geq P^*\end{aligned}$$

Define the discounted sum of profits from a sequence $P_t=P^*$ and $f_t=f^*$ for all t as:

$$\begin{aligned}\Pi(P^*) &= \sum_{t=0}\beta^tP^*(1-P^*)M_t\\ &\quad st\\ M_0 &= 1\\ M_{t+1} &= \left[\left(1-\frac{z_2}{z_2-z_1}P^*\right)\frac{z_2}{z_1}+\frac{z_2}{z_2-z_1}P^*\right]M_t\end{aligned}$$

From the optimality of $\{P_t^*\}$ and continuity of V in F $\lim_{t\to\infty} V(F_t^*) = V(F^*) = \Pi(P^*)$. Therefore for any $\varepsilon > 0$ t can be chosen high enough such that $\Pi(P^*) + \varepsilon > V(F_t^*) > \Pi(P^*) - \varepsilon$.

Now consider the following one period deviation from $\{P_t^*\}$, in period t charge $P_t^* - \delta$. This strategy cannot be better than $\{P_t^*\}$ so:

$$(P_t^* - \delta)(1 - (P_t^* - \delta)) + \beta\left((1 - F_t^*(P_t^* - \delta))\frac{z_2}{z_1} + F_t^*(P_t^* - \delta)\right)V(F_{\delta,t+1}) - V(F_t^*) \le 0$$

where $F_{\delta,t+1} = \Gamma_F(F_t^*, P_t^* - \delta)$.

$F_t^* \to F^*$ $P_t^* \to P^*$ so for any $\sigma > 0$ t may be chosen large enough such that $F_t^*(P_t^*) - F_t^*(P_t^* - \delta) \ge \frac{z_2}{z_2 - z_1} - \sigma$. Since $P_t^* - \delta < P_t^*$, F_δ FOSD F_{t+1}^* and $V(F_{\delta,t+1}) \ge V(F_{t+1}^*)$. Combining these two facts the following is true:

$$-\delta(1 - 2P_t) - \delta^2 - \beta\left(\delta\left(\frac{z_2}{z_2 - z_1} - \sigma\right)\left(\frac{z_2}{z_1} - 1\right)\right)V(F_{t+1}^*) \ge 0$$

rearranging

$$-\delta(1 - 2P_t) - \delta^2 - \beta\left(\delta\left(\frac{z_2}{z_1} - \sigma\left(\frac{z_2}{z_1} - 1\right)\right)\right)V(F_{t+1}^*) \ge 0$$

where the first two terms are the change in this periods profits and the second term is a lower bound on the change in future profits from selling to more people today. Finally the continuity of V implies that for any $\varepsilon > 0$, t may be chosen large enough such that $\Pi(P^*) - \varepsilon \le V(F_{t+1}^*)$

$$\Longrightarrow \beta \le \frac{1 - 2P_t}{\left(\frac{z_2}{z_1} - \sigma\left(\frac{z_2}{z_1} - 1\right)\right)(\Pi(P^*) - \varepsilon)} - \frac{\delta}{\left(\frac{z_2}{z_1} - \sigma\left(\frac{z_2}{z_1} - 1\right)\right)(\Pi(P^*) - \varepsilon)}$$

for any $\omega > 0$ $\exists \delta, \varepsilon, \sigma > 0$ such that

$$\begin{aligned} & \frac{1 - 2P_t}{\left(\frac{z_2}{z_1} - \sigma\left(\frac{z_2}{z_1} - 1\right)\right)(\Pi(P^*) - \varepsilon)} - \frac{\delta}{\left(\frac{z_2}{z_1} - \sigma\left(\frac{z_2}{z_1} - 1\right)\right)(\Pi(P^*) - \varepsilon)} \\ \le\; & \frac{1 - 2P_t}{\frac{z_2}{z_1}\Pi(P^*)} - \omega \end{aligned}$$

so

$$\beta \leq \frac{1-2P_t}{\frac{z_2}{z_1}\Pi\left(P^*\right)} - \omega \tag{1.9}$$

for any $\omega > 0$. Note for $\beta > 0$ this also rules out $P^* = \frac{1}{2}$.

Now consider a different deviation during period t to $P_t^* + \delta$:

$$\left(P_t+\delta\right)\left(1-\left(P_t+\delta\right)\right)+\beta\left(\left(1-F^*\left(P_t+\delta\right)\right)\frac{z_2}{z_1}+F^*\left(P_t+\delta\right)\right)V\left(F_{\delta,t+1}\right)-V\left(F_t^*\right)$$

Note

$$F^* = \Gamma_F\left(\Gamma_F\left(F^*, P_t^*+\delta\right), P^*\right)$$

now $V\left(F_{\delta,t+1}\right) > \Pi\left(P^*\right) - \varepsilon$ since a feasible strategy is to charge P^* in every period after $P_t^*+\delta$. Since Γ_F, V are continuous for any $\sigma, \varepsilon > 0$ t can be chosen high enough such that $\left|F^* - \Gamma_F\left(\Gamma_F\left(F_t^*, P_t^*+\delta\right), P^*\right)\right| < \sigma$ and hence that $V\left(F_{\delta,t+1}\right) > \Pi\left(P^*\right) - \varepsilon$

$$\begin{aligned}
&\delta\left(1-2P_t\right)+\delta^2+\beta\delta\left(\frac{z_2-\frac{z_1}{1-P^*}}{z_2-z_1}-\sigma\right)\left(\frac{z_2}{z_1}-1\right)\left(\Pi\left(P^*\right)-\varepsilon\right)\\
\geq\ & 0\\
\Longrightarrow\ & \beta \geq \frac{1-2P_t}{\left(\frac{z_2-\frac{z_1}{1-P^*}}{z_2-z_1}-\sigma\right)\left(\frac{z_2}{z_1}-1\right)\left(\Pi\left(P^*\right)-\varepsilon\right)} - \frac{\delta}{\left(\frac{z_2-\frac{z_1}{1-P^*}}{z_2-z_1}-\sigma\right)\left(\frac{z_2}{z_1}-1\right)\left(\Pi\left(P^*\right)-\varepsilon\right)}
\end{aligned}$$

for any $\omega > 0$ $\exists\delta, \varepsilon, \sigma > 0$ such that

$$\begin{aligned}
&\frac{1-2P_t}{\left(\frac{z_2-\frac{z_1}{1-P^*}}{z_2-z_1}-\sigma\right)\left(\frac{z_2}{z_1}-1\right)\left(\Pi\left(P^*\right)-\varepsilon\right)} - \frac{\delta}{\left(\frac{z_2-\frac{z_1}{1-P^*}}{z_2-z_1}-\sigma\right)\left(\frac{z_2}{z_1}-1\right)\left(\Pi\left(P^*\right)-\varepsilon\right)}\\
\geq\ & \frac{1-2P_t}{\left(\frac{z_2-\frac{z_1}{1-P^*}}{z_1}\right)\Pi\left(P^*\right)} - \omega
\end{aligned}$$

so

$$\beta \geq \frac{1-2P_t}{\frac{z_2-\frac{z_1}{1-P^*}}{z_1}\Pi\left(P^*\right)} - \omega \tag{1.10}$$

For

$$\omega < \frac{\left(1-2P_t\right)\left(z_1\right)^2}{\Pi\left(P^*\right)\left(1-P^*\right)z_2\left(z_2-z_1\right)}$$

both conditions given by equations 1.9 and 1.10 cannot be met so either $P_t+\delta$ or $P_t-\delta$

is a profitable deviation which is a contradiction that there exists a T such that $\{P_t^*\}$ is weakly increasing or decreasing for all $t \geq T$.

Proof of Theorem 8

For all (ω, P) *except* $(0, P^{crit})$ *the profit function is continuous and differentiable with respect to both price and advertising, and* $\lim_{(\omega,P)\to(0,P^{crit})} \pi(\omega, P) = 0$

Proof. Provided H_0 is differentiable with respect to P, ω then so is π.

$$H_0(1-\omega, P) = P + (1-\omega)(1-P)\sum p_k^k (u^*)^k$$

where $u^* = H_1(1, P)$ is the smallest non-negative solution to

$$u^* = P + (1-\omega)(1-P)\sum \frac{kp_k^{k-1}(u^*)^{k-1}}{z}$$

H_0 is differentiable if u^* is differentiable in ω and P. The right hand side of the equation for u^* is continuous, increasing and convex in u^*. I will show the differentiability of u^* for the 3 cases $\omega > 0$; $P > P^{crit}$;and $P < P^{crit}$.

Using the implicit function theorem we have

$$\begin{aligned} \frac{du^*}{dP} &= \frac{1 - G_1(u^*)}{1-(1-\omega)(1-P)G_1'(u^*)} \\ \frac{du^*}{d\omega} &= \frac{-(1-P)G_1(u^*)}{1-(1-\omega)(1-P)G_1'(u^*)} \end{aligned}$$

$\frac{du^*}{dP}$ and $\frac{du^*}{d\omega}$ exist provided $1-(1-\omega)(1-P)G_1'(u^*) > 0$

When $\omega > 0$ the right-hand side of $u = P + (1-\omega)(1-P)\sum \frac{kp_k u^{k-1}}{z}$ is strictly less than 1 if $u = 1$ and strictly greater than 0 when $u = 0$. Therefore the solution is strictly less than 1, and at the solution $1-(1-\omega)(1-P)G_1'(u) > 0$.

When $P > P^{crit}$ by the definition of P^{crit}

$$\begin{aligned} P^{crit} &= 1 - \frac{1}{G_1'(1)} \\ &\Rightarrow (1-P)G_1'(u) < 1 \text{ for } u \leq 1 \text{ and } P > P^{crit} \end{aligned}$$

so again $1-(1-\omega)(1-P)G_1'(u)>0$ and $\frac{du}{dP}$ and $\frac{du}{d\omega}$ exist.

When $P<P^{crit}$ consider $P+(1-\omega)(1-P)\sum\frac{kp_k u^{k-1}}{z}$. This is strictly convex in u for $0\le P<P^{crit}$ and equal to 1 at $u=1$. At any solution $u^*<1$ $(1-\omega)(1-P)G_1'(u^*)<1$ otherwise $P+(1-\omega)(1-P)G_1(1)\neq 1$. Hence $1-(1-\omega)(1-P)G_1'(u)>0$ and $\frac{du}{dP}$ and $\frac{du}{d\omega}$ exist.

Finally consider

$$\begin{aligned}\lim_{(\omega,P)\to(0,P^{crit})}\pi(\omega,P) &= \lim_{(\omega,P)\to(0,P^{crit})}(P-c)(1-H_0(1-\omega,P))-\alpha\omega\\ &= \left(P^{crit}-c\right)\left(1-\lim_{(\omega,P)\to(0,P^{crit})}H_0(1-\omega,P)\right)\end{aligned}$$

It was shown in Theorem 2 that for the case $\omega=0$ $\lim_{P\to P^{crit}}1-H_0(1,P)=0$ so the theorem holds for this case. Now considering the case $\omega>0$ take any sequence $(\omega,P)\to(0,P^{crit})$ where $\omega>0$ the expression $P+(1-\omega)(1-P)\sum\frac{kp_k^{k-1}(u)^{k-1}}{z}<1$ at $u=1$ and $\lim_{(\omega,P)\to(0,P^{crit})}P+(1-\omega)(1-P)\sum\frac{kp_k^{k-1}(u)^{k-1}}{z}=1$ furthermore $\lim_{(\omega,P)\to(0,P^{crit})}u^*=1$. Hence $\lim_{(\omega,P)\to(0,P^{crit})}H_0(1-\omega,P)=1$ and $\lim_{(\omega,P)\to(0,P^{crit})}\pi(\omega,P)=0$. ■

Proof of Corollary 4

If $\pi(\omega,P)>0$ for some (ω',P') then $\exists\varepsilon>0$ such that for all $(\omega,P)\in B_\varepsilon(0,P^{crit})$ where B_ε is an open ball $\pi(\omega,P)<\pi(\omega',P')$

Proof. From theorem 8 the following two properties hold for profit $\pi(\omega,P)$ is continuous in (ω,P) for $(\omega,P)\neq(0,P^{crit})$ and $\lim_{(\omega,P)\to(0,P^{crit})}\pi(\omega,P)=0$. The result follows immediately for ε small enough $(\omega,P)\in B_\varepsilon(0,P^{crit})\Rightarrow\pi(\omega,P)<\pi(\omega',P')$. ■

Proof of Theorem 9

If $0<P^{crit}<1$, for any sequence of strategies $\lim_{(\omega,P)\to(0,P^{crit})}H_0'(1,P)=\infty$

Proof. $H_0'(1,P)$ is given by:

$$H_0'\left(\omega,P^{crit}\right)=(1-P)\left[G_0(H_1(1,P))+\frac{z_1(1-P)\left[G_1'(H_1(1,P))\right]^2}{1-(1-\omega)(1-P)G_1'(H_1(1,P))}\right]$$

where $H_1(1,P)$ is the smallest non-negative solution to

$$H_1(1,P)=P+(1-P)(1-\omega)G_1(H_1(1,P))$$

Theorem 8 proves that $H_0'(\omega, P)$ is defined everywhere except $(0, P^{crit})$. Now consider any sequence $\{(\omega, P)\} \to (0, P^{crit})$ then

$$\lim_{\{(\omega,P)\}\to(0,P^{crit})} (1-P)\left[G_0(H_1(1,P)) + \frac{z_1(1-P)\left[G_1'(H_1(1,P))\right]^2}{1-(1-\omega)(1-P)G_1'(H_1(1,P))}\right]$$
$$= (1-P^{crit})\left[G_0(1) + \frac{z_1(1-P^{crit})\left[G_1'(1)\right]^2}{1-(1-P^{crit})G_1'(1)}\right]$$

where $(1-P^{crit}), G_0(1)$ and $z_1(1-P^{crit})\left[G_1'(1)\right]^2$ are finite and from the definition of P^{crit} $1-(1-P^{crit})G_1'(1) = 0$. Hence $\lim_{(\omega,P)\to(0,P^{crit})} H_0'(1,P) = \infty$. ■

Proof of Theorem 10

Advertising exhibits decreasing and convex marginal returns

Proof. Returns to advertising are given by $H_0'(1-w, P)$. The rate of change of the returns with respect to advertising level w is given by $\frac{dH_0'(1-w,P)}{dw} = -H_0''(1-w,P)$ where $-H_0''(1-w,P) < 0$ and $H_0'''(1-w,P) > 0$ because $H_0''(1-w,P)$ is a polynomial in $(1-w)$with positive coefficients. ■

Proof of Theorem 11

Assuming $p_k > 0$ for all k the highest return type of individual k^* is found as the solution to:

$$k^* \in \{\lfloor k^{**} \rfloor, \lceil k^{**} \rceil\} \text{ for } P < P^{crit}$$

where

$$k^{**} = \max\left\{0, -\left(\frac{1}{\ln H_1(1,P)} + \frac{H_1(1,P)}{H_1'(1,P)}\right)\right\}$$

Proof. The probability generating function of component sizes an individual with k friends belongs to, conditional on not being in the giant component, is given by $\left(\frac{H_1(x,P)}{H_1(1,P)}\right)^k$. The expected component size is $1 + k\frac{H_1'(1,P)}{H_1(1,P)}\left(\frac{H_1(1,P)}{H_1(1,P)}\right)^{k-1} = 1 + k\frac{H_1'(1,P)}{H_1(1,P)}$. Also the probability a person with k friends is not in the giant component is $H_1(1,P)^k$. Therefore

$$k^* = \arg\max_{k\in\{0,1\ldots\}} \left(1 + k\frac{H_1'(1,P)}{H_1(1,P)}\right) H_1(1,P)^k$$

note that for $0 < H_1(1,P) < 1$ $b > 0$ the function $f(k) = (1+kb)H_1(1,P)^k$ is continuous in k; has a maximum at $k^{**} = \max_{k\geq 0}\left\{0, -\left(\frac{1}{\ln H_1(1,P)} + \frac{1}{b}\right)\right\}$ and $f'(k) > 0$ for $k < k^{**}$ and $f'(k) < 0$ for $k > k^{**}$. Hence k^* is either the greatest integer below $\lfloor k^{**} \rfloor$ or the smallest integer above k^{**}, $\lceil k^{**} \rceil$. Thus

$$k^* \in \{\lfloor k^{**} \rfloor, \lceil k^{**} \rceil\} \text{ for } P < P^{crit}$$

■

Proof of Corollary 5

The optimal target k^{**} is continuous in $H_1(1,P)$ for $H_1(1,P) < 1$, $\lim_{P\to P^{crit}} k^{**} = \infty$, $k^{**} \leq \frac{-1}{\ln(H_1(1,P))}$ for $P < P^{crit}$

Proof. We have

$$\begin{aligned}
k^{**} &= -\left(\frac{1}{\ln H_1(1,P)} + \frac{H_1(1,P)}{H_1'(1,P)}\right) \\
&= -\left(\frac{1}{\ln H_1(1,P)} + \left(\frac{(1-P)\,G_1(H_1(1,P))}{H_1(1,P)\,(1-(1-P)\,G_1'(H_1(1,P)))}\right)^{-1}\right) \\
&= -\left(\frac{1}{\ln H_1(1,P)} + \left(\frac{(1-P)\sum kp_k H_1(1,P)^{k-2}}{\left(z_1 - (1-P)\sum k(k-1)\,p_k H_1(1,P)^{k-2}\right)}\right)^{-1}\right)
\end{aligned}$$

where $\frac{1}{H_1'(1,P^{crit})}$ is finite so immediately

$$\lim_{P\to P^{crit}} \frac{-1}{\ln H_1(1,P)} = \infty$$

$$\Rightarrow \lim_{P\to P^{crit}} k^* = \infty$$

For $P < P^{crit}$

$$z_1 - (1-P)\sum k(k-1)\,p_k H_1(1,P)^{k-2} > 0$$

For $H_1(1,P) > 0$

$H_1(1,P) > 0$ and $(1-P)\sum kp_k H_1(1,P)^{k-2} > 0$ so k^* is continuous in $H_1(1,P)$ and hence P for $P < P^{crit}$. Finally $\frac{(1-P)\sum kp_k H_1(1,P)^{k-2}}{\left(z_1-(1-P)\sum k(k-1)p_k H_1(1,P)^{k-2}\right)} > 0$ so $-\frac{1}{\ln H_1(1,P)}$ is an upper bound on k^{**}. ■

Proof of Theorem 12

If $P^{j^{**}} > P^{crit}$ then $\exists \widehat{k}$ such that for $k \geq \widehat{k}$ $j^{**} = j^{*k}$

Proof. If $P^{j^{**}} > P^{crit}$ then

$$H_1\left(1, P^{j**}\right) = 1$$

and

$$H_0'\left(1, P^{j^{**}}\right) = \sum p_k \left(1 - P^{j^{**}}\right)\left(1 + kH_1'\left(1, P^{j^{**}}\right)\right)$$

For any $P^j > P^{j*}$

$$\left(1 - P^{j^{**}}\right)\left(1 + kH_1'\left(1, P^{j^{**}}\right)\right) > \left(1 - P^j\right)\left(1 + kH_1'\left(1, P^j\right)\right) \text{ for all } k$$

because $H_1'(1, P)$ is decreasing in P. For any $P^j < P^{crit}$:

$$H_1(1, P^{j^{*k}}) < 1$$

because k increases the returns to advertising to a person with k friends and it has the following properties

$$\lim_{k \to \infty} \left(H_1\left(1, P^j\right)\right)^k \left(1 - P^j\right)\left(1 + k\frac{H_1'\left(1, P^j\right)}{H_1\left(1, P^j\right)}\right) = 0$$

and

$$\frac{\partial\left(\left(H_1\left(1, P^j\right)\right)^k \left(1 - P^j\right)\left(1 + k\frac{H_1'\left(1, P^j\right)}{H_1\left(1, P^j\right)}\right)\right)}{\partial k} < 0$$

$$\text{for all } k > -\left(\left(\frac{H_1'\left(1, P^j\right)}{H_1\left(1, P^j\right)}\right)^{-1} - \left(\ln H_1\left(1, P\right)\right)^{-1}\right)$$

which implies that $\exists \widehat{k}$ such that

$$\left(1 - P^{j^{**}}\right)\left(1 + kH_1'\left(1, P^{j^{**}}\right)\right) > \left(H_1\left(1, P^j\right)\right)^k \left(1 - P^j\right)\left(1 + k\frac{H_1'\left(1, P^j\right)}{H_1\left(1, P^j\right)}\right) \text{ for all } k \geq \widehat{k}$$

■

Chapter 2

Signaling in Social Networks

2.1 Introduction

Social networks are important in a variety of environments. The importance of these networks has been highlighted theoretically and empirically in many areas including risk sharing (Fafchamps and Lund (2003); Bloch, Genicot and Ray (2005)), diffusion of innovation (Young 2000), adoption/diffusion of behaviour (Glaeser, Sacerdote and Scheinkman (1996); Jackson and Yuriev (2006)), trust and social capital (Mobius and Szeidl (2006)) and search in labour markets (Calvo-Armengol and Jackson (2004)). These models, for the most part, have treated the network structure as exogenous and studied how in these various environments the characteristics of any given network affect individual and social welfare. Typically these papers have implications for policy for improving aggregate welfare in the form of changes to the social network. Models of network formation are needed for designing policies and assessing their likely effects.

In this paper I provide a rationale for why individuals may search for and form friendships with individuals with whom they share a friend in common. I present a model of cooperation in a social network in which the network is growing over time. In this environment new players decide how to search for new friends. The way in which players search for and then form friendships reveals how willing they are to engage in cooperation with a potential friend.

In the model, a friendship allows individuals to interact in two ways. Firstly the two individuals may derive utility from the friendship by engaging in a low or high stakes prisoner's dilemma game during each period. Secondly individuals may also communicate with one another, in particular, it allows either individual to pass a warning to the other if a friend they share in common has not cooperated during an earlier prisoner's dilemma game. There are two types of people in the population one type which is prepared to cooperate and another who is not, these are modelled through the discount factor whereby cooperating individuals have a high discount factor and individuals who do not cooperate have a low discount factor. The equilibrium is semi-separating whereby cooperative (patient) agents are able to signal their type through choosing to become friends with people who know one another. The threat of communication between people who know one another means that uncooperative (impatient) agents do not choose to form friendships in the same way. This signaling results in people trusting each other more when they share a friend in common.

The understanding of how and why social networks form has come from two different strands of literature. The random graph literature and the economic literature. The former employs statistical tools to describe the construction of a network according to some mechanical algorithm and the subsequent properties of the resulting network. This strand of literature has been successful in showing how characteristics of observed networks can result from some elementary stochastic or mechanical process. What it often does not provide is a rationale for why the network forms through a particular process and not another. This is especially important in the case of social networks when individuals are making choices about how to find and become friends with other individuals. In a recent paper Jackson and Rogers (JR) (2007) show how a process of network growth, that incorporates random and network (meeting friends of friends) based meetings, produces networks which exhibit many of the stylized characteristics associated with social networks as the parameters describing the formation process are changed. On the other hand the economic literature explains network structure from a game theoretic perspective. This literature allows links to form in a network at the discretion of economic agents who are or control the nodes of the network. It explicitly incorporates costs and benefits of network formation to agents and is able to characterize a network as an equilibrium in agents' actions. This allows one to analyze the networks which form in terms of efficiency and welfare properties. By explicitly incorporating agents' decisions this literature gives us a good deal of information about what networks are stable but has thus far generally stopped short of giving predictions about degree distributions, clustering coefficients etc. that one can match up to observed data.

This paper is related to both literatures in that the links are formed and broken at the discretion of economic agents, akin to the economics literature, such that the agents' actions constitute a process of network formation equivalent to a process from the random graph literature. The model builds on the algorithm of network formation proposed by JR, from the random networks literature, where the probability an agent receives a new link is in part random and in part proportional to the number of existing friends of an agent. The motivation for the model comes from the observation that people generally "trust" people with whom they share friends in common. In the model "trust" arises endogenously in that when two people share a friend in common each believes with probability 1 that the other will cooperate in a repeated prisoners dilemma. The equilibrium of the model is one in which players signal that they are willing to cooperate in a repeated prisoner's dilemma through their choice of friends.

The model illustrates how patient players can avoid an initial period of screening (playing a low stakes game), when establishing cooperation with another patient player, by sharing a friend in common with that player. The key to achieving this is that the social network facilitates communication among connected agents as well as the prospect to cooperate. In this environment impatient types (who prefer to always defect) would prefer to defect against multiple players playing in a low stakes game than successfully defect once in a high stakes game and have all their other friends find out. It is this threat of communication which allows patient players to credibly signal their type. Patient players prefer to cooperate, and are thus unconcerned of the threat posed by communication, so can credibly signal their type by choosing friends who know one another.

The purpose of this paper is to propose a model which firstly provides a motivation for why individuals choose to become friends with certain people within the social network and secondly to use this model to explain how observed characteristics of social networks may change in different environments or change as a result of some policy. Thus far there are few papers which have been able to do both. One exception is Currarini, Jackson and Pin (2008) which derives a model of network formation to describe segregation patterns in a population of heterogenous groups of agents. In this paper the focus is primarily on the distribution of friendships and clustering aspects of a homogenous population whereas Currarini, Jackson and Pin (2008) focus on the differences between the various heterogenous groups in the population in terms of the number of friendships formed, the relative number of same-type versus other-type friends and the relative same type bias.

2.2 Model

Let time be denoted by $t = 0, ..., \infty$.

2.2.1 Social network

In period $t = 0$ there is a social network with N players. This network is represented by an $N \times N$ matrix G^0 where an element of G^0, $g_{ij}^0 = 1$ indicates that player i has established a friendship with player j. Thus the social network is directed.

Each period a new player is added to the social network and forms links with the existing players. Each successive network is described by an $(N+t) \times (N+t)$ matrix G^t.

Players have some knowledge of their local area of the network. Define Q_i^t the neighborhood of player i at time t where $Q_i^t = \{j \mid \max\{g_{ij}^t, g_{ji}^t\} = 1\}$. Let $K_i^t, K_{ij}^t, K_{ijk}^t$ be $K_i^t = \{\max\{g_{kk'}^t, g_{k'k}^t\} \mid (k,k') \in Q_i^t \times Q_i^t\}$, $K_{ij}^t = \{\max\{g_{jk'}^t, g_{k'j}^t\} \mid k' \in Q_i^t\}$ for a $j \in Q_i^t$ and $K_{ijk}^t = \max\{g_{jk}^t, g_{kj}^t\}$ for all $(j,k) \in Q_i^t \times Q_i^t$. A player i at time t knows $\cup_{N-i \leq t' \leq t} Q_i^t$ and $\cup_{N-i \leq t' \leq t} K_i^t$.

These assumptions are simply that a player knows the identity of his/her friends and furthermore if any of his/her friends know one another and the history of the existence of these relationships since she was born.

Prisoner's Dilemma

If a friendship exists between two players ($\max\{g_{ij}^t, g_{ji}^t\} = 1$) these players participate in a high or low stakes $s_{ij}^t = H, L$ prisoners dilemma (PD) game during each period $t = 0, 1...\infty$ of play. The stakes of the games a player i participates in during period t is $S_i^t = \{s_{ij}^t | j \in Q_i^t\}$. The stake of the game is chosen by one of the players which I assume to be $\min\{i, j\}$ which is the older of the two players. A player i then chooses an action $a_{ij}^t = \{C, D\}$ when facing player j during period t, the set of actions a player uses during a period t is given by $A_i^t = \{a_{ij}^t | j \in Q_i^t\}$. The row player payoffs for this game are:

	C	D
C	z_s	$-y_s$
D	x_s	0

The payoffs of the game are all increasing in the stake and the dominant strategy of the one shot game is to play D ($x_s > z_s$ and $y_s > x_s > 0$). Also define $\Delta z = z_H - z_L$ $\Delta x = x_H - x_L$ and $\Delta y = y_H - y_L$.

Communication

A friendship also allows two players to communicate with one another about the behavior of players whom they both know for i and j this set of players is $Q_i^t \cap Q_j^t$. In the context of this model I limit communication to player j sending player i a verifiable message about all players $k \in Q_i^t \cap Q_j^t$ reporting $w_k^{ijt} = 1$ if player k played D when j played C during period t. I assume that upon receiving this information a player can then pass it on to any other friend $i' \in Q_i^t$ that also knows k, $k \in Q_i^t \cap Q_{i'}^t$. The set of communications a player i receives in period t is denoted by W_i^t.

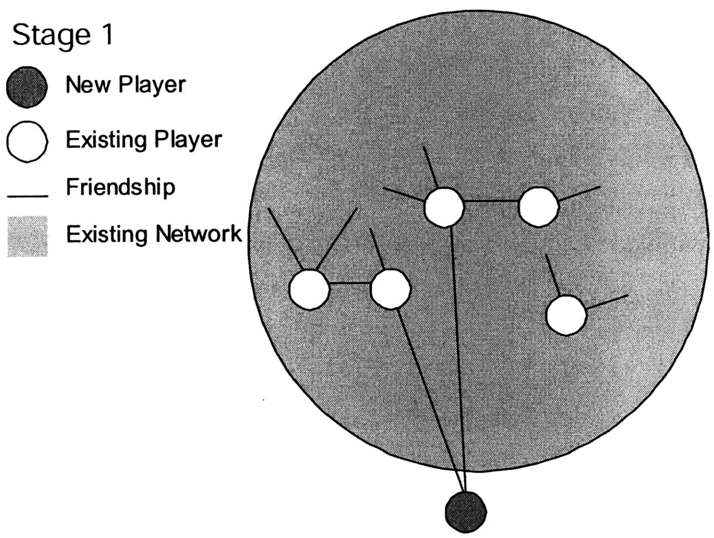
Types of players

There are two types of players patient and impatient. The discount factor δ of patient players is $\delta_P \approx 1$ and for impatient players $\delta_I \approx 0$.

Choice of friends

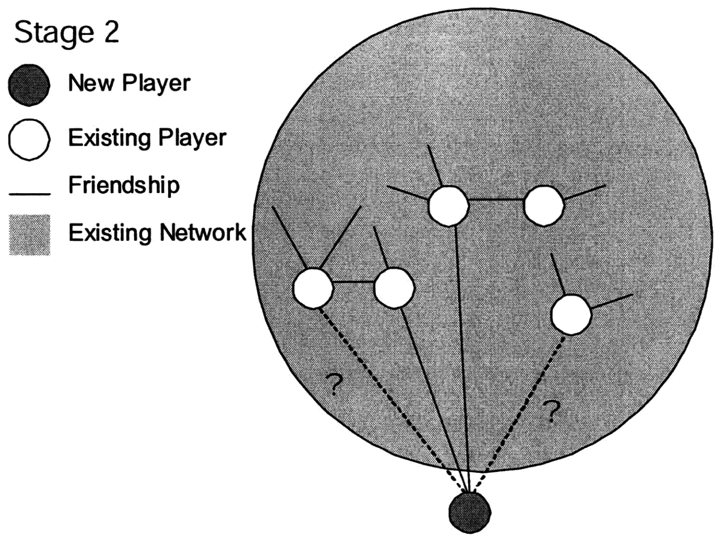
Each period a new player is born and enters the social network. When a player enters the social network she forms $M << N$ links with the existing network. A proportion $\gamma \in \left[0, \frac{1}{2}\right]$ of these connections are chosen by the player and the remaining $(1-\gamma) M$ connections are established at random. For simplicity imagine the following two stage procedure:

Stage 1 A player joining the network, randomly meets $(1-\gamma) M$ individuals. Figure 2.2.1 shows stage one for the parameter values $M = 3$ and $\gamma = \frac{1}{3}$:



Stage 1

Stage 2 In stage 2 the player has a choice. A player can either connect to γM additional individuals, where each one of these additional individuals is a friend of one of the $(1-\gamma) M$ individuals met in stage 1 (network-based meetings); or, alternatively, a player can connect to γM randomly chosen individuals from the population (who may or may not be friends with the original $(1-\gamma) M$ individuals, again I assume that if they are met randomly the probability that any two are themselves friends is approximately 0) (random meetings). I assume that if player j was met during stage 1 amongst his/her friends only those for whom $g_{jk} = 1$ (not those for whom $g_{kj} = 1$[1]) can be met in stage 2 each with equal probability $\left(\frac{1}{M}\right)$ if the player chooses to connect through network-based meetings. Figure 2.2.1 shows a new player with the choice of connecting to a friend of a friend or another randomly chosen individual.



Stage 2

The existing players in the network do not observe whether or not it was possible for the new player to choose to connect to one of their friends, they only know that the new player will have the opportunity to do this with γM individuals out of the $(1-\gamma)M$ original individuals that the new player initially met randomly. In particular they don't know if they were part of the γM players whose friends the new player could have chosen to connect to. Therefore, when an existing player meets a new player and that new player does not also connect to one of her friends, she does not know whether the new player chose not to do so, or whether she never had the chance to. Looking ahead in the model patient types can signal their type by choosing to connect to γM friends of some of the people she met in the network, while short lived types will choose to connect to random individuals.

Finally I assume that a player with no connections is removed from the social network. This allows one to focus on the signaling incentives of new players when choosing whether

[1] A similar assumption is needed in JR so that the alogirthm of network formation is sufficiently tractible to derive the characteristics of the social network and is maintained here for the same reason.

or not to search for friends or friends. If impatient players remain in the social network then by searching for friends of friends this has an additional benefit of potentially identifying individuals who do not have any friends and are therefore an impatient type. This may be a realistic reason for doing so however here I would like to focus on the signaling achieved through the choice of friends. By removing these impatient types, a new player believes that when it is born and forms friendships any player it finds is patient. If the new player is patient she is therefore only concerned about convincing their friends of this through their choice of random versus network based meetings.

Sequence of play

There is an infinite number of periods of play. During each period, play proceeds as follows:

1. A new player $i = N + t$ is born and connects to the network as described above, and all players $j < i$ update their knowledge of the network Q_j^t, K_j^t
2. Links are ordered randomly.
3. On the first link the oldest player $\min\{j, k\}$ (of the two, one at either end) chooses whether to play a game for high or low stakes these players update S_i^t.
4. The prisoner's dilemma stage game is then played, players update A_i^t.
5. Communication takes place.
6. Repeat steps 3-5 for the second, third ... links until every game has taken place.

Players update their personal history $h_i^t = \cup_{N-i \le t' \le t} \left(A_i^{t'}, S_i^{t'}, W_i^{t'}, Q_i^{t'}, K_i^{t'}\right)$ as events occur.

2.3 A sequential equilibrium of the model

In this section I present a semi-separating sequential equilibrium of the model where patient players signal their type through choosing network-based meetings during stage

2 of the joining process. In the equilibrium a patient player is able to gain the trust of a proportion γ of friends by choosing to share friends in common. This proportion γ of the friends then believe the player is patient with probability ≈ 1 and is willing to cooperate and therefore agree to play a game for high stakes in the first period of interaction.

Theorem 13 *Suppose* $x_H + z_H < 2x_L$, *then* $\exists\ \underline{\delta} < 1$ *and* $\overline{\delta} > 0$ *such that if* $\delta_P \in (\underline{\delta}, 1)$ *and* $\delta_I \in (0, \overline{\delta})$ *there exists a sequential equilibrium in which patient players choose network based meetings and an existing player i has beliefs, upon meeting, a new player j* $\lim_{t\to\infty} \Pr(\delta_j = \delta_P | Q_i^t \cap Q_j^t \neq \varnothing) = 1$.

I will relegate the proof and the description of equilibrium beliefs and strategies to an appendix. Instead I will give a sketch of the equilibrium and intuition for the result.

When two players meet and share a friend in common they play a high stakes game from the first period of interaction. If the two players do not share a friend in common then a low stakes game is played in the first period and if cooperation occurs a high stakes game is played thereafter. In the equilibrium patient players choose to make network based connections when joining the network to avoid playing a low stakes game in the first period of interaction. On the other hand impatient players choose to make random connections when joining the network and defect in the low stakes game offered during the first period of interaction in all the friendships which they form. The threat of communication between two players who know one another prevents impatient types making network connections.

The choice of how to form friendships by incoming players results in existing players in the network holding beliefs $\Pr(\text{Patient}) \approx 1$ when they meet a player with whom they share a friend in common and beliefs $\Pr(\text{Impatient}) = \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}$ when they do not. Players update their beliefs to $\Pr(\text{Patient}) = 1$ if a player cooperates in the first period of interaction. If a player receives a warning about another player then they will update their beliefs to $\Pr(\text{Impatient}|\text{warning}) = 1$.

A patient player chooses to make network based connections because it signals to the other player that they are patient. This allows the patient player to avoid playing a low stakes game with the players with whom they share a friend in common. An existing patient player will choose to cooperate with a player when they do not share a friend in common provided:

$$z_L + \frac{\delta_P z_H}{1-\delta_P} - \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)} \left(y_L - z_L - \frac{\delta_P z_H}{1-\delta_P} \right) > 0$$

which will be the case for δ_P sufficiently close to 1.

Impatient players choose random based connections because this type of connection prevents communication between two players who will play the impatient player. In the event that the impatient player did make a network based meeting the best she could do during the subsequent period of game play would be to cooperate with the first of the two players and defect on the second. Mimicking the patient players' friendship strategy is not better than the strategy of impatient types when defecting on two opponents in a low stakes game is better than cooperating and defecting in a high stakes game. This is the condition in the theorem, $2x_L > x_H + z_H$. Impatient types will always defect on equilibrium when they are sufficiently impatient. Off equilibrium for a sufficiently large number of people $\zeta^* > M$ who all know the impatient player and each other, the impatient player will in fact choose to cooperate because the threat of punishment facilitated by communication amongst this entire group is very large. Of course the impatient player would have to cooperate for a long period of time to acquire this number of friends so it does not occur on equilibrium.

The sequential equilibrium is one in which patient players can signal their type through sharing friends in common with their friends. The intuition for the result is that the threat to impatient players of communication and group sanctioning provided by the social network allows patient players to credibly signal their own type. This signaling motive influences the patient players pattern of friendships which they choose and is revealed in the characteristics of the network itself such as degree distributions and clustering coefficient.

The incentives which influence a player's choice of friends have implications for the dynamic process of network growth. The incentives which govern the choices of the patient players (who are the only types which remain in the network beyond the age of 1 period) determine the evolution of the social network. I consider the implications of the model as the network becomes very large. The growth process that results as an equilibrium is such that all players have an equal probability of getting a random link and a probability proportional to their in-degree of receiving network based links.

Corollary 6 *In each period the probability an existing node has of obtaining a new link to a patient player is approximately* $(1-\lambda)\left(\frac{(1-\gamma)M}{t+N}+\frac{d_i(t)\gamma}{t+N}\right)$ [2]

Proof:

Note

$$\begin{aligned} & \Pr(\text{New link from a patient player}) \\ = \; & \Pr(\text{Patient player}) \times \Pr(\text{New link}|\text{Patient player}). \end{aligned}$$

Now

$$\Pr(\text{Patient player}) = (1-\lambda);$$

and

$$\Pr(\text{New link}|\text{Patient player}) \approx \frac{(1-\gamma)M}{t+N} + \frac{(1-\gamma)Md_i(t)}{t+N}\frac{\frac{\gamma M}{(1-\gamma)M}}{M}$$

where the first term is the probability of being selected at random and the second term is the probability that a new player chooses to link to the node. The second term consists of two parts, the first is $\frac{(1-\gamma)Md_i(t)}{t+N}$ which is the probability that one of the node's neighbors is found at random and the second part is $\frac{\frac{\gamma M}{(1-\gamma)M}}{M}$ which is the probability that the node is then chosen to be linked to. This then simplifies to $\frac{(1-\gamma)M}{t+N}+\frac{d_i(t)\gamma}{t+N}$.

QED

The growth process specified here has the same characteristics of network formation as JR. That is all individuals can meet new friends through two channels: randomly meeting people which occurs with equal probability (independent of the number of friends) over all individuals; and meeting people through other friends which occurs with a probability

[2] *The probablity in the theorem is approximate because it ignores the possibility some of the randomly met players are in each others' neighbourhood, or that a player could be met more than once. It is very accurate when we assume that the network is large compared to the out-degree of players $N >> M$ because the adjustments for these eventualities go to 0.*

proportional to the number of friends the player has. This combination of randomness and network based meetings which enables JR to explain many of the stylized facts about social networks and arises, in this model, as a consequence of people trusting others with whom they share friends in common.

The process described in JR incorporates several parameters to give it the flexibility to be fitted to data on a wide variety of existing social and physical networks. JR allow four parameters m_R, p_R, m_N, p_N where m_i is the number of friends identified in each stage of the joining process (R corresponds to a random meeting in stage 1 and N corresponds to a network meeting made during stage 2) and p_i is the probability each of the people met during stages 1 and 2 then becomes a friend. The equilibrium presented here encompasses the case where $p_R = p_N = 1 - \lambda$ and $(1-\gamma) M \geq \gamma M$.

2.4 Characteristics of the social network

In this section I present how the distribution of friendships and amount of clustering, in the social network which forms in the equilibrium of the previous section, relative to the primitives of the micro-foundation in this paper. This is done to illustrate how the primitives of the model can affect these characteristics.

2.4.1 Distribution of friendships

To derive characteristics of the underlying social structure I now ignore the entry of low types because these do not survive in the social network longer than 1 period. Denote time by t and a player which enters at time t by $i = t$. I rescale the time intervals from the previous model so a high type enters in each time period t and ignore the entry of low types since they do not survive in the social network. The probability that an existing node i with in-degree $d_i(t)$ gets a new link (in the next period when a High type enters) is approximately:

$$\frac{(1-\gamma) M}{t+N} + \frac{d_i(t)\gamma}{t+N}$$

Theorem 14 *The degree distribution from a mean field approximation of the network formation process is* $\lim_{t\to\infty} F_t(d) = 1 - \left(\frac{M}{\frac{\gamma}{1-\gamma}d+M}\right)^{\frac{1}{\gamma}}$

Proof:

JR provide a proof that a process where the degree of a node born at time i has initial degree d_0 and evolves according to

$$\frac{dd_i(t)}{dt} = \frac{ad_i(t)}{t} + \frac{b}{t} + c$$

when $a > 0$ and $c = 0$ or $a \neq 1$, then the complementary cdf is

$$1 - F_t(d) = \left(\frac{d_0 + \frac{d}{a} - \frac{ct}{1-a}}{d + \frac{b}{a} - \frac{ct}{1-a}}\right)^{1/a}$$

In the setting of this paper $d_0 = 0$, $a = \gamma$, $b = (1-\gamma)M$ and $c = 0$. So treating this as a continuous process then we have the differential equation:

$$\frac{dd_i(t)}{dt} = \frac{d_i(t)\gamma}{t+N} + \frac{(1-\gamma)M}{t+N}$$

we can solve this equation to get:

$$d_i(t) = \frac{1-\gamma}{\gamma}M\left(\frac{t+N}{i}\right)^{\gamma} - \frac{1-\gamma}{\gamma}M$$

At time t $1 - F_t(d)$ is the fraction of individuals with in-degree greater than d. If we solve the above expression for i such that $d_i(t) = d$ this then corresponds to the number of individuals who have a greater in-degree than d. If $i^*(d)$ is such that $d_{i^*(d)}(d) = d$ then

$$1 - F_t(d) = \frac{i^*(d)}{t+N} \text{ for all } d \text{ such that } i^*(d) > N$$

we can then derive the in-degree distribution as:

$$F_t(d) = 1 - \left(\frac{M}{\frac{\gamma}{1-\gamma}d + M}\right)^{\frac{1}{\gamma}} \text{ for all } d \text{ such that } i^*(d) > N$$

The fraction of individuals not described by this distribution $\frac{N}{t+N} \to 0$ as $t \to \infty$.

QED

To see how this relates to a scale free distribution we write this as the complimentary cdf:

$$1 - F_t(d) = \left(\frac{M}{\frac{\gamma}{1-\gamma} d + M} \right)^{\frac{1}{\gamma}}$$

If I now take logs of both sides we can see that this exhibits scale free properties for d's which are large relative to $\frac{1-\gamma}{\gamma} M$

$$\log(1 - F_t(d)) = \frac{1}{\gamma} \left[\log(M) - \log\left(\frac{\gamma}{1-\gamma} d + M \right) \right].$$

The most important property to note is that decreasing γ results in a second order stochastic dominant shift in the degree distribution. Intuitively this is because it puts additional weight on the network meetings process which in turn biases the probability of gaining an additional connection towards those with more existing connections. This has the effect of spreading out the distribution, giving it fatter tails, relative to a distribution derived from purely random meetings.

2.4.2 Clustering

I present results for three common measures of clustering. The first is the fraction of "transitive triples." This represents the fractions of times in a network where given that i knows j and j knows k that then i also knows k. The fraction is given by

$$C^{TT}(g) = \frac{\sum_{i;j\neq i;k\neq j,i} g_{ij} g_{jk} g_{ik}}{\sum_{i;j\neq i;k\neq j,i} g_{ij} g_{jk}}.$$

A second standard measure ignores the directed nature of the above relationships between individuals. This is a setting in which $\widehat{g}_{ij} = \max\{g_{ij}, g_{ji}\}$. A measure where only the existence of the relationship rather than the directed nature of it is important is

$$C(g) = \frac{\sum_{i;j\neq i;k\neq j,i} \widehat{g}_{ij} \widehat{g}_{jk} \widehat{g}_{ik}}{\sum_{i;j\neq i;k\neq j,i} \widehat{g}_{ij} \widehat{g}_{jk}}.$$

A further variation is one in which the above $C(g)$ is calculated on a node by node basis and the average is taken across all nodes. This measure is calculated as

$$C^{Avg}(g) = \frac{1}{n}\sum_i \frac{\sum_{i;j\neq i;k\neq j,i} \widehat{g}_{ij}\widehat{g}_{jk}\widehat{g}_{ik}}{\sum_{i;j\neq i;k\neq j,i} \widehat{g}_{ij}\widehat{g}_{jk}}.$$

This puts relatively less weight on nodes with high degrees and more weight on low degree nodes compared with the first two measures.

Theorem 15 *The Fraction of Transitive Triples, $C^{TT}(g)$ tends to:*

$$\frac{\gamma}{M}.$$

Total Clustering, $C(g)$ tends to:

$$\frac{6\gamma^2}{5M - 2 - 2\gamma(4M-2)}$$

and Average Clustering, $C^{Avg}(g)$ tends to:

$$\int_0^\infty \frac{1}{1-\gamma} M^{\frac{1}{\gamma}} \left(\frac{\gamma}{1-\gamma} d + M\right)^{-\frac{1+\gamma}{\gamma}} \times \left(\frac{\left(2\gamma M + 4\left(\frac{1-2\gamma}{2\gamma}\right)\left(\frac{1-\gamma}{\gamma}\right) M \log\left(\frac{d(1-\gamma)M}{\gamma} + 1\right) + 2d\right)}{(d+M)(d+M-1)} \right) dd$$

Proof:

The growth process which describes this model is for a subset of possible parameter values from the process in JR.

JR prove that C^{TT} tends to:

$$\frac{p_R}{m(1+r)}$$

if $\frac{p_R}{r} \leq 1$. $C(g)$ tends to:

$$\frac{6p_R}{(1+r)\left[(3m-2)(r-1)+2mr\right]}$$

if $r > 1$. $C^{\text{Avg}}(g)$ tends to:

$$\int_0^{\infty} \left[\frac{(rm)^{r+1}(r+1)}{(d+rm)^{r+2}} \right] \left(\frac{1}{(d+M)(d+M-1)/2} \right) \times \left(\begin{array}{c} m^2 C^{TT} \left(1 + \frac{2d(1+r)}{m}\right) - p_R d \\ +rm \left[\log\left(\frac{d}{rm}+1\right)\right] \left(\frac{p_R}{r} + p_R - 2C^{TT} m (1+r)\right) \end{array} \right) dd$$

In terms of the parameters in this model $p_R = 1$ $m = M$ and $r = \frac{1-\gamma}{\gamma} > 1$ since $\gamma \leq \frac{1}{2}$. Making these substitutions the result follows.

QED

The most important property to observe is that the first two measures of clustering C^{TT} and C are monotone increasing in the number of network based connections γ. The intuition for this is that increasing γ increases the proportion of people met during stage one for whom an additional friend is met which increases the number of triads (three people who all know one another) in the network.

2.4.3 Number of non-random connections

In this section I allow the players to choose the fraction of non-random connections through costly search effort. One interpretation is that players must expend additional effort to systematically navigate/search to find individuals with whom a friend is shared. The cost of this effort $C(\gamma M)$ is balanced against the gains from being able to establish a high level of cooperation in the first period of interaction.

I assume the marginal cost of this effort is increasing and convex $C', C'' > 0$ and at some point exceeds the net signaling benefit of another friend of a friend $\lim_{\gamma \to \frac{1}{2}} C'(\gamma M) = \infty$. In this section I define the value of signaling in terms of three underlying parameters, δ_I, δ_P and Δz and describe how each effects the number of non-random connections γ. This provides a framework to study the impact of proposed policies which may affect the social network and allows one to formulate testable hypotheses of how network characteristics should vary across different environments. Throughout this analysis I am holding M (the number of friendships a new player makes) constant.

I will illustrate how the benefits from signaling that you are a patient type are increasing in δ_I, δ_P and Δz. In the equilibrium described in the previous section the value of signaling is the additional benefit a patient player receives by avoiding the period of screening Δz. For impatient players to prefer to defect initially, rather than wait one period to defect in a high stakes game, it was assumed that

$$x_L > z_L + \delta_I x_H$$

which is true for $\delta_I \to 0$. If however the discount factor of impatient types (δ_I) is too high this is no longer true. To maintain a signaling equilibrium where there is screening of impatient types the number of periods for which players play the low stakes game is increased. I assume that the existing players in the network can commit to the stake of games in future periods conditional on cooperation in prior periods. If this is the case then for an impatient type to defect in the first period the value of defecting immediately exceeds the value of waiting until a high stakes game is offered and defecting then. So the number of periods in which a low stakes game is offered must be at least large enough such that

$$x_L \geq \frac{1-\delta_I^n}{1-\delta_I} z_L + \delta_I^n x_H$$

where the value on the left is the utility from defecting immediately and the value of the right is the value from cooperating for n periods before defecting in the first high stakes game. Now defining $\underline{n}(\delta_I)$ implicitly by[3]:

$$x_L = \frac{1-\delta_I^{\underline{n}}}{1-\delta_I} z_L + \delta_I^{\underline{n}} x_H$$

then $\underline{n}(\delta_I)$ is increasing in δ_I and goes to ∞ as $\delta_I \to 1 - \frac{z_L}{x_L}$. Therefore for a value of δ_I,

[3]In principle the right hand side of this expression should include a term incorporating the probability of obtaining an additional friendship, which becomes increasingly likely as $\underline{n}$ becomes large. However I will assume that the frequency of the arrival of new players decreases as δ_L increases. That is in effect the rate of time preference is not changing but rather the frequency of interaction increases as δ increases.

the value of signaling V_{Signal} is:

$$V_{\text{Signal}} = 2\frac{1-\delta_P^{\underline{n}}}{1-\delta_P}\Delta z$$

where the 2 comes from having an additional friend of a friend signals to both friends one's own type. If the marginal cost of obtaining friends of friends is convex then patient players will include as many friends of friends amongst the M friendships they establish, when they are born, such that the marginal cost of doing so is less than $2\frac{1-\delta_H^{\underline{n}}}{1-\delta_H}\Delta z$. Assuming an interior solution (this number is greater than 0 and less than $\frac{M}{2}$) then $\gamma(\delta_I, \delta_P)$ is given by:

$$2\frac{1-\delta_P^{\underline{n}(\delta_I)}}{1-\delta_P}\Delta z = C'(\gamma M)$$

The main theorem in this section describes the comparative statics of this relationship.

Theorem 16 *The number of network meetings a patient type chooses, $\gamma(\delta_I, \delta_P) M$, is increasing in Δz, δ_I and δ_P.*

Proof:

It suffices to show γ is increasing in Δz, δ_I and δ_P. First note that V_{Signal} is increasing in δ_I, since:

$$\begin{aligned} \frac{d\underline{n}}{d\delta_I} &> 0; \frac{\partial V_{\text{Signal}}}{\partial \underline{n}} > 0 \\ &\Longrightarrow \frac{\partial V_{\text{Signal}}}{\partial \underline{n}} \frac{d\underline{n}}{d\delta_I} = \frac{dV_{\text{Signal}}}{d\delta_I} > 0; \end{aligned}$$

Furthermore note that V_{Signal} is increasing in both Δz and δ_P; and MC_{Search} is increasing in γ. Now defining:

$$F(\gamma, \Delta z, \delta_I, \delta_P) = 2\frac{1-\delta_P^{\underline{n}(\delta_I)}}{1-\delta_P}\Delta z - C'(\gamma M)$$

and noting

$$\frac{dV_{\text{Signal}}}{d\delta_I} > 0; \frac{dV_{\text{Signal}}}{d\delta_P} > 0; \frac{dV_{\text{Signal}}}{d\Delta z} > 0; C'' > 0$$

we can sign $\frac{d\gamma}{d\delta_I}, \frac{d\gamma}{d\delta_P}, \frac{d\gamma}{d\Delta z}$ by:

$$\begin{aligned}
sign\left(\frac{d\gamma}{d\delta_I}\right) &= -sign\left(\frac{\frac{\partial F}{\partial \delta_I}}{\frac{\partial F}{\partial \gamma}}\right) = \frac{sign\left(\frac{dV_{\text{Signal}}}{d\delta_I}\right)}{sign\left(C''\left(\gamma M\right)\right)} > 0 \\
\frac{d\gamma}{d\delta_P} &= -\frac{\frac{\partial F}{\partial \delta_P}}{\frac{\partial F}{\partial \gamma}} = \frac{sign\left(\frac{dV_{\text{Signal}}}{d\delta_P}\right)}{sign\left(C''\left(\gamma M\right)\right)} > 0 \\
\frac{d\gamma}{d\Delta z} &= -\frac{\frac{\partial F}{\partial \Delta z}}{\frac{\partial F}{\partial \gamma}} = \frac{sign\left(\frac{dV_{\text{Signal}}}{d\Delta z}\right)}{sign\left(C''\left(\gamma M\right)\right)} > 0
\end{aligned}$$

QED

This allows one to analyze how a broadbased policy designed to affect the social network will impact it through the changes it has on Δz, δ_I and δ_P. It also gives one a basis for predicting in which settings networks with higher γ's will exist compared to others.

2.4.4 Policy analysis

The purpose of this section is to illustrate how the signaling motive provided by the model can be used to analyze policies designed to alter the structure of a social network. Specifically we are interested in how a policy will affect a new player's choice of γ. Provided that the policy change is not so great that the conditions required for the signaling equilibrium to exist are not violated. The effect of a new policy can be inferred from its impact on V_{Signal} through the underlying parameters Δz, δ_P, and δ_I. If the impact is positive then from Theorem 16 the impact is also positive on γ and the resulting social network will SOSD (Theorem 14) and have greater clustering (Theorem 15) than a social network absent the policy change.

For example in a given network consider a policy which enables more frequent interaction among agents. The effect of this can be interpreted as a shift in the discount rate of both patient and impatient agents. Now from Theorem 16 this will increase γ which increases both the level of clustering and the fat tailed nature of the degree distribution.

2.4.5 Comparison of networks

The model also permits one to compare different social networks on the basis of the relative proportion of network meetings γ and infer the relative value of signaling in each network. That is assuming the costs of searching are similar across networks, a network with a greater amount of network based meetings is one in which the value of signaling is greater as well. This may then allow one to make inferences about the relative frequency of interactions or the value derived from cooperation. Furthermore it gives a number of testable hypotheses about the correlation between social network characteristics (clustering coefficients and degree distributions) and a number of potentially observable parameters of relationships (frequency of interaction, benefits from cooperation, incentives for defection).

2.4.6 Welfare

Inefficiently low network based meetings

The utilitarian social welfare maximizing level of signaling is less than optimal in equilibrium. In the model agents only incorporate one half of the benefits from signaling into their choice of meeting friends of friends. That is they ignore the benefit that the old player gets from meeting a new player through a network based meeting compared to a random meeting. The benefit to society of one additional network based meeting is in fact $2V_{\text{Signal}}$ since the new player and older player avoids a period(s) of screening.

Trade-off between efficiency of network and equality

To the extent that inequality is a concern, increases in the efficiency of the social network through greater levels of signaling will also result in greater inequality across the society. Each additional friendship between two patient types benefits both agents and so the agents with the greatest number of friendships are also the agents that derive the greatest level of utility in the society. So from this point of view there is a trade off between improving equality in the population and improving the level of trust. It is important to note at this point that exogenously changing the network through targeted addition and/or deletion of friendships is not an instrument available to a benevolent social planner. Rather, when the social planner is restricted to manipulations to the network through the parameters affecting the value of signaling then there is necessarily a tension between improving efficiency and reducing inequality. Reducing inequality comes at the expense of efficiency and vice versa.

2.4.7 Renegotiation of the stake of the game

When two players do not share a friend in common we may be concerned that in the previous section the sequence of stakes offered may not be renegotiation proof. Specifically that after the first period, when patient types reveal themselves by cooperating, then the two players share a common belief that the other is patient. If this is the case then both players will prefer to switch to a high stakes game in the second period. However if it is not possible to commit to the sequence of stakes which will be offered then it is not an equilibrium for the impatient type to defect for sure in the first period. Rather they would strictly prefer to cooperate and renegotiate a high stakes game in the second period in which they would subsequently defect. In this section I demonstrate that there exists a renegotiation proof sequential equilibrium of the two player relationship which exhibits the qualitative features of the equilibrium in the commitment case. That is the expected payoff for the impatient new player is the same, x_L, the expected payoff for the patient new player is $\frac{z_L}{1-\delta_P} + \frac{\Delta z \delta_P \beta(\delta_I)}{(1-\delta_P)(1-\delta_P(1-\beta(\delta_I)))}$ and the comparative statics of V_{Signal} in Theorem 5 continue to hold in the renegotiation setting.

I implement a renegotiation procedure closely related to one used in Watson (1999) whereby the renegotiation is limited to a local escalation or de-escalation in the timing of the current regime. As is standard in the literature on renegotiation I assume players vote on whether to abandon one regime in favour of another. The renegotiation criterion considers *jump alterations* and *stall alterations.* In a *jump alteration* at a time t players agree to continue the current regime as if they were continuing from a time $s > t$. In a *stall alteration* players agree to resume the current regime as planned after a delay until time $s > t$. These jumps and stalls are incentive compatible provided that the new regime is an equilibrium (incentive-compatible) continuation of the game.

Definition 4 *An equilibrium regime is called alteration proof if for every $t \geq 0$ every incentive compatible alteration is defeated.*

For simplicity suppose we focus on a simple two player friendship where the players do not share any friends in common and one of the players can be either patient or impatient. The ex ante probability of this player being impatient is $\frac{\lambda}{[(1-2\lambda)(1-\gamma)+\lambda]}$. In this setting the only way to learn that a player is impatient is if the player defects and thereby ends the friendship. Consider the following characteristics of a conjectured equilibrium:

1. A low stakes game is offered in the first period
2. Impatient types mix in the first period between defecting and cooperating with probability μ
3. In period 2 the older player mixes with probability β between offering a high and low stakes game
4. From period 2 onwards if a high stakes game is offered the impatient type defects and if a low stakes game is offer the impatient type cooperates.
5. From period 3 onwards the older player either offers a high stakes game if a high stakes game was played previously or mixes with probability β if only low stakes games have been offered.

Theorem 17 *Suppose $\lambda \geq \frac{(1-\gamma)\Delta z}{z_L + y_H(1-\delta_P) + 2\Delta z(1-\gamma)}$, $\beta = \frac{1}{(x_H - x_L)}\left(\frac{(x_L - z_L)}{\delta_I} - x_L\right)$, μ satisfies $\frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}\frac{z_H}{1-\delta_P} + \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}(-y_H) = \frac{z_L}{1-\delta_P}$, $1 - \max\left\{\frac{z_L}{x_L}, \frac{z_H}{x_H}\right\} > \delta_I > \frac{x_L - z_L}{x_H}$ then $\exists \underline{\delta}$ such that there is an alteration proof equilibrium of the two person repeated prisoners dilemma for $\delta_P \in [\underline{\delta}, 1]$ which satisfies the above characteristics.*

I will leave the proof for the appendix but will give an informal description of why the equilibrium is renegotiation proof. It satisfies alteration proofness because, in the case of delays, this sequence of stakes is weakly increasing and neither agent would like to delay the revelation of information. In the case of jumps, the only jump which is weakly preferred from the older players point of view is a jump to playing high stakes game. This is not incentive compatible because it requires all impatient types to have defected by the previous period which is not incentive compatible for impatient types when $\delta_I > \frac{x_L - z_L}{x_H}$.

In this equilibrium the expected payoff of the impatient new player is x_L because the equilibrium is such that the impatient player is indifferent between defecting in the first period and continuing with the relationship by cooperating. The payoff of the patient new player is $\frac{z_L}{1-\delta_P} + \frac{\Delta z \delta_P \beta(\delta_I)}{(1-\delta_P)(1-\delta_P(1-\beta(\delta_I)))}$ and the value of signaling is

$$\begin{aligned} V_{\text{Signal}} &= \frac{z_H}{1-\delta_P} - \frac{z_L}{1-\delta_P} - \frac{\Delta z \delta_P \beta(\delta_I)}{(1-\delta_P)(1-\delta_P(1-\beta(\delta_I)))} \\ &= \frac{\Delta z}{(1-\delta_P(1-\beta(\delta_I)))} \end{aligned}$$

where $\beta(\delta_I)$ is decreasing in δ_I so the payoff to the patient player is also decreasing in δ_I and we can conclude that the comparative statics in Theorem 5 are unchanged for δ_I, δ_P, and Δz.

I claim without proof that modifying the commitment equilibrium from the previous section, by having players play the renegotiation equilibrium presented here in the instances where players do not share a friend in common, will not qualitatively change any of the predictions of the model. The difficulty in doing this explicitly is that the values of β and μ must necessarily include the probabilities of obtaining further friends through maintaining the friendship since despite these terms approaching zero as $N \to \infty$ β and μ are set to keep the two players indifferent along the on-equilibrium path where low stakes games are being offered. For expositional simplicity the analysis here effectively ignores these small adjustments and presents the equilibrium of the repeated interaction when there is no prospective of gaining additional friendships.

2.5 Conclusion

This paper begins with the observation that sharing a friend in common can be valuable in establishing trust, especially early on in a relationship. It builds on this observation to develop a model of network formation as an equilibrium in individuals' actions, whereby players' believe players with whom they share a friend will be prepared to cooperate in a prisoner's dilemma game. Furthermore it shows that this equilibrium is equivalent to a process of network formation derived by JR which can explain a number of the characteristics of real world networks.

There are a number of advantages to nesting a process of network formation in an equilibrium. First it allows key characteristics of social networks, degree distributions and clustering coefficients, to be related to underlying properties of the environment. Second it provides a framework for understanding how potential policies can change the underlying social structure or more importantly how to design policies to achieve certain policy objectives through changing the social network. Finally on an empirical note it gives a number of testable hypotheses about the correlation between social network characteristics (clustering coefficients and degree distributions) and a number of potentially observable parameters of relationships (frequency of interaction, benefits from cooperation, incentives for defection).

2.6 Appendix A: Proofs

Proof of Theorem 13 and description of equilibrium

Before giving the proof I will describe the strategies and beliefs which support the equilibrium. It will be useful to define a partition Π_i^t over the friends of player i at time t. This partition groups individuals together if a warning about player i can spread from anyone of them to the rest. Define the group of player j in the partition of player i's neighbors by

$$\pi_i^t(j) = \{j\} \cup \left\{ \begin{array}{c} k_n | \exists k_1, ...k_n \in Q_i^t, n > 0 : \\ \max\{g_{jk_1}^t, g_{k_1 j}^t\} = 1, \max\left\{g_{k_w k_{w+1}}^t, g_{k_{w+1} k_w}^t\right\} = 1 \text{ for all } w \leq n \end{array} \right\}$$

The following strategies and beliefs constitute a sequential equilibrium.

Patient player i strategy

- *Choice of friends* {Network,Random}
  - Player i chooses Network based meetings.
- *Choice of stake* $s_{ij}^t \in \{\text{High}, \text{Low}\}$ *in period* t *against a player* j.
  - Player i specifies a high stakes game in period t with player j:
  - $t > N + j$
    * Warnings - $w_j^{ikt'} = 0$: for all $t' \leq t$ and $k \in \pi_i(j)$; and
    * Stakes - Stakes have been on-equilibrium for all $t' < t$; and
    * Actions - $\left(a_{ik}^{t'}, a_{ki}^{t'}\right) = (C, C)$ for all $t' < t$ and $k \in \pi_i(j)$; and
    * Connections - No changes in connections K_{ij}^t not accompanied by a warning.
  - $t = N + j$
    * Warnings - $w_j^{ikt'} = 0$: for all $t' \leq t$ and $k \in \pi_i(j)$; and
    * Stakes - No history of past stakes S; and
    * Actions - $\left(a_{ik}^{t'}, a_{ki}^{t'}\right) = (C, C)$ for all $t' < t$ and $k \in \pi_i(j)$; and
    * Connections - $\exists k \in Q_i^t : \max\{g_{jk}^t, g_{k,j}^t\} = 1$;no changes in connections K_{ij}^t not accompanied by a warning.
  - Player i chooses a low stakes game otherwise
- *Choice of strategy* $a_{ij}^t \in \{C, D\}$ *against a player* j *in period* t.

- Player i plays C when:
  - Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$: for all $t' \leq t$ and $k \in \pi_i(j)$; and
  - Stakes have been on equilibrium; and
  - Actions - $\left(a_{ij}^{t'}, a_{ji}^{t'}\right) = (C, C)$ for all $t' < t$; and
  - Connections - No changes in connections K_{ij}^t K_{ji}^t not accompanied by a warning.
- Player i plays D otherwise.

- *Choice of sending warning about player j*
  - If player j deviated player i always sends a warning

Impatient player i strategy

- *Choice of friends* {Network,Random}
  - Player i chooses to connect to random players.

- *Choice of stake in each period t against a player j for which $g_{ij}^t = 1$, $s_{ij}^t \in$* $\{\text{High}, \text{Low}\}$.
  - Player i specifies a high stakes game in period t with player j:
  - $t > N + j$
    - Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$: for all $t' \leq t$ and $k \in \pi_i(j)$; and
    - Past stakes have been on equilibrium; and
    - Actions - $\left(a_{ij}^{t'}, a_{ji}^{t'}\right) = (C, C)$ for all $t' < t$; and
    - Connections - No observed changes in connections for K_{ij}^t not accompanied by a warning.
  - $t = N + j$
    - Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$: for all $t' \leq t$ and $k \in \pi_i(j)$; and
    - Stakes - No history of past stakes; and

    * Actions - No history of past actions; and
    * Connections - $\exists k \in Q_i^t : \max\{g_{jk}, g_{k,j}\} = 1$;no changes in connections K_{ij}^t not accompanied by a warning.
  - Player i chooses a low stakes game otherwise

- *Choice of strategy $a_{ij}^t \in \{C, D\}$ against a player j in period t.*
  - Play C:
    * Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$: for all $t' \leq t$ and $k \in \pi_i(j)$
    * Past stakes have been on equilibrium;
    * No changes in connections K_{ij}^t not accompanied by a warning; and
    * i is yet to play j' in period t; or
    * $|\pi_i(j)| > \zeta^*(t) > M$
  - Play D otherwise

- *Choice of sending warning about player j*
  - If player j deviates player i always sends a warning

Player's Beliefs

Observing network structure:

- The belief of a player with degree d at time t when that player meets a new player k who is not a friend of a friend (stranger) and I have not received a warning about is

$$\Pr\left(\delta_j = \delta_I | w_j = 0; \nexists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right)$$
$$= \frac{\lambda\left(1 - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right)}{\lambda\left(1 - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right) + (1-\lambda)(1-2\gamma)\left(\Pi_{i=0}^{(1-2\gamma)M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right)}.$$

- where

$$\lim_{t\to\infty} \Pr\left(\delta_j = \delta_I | w_j = 0; \nexists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) = \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}$$

- The belief of a player with degree d at time t when that player meets a new player k with whom a friend is shared and they have not received a warning about is:

$$\Pr\left(\delta_k = \delta_I | w_j = 0; \exists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) = \frac{\lambda \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)}{\lambda \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right) + 2\gamma(1-\lambda)}.$$

  where

$$\lim_{t\to\infty} \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right) = 0$$

Receiving warnings:

- A player's belief about another player j in any history when he/she receives warning about them $w_j^{ikt} = 1$:

$$\Pr_i\left(\delta_j = \delta_I | w_j^{ikt} = 1 \text{ for some } k \in Q_i^t\right) = 1$$

- A player's belief about another player j in any history when he/she receives warning from them $w_k^{ijt} = 1$:

$$\Pr_i\left(\delta_j = \delta_I | w_k^{ijt} = 1 \text{ for some } k \in Q_i^t\right) = 1$$

Observing actions:

- The belief about another player after she has cooperated at least once and there has been no warning received about that player

$$\Pr\left(\delta_j = \delta_I | a_{ji}^t = C \text{ for all } t \geq j - N; w_j^{ikt} = 0\right) = 0.$$

Observing choice of stakes by an older player:

- An off equilibrium increase in the stake

$$\Pr_i\left(\delta_j = \delta_I | s_{ji}^t = High; j < i; t = i; Q_i \cap Q_j = \varnothing\right) = 1$$

- An off equilibrium decrease in the stake does not affect beliefs.

Observing off equilibrium changes in K_i^t:

- Observing changes in K_i^t without receiving a warning results in beliefs

$$\Pr\left(\delta_j = \delta_I | K_{ijk}^{t'} \neq K_{ijk}^t \text{ for some } t' \leq t; w_j^{ikt'} = 0 \text{ and } w_k^{ijt'} = 0 \text{ for all } t' \leq t\right) = 1.$$

Theorem 13 *Suppose* $x_H + z_H < 2x_L$, *then* $\exists\ \underline{\delta} < 1$ *and* $\overline{\delta} > 0$ *such that if* $\delta_P \in (\underline{\delta}, 1)$ *and* $\delta_I \in (0, \overline{\delta})$ *there exists a sequential equilibrium in which patient players choose network based meetings and an existing player* i *has beliefs, upon meeting, a new player* j $\lim_{t\to\infty} \Pr(\delta_j = \delta_P | Q_i^t \cap Q_j^t \neq \varnothing) = 1$.

Proof:

Patient Player i

Choice of action in stage game (against player j*)*

- Playing C occurs when:
  -
    - Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$: for all $t' \leq t$ and $k \in \pi_i(j)$; and
    - Stakes - no off-equilibrium increase in the stake; and
    - Actions - $\left(a_{ij}^{t'}, a_{ji}^{t'}\right) = (C, C)$ for all $t' < t$; and
    - Connections - No changes in connections K_{ij}^t not accompanied by a warning.

Case 1:

If $t > N - j$ then the patient players beliefs are

$$\Pr\left(\delta_j = \delta^I | a_{ji}^t = C \text{ for all } t \geq j - N; w_j^{ikt} = 0\right) = 0$$

and the game is a high stakes game. Thus provided

$$\frac{z_H}{1-\delta_P} \geq x_H$$

is satisfied then this is the best action the agent can take.

Case 2:

If $t = N - j$ and the players share a friend in common, the older players beliefs are

$$\Pr\left(\text{impatient}|w_k = 0; \exists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) = \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)$$

and the game is a high stakes game. Thus provided

$$\begin{aligned} &\left(1 - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right)\frac{z_H}{1-\delta_P} - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)y_H \\ \geq\ &\left(1 - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right)x_H \end{aligned}$$

is satisfied then this is the best action the agent can take. For the younger player beliefs are

$$\Pr\left(\text{impatient}|w_k = 0; \exists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1, j < i\right) = 0$$

Thus provided

$$\frac{z_H}{1-\delta_P} \geq x_H$$

is satisfied then this is the best action the agent can take.

Case 3:

If $t = N - j$ and $Q_i^t \cap Q_j^t = \varnothing$ then the older players beliefs are

$$\Pr\left(\text{impatient}|w_k = 0; \nexists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) \approx \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}$$

Thus provided

$$\left(1-\frac{\lambda}{\lambda+(1-\lambda)(1-2\gamma)}\right)\left(\frac{\delta_P z_H}{1-\delta_P}+z_L\right)-\frac{\lambda}{\lambda+(1-\lambda)(1-2\gamma)}y_L$$
$$\geq\left(1-\frac{\lambda}{\lambda+(1-\lambda)(1-2\gamma)}\right)x_L$$

is satisfied then this is the best action the agent can take. If the player is the new player then the belief it has is

$$\Pr\left(\text{impatient}|w_k=0;\exists k'\in Q_i^t:\max\left\{g_{k,k'}^t,g_{k',k}^t\right\}=1,j<i\right)=0$$

and the strategy is optimal provided:

$$(1-\lambda)\left(\frac{\delta_P z_H}{1-\delta_P}+z_L\right)-\lambda y_L\geq(1-\lambda)x_L$$

$\exists\underline{\delta}$ such that $\delta_P\in[\underline{\delta},1]$ satisfies all of the above conditions.

- Play D otherwise
  - Means at least one of the following has occurred
    - Warnings - $w_j^{ikt'}\neq 0$ or $w_i^{jkt'}\neq 0$: for a $t'\leq t$ and $k\in\pi_i(j)$; or
    - Stakes - An off equilibrium increase in the stake;
    - Actions - either i or j has played D previously; or
    - Connections - A change in connections K_{ij}^t or K_{ji}^tnot accompanied by a warning.

In all instances the updated beliefs of at least one of the individuals are

$$\Pr(\text{impatient})=1$$

and the other player knows this so the optimal action is D.

- *Choice of stake in game*

- An off-equilibrium choice of a high stake game results in the new player believing $\Pr(\text{impatient}) = 1$ and in the subsequent game the new player will play defect. The most the patient player can get is therefore 0. This is not optimal since the patient player can receive at least this by choosing an on-equilibrium low stake game and an on-equilibrium action, an on-equilibrium defect also results in a payoff of 0 but an on-equilibrium cooperate has a positive expected payoff. An off-equilibrium choice of a low stakes game will not change the other players choice of action so is worse.

- *Choice of links*
  - For a patient player choosing to signal that they are patient by establishing links with friends of friends is beneficial because it avoids the low stakes screening game in the first period. Thus gains a benefit Δz on each relationship.

- *Choice to send warning*
  - Sending a warning is costless and does not affect a player's payoffs so is always optimal.

Impatient Player

- *Choice of action in stage game*
  - Play C:
    - Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$: for all $t' \leq t$ and $k \in \pi_i(j)$
    - Past stakes have been on equilibrium;
    - No changes in connections K_{ij}^t not accompanied by a warning; and
    - $\exists j' \in \pi_i(j)$ who player i is yet to play in period t; or
    - $|\pi_i(j)| > \zeta^*(t) > M$

2 cases.

1) $\exists j' \in \pi_i(j)$ who player i is yet to play in period t.

The times when the impatient player chooses to play C against a player j are when there is another $j' \in \pi_i(j)$ who would receive a warning, in which case the impatient player is better of choosing to cooperate with j and then play D later in the period.

$$\frac{z_H}{1-\delta_I}$$

2) The other time an impatient player i will cooperate is if the shared neighborhood $\pi_i(j)$ is sufficiently large. For any δ_I there will exist a shared neighborhood $\pi_i(j)$ such that the player will prefer to cooperate. The key to the proof is to show that the size of this neighborhood is $> M$ and will therefore never eventuate on the equilibrium path. The impatient player will destroy all of her initial friendships so even if by chance these individuals know one another the player will still choose to defect on one of them in the first period.

For any $\delta > 0$ there is a size of neighborhood $\zeta^* = |\pi_i^*(j)|$ such that the payoff from cooperating forever exceeds the payoff from defecting. Ignoring the increased probability of meeting future players from maintaining connections today this would be:

$$\frac{\zeta^*(\delta)\,\delta z_H}{1-\delta} + z_H = x_H$$

including the payoffs from future meetings increases the left hand side. Hence there is a range of neighborhoods $|\pi_i(j)| > \zeta^*$ where the impatient player will prefer to cooperate with players $k \in \pi_i(j)$. The remainder of the proof is to show that $\exists\delta > 0$ such that a neighborhood of size M is not sufficiently large to induce the impatient player to cooperate. This is the largest possible neighborhood an impatient player may have in the first period it is born. Given on-equilibrium actions the impatient player will not gain more connections than this because it will be removed from the network after 2 periods.

A player may obtain future benefits from meeting incoming patient players in the future if it maintains a friendship today. This effect lowers the threshold size ζ^*. Provided that the threshold is never $\leq M$ then this will never eventuate on-equilibrium. The expected number of future in-degree friendships a player will obtain over time as the population grows is:

$$\frac{dd_\imath(t)}{dt} = \frac{d_i(t)\gamma}{N+t} + \frac{(1-\gamma)M}{N+t}$$

with solution

$$d_i(t) = \frac{1-\gamma}{\gamma}M\left(\frac{N+t}{i}\right)^\gamma - \frac{1-\gamma}{\gamma}M$$

The largest number of in-links at time t is therefore $\frac{1-\gamma}{\gamma}M\left(\left(1+\frac{t}{N}\right)^\gamma - 1\right)$

The expected profits from future connections if the player cooperates is then

$$M\frac{1-\gamma}{\gamma}z_H\sum_{t=1}^{\infty}\delta^t\left(\left(1+\frac{t}{N}\right)^\gamma - 1\right)$$

for $\delta, \gamma < 1$ this is finite and $\lim_{\delta\to 0} M\frac{1-\gamma}{\gamma}z_H\sum_{t=1}^{\infty}\delta^t\left(\left(1+\frac{t}{N}\right)^\gamma - 1\right) = 0$. Hence if an impatient player i is facing a player j and has $|\pi_i(j)| = M$ then payoff from cooperating is at most

$$\frac{\delta M z_H}{1-\delta} + z_H + M\frac{1-\gamma}{\gamma}z_H\sum_{t=1}^{\infty}\delta^t\left(\left(1+\frac{t}{N}\right)^\gamma - 1\right)$$

and in the limit as $\delta \to 0$ is z_H. Hence for $\exists\delta : \delta_I \in [0,\delta]$ the impatient player will prefer to play D.

- *Choice of stake in game*
  - When facing a new player $\jmath$ where $Q_\imath \cap Q_\jmath = \varnothing$ choosing a high stakes game instead of a low stakes game will result in

    $$\Pr_\jmath(\delta_\imath = \delta_I | \text{Off equilibrium stake choice}) = 1$$

    and the new player will play D. This is optimal for will not effect any of j's other relationships because On the other hand an off-equilibrium decrease

- *Choice of links*

– If impatient players choose to mimic patient players by connecting to two players who know each other in the first period of interaction in the stage game they will cooperate with the first player and defect on the second player. In the second period of interaction they will only have the opportunity to interact with the first of the players since the relationship with the second player is destroyed when the impatient player defects in the first period. However by the start of the second period this first player will hold beliefs $\Pr(\text{impatient}) = 1$ about the impatient player because they will have received a warning about him/her in the previous period from the second player whom the impatient player played defect against. In the second period of interaction the impatient player and the first player will both play defect. Therefore the most an impatient player can obtain by mimicking a patient player is $z_H + x_H$. It will not be optimal for the impatient player to do this if the following signaling condition holds

$$2x_L > z_H + x_H.$$

The left-hand side is the payoff from successfully deviating on two opponents in a low stakes game and the right is the payoff from cooperating with the first and deviating on the second in a high stakes game.

- *Choice of whether to send a warning*

  – Sending a warning is costless and does not effect a player's payoffs so is also a best response.

QED

Proof of Theorem 17 and description of renegotiation equilibrium

I will first describe the renegotiation concept which is similar to the renegotiation concept used in Watson (1999). Define the sequence of stakes offered by the probability a high stake game is played $\alpha(t)$. Also define the probability that the new player has defected by time t as $\theta(t)$. A jump alteration $\widehat{\alpha}, \widehat{\theta}$ prescribes from period t what the original regime prescribed from $t+\Delta$. Thus $\widehat{\theta}$ and $\widehat{\alpha}$ are specified such that $\widehat{\theta}(w) = \theta(w+\Delta)$ and $\widehat{\alpha}(w) = \alpha(w+\Delta)$ for all $w > t$. Note that $\widehat{\alpha}$ must be such that the right probability mass betrays at time $t-1$ so that $\widehat{\theta}(t) = \theta(t+\Delta)$. If such a $\widehat{\alpha}$ exists then the new regime is an equilibrium (incentive-compatible) continuation. A stall alteration at t defined by $\widetilde{\alpha}$ and $\widetilde{\theta}$ prescribes from $t+\Delta$ what the original regime prescribed from time t. In between times t and $t+\Delta$, $\widetilde{\alpha}$ is set to preserve the current beliefs of the players about each other over the intervening period. Thus $\widetilde{\alpha}$ and $\widetilde{\theta}$ satisfy $\widetilde{\alpha}(w) = \alpha(w-\Delta)$ and $\widetilde{\theta}(w) = \theta(w-\Delta)$ for every $w \geq t+\Delta$, also $\widetilde{\theta}(w) = \theta(t)$ for every $w \in [t, t+\Delta-1]$. $\widetilde{\alpha}$ on $[t, t+\Delta-1]$ is arbitrary however for it to be incentive-compatible no type assigned positive probability at time t can now have an incentive to defect during $[t, t+\Delta-1]$.

Before the start of each period players may reconsider the continuation of their regime. Players decide by voting for or against an exogenously given incentive-compatible alteration. The players simultaneously vote for or against the alteration. Additionally I assume that no information is revealed in the renegotiation process about players' types. A regime is abandoned in favour of another if in the case of a jump both players accept it or in the case of a stall at least one player accepts it.

Now consider the following strategies for two friends and suppose they do not share a friend in common. I will assume player 1 is the older player and is patient. Player 2 however is a new player and may be patient or impatient.

Player 1

- *Choice of stake*
  - First period of interaction offer a low stakes game.
  - Second period of interaction mix with probability β of a high stakes game.
  - Thereafter mix if in all previous periods the stake was low and (C, C) has been the history of play.
  - Offer a high stakes game when the previous period the stake was high and cooperation has been the history of play.
- *Choice of strategy*

- Play C

Patient Player 2

- *Choice of strategy*
  - Play C

Impatient Player 2

- *Choice of strategy*
  - When the stake of the game is high
    * Play D
  - When the stake of the game is low
    * Mix with probability μ of cooperating in the first period
    * Play C in all subsequent periods

Beliefs

- The belief of the new player is always

$$\Pr(\text{Patient}) = 1$$

- The belief of the older player before the first period is

$$\Pr(\text{Impatient}) = \frac{\lambda}{[(1-2\lambda)(1-\gamma)+\lambda]}$$

- The belief of the older player when the history includes at least one high stakes game is

$$\Pr(\text{Patient}) = 1$$

- The belief of the older player after a history of only low stake games is

$$\Pr(\text{Impatient}) = \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\mu\lambda]}$$

Theorem 17 *Suppose* $\lambda \geq \frac{(1-\gamma)\Delta z}{z_L+y_H(1-\delta_P)+2\Delta z(1-\gamma)}$, $\beta = \frac{1}{(x_H-x_L)}\left(\frac{(x_L-z_L)}{\delta_I} - x_L\right)$, μ *satisfies* $\frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}\frac{z_H}{1-\delta_P} + \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}(-y_H) = \frac{z_L}{1-\delta_P}$, $1-\max\left\{\frac{z_L}{x_L}, \frac{z_H}{x_H}\right\} > \delta_I > \frac{x_L-z_L}{x_H}$ *then* $\exists\underline{\delta}$ *such that these strategies are an alteration proof equilibrium of the two person repeated prisoners dilemma for* $\delta_P \in [\underline{\delta}, 1]$.

Proof of equilibrium:

There is always a $\underline{\delta}$ close enough to 1 such that it is optimal for the patient player to cooperate provided there is a positive probability the other player is also patient. To see this note that $\lim_{\delta\to 1}$ of the left hand side of the following equation

$$\Pr(Patient)\frac{z_L}{1-\delta} - (1-\Pr(Patient))\,y_H \geq \Pr(Patient)\,x$$

is ∞ provided $\Pr(Patient) > 0$. Where the left hand side is a lower bound on the payoff from cooperation and the right hand side is the payoff from defecting.

The only way an impatient type is revealed given the equilibrium strategies is through playing defect and thereby destroying the friendship. After the first period this only occurs when a high stakes game is played since both patient and impatient players cooperate in low stakes games from period two onwards. It is only when the players interact in a high stakes game that the older player finds out if the other player is patient or impatient.

In periods $t = 1, ..., \infty$ the impatient player is indifferent between cooperating and defecting when the stake of the game is low. The payoff from defecting this period is the same as cooperating and waiting until the following period

$$x_L = z_L + \delta_I[\beta x_H + (1-\beta)x_L]$$

$$\Rightarrow \beta = \frac{1}{(x_H-x_L)}\left(\frac{(x_L-z_L)}{\delta_I} - x_L\right)$$

. $0 < \beta < 1$ when $\delta_I > \frac{x_L-z_L}{x_H}$. The impatient player will not prefer to cooperate provided that

$$x_H > \frac{z_H}{1-\delta_I}$$

In periods $t = 2, ..., \infty$ the older player who chooses the stake of the game will be indifferent between offering a high or a low stakes game provided that

$$\Pr(\text{Patient}) \frac{z_H}{1-\delta_P} + (1 - \Pr(\text{Patient}))(-y_H) = \frac{z_L}{1-\delta_P}$$

where

$$\Pr(\text{Patient}) = \frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}$$

if the impatient player mixes in period 1 with probability μ of cooperating and $(1-\mu)$ defecting

$$\frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma)+\lambda\mu]} \frac{z_H}{1-\delta_P} + \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}(-y_H) = \frac{z_L}{1-\delta_P}$$

Also if μ satisfies $\frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma)+\lambda\mu]} \frac{z_H}{1-\delta_P} + \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}(-y_H) = \frac{z_L}{1-\delta_P}$ then the older player is indifferent between offering a high and low stakes game in the second period and any future period as long as only low stakes games have been offered previously. For $\mu \in [0,1]$ to exist then $\lambda \geq \frac{(1-\gamma)\Delta z}{z_L + y_H(1-\delta_P) + 2\Delta z(1-\gamma)}$.

I will now prove the equilibrium is alteration proof.

Delays

Given the repetitive nature of the equilibrium the only times a delay is meaningful is between the first and second period and in the instance when a high stakes game has been played and information has been revealed about the type of the unknown player. Firstly consider the incentives for the unknown player to delay. The sequence of stakes is weakly increasing so both types of the unknown player never want to delay an increase in the stake of the game as both benefit from this. Now consider the incentives of the older player for a delay alteration. A delay between the first and second periods is not strictly preferred since in the second period the older player is indifferent between a low or high stake game. In the sequence of stakes, which are offered by the older player, the first high stakes game which arises as a result of mixing induces information about the unknown player to be revealed. A delay after this high stakes game has been played is not preferred by the older player, since the player knows for sure that the other player is patient since the relationship survived the high stake game in the previous period and will prefer to play high stake games.

Jumps

In this equilibrium only three different jumps can occur from a low stake to the mixed stake, from a low stake to a high stake, and from the mixed stake to a high stake. Jumps ahead from the low stake offered in the first period are not preferred by the older player if the payoff from offering a high stake are worse than a low stake. This will not be the case if $\mu \in [0,1]$ since the only difference between the first and second periods is that the older player places a lower probability on the unknown player being impatient. $\lambda \geq \frac{(1-\gamma)\Delta z}{z_L + y_H(1-\delta_P) + 2\Delta z(1-\gamma)}$ guarantees that $\mu \in [0,1]$. The only jumps which are possible are jumps from periods in which the older player is required to mix to periods in which he/she offer a high stake for sure. For this type of jump to be incentive compatible the impatient type must defect for sure in the period immediately prior which is not incentive compatible when $\delta_I > \frac{x_L - z_L}{x_H}$ because the impatient type will prefer to wait.

Chapter 3

Government Support for Intermittent Renewable Generation Technologies

3.1 Introduction

This paper analyzes the effects of government support for intermittent electricity generating technologies on the long-run incentives for investment and use of dispatchable (non-intermittent) electricity generation technologies. Currently there are a broad range of policies supporting a number of intermittent renewable technologies (e.g. wind, solar) by governments to deal with the prospect of climate change. I analyze how and when support for a clean intermittent technology is likely to be effective at reducing greenhouse gas (GHG) emissions from electricity generation. For instance proponents of wind generation argue that they promote investment in a clean source of electricity and therefore reduce greenhouse gas emissions. I derive the conditions under which this may or may not be true.

I find that, in the long run, intermittent generating technologies are not pure substitutes for dispatchable technologies. Rather the relationship between the intermittency of the energy resource associated with the technology (i.e. wind or sunlight) and demand for electricity determines the substitutability of output from an intermittent technology for dispatchable technologies. If the energy available from the intermittent resource is high when electricity demand is low, then increased output from an intermittent technology may be complementary with output from peaking and intermediate dispatchable technologies in the long term and in the with output from existing generators (as opposed to new technologies) in the medium term. In this case the net effect on carbon emissions from a government policy supporting the intermittent technology is ambiguous. It will be determined by the relative carbon intensities of the various technologies peaking/intermediate vs baseload and new vs old, and the electricity offset by the intermittent resource.

Absent government intervention, the long run mix of technologies in deregulated electricity markets will reflect the variability of electricity demand. That is, if demand is highly variable then the mix of generation will consist of a greater fraction of peaking and intermediate generation than it will if demand is less variable. When an intermittent technology with low variable costs is introduced, it will operate whenever the power source is available. In this case investment in technologies other than the intermittent technology will reflect the shape of electricity demand minus intermittent generation. Both electricity demand and intermittent generation exhibit variability over time. When the intermittent generation is not too positively correlated with electricity demand the difference between the two will exhibit a greater variance than either one individually. This increase in variability will be reflected in the pattern of investment and will lead to a greater amount of peaking and intermediate investment relative to baseload investment. In the long run, the entire mix of generating technologies which supply electricity will be affected. The flow on effect to carbon emissions of support for the intermittent technology through the change in the mix of investment may be positive or negative.

This paper gives a characterization of the long run impact on emissions of government support for intermittent renewable technologies on the investment mix of other technologies. There is work that has aimed to do this empirically. DeCarolis and Keith (2006) look at the economics of wind power for reducing GHG emissions by employing a greenfield optimization model that determines the optimal mix of wind, gas, turbine, storage and transmission capacities in a hypothetical electricity system under a carbon tax, using data from 5 sites in the mid-west. Lamont (2008) determines the long-term system value of intermittent electric generation and in particular the marginal value of an intermittent resource as a function of the correlation between the intermittency and system marginal cost. A number of other papers have also looked at some of the relative costs and benefits associated with intermittent renewable technologies, such as wind and solar power, for various jurisdictions around the world. For instance Borenstein (2008) calculates the net social return from investment in solar photovoltaic to include benefits from its availability at times of peak demand and avoided transmission losses as well as the social benefits it may provide through avoided GHG emissions. Strbac (2002) calculates the additional system operation costs associated with a policy to increase the amount of electricity generated by renewables in Great Britain from 10% to 20% or 30% by 2020. Holttinen and Tuhkanen (2004) simulate the effects of large-scale wind production on CO_2 abatement in Nordic Countries. Denny and Malley (2006) consider a "forecasted" approach to system dispatch to estimate the emissions impact from the introduction of wind generation through its effects on power system operation due to ramping and reliability issues. Keith et al. (2004) evaluates a number of methodologies for estimating the amount emissions displaced by new investments including intermittent sources. There are also a range of methodologies and analyses which analyze the effects of wind energy projects on not only GHG emissions but also ecological and human development, the National Research Council (2007) reviews the literature on these broader environmental impacts of wind energy. In contrast to this previous work the approach of this paper theoretic. It focuses on the long-run impact on GHG emissions from the support for intermittent technologies through their effect on the mix of investments and their subsequent use in an electricity system. A theoretic approach offers the advantage that it provides a framework for understanding the conditions under which an intermittent technology will be more or less effective at reducing GHG. This is particularly important when there is little experience with large-scale deployment of intermittent generation in electricity system as is the case today.

The mechanism for the change in investment analyzed in this paper is the change in the annual profile of electricity demand met from dispatchable technologies. Borenstein

(2005) also considers the change in investment through this same mechanism as a result of a change from uniform to real-time pricing.

Section 2 introduces a model of a competitive electricity market with long run investment in different generating technologies, Section 3 & 4 analyze the effects on investment from supporting an intermittent technology on the mix of baseload vs peaking/intermediate technologies and old vs new technologies respectively, Section 5 discusses the impact on emissions of greenhouse gases and Section 6 concludes.

3.2 Models

In this section I introduce two models to analyze the effect on long run incentives for investment in generation in an electricity market, when there is government policy supporting an intermittent technology. The first model addresses the effect of these policies on the efficient long run *mix* of generating technologies. The second model addresses the effect of these policies on the scrapping/investment decision between old and new technologies.

3.2.1 Demand

I assume that consumers have a perfectly inelastic demand for electricity. This analysis may be extended to include more elastic demand, the inelastic case is a standard assumption in many empirical analyses of electricity markets and will provide a transparent exposition of how government support for intermittent technologies affect investment in other types of generation.

Electricity demand x varies over time. I describe the distribution of electricity demand by the cdf $F^d(x)$ and associated pdf $f^d(x)$ where x is the ratio of actual demand to average demand and normalize average demand for electricity to 1 such that $\int x dF^d(x) = 1$. I denote peak demand $\inf \{x | F^d(x) = 1\}$ by $\overline{x}$.

3.2.2 Intermittent generating technology

There is an intermittent renewable energy resource located at a number of different places denoted by $g = 1, ..., G$. The investment decision $w = (w_1, ..., w_G) \in [0,1]^G$ for the intermittent technology is whether to exploit each of these locations by building a generating facility, $w_g = 1$ indicates 100% exploitation of the resource at location g. I assume that the intermittent technology has zero variable costs, but relatively high fixed costs K_w and creates no emissions $EI_w = 0$. The available energy at each location a_g changes over time. The distribution of energy at a site is given by cdf $F^{a_g}(a_g)$ with associated pdf $f^{a_g}(a_g)$. The joint distribution of electricity demand and availability of the intermittent resources is denoted by $F^{da}(x, a_1, ..., a_G)$, and $f^{da}(x, a_1, ..., a_G)$. The state of the world in this model is described by the vector of electricity demand and availability of the intermittent $(x, a_1, ..., a_G)$ which I will denote by s. Thus the relative frequencies of different states is given by f^{da}.

3.2.3 Net load

For a given set of investments w in the intermittent technology the joint distribution of electricity demand and intermittent output is denoted by $F^{dw}(x,a), f^{dw}(x,a)$ where $a = \sum_g w_g a_g$.[1] For this set of investments the the net load may be found. I denote the shape of net load b by $F^{nw}(b), f^{nw}(b)$. Where

$$b = x - a$$

so that

$$\begin{aligned} f^{nw}(0) \text{ is a mass point} &= \int_0^{\infty}\int_x^{\infty} f^{dw}(x,a)\,dadx \text{ for } b = 0 \\ f^{nw}(b) &= \int_0^{\infty} f^{dw}(b+a,a)\,da \text{ for } b > 0 \end{aligned}$$

[1] This can be calculated as $F^{dw}(x,a) = \int_0^{a-w_2a_2} ... \int_0^{a-\sum_{g=2}^{G-1} w_g a_g} f^{da}\left(x, a - \sum_{g=2}^{G} w_g a_g, a_2, ..., a_G\right) da_2...da_G$.

and

$$F^{nw}(b) = \int_0^b \int_0^\infty f^{dw}(y+a,a)\,dady + f^{nw}(0)$$

Note that at the mass point $b = 0$ wind output is greater than total electricity demand and must be curtailed.

3.2.4 Conventional generation

I assume there is a set of dispatchable technologies denoted by $i = 1, 2...B$ characterized by a constant marginal cost of production c_i (\$/MWh increasing in i), per year capital cost K_i (\$/MW decreasing in i) and carbon emission intensity EI_i (tonnes of CO_2 equivalent emissions per MWh). I assume that these technologies are dispatchable (not intermittent) and may produce output less than or equal to their total capacity at any point in time.[2] Also I assume no two technologies have identical marginal and capital costs and that $K_1 = 0$ where c_1 can be thought of the marginal costs of demand side management or the value of lost load.

Amongst the set of potential investments the least cost technology(s) to build if it is utilized a fraction u of the time is $H(u)$ where

$$H(u) = \arg\min_i K_i + uc_i$$

The lower and upper bounds of utilization for a technology i by u_l^i, u_h^i such that $[u_l^i, u_h^i] = H^{-1}(i)$ which is a closed interval of rates of utilization over which the technology i is least cost. The set of least cost technologies are $G^* = \{i | i \in H(u) \text{ for some } u \in [0,1]\}$. I describe the technology $B = H(1)$ as a baseload technology ($H^{-1}(B) = [u_l^B, 1]$) and the set of technologies $G^*/H(1)$ as peaking/intermediate technologies.

[2]The analysis does not consider start up costs or ramping rates of different technologies.

3.2.5 Market design

I assume that the market is such that the incentives for investment result in the least cost mix of generation investment to meet electricity demand subject to any constraints imposed by government policy. The least cost mix of investment is given by the solution to:

$$\min_{w,MW^{\imath}(s)Cap^{\imath}} \sum_i \left(K_{\imath} Cap_i + \int c_{\imath} MW_i(s)\, dF^{da}(s) \right) + \sum_g w_g K_w$$

subject to the energy balance constraint

$$x \leq \sum_i MW_i(s) + \sum_g w_g a_g$$

capacity constraint

$$Cap_i \geq MW_{\imath}(s) \geq 0$$

where $MW_i(s)$ is the level of output from technology i when the state of the world is s. If there is a government policy in place to support the intermittent technology an additional constraint on the amount of intermittent output:

$$\text{Target} \leq \int \min \left\{ \sum_g w_g a_g, x \right\} dF^{da}(s)$$

Where "Target" is the minimum fraction of total electricity which the policy states must come be generated from an intermittent source.

Lemma 9 *Suppose the shadow price on the energy balance constraint is given by* $\lambda(s)$ *and there is no government policy supporting the intermittent technology. Then the least cost operation* $\{MW_i^*(s)\}$ *and set of investments* Cap_i^* *and* w_g^* *satisfy:*
Merit Order Dispatch

$$\begin{aligned} MW_i^*(s) &= Cap_i^* && \textit{if } \lambda(s) > c_i \\ MW_i^*(s) &= 0 && \textit{if } \lambda(s) < c_i \\ MW_i^*(s) &\in (0, Cap_i^*) && \textit{if } \lambda(s) = c_i \end{aligned} \tag{3.1}$$

$$Cap_i^* = F^{nw^*-1}\left(1-u_l^i\right) - F^{nw^*-1}\left(1-u_h^i\right) \tag{3.2}$$

$$\begin{array}{ll} w_g^* = 1 & \text{if } \int \lambda(s)\, a_g(s)\, dF(s) > K_w \\ w_g^* = 0 & \text{if } \int \lambda(s)\, a_g(s)\, dF(s) < K_w \\ w_g^* \in (0,1) & \text{if } \int \lambda(s)\, a_g(s)\, dF(s) = K_w \end{array} \tag{3.3}$$

Proof. The Lagrangian for the least cost means of meeting electricity demand is given by:

$$\begin{aligned} \mathcal{L} = & \sum_i \left(K_i Cap_i + \int c_i MW_i(s)\, dF(s)\right) + \sum_g w_g K_w \\ & + \lambda(s)\left[x - \sum_i MW_i(s) - \sum_g w_g a_g\right] \\ & + \overline{\gamma}_i(s)\left[Cap_i - MW_i(s)\right] + \underline{\gamma}_i(s)\left[MW_i(s)\right] \\ & + \underline{\tau}_i Cap_i + \overline{\eta}\left[1 - w_g\right] + \underline{\eta} w_g \end{aligned}$$

First order condition with respect to $MW_i(s)$ gives

$$c_i - \lambda(s) - \overline{\gamma}_i(s) + \underline{\gamma}_i(s) = 0$$

which implies

$$\begin{array}{ll} \lambda(s) = c_i & \text{if } 0 < MW_i(s) < Cap_i \\ \lambda(s) > c_i & \text{if } MW_i(s) = Cap_i \\ \lambda(s) < c_i & \text{if } MW_i(s) = 0 \end{array}$$

which is satisfied by merit order dispatch. The first order conditions with respect to Cap_i :

$$K_i - \int \overline{\gamma}_i(s)\, dF(s) = 0$$

which implies that:

$$K_i = \int \max\{\lambda(s) - c_i, 0\}\, dF^{nw^*}(s)$$

holds at the optimum. To see this is satisfied by merit order dispatch and efficient investment

$$
\begin{aligned}
&\int \max\left\{\lambda(s) - c_i, 0\right\} dF^{nw^*}(s) \\
= &\sum_{j=1}^{i-1} c_j \left[u_h^j - u_l^j\right] \\
= &\sum_{j=1}^{i-1} c_j \left[u_l^{j+1} - u_l^j\right] = \sum_{j=2}^{i} u_l^j \left[c_{j-1} - c_j\right] \\
= &\sum_{j=2}^{i} K_j - K_{j-1} = K_i
\end{aligned}
$$

where the first step follows from merit order dispatch equation 3.1 and the efficient investment condition equation 3.2. The second step follows from $u_h^i = u_l^{i+1}$. The third step follows from $\left[c_{n-1} - c_n\right] u_l^n = K_n - K_{n-1}$ and $K_1 = 0$.

First order condition with respect to w_g :

$$
K_w - \int \lambda(s)\, a_g dF(s) - \overline{\eta} + \underline{\eta} = 0
$$

which implies

$$
\begin{aligned}
K_w &= \textstyle\int \lambda(s)\, a_g dF(s) \quad \text{if } 0 < w_g < 1 \\
K_w &< \textstyle\int \lambda(s)\, a_g dF(s) \quad \text{if } w_g = 1 \\
K_w &> \textstyle\int \lambda(s)\, a_g dF(s) \quad \text{if } w_g = 0
\end{aligned}
$$

■

Merit order dispatch states that each technology is dispatched in order of their respective marginal costs up to the installed capacity of the technology. Efficient investment for dispatchable technologies ensures that a unit of capacity which is used as much or as little as possible within the scope of efficient dispatch will have a level of utilization $u^i \in \left[u_l^i, u_h^i\right]$. Efficient investment for the intermittent technology ensures that given spot prices equal to the shadow price on the energy balance constraint $\lambda(s) = c_i$ where i is the marginal generation unit then the intermittent investment does not make a loss. With the inclusion of the government policy supporting the intermittent technology the condition for investment in the intermittent technology changes to

$$
\begin{array}{ll}
K_w = \int (\lambda(s) + \chi) a_g dF(s) & \text{if } 0 < w_g < 1 \\
K_w < \int (\lambda(s) + \chi) a_g dF(s) & \text{if } w_g = 1 \\
K_w > \int (\lambda(s) + \chi) a_g dF(s) & \text{if } w_g = 0
\end{array}
$$

where χ is the shadow price on the policy constraint. The quantity χ has the natural interpretation as the price of renewable energy certificates if the policy is a renewable portfolio standard or the level of a feed-in tariff to induce investment to meet the policy target. The only changes to the operation and investment of the dispatchable technologies comes through the change in the investment in the intermittent technology w^* which affects investment through the change in the distribution of net load F^{nw^*} which in turn affects the amount of output from each technology.

3.3 Baseload vs peaking/intermediate technologies

In this section I present results on the impact of a government policy supporting the intermittent technology on investment and output in baseload and peaking/intermediate technologies. Provided government support for the intermittent technology does not discourage investment at locations g then investment in baseload technologies will decrease. On the other hand investment in and output from peaking/intermediate technologies may increase or decrease with government support. I will denote investment and output variables by ˜ if there is a policy in place. Hence investment in the intermittent technology is given by w_g^* with no policy and $\widetilde{w}_g^*$ with a policy.

Proposition 1 *Suppose $\widetilde{w}_g^* \geq w_g^*$ for all g then $Cap_B \geq \widetilde{Cap}_B$ and $\int MW_i(s)\, dF(s) \geq \int \widetilde{MW}_i(s)\, dF^{da}(s)$.*

Proof. When $\widetilde{w}_g^* \geq w_g^*$, F^{nw^*} first order stochastically dominates $F^{n\widetilde{w}^*}$. The total capacity investment in the baseload technology is determined by the cumulative distribution of net load $F^{nw^*}(b)$ and the utilization rate above which the baseload technology is the most efficient u_B. The efficient level of investment in baseload capacity Cap_B satisfies

$$F^{nw^*}\left(Cap_B\right) = 1 - u_B$$

Similarly investment in the baseload technology under government support for an intermittent technology is determined by the cumulative distribution of the residual demand $F^{n\widetilde{w}^*}\left(\widetilde{Cap}_B\right)$ and u_B.

$$F^{n\widetilde{w}^*}\left(\widetilde{Cap}_B\right) = 1 - u_B$$

When $\widetilde{w}_g^* \geq w_g^*$ F^{nw^*} first order stochastically dominates $F^{n\widetilde{w}^*}$. This immediately implies $Cap_B \geq \widetilde{Cap}_B$ and so the total capacity of baseload technology will weakly decrease under government support for an intermittent technology. Output without the policy is given by

$$\int_0^{u_B} u \frac{1}{f^{nw^*}\left(F^{nw^*\,-1}\left(1-u\right)\right)} du$$

output with the policy is given by

$$\int_0^{u_B} u \frac{1}{f^{n\widetilde{w}^*}\left(F^{n\widetilde{w}^*\,-1}\left(1-u\right)\right)} du$$

Output decreases as an immediate consequence of F^{nw^*} first order stochastically dominating $F^{n\widetilde{w}^*}$. ■

These results show that under relatively weak assumptions on the availability of the intermittent resource the intermittent technology is substituted for the baseload technology when there is a government policy supporting it. The same is not necessarily true for peaking/intermediate technologies which may be complements or substitutes for the intermittent technology.

Proposition 2 *For technologies other than the baseload technology a government policy supporting the intermittent technology may lead to an increase or decrease in capacity* $Cap_i \lesseqgtr \widetilde{Cap}_i$ *and output* $\int MW_i\left(s\right) dF\left(s\right) \lesseqgtr \int \widetilde{MW}_i\left(s\right) dF\left(s\right)$ *for* $i \neq B$.

I will give an example of capacity/output increasing and an example of it decreasing. Assume $G = 1$ and K_w is large such that without the government policy $w_1^* = 0$. Assume

$f^d(x) = 1$ for $x \in [0.5, 1.5]$ and $f^a(a_1) = 5$ for $a_1 \in [0, 0.2]$. I consider the two cases where the correlation ρ between x and a_1 is 1 and -1. Assume the target of the government policy is 0.1 such that $\widetilde{w}_1^* = 1$.

Example 1 $\rho = 1$

In this case net load is $f(b) = f(x - a) = 1.2$ for $b \in [0.5, 1.3]$.

$$Cap_i^* - \widetilde{Cap}_i^* = (u_h^i - u_l^i) - 0.8 (u_h^i - u_l^i) > 0$$

and the change in output is given by:

$$\int \left(MW_i^*(s) - \widetilde{MW}^*(s)\right) dF^d(s) = \frac{(u_h^i - u_l^i)}{2} \left[(u_h^i - u_l^i) - 0.8 (u_h^i - u_l^i)\right] > 0$$

Example 2 $\rho = -1$

The density of net load is $f(b) = f(x - a_1) = 0.8$ for $b \in [0.3, 1.5]$The change in capacity is therefore given by:

$$Cap_i^* - \widetilde{Cap}_i^* = (u_h^i - u_l^i) - 1.2 (u_h^i - u_l^i) < 0$$

and the change in output is given by:

$$\int \left(MW_i^*(s) - \widetilde{MW}^*(s)\right) dF(s) = \frac{(u_h^i - u_l^i)}{2} \left[(u_h^i - u_l^i) - 1.2 (u_h^i - u_l^i)\right] < 0$$

These two examples highlight the correlation between the availability of the intermittent resource and electricity demand in determining whether the peaking/intermediate technology is a substitute or complement for the intermittent technology. Figure 3-1 illustrates the change in output of peaking/intermediate technologies from the introduction of a government policy supporting the intermittent technology. The vertical axis is MW and the horizontal axis is a fraction. The figure shows the load duration curve (demand ordered from highest to lowest) and net load duration curve (net load ordered from highest to lowest) when there is a government policy. The dark shaded area is a decrease in output from the peaking/intermediate technology and the lighter area is an increase. The net of

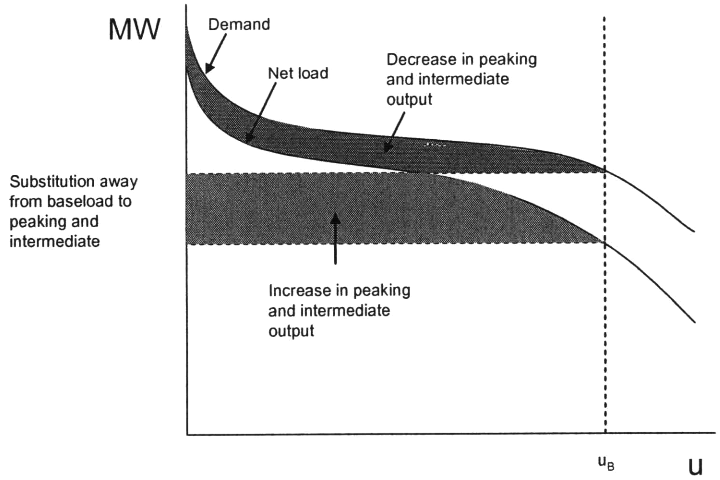


Figure 3-1: Change in output by peaking/intermediate technologies from the introduction of a government policy

the two is the total change as a result of a government policy. The scenario as illustrated in the figure is one in which a significant amount of intermittent output occurs during the lowest hours of demand. In this case a combination of the intermittent technology, during offpeak hours and peaking/mid merit technology, peak hours, replaces baseload output. This may have adverse effects for greenhouse gas emissions if the baseload plant that would have been invested in is cleaner than the combination of peaking/intermediate and intermittent technology.

3.4 Old and new generation

In this section the usage/scrapping decision of the existing stock of old generators is incorporated into the framework. The stock of these old generators is given by Cap_{old} which is fixed. The capital costs associated with the existing stock of old generators is assumed to be sunk. The on-going fixed costs of these generators is small and for the purposes of the analysis are assumed to be zero. Further for illustrative purposes I assume there is one dispatchable technology other than the technology $i = 1$ with zero fixed costs and the intermittent resource which has zero variable costs. I assume that the variable costs of the new technology c_{new} are less than the old technology, thus $c_1 > c_{old} > c_{new} > c_{intermittent} = 0$. The minimum cost to meet demand is the solution to:

$$\min_{\substack{w, MW_{old}(s),\\ MW_{new}(s), Cap_{new}}} K_{new}Cap_{new} + \int \left(c_{new}MW_{new}\left(s\right) + c_{old}MW_{old}\left(s\right)\right) dF\left(s\right) + \sum_g w_g K_w$$

subject to the energy balance constraint

$$x \leq MW_{new}\left(s\right) + MW_{old}\left(s\right) + \sum_g w_g a_g \text{ for all } \left(x, a_1, ...a_G\right)$$

capacity constraint

$$\begin{aligned} Cap_{New} &\geq MW_{New}\left(s\right) \geq 0 \\ Cap_{New} &\geq MW_{New}\left(s\right) \geq 0 \end{aligned}$$

and if there is a government policy in place to support the intermittent technology there is an additional constraint:

$$\text{Target} = \int \min\left\{\sum_g w_g a_g, x\right\} dF\left(s\right)$$

Define a function $u^*\left(u_1\right) : [0,1] \rightarrow \mathbb{R}$ by:

$$u^*\left(u_1\right) = \frac{K_{new}}{c_{old} - c_{new}} - u_1\frac{c_1 - c_{new}}{c_{old} - c_{new}}$$

The function u^* returns the fraction of the time the old technology must be marginal for the new technology to break even if technology $i = 1$ is marginal for a fraction u_1 of the time. This function is useful to pin down efficient investment in the following lemma.

Lemma 10 *Suppose $u^* < 1$ and the shadow price on the energy balance constraint is given by $\lambda\left(s\right)$ for each state of the world and there is no government policy supporting the intermittent technology the least cost operation $\left\{MW_i^*\left(s\right)\right\}$ and set of investments Cap_{new}^**

and w_g^* *satisfy:*

Merit order dispatch

$$\begin{array}{ll} MW_i^*(s) = Cap_i^* & \text{if } \lambda(s) > c_i \\ MW_i^*(s) = 0 & \text{if } \lambda(s) < c_i \\ MW_i^*(s) \in (0, Cap_i^*) & \text{if } \lambda(s) = c_i \end{array} \tag{3.4}$$

Efficient investment

$$Cap_1^* = F^{nw^*-1}(1) - F^{nw^*-1}(1 - u_1^*) \tag{3.5}$$

$$Cap_{new}^* = F^{nw^*-1}(1 - u^*(u_1^*)) \tag{3.6}$$

where u_1^* *satisfies*

$$u_1^* = 0 \quad \text{if } F^{nw^*-1}(1) - F^{nw^*-1}(1 - u^*(0)) \geq Cap_{old}$$

$$F^{nw^*-1}(1 - u_1^*) - F^{nw^*-1}(1 - u^*(u_1^*)) = Cap_{old} \quad \text{otherwise}$$

$$\begin{array}{ll} w_g^* = 1 & \text{if } \int \lambda(s)\, a_g dF(s) > K_w \\ w_g^* = 0 & \text{if } \int \lambda(s)\, a_g dF(s) < K_w \\ w_g^* \in (0,1) & \text{if } \int \lambda(s)\, a_g dF(s) = K_w \end{array} \tag{3.7}$$

Proof. The proof is the same as for Lemma 9 except for the condition for investment in capacity for the new technology. The first order condition with respect to Cap_{new}

$$K_{new} = \int \max\{\lambda(s) - c_{new}, 0\}\, dF(s)$$

If $\lambda(s) = c_i(s)$ where $c_i(s)$ is the marginal generation unit in the merit order, then:

$$\begin{aligned} & \int \max\{\lambda(s) - c_{new}, 0\}\, dF(s) \\ = \;& c_{old} \max\left\{1 - F^{nw^*}(Cap_{new}^*), Cap_{old}\right\} + c_1\left(1 - F^{nw^*}(Cap_{new}^* + Cap_{old}^*)\right) \\ = \;& c_1 u_1^* + c_{old}(u^*(u_1^*) - u_1^*) \\ = \;& K_{new} \end{aligned}$$

where the first step follows from merit order dispatch and the second and third steps follow from the definitions of Cap^*_{new}, u^*_1 and u^*. ■

In the lemma there are two cases, one where some capacity of the old technology is never used, which one could consider as being scrapped ($u^*_1 = 0$), and another case where all of the old technology is used and there is some demand side management through technology $i = 1$ ($u^*_1 > 0$) This lemma illustrates that the old technology is similar to a peaking/intermediate technology in that it has small ongoing fixed costs and high variable costs relative to the new technology and is thus utilized relatively infrequently compared to the new technology which is more similar to the baseload technology from the previous section. Indeed similarly to peaking/intermediate technologies the output from the old technology may increase with the introduction of a policy supporting the intermittent technology. This is shown in the following result.

Proposition 3 *Output from the old technology may increase or decrease with the introduction of a policy to support the intermittent technology*

I will give an example of output increasing and an example of it decreasing. Assume $G = 1$ and K_w is large such that without the government policy $w^*_1 = 0$. Assume $f(x) = 1$ for $x \in [0.5, 1.5]$ and $f(a_1) = 5$ for $a_1 \in [0, 0.2]$. I consider the two cases where the correlation ρ is 1 and -1. Assume the target of the government policy is 0.1 such that $\widetilde{w}^*_1 = 1$. Finally assume $Cap_{old} > F^{d\ -1}(1) - F^{d\ -1}(1 - u^*(0))$. When this is the case the capacity investment for the new technology is $F^{nw^*\ -1}(1 - u^*(0))$.

Example 1 $\rho = 1$

The density of net load is $f^n(b) = f(x - a_1) = 1.2$ for $b \in [0.5, 1.3]$. The change in output is given by:

$$\int \left(MW^*_{old}(s) - \widetilde{MW}^*_{old}(s) \right) dF(s) = \frac{u^*(0)}{2} \left[u^*(0) - 0.8u^*(0) \right] > 0$$

Example 2 $\rho = -1$

The density of net load is $f^n(b) = f(x - a_1) = 0.8$ for $b \in [0.3, 1.5]$. The change in output is given by:

$$\int \left(MW_i^*(s) - \widetilde{MW}^*(s)\right) dF(s) = \frac{u^*(0)}{2}\left[u^*(0) - 1.2u^*(0)\right] < 0$$

It is the cost characteristics of the old technology, low fixed and high variable costs which make it an attractive option for supplying electricity relatively infrequently during times of peak electricity demand. When the availability of the intermittent resource is such that there is a great deal of capacity required to operate relatively infrequently, when there is significant intermittent investment, then output from this older technology increases in the presence of government support.

3.4.1 Carbon emissions

In this section I derive the conditions under which government support for an intermittent technology will increase carbon emissions. It is informative to discuss carbon emissions in terms of the relative carbon intensities of the mix of dispatchable (non-intermittent) technologies with and without government support for the intermittent technology. The carbon intensity is the amount of emissions per unit of demand met by dispatchable technologies. Denoting the emission intensity by λ this is:

$$\begin{aligned}\lambda &= \frac{Emissions}{Output} \\ &= \frac{\int \left(\sum_i EI_i MW_i(s)\right) dF^{da}(s)}{\int \left(\sum_i MW_i(s)\right) dF^{da}(s)}\end{aligned}$$

I will use $\lambda^{NoPolicy}$ as the emission intensity with no government policy and λ^{Policy} with a government policy.

Comparing the emissions outcomes with and without government support amounts to determining whether the carbon intensity per MWh $\lambda^{NoPolicy}$ from the mix of technologies when there is no support for the intermittent technology is larger or smaller than $(1 - \text{Target})$ times the emissions intensity per MWh with support for the intermittent technology λ^{Policy}. Government support for an intermittent technology will increase carbon emissions when:

$$\text{Target} \times \lambda^{NoPolicy} < (1 - \text{Target}) \left(\lambda^{Policy} - \lambda^{NoPolicy}\right)$$

This highlights the two potential opposing effects from supporting a clean but intermittent technology. The first is the direct effect from having a carbon free technology generate $s\%$ of electricity (the left-hand side), the second and potentially adverse effect comes through changing the efficient mix of non-renewable technologies over the remaining $(1 - \text{Target})\,\%$ of electricity (the right hand side). If the intermittent generation results in a shift towards technologies with relatively high EI_i then the increase in emissions due to this substitution can more than offset the gains from the direct effect.

Currently there is insufficient experience with large scale deployment of intermittent technologies to empirically identify an effect on investment behavior in a region from an increase in intermittent investment. However the message from the theory is that policies supporting intermittent technologies are inconsistent with or costly approaches to addressing climate change objectives when:

1. the availability of the intermittent resource is higher when electricity demand is lower; and,

2. new baseload technologies are less carbon intensive than the existing stock of technologies and new peaking/intermediate technologies.

The first of these conditions can be tested in different regions of the world through measurement of the resource (hourly wind speeds and hourly intensity of sunlight in the case of wind and solar) and electricity demand over time. The second condition calls for a scenario-based analysis as there is considerable uncertainty about the future costs associated with various technological options for supplying electricity.

3.5 Conclusion

The contribution of this paper is to highlight that policies which explicitly support intermittent technologies, over other technological options, may have effects beyond the electricity that is displace by the intermittent output in the short run. In particular the results highlight that when considering GHG emissions the short run effect from the electricity displaced may be offset by long run investment changes. Thus extrapolating short run benefits for calculating long term effects can be misleading. This effect comes through the effect on the investment decisions of new generators which are not covered by the government policy. The variability of both electricity and the availability of the intermittent resource may result in some technologies being complementary in the long run to the intermittent technology. A policy which supports the intermittent technology may therefore increase the amount of investment and output from these technologies. These technologies aren't covered by the government policy and hence need not have low GHG emissions.

In the first part of the paper the focus is on the type of technological investment which takes place. Here I contrast peaking/intermediate technologies with baseload technologies. I find that baseload technologies are, under mild conditions, substitutes for an intermittent technology. No such distinction is true for peaking/intermediate technologies which may be complements or substitutes. If the availability of the intermittent resource is sufficiently negatively correlated with electricity demand then investment/output from peaking/intermediate technologies may increase.

In the second part of the paper I focus on the amount of investment in new technologies when there is an existing stock of old technologies. Here I find that if the availability of the intermittent resource is sufficiently negatively correlated with electricity demand then output from the old technology may increase when there is a policy which supports the intermittent technology.

When peaking/intermediate and/or older technologies are relatively carbon intensive compared to new baseload technologies it raises concerns about the effectiveness of these types of policies for addressing climate change objectives. In particular one may be concerned that the support for an intermittent technology reduces the investment in and output from nuclear and/or coal with carbon capture and sequestration by amounts even greater than the electricity included in the policy.

Bibliography

[1] Ballester, C., Calvo-Armengol, A, and Zenou, Y. (2006) "Who's Who in Networks. Wanted: The Key Player," *Econometrica*, 74 (4), pp1403-1418.

[2] Banerji, A., and Dutta, B. (2005) "Local Network Externalities and Market Segmentation," mimeo.

[3] Berge, C. (1963) *Topological Spaces*, Edinburgh: Oliver and Boyd Ltd.

[4] Bloch, F., Genicot, G. and Ray, D. (2005) "Informal Insurance in Social Networks," mimeo.

[5] Borenstein, S. (2005) "The long-run efficiency of real-time electricity pricing," *Energy Journal*, 26 (3), pp93-116.

[6] Borenstein, S. (2008) "The market value and cost of solar photovoltaic electricity production," CSEM WP 176.

[7] Callaway, D., Newman, M., Strogatz, S., and Watts, D. (2000) "Network Robustness and Fragility: Percolation on Random Graphs," *Physical Review Letters*, 85 (25), pp5468-5471.

[8] Calvo-Armengol, A. and Jackson, M. (2004) "The Effects of Social Networks on Employment and Inequality," *American Economic Review,* 94, pp426-454.

[9] Chevalier, J., and Mayzlin, D. (2006) "The Effect of Word of Mouth Online: Online Book Reviews," *Journal of Marketing Research*, 43 (3), pp345-354.

[10] Currarini, S., Jackson, M. and Pin, P. (2008) "An Economic Model of Friendship: Homophily, Minorities and Segregation," forthcoming *Econometrica.*

[11] DeCarolis, J.F., and Keith, D.W. (2006) "The economics of large-scale wind power in a carbon constrained world," *Energy Policy*, 34 (4), pp395-410.

[12] Denny, E. and O'Malley, M. (2006) "Wind generation, power system operation, and emissions reduction," *IEEE Transactions on Power Systems*, 21 (1), pp341-347.

[13] Ellison, G. (1993) "Learning, Local interaction, and Coordination," *Econometrica*, 61, pp1047-1071.

[14] Erdös, P., and Renyí, A (1959) "On Random Graphs," *Publicationes Mathematicae*, 6, pp290-297.

[15] Erdös, P., and Renyí, A (1960) "On the Evolution of Random Graphs," *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 5, pp17-61.

[16] Erdös, P., and Renyí, A (1961) "On the Strength of Connectedness of a Random Graph," *Acta Mathematica Scienta Hungary*, 12, pp261-267.

[17] Fafchamps, M. and Lund, S. (2003) "Risk-Sharing Networks in Rural Philippines," *Journal of Development Economics*, 71 (2), pp261-287.

[18] Feick, L. and Price, L. (1987) "The Market Maven: A Diffuser of Marketplace Information," *Journal of Marketing*, 51 (1), pp83-97.

[19] Galleatti, A. and Mattozzi, A. (2008) "Personal Influence: Social Context and Political Competition," mimeo.

[20] Gil,H. and Joos, G. (2007) "Generalized estimation of average displaced emissions by wind generation," *IEEE Transactions on Power Systems*, 22 (3), pp1035–1044.

[21] Gladwell, M. (2000) *The tipping point: How little things can make a big difference,* New York: Little, Brown and Company.

[22] Glaeser, E., Sacerdote, B. and Scheinkman, J. (1996) "Crime and Social Interactions," *Quarterly Journal of Economics,* 111 (2), pp507-548.

[23] Godes, D., and Mayzlin, D. (2009) "Using Online Conversations to Study Word of Mouth Communication," *Marketing Science*, 23 (4), pp545-560.

[24] Goyal, S., and Galleotti, A. (2009) "A Theory of Strategic Diffusion," Forthcoming *RAND Journal of Economics.*

[25] Holttinen, H. and Tuhkanen, S. (2004) "The effect of wind power on CO_2 abatement in Nordic Countries," *Energy Policy*, 32, pp1639-1652.

[26] Jackson, M. (2008) *Social and Economics Networks,* Princeton University Press.

[27] Jackson, M. and Rogers, B. (2007) "Meeting Strangers and Friends of Friends: How Random are Social Networks," *American Economic Review,* 97 (3), pp890-915.

[28] Jackson, M. and Yuriev, L. (2006) "Diffusion of Behavior and Equilibrium Properties in Network Games," *American Economic Review (Papers and Proceedings)*, 97 (2), pp92-98.

[29] Keith, G. Biewald, B. and White, D. (2004) *Evaluating simplified methods of estimating displaced emissions in electricity power systems: What works and what doesn't,* Synapse Energy Economics Report.

[30] Lamont, A. (2008) "Assessing the long-term system value of intermittent electric generation technologies," *Energy Economics*, 30 (3), pp1208-1231.

[31] Mobius, M. and Szeidl, A. (2006) "Trust and Social Collateral," mimeo.

[32] Mobius, M., Niehaus, P., and Rosenblat, T. (2006) "Social Learning and Consumer Demand," mimeo.

[33] Moore, C. and Newman, M. (2000) "Epidemics and Percolation in Small-World Networks," *Physical Review E*, 61 (5), pp5678-5682.

[34] Morris, S. (2000) "Contagion," *Review of Economic Studies*, 67, pp57-78.

[35] National Research Council, (2007) *Environmental impacts of wind-energy projects*, The National Academies Press.

[36] Newman, M. (2002) "The Spread of Epidemic Diseases on Networks," *Physical Reveiw E*, 66 (1), art. no. 016128.

[37] Newman, M. (2007) "Component Sizes in Networks with Arbitrary Degree Distributions," *Physical Review E,* 76, art. no. 045101.

[38] Newman, M., Strogatz, S., and Watts, D. (2001) "Random Graphs with Arbitrary Degree Distributions and their Applications," *Physical Review E*, 64, art. no. 026118.

[39] Newman, M., and Watts, D. (1999) "Scaling and Percolation in the Small-World Network Model," *Physical Review E*, 60 (6), pp7332-7342.

[40] Park, J., and Newman, M. (2005) "Solution for the Properties of a Clustered Network," *Physical Review E*, 72, Art No. 026136.

[41] Reichheld, F. (2003) "The One Number You Need to Grow," *Harvard Business Review*, 81 (12), pp47-54.

[42] Sander, L., Warren, C., Sokolov, I., Simon, C., and Koopman, J. (2002) "Percolation on Heterogeneous Networks as a Model for Epidemics," *Mathematical Biosciences,* 180, Issue 1-2, pp293-305.

[43] Strbac, G. (2002) *Quantifying the system cost of additional renewables in 2020*, A report to the Department of Trade and Industry, [online] Available: http://www.berr.gov.uk/files/file21352.pdf (accessed April 2009).

[44] Watson, J. (1999) "Starting Small and Renegotiation," *Journal of Economic Theory,* 85, pp52-90.

[45] Watson, J. (2002) "Starting Small and Commitment," *Games and Economic Behaviour*, 38 (1), pp176-199.

[46] Watts, D. (2002) "A Simple Model of Cascades on Random Networks," *Proceedings of the National Academy of Sciences of the United States of America*, 99 (9) pp5766-5771.

[47] Watts, D., Perretti, J., and Fumin, M. (2007) "Viral Marketing for the Real World," mimeo.

[48] Young, P. (2002) "Diffusion of Innovations in Social Networks," mimeo.