ON THE EVALUATION OF INTEGRALS OF THE TYPE

$$f(\tau_1, \tau_2, \ldots, \tau_n) = \frac{1}{2\pi i} \int F(s)\, e^{W(s, \tau_1, \tau_2, \ldots, \tau_n)}\, ds$$

AND THE MECHANISM OF FORMATION OF TRANSIENT PHENOMENA

Series 55 — Part 2a

AN ELEMENTARY INTRODUCTION TO THE THEORY OF THE SADDLEPOINT METHOD OF INTEGRATION

MANUEL V. CERRILLO

TECHNICAL REPORT NO. 55: 2a

MAY 3, 1950

RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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ON THE EVALUATION OF INTEGRALS OF THE TYPE $f(\tau_1, \tau_2, \ldots, \tau_n) = \frac{1}{2\pi i} \int F(s) e^{W(s, \tau_1, \tau_2, \ldots, \tau_n)} ds$ AND THE MECHANISM OF FORMATION OF TRANSIENT PHENOMENA

Part 2a

ELEMENTARY INTRODUCTION TO THE THEORY OF THE SADDLEPOINT METHOD OF INTEGRATION

Manuel V. Cerrillo

Abstract

This report is a member of a group of reports that are to be published under the generic number 55. It is primarily concerned with the ideas that serve as a foundation for the Saddlepoint Method of Integration. This method is also known by other names; particularly, as the Method of Steepest Descent and the Method of The Points of Stagnation.

The method and ideas presented here will serve as an introductory presentation to a more complete theory of integration which will be discussed in subsequent reports of this series.

The material covers the following condensed description:

(a) Definition and properties of saddlepoints of different orders.

(b) Basic procedure of integration through saddlepoints of the second order.

(c) Definition of the family of integral approximations of the resonant type. The pole, zero, and dipole solutions.

(d) Integration in the case of composite behavior.

(e) Introduction to the idea of "Generating Function" associated with second-order

saddlepoints. The Fresnel solid.

(f) The case of saddlepoints of higher order and the theory of transition for saddlepoints of the third order.

(g) Zero, pole, and dipole types of integral.

(h) The Airy-Hardy type of "Generating Function." The corresponding generating solid.

(i) The mixed type of integral and composition of solutions.

(j) A few comparative typical solutions for transitions of higher order corresponding to the pole, zero, and dipole cases.

Note: The manuscript was originally written for use in certain applications in circuit theory. Because of this objective some specific terminology was used: "resonant transients," "transitional transients," etc. This nomenclature is, of course, not quite suitable for other applications of the method of integration. We therefore advise the reader to disregard the terms "resonant and transitional transients" and use instead the terms "second-order, third-order solutions."

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ON THE EVALUATION OF INTEGRALS OF THE TYPE

$$f(\tau_1, \tau_2, \ldots, \tau_n) = \frac{1}{2\pi i} \int F(s) e^{W(s, \tau_1, \tau_2, \ldots, \tau_n)} ds$$

AND THE MECHANISM OF FORMATION OF TRANSIENT PHENOMENA

Part 2a Elementary Introduction to the Theory of the Saddlepoint Method of Integration

INTRODUCTION

The present report, Technical Report No. 55:2a, is a member of a general group of reports which is to be published under the generic number 55.

Report No. 55 is in general concerned with

1. the particular theories which have been evolved in connection with the approximate integration of a large family of integrals, whose prototype is described by

$$f(\tau_1, \tau_2, \ldots, \tau_n) = \frac{1}{2\pi i} \int_{\gamma_s} F(s) e^{W(s, \tau_1, \tau_2, \ldots, \tau_n)} ds$$

   where s is a complex variable,

   γ_s denotes the contour in the s plane along which the function $f(\tau_1, \ldots, \tau_n)$ is defined by the integral,

   $F(s)$ and $W(s, \tau_1, \ldots, \tau_n)$ constitute integrand functions whose analytic characterizations are not given here,† and

   $\tau_1, \tau_2, \ldots, \tau_n$ are a set of variable parameters which are all independent of s.

2. the production of a theory which attempts to unify several of the methods of integration, such as the stationary phase method, the saddlepoint method, the cliff and the pocket methods. These methods have proved to be quite suitable for obtaining constructive integral solutions of a large number of integrals which are contained in the prototype integral given above.

In the original manuscript for Report No. 55, the present report, No. 55:2a, was contained as a single chapter. This chapter was intended as an elementary introduction to the basic ideas, aims, procedure, and results of the saddlepoint method of integration.†† As the material contained in this chapter was rather voluminous it was decided to publish it as a separate report.

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†See Report No. 55:1 and subsequent reports in the 55 series, in which such characterizations are made.

††This method is also known by other names: the method of steepest descent, the method of stagnation points.

We will give a brief outline of this report:

1. An elementary definition of saddlepoints, their orbits, and the lines of steepest descent in the neighborhood of the saddlepoints.
2. Typical expansions of the integral functions around the saddlepoints.
3. A typical procedure of integration for saddlepoints of the first order, when the saddlepoints move far away from other singularities of the remaining integrand.
4. Typical procedures of integration, when the saddlepoints move in the vicinity of poles, zeros, and dipoles.
5. Introductory discussion concerning saddlepoints of higher order, particularly in connection with transition and confluence of saddlepoints.
6. Elementary discussion of integrals which can be reduced to second-order saddlepoints; transition solutions; definition of the function $Ah_{1,2,3}$.
7. Mixed transitional transients.

Since the aim of this report is to give an elementary illustration of the saddlepoint method of integration, we have omitted the general theory of saddlepoint transition in the presence or absence of poles, branch points, and the like. The general theory is carefully discussed in other parts of Report No. 55. Also, the question of measure of approximation is omitted here because it is more suitable to treat this question in connection with the general theory.

Report No. 55:1 contains a discussion of the requirements of a general character which a method of integration has to satisfy. The requirements serve as a basis for the definition of the so-called constructive methods. The ideas may be difficult to visualize for persons with no previous experience with approximate integration. Therefore, the discussion and the results of this report are intended to show that, by means of particular cases, it is possible to design methods of approximate integration along the lines indicated in Report No. 55:1.

CHAPTER I

INTEGRATION THROUGH SADDLEPOINTS OF THE SECOND ORDER

SECTION I. 1 SADDLEPOINTS. CONTOURS OF INTEGRATION.

Let

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} F(s)\, e^{W(s,t)}\, ds; \quad W(s,t) = st - \phi(s) \qquad 1(I.1)$$

F(s) is supposed to be free from terms of exponential behavior. For the moment let us assume that F(s) is a rational or algebraic function.

I. 10 DEFINITION OF A SADDLEPOINT OF THE SECOND ORDER.

$$\frac{dW}{ds} = 0 \text{ or } t = \phi'(s) \qquad 2(I.1)$$

The solutions of 2(I. 1) are called saddlepoints of W(s, t). Let them be expressed by the symbol

$$s_s = s_s(t) \qquad 3(I.1)$$

I. 11 ORBITS. Consider one of the saddlepoints of W(s, t):

$$s_s = s_s(t)$$

defines a curve, the orbit of s_s, in the s plane. The saddlepoint describes this trajectory as t changes. See Fig. 1(I. 1).

I. 12 LINES OF STEEPEST DESCENT OF s_s. The value of the function W(s, t) at a saddlepoint is indicated by $W(s_s, t)$.

Let the difference

$$W(s,t) - W(s_s,t) = M = P(\sigma,\omega,t) + i\,Q(\sigma,\omega,t) \qquad 4(I.1)$$

Let t be fixed. The line, or lines, defined by

$$P = 0 \qquad Q = 0 \qquad 5(I.1)$$

are called lines of steepest descent of the saddlepoint corresponding to this particular value of t. Take for example

$$P(\sigma,\omega,t) = 0 \qquad 6(I.1)$$

The equation may have one or more solutions. Let us designate them by

$$\left.\begin{aligned} &\sigma = \phi_\kappa(\omega,t) \qquad \kappa = 1,2,\ldots \\ &\text{Similarly, from } Q = 0 \\ &\sigma = \psi_\kappa(\omega,t) \end{aligned}\right\} \quad 7(I.1)$$

Equation 7(I. 1) explicitly defines the lines of steepest descent.

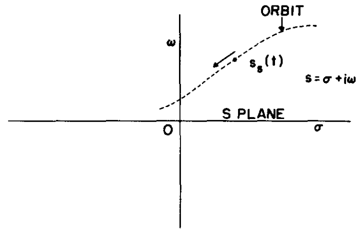

Fig. 1(I. 1)

THEOREM I. The lines of steepest descent, corresponding to a given saddlepoint, pass through the saddlepoint. The proof is simple.

I. 13 BEHAVIOR OF W(s, t) IN THE VICINITY OF A SADDLEPOINT OF THE SECOND ORDER. For the moment, we shall not consider the possibility that W(s, t) possesses poles in the vicinity of s_s. The case of poles will be considered later, when we describe more advanced methods of integration (pocket, substitution, essential methods).

Under the assumption given above, W(s, t) can be expanded in Taylor series in the vicinity of s_s.

$$W(s,t) = W(s_s, t) + \frac{(s - s_s)^2}{2!} W^{II}(s_s, t) + \frac{(s - s_s)^3}{3!} W^{III}(s_s, t) + \ldots \qquad 8(I.\ 1)$$

In the immediate vicinity of s_s we can use

$$W(s,t) - W(s_s, t) \approx \frac{(s - s_s)^2}{2!} W^{II}(s_s, t) \qquad 9(I.\ 1)$$

Momentarily let

$$\frac{W^{II}(s_s, t)}{2!} = A\, e^{i\alpha_2} \qquad 10(I.\ 1)$$

and use the transformation

$$z = s - s_s = r\, e^{i\theta} \qquad 11(I.\ 1)$$

Then

$$W(s,t) - W(s_s, t) = P + iQ \approx A\, r^2 \cos(2\theta + \alpha_2) + iA\, r^2 \sin(2\theta + \alpha_2) \qquad 12(I.\ 1)$$

The lines of steepest descent are given by

P lines	Q lines
$\cos(2\theta + \alpha_2) = 0$	$\sin(2\theta + \alpha_2) = 0$

Therefore

$$\theta_P = \frac{\pi}{4} - \frac{\alpha_2}{2} + \kappa\frac{\pi}{2} \qquad 13(I.\ 1)$$

κ integer $0 \leqslant \kappa \leqslant 3$

$$\theta_Q = -\frac{\alpha_2}{2} + \kappa\frac{\pi}{2} \qquad 14(I.\ 1)$$

κ integer $0 \leqslant \kappa \leqslant 3$

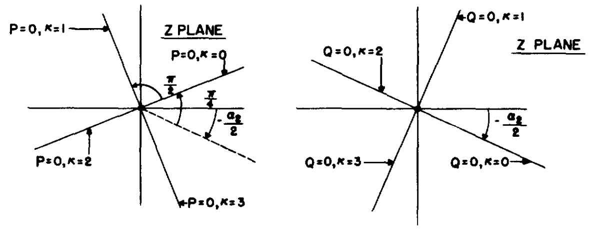

Fig. 2(I. 1)

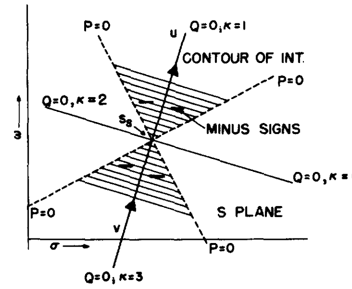

Fig. 3(I. 1)

In general, α_2 is a function of time; therefore the lines rotate accordingly (see Fig. 2(I.1)).

I. 14 CONTOUR OF INTEGRATION. In the saddlepoint methods of integration, the contour of integration is deformed along the lines Q = 0 of steepest descent and such that P attains the largest possible negative values. A simple computation shows us where such a line is placed. See Fig. 3(I. 1).

Along the lines Q = 0 the value of the exponential function is given by

$$W(s, t) - W(s_s, t) = A\, r^2 \cos \kappa\pi = A\, r^2 (-1)^{\kappa} \qquad 15(I.1)$$

It will be negative for $\kappa = 1$, $\kappa = 3$.

SECTION I. 2 SADDLEPOINT METHOD OF APPROXIMATE INTEGRATION THROUGH A SADDLEPOINT OF THE SECOND ORDER.

I. 20 BASIC REQUIREMENTS ON THE CONTOUR OF INTEGRATION. In the saddlepoint method of integration, the contour γ_s must be deformed along the lines of steepest descent with maximum negative P. Since the deformed contour must be equivalent to the original contour γ_s in 1(I. 1), one must proceed as follows:

1. Find the corresponding situation of the lines of steepest descent on the different saddlepoints of W(s, t).
2. In each saddlepoint find the lines Q = 0 for which P is maximum negative. Shadow the corresponding areas in which the real part of $W(s, t) - W(s_s, t)$ is negative.
3. Use only those saddlepoints for which the negative areas overlap in such a way that we can deform the original contour γ_s into the lines Q = 0 of steepest descent which lie continuously in shaded areas. The saddlepoints so selected are called primary saddlepoints.

If these requirements cannot be met, then the saddlepoint method of integration cannot be used. We will assume that such deformation is possible and consider momentarily that the reader is acquainted with the procedure given above.

I. 21 BASIC REQUIREMENTS ON $W^{II}(s_s, t)$. ASYMPTOTIC CHARACTER OF THE SOLUTION. Let us assume that

1. $W^{II}(s_s, t)$ does not vanish in the interval $t_a \leq t \leq t_b$.

2. $\left|W^{II}(s_s, t)\right|$ is sufficiently large in the vicinity of s_s and for $t_a \leq t \leq t_b$. The term "sufficiently large" has a rather vague meaning since we do not establish a lower bound of comparison. For the moment, we will settle this situation as follows. Let us plot the value of

$$E = e^{W(s,t) - W(s_s,t)} = e^{-r^2 \left|\frac{W^{II}(s_s,t)}{2!}\right|} \qquad 1(I.\,2)$$

where r is the distance along the chosen Q = 0 line of steepest descent. For very large values of

$$\left|\frac{W^{II}(s_s,t)}{2!}\right|$$

the function 1(I. 2) has the appearance of curve 1 in Fig. 1(I. 2). In a small interval of r, say $2\Delta r$, the function E changes suddenly from one to almost zero. Curves 2 and 3 show this behavior when

$$\left|\frac{W^{II}(s_s,t)}{2!}\right|$$

is small. For small values of

$$\left|\frac{W^{II}(s_s,t)}{2!}\right|$$

as in curve 3, E is practically constant in the interval $2\Delta r$.

One will require that E show a pronounced behavior, as in curve 1, Fig. 1(I. 2). For the moment we are unable to deduce a quantitative criterion for this situation.† The point will be touched later.

If the function E suffers a large variation in the vicinity of s_s, then the main contribution of the contour in the integral 1(I. 1) comes in the vicinity of s_s. This is the basic idea of the saddlepoint method.

The asymptotic character of the solution comes from the requirement that $\left|W^{II}(s_s, t)\right|$ must be large. The second derivative must be large, but it does not necessarily follow that t must also be large.

In the extended saddlepoint method of integration the peak configuration of E is forced artificially by means of the so-called condensation process. The solutions, then, are not

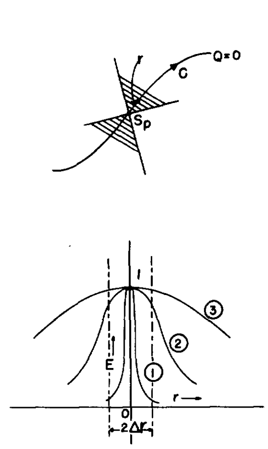

Fig. 1(I. 2)

†The following criterion will be established later: Let k be the percentage of error which we can admit in the integral approximation. The interval of time in which this is true is given by

$$k/100 \leq \left|W^{III}(s_s, t)\right| / 3\sqrt{2\pi W^{II}(s_s, t)}$$

The values of t, on using the equality, mark the limits of the interval.

necessarily asymptotic. In the extended saddlepoint methods (ESPM) the criterion given in the footnote is not necessarily valid.

I.22 THE BASIC INTEGRAL IN THE SADDLEPOINT METHOD. The integral

$$\int_0^\infty s^\nu e^{-as^p} ds = \frac{\Gamma\left(\frac{\nu+1}{p}\right)}{p\, a^{\left(\frac{\nu+1}{p}\right)}} \qquad \nu > -1 \qquad 2(I.2)$$

is called the basic "substitution" integral, or the integral of comparison. (The denomination "substitution" comes from some more advanced and powerful methods of integration.)

All that we do is to reduce 1(I. 1) by a pertinent method of approximation, to one or more integrals of the type 2(I. 2).

SECTION I. 3 THE RESONANCE FAMILY.

I. 30 FIRST TRANSIENT CLASSIFICATION. DEFINITION OF THE RESONANCE FAMILY. Let us assume that

a. $\left|W^{II}(s_s, t)\right|$ is "sufficiently large" in the interval

$$t_a \leqslant t \leqslant t_b$$

b. The function W(s, t) can be expressed approximately as

$$W(s,t) \approx W^*(s,t) = W(s_s,t) + \frac{(s - s_s)^2}{2!} W^{II}(s_s, t) \qquad 1(I.3)$$

Then, under the requirements given above, one approximates the integral 1(I. 1)

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} F(s)\, e^{W(s,t)}\, ds \qquad 1(I.1)$$

by

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} F(s)\, e^{W^*(s,t)}\, ds \qquad 2(I.3)$$

DEFINITION. If F(s) is a rational function of s and $W^*(s, t)$ is at most a polynomial of second degree in s, then the family of transients whose mathematical expression is given by 2(I. 3) is called THE RESONANCE FAMILY.

In the general case of the integral 1(I. 1) one says that the transient has a resonant behavior in the interval $t_a \leqslant t \leqslant t_b$, if 1(I. 1) can be reduced to 2(I. 3) in the same interval.

I. 31 PRINCIPAL MEMBER OF THE RESONANCE FAMILY (order 2). Let us plot the orbits of the saddlepoints of W(s, t) and the position of the poles and zeros of F(s), which we assume to be a rational function. See Fig. 1(I. 3).

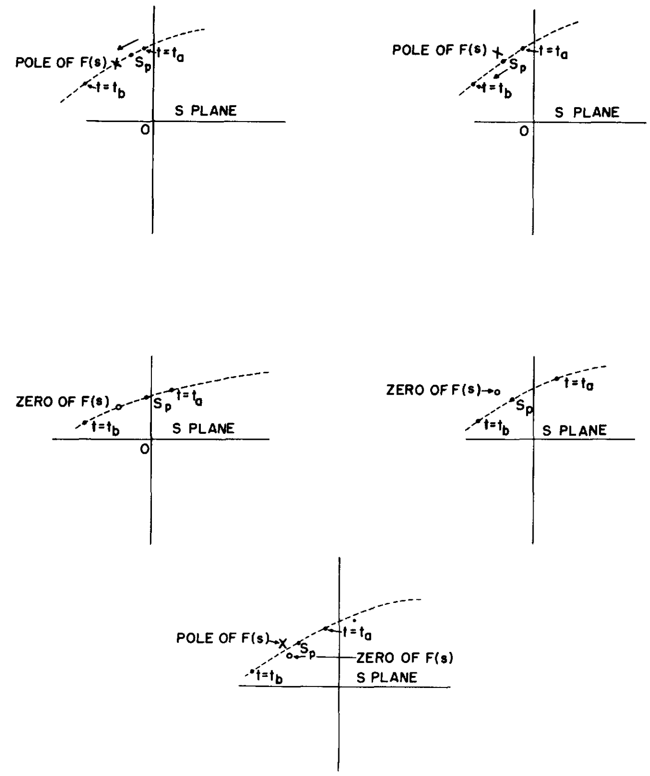

Fig. 1(I. 3)

The following cases are of special interest.

The next definitions are associated with the time interval $t_a \leqslant t \leqslant t_b$, which will be indicated by [a, b].

Pole

- Case A_1 — A primary saddlepoint passes exactly over a pole of F(s) at some time inside the interval [a, b]. Coinciding pole transient.
- Case A_2 — A primary saddlepoint passes close to but not necessarily over a pole of F(s) at some time inside [a, b]. Deviated pole transient.

Zero

- Case B_1 — A primary saddlepoint passes exactly over a zero of F(s) at some time inside [a, b]. Coinciding zero transient.
- Case B_2 — A primary saddlepoint passes close to a zero of F(s) at some time inside [a, b]. Deviated zero transient.

Dipole

- Case C_1 — A primary saddlepoint passes in the vicinity of a pole and a zero of F(s) at some time inside [a, b]. Dipole transient.
- Case C_2 — The pole and zero of Case C_1 coincide. Pure resonance transient.

These classifications are important because:

a. More complicated cases can be reduced to them.

b. They play an important role in the "composition of transient waves."

c. They show the combined effect on f(t) of F(s) and $W^*(s,t)$.

d. They justify the procedure of computation.

I. 32 COMPOSITION OF TRANSIENTS IN SUCCESSIVE ZERO COMPONENTS. Let us assume in this paragraph that in the neighborhood of s_s, and for t inside [a, b] there are no poles of F(s). Then it can be expanded around s_s as a Taylor series as

$$F(s) = \sum_{0}^{\infty} \frac{F^{(\nu)}(s_s)}{\nu!} (s - s_s)^{\nu} \tag{3(I. 3)}$$

Substitution of 3(I. 3) and 8(I. 1) in 1(I. 1) leads to

$$f_n(t) = e^{W(s_s, t)} \sum_{\nu=0}^{\infty} \frac{F^{(\nu)}(s_s)}{\nu!} \frac{1}{2\pi i} \int_{\gamma_s} (s - s_s)^{\nu} e^{\frac{(s - s_s)^2}{2!} W^{II}(s_s, t) + \dots} ds \tag{4(I. 3)}$$

This expression is only valid, in general, in the vicinity of a given saddlepoint. It is then necessary to obtain a similar expression for each primary saddlepoint. The index n means the number of the needed primary saddlepoints through which the original contour must be deformed; $f_n(t)$ denotes the contribution to f(t) as supplied by the segment of γ_s in the neighborhood of the n primary saddlepoints. The complete solution for f(t) is the sum of $f_n(t)$. There are just a few primary saddlepoints and they may appear quite often in conjugate pairs, and therefore the whole situation is very simple to handle. Since f(t) must be real, then we can easily understand why primary saddlepoints appear frequently in conjugate pairs.

The prototype integral in 4(I. 3) is given by

$$I_{\nu}^{*} = \frac{1}{2\pi i} \int_{\gamma_s} (s - s_s)^{\nu} e^{\frac{1}{2!} W^{II}(s_s, t)(s - s_s)^2} ds \tag{5(I. 3)}$$

The integration can be performed as follows:†

One can write (see Fig. 2(I. 3))

$$\int_{\gamma_s} = \int_{s_s}^{u} - \int_{s_s}^{v} \tag{6(I. 3)}$$

---

†We will perform the integration in the s plane. However, it is advisable, in general, to introduce in 1(I. 1) a pertinent convenient transformation, which guarantees the fulfillment of the requirements needed for a good second-order saddlepoint integration. We will describe these transformations in the "Extended Saddlepoint Method of Integration."

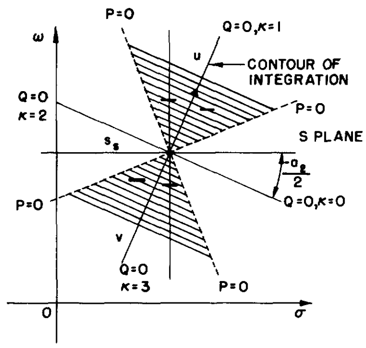

Fig. 2(I. 3)

In the first integral: $\kappa = 1$.

Let

$$s - s_s = re^{i\theta} = re^{i\left(-\frac{a_2}{2} + \frac{\pi}{2}\right)}$$

and

$$ds = e^{i\left(\frac{\pi}{2} - \frac{a_2}{2}\right)} dr = ie^{-\frac{a_2}{2}} dr$$

See Eq. 14(I. 1).

In the second integral: $\kappa = 3$. See 14(I. 1).

$$s - s_s = re^{i\left(\frac{3\pi}{2} - \frac{a_2}{2}\right)} = -re^{i\left(\frac{\pi}{2} - \frac{a_2}{2}\right)}$$

$$ds = -e^{i\left(\frac{\pi}{2} - \frac{a_2}{2}\right)} dr = -ie^{-\frac{a_2}{2}} dr$$

Then

$$I_\nu^* = \frac{1}{2\pi i} e^{(\nu+1)i\left[\frac{\pi}{2} - \frac{a_2}{2}\right]} \int_0^r r^\nu e^{-\frac{\left|W^{II}(s_s, t)\right|}{2!} r^2} dr$$

$$- \frac{(-1)^{\nu+1}}{2\pi i} e^{(\nu+1)i\left[\frac{\pi}{2} - \frac{a_2}{2}\right]} \int_0^r r^\nu e^{-\frac{\left|W^{II}(s_s, t)\right|}{2!} r^2} dr \qquad 7(I. 3)$$

or

$$I_\nu^* = \frac{i^\nu}{2\pi} e^{-i(\nu+1)\frac{a_2}{2}} \left\{1 - (-1)^{\nu+1}\right\} \int_0^r r^\nu e^{-ar^2} dr \qquad 8(I. 3)$$

Since for very large values of r the integrand becomes negligibly small, one then can write

$$I_\nu^* \approx \frac{i^\nu}{2\pi} e^{-i(\nu+1)\frac{\alpha_2}{2}} \left\{1 - (-1)^{\nu+1}\right\} \int_0^\infty r^\nu e^{-ar^2} dr$$

$$= \frac{i^\nu}{2\pi} e^{-i(\nu+1)\frac{\alpha_2}{2}} \left\{1 - (-1)^{\nu+1}\right\} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{2\left|\frac{W^{II}(s,t)}{2!}\right|^{\left(\frac{\nu+1}{2}\right)}}$$

$$= \frac{i^\nu}{2\pi} \frac{\left\{1 - (-1)^{\nu+1}\right\}}{2} \Gamma\left(\frac{\nu+1}{2}\right) \frac{1}{\left(\frac{1}{2!} W^{II}(s_s, t)\right)^{\left(\frac{\nu+1}{2}\right)}} \qquad 9(I.3)$$

This last result was obtained by using 2(I.2) and

$$\frac{e^{-i\left(\frac{\nu+1}{2}\right)\alpha_2}}{\left|W^{II}(s_s, t)\right|^{\frac{\nu+1}{2}}} = \frac{1}{\left(W^{II}(s_s, t)\right)^{\frac{\nu+1}{2}}}$$

By substitution in 4(I.3) one finally gets

$$f_n^*(t) = e^{W(s_s, t)} \left\{ \frac{1}{2\pi} \sum_{\nu=0}^{\infty} (i)^\nu \frac{\left\{1 - (-1)^{\nu+1}\right\}}{2} \Gamma\left(\frac{\nu+1}{2}\right) \frac{F^{(\nu)}(s_s)}{\nu!} \frac{1}{\left(\frac{1}{2!} W^{II}(s_s, t)\right)^{\frac{\nu+1}{2}}} \right\}$$

$$f_n^*(t) = e^{W(s_s, t)} \left\{ \frac{1}{2\pi} \sum_{\mu=0}^{\infty} (-1)^\mu \Gamma\left(\frac{1}{2} + \mu\right) \frac{F^{(2\mu)}(s_s)}{(2\mu)!} \frac{1}{\left(\frac{1}{2!} W^{II}(s_s, t)\right)^{\mu+\frac{1}{2}}} \right\}$$

since the term $\left\{1 - (-1)^{\nu+1}\right\}$ vanishes for ν odd and $2\mu = \nu$.
The final result is usually written in a more convenient form

$$f_n^*(t) = e^{W(s_s, t)} \frac{1}{\left(2\pi W^{II}(s_s, t)\right)^{\frac{1}{2}}} \left\{ \sum_{\mu=0}^{\infty} \frac{(-1)^\mu \Gamma\left(\frac{1}{2} + \mu\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{F^{(2\mu)}(s_s)}{(2\mu)!} \frac{1}{\left(\frac{1}{2!} W^{II}(s_s, t)\right)^\mu} \right\} \qquad 10(I.3)$$

The requirement that $\left|W^{II}(s_s, t)\right|$ must be large produces a fast convergence of 10(I.3). Practically, few terms of the series are needed. By appropriate transformation in the integral 1(I.1), it is possible — by the introduction of the so-called condensation process — to make so rapid a convergence of 10(I.3) that only the first term (and the second for very accurate results) are needed to obtain a good degree of approximation. For large $W^{II}(s_s, t)$ one gets the very simple expression

$$f_n^*(t) \approx \frac{e^{W(s_s, t)}}{\sqrt{2\pi W^{II}(s_s, t)}} \qquad 11(I.3)$$

The terms of 10(I. 3) for $\mu \geq 1$ are called, for reasons which appear later, "coinciding zero transient components of order μ." That is why we say that the solution 10(I. 3) is given in "terms of successive zero components."

Four main features of the solution 10(I. 3) must be noted.

a. The exponential term gives the instantaneous transient oscillation and main decay.† The instantaneous frequency of the transient is given by the position of the saddlepoint.

b. The bracket term of 10(I. 3) multiplied by

$$\left(2\pi W^{II}(s_s, t)\right)^{-\frac{1}{2}}$$

gives the envelope function.

c. The solution diverges, and consequently is meaningless, if $W^{II}(s_s, t) = 0$ at some point or points in the interval $t_a \leqslant t \leqslant t_b$.†† Except at these points, the solution is convergent. The number of terms needed to get a good approximation clearly depends on $\left|W^{II}(s_s, t)\right|$.

d. By simple reasoning of a physical nature, we shall show that the points for which $\left|W^{II}(s, t)\right| = 0$ must be associated with points of maxima of the envelope wave. A more advanced method of integration reveals that it is the case.††† We have, therefore, a simple criterion for locating the maxima of the amplitude wave: Find the values of t which make

$$W^{II}(s, t) = 0 \quad \text{or very small}$$

for time in the interval $t_a \leqslant t \leqslant t_b$. If the above equation does not have a solution in $t_a \leqslant t \leqslant t_b$, then the transient envelope is monotonic inside this time interval.

I. 33 COINCIDING POLE SOLUTION. POLE RESONANCE. In this paragraph, we will start with the discussion of the effect of the poles of F(s) upon the transient waves.

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†Let $W(s_s, t) = \text{Real } W(s_s, t) + i \text{ Imag } W(s_s, t)$

Then

$$e^{W(s_s, t)} = \underbrace{e^{\text{Real } W(s_s, t)}}_{\text{decay}} \cdot \underbrace{e^{i \text{ Imag } W(s_s, t)}}_{\text{oscillations}} .$$

††Later we shall produce solutions which are free of this handicap. Transients in which $W^{II}(s, t) = 0$ at certain points of t belong to the so-called class of transitional transients. We do not now undertake the solution for more complicated cases because serious confusion may appear in the understanding of the mechanism of transient formation.

†††In fact, at the points $W^{II}(s_s, t) = 0$ the transient wave changes from one state of oscillation to another. The proof is given in the chapter on transitional transients. The maxima of the envelope are somewhere in the vicinity of these values of t which make $W^{II}(s_s, t) = 0$.

The problem will be tackled in its simplest elements in order to illustrate more clearly some typical mathematical steps needed frequently in the theory of approximation.

Let s_k be a pole of first order of $F(s)$ which, by hypothesis, is not a pole of $W(s,t)$. In the vicinity of $s = s_k$, $F(s)$ can be expanded in Laurent series and $W(s,t)$ in Taylor series.

$$\left.\begin{aligned} F(s) &= \frac{R_k}{s - s_k} + \phi(s) \qquad\qquad \phi(s) \text{ analytic at } s = s_k \\ W(s,t) &= W(s_k,t) + W^{I}(s_k,t)(s - s_k) + \frac{W^{II}(s_k,t)}{2!}(s - s_k)^2 + \dots \end{aligned}\right\} \qquad 12(I.3)$$

We will now assume that: At a certain value of t, inside the interval $t_a \leqslant t \leqslant t_b$, the saddlepoints pass close to, or exactly over, s_k. See Fig. 3(I.3).

The contribution of the integral 1(I.1) in the vicinity of s_k, can now be written as

$$f_n(t) = \frac{R_k e^{W(s_k,t)}}{2\pi i} \int_\gamma \frac{e^{a(s-s_k)+b(s-s_k)^2+\dots}}{s - s_k}\, ds + \frac{e^{W(s_k,t)}}{2\pi i} \int_\gamma \phi(s)\, e^{a(s-s_k)+b(s-s_k)^2+\dots}\, ds$$

where $\quad a = W^{I}(s_k,t); \quad b = \frac{1}{2!} W^{II}(s_k,t)$ 13(I.3)

The second integral of 13(I.3) will not be considered, since we have dealt with this type in subsection I.32.† We will concentrate our attention on the integral

$$I_k = \frac{1}{2\pi i} \int \frac{e^{a(s - s_k) + b(s - s_k)^2}}{s - s_k}\, ds \qquad 14(I.3)$$

It can be noted that the prototype integral 2(I.2) cannot be used because it is valid only for $\nu > -1$. Here $\nu = -1$.

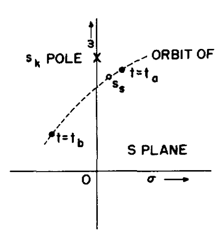

Fig. 3(I.3)

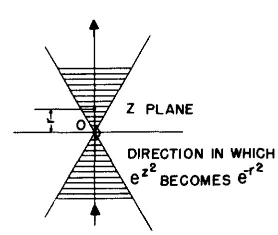

Fig. 4(I.3)

†We will consider this integral again in a future paragraph in the case in which $W^{II}(s_s, t) = 0$ at some point of this interval.

Let us introduce the transformation

and let

$$\left.\begin{aligned} s - s_k &= \frac{z}{\sqrt{b}} \\ \frac{a}{\sqrt{b}} &= X \end{aligned}\right\} \qquad 15(I.3)$$

One gets

$$I_k = \frac{1}{2\pi i} \int_{\gamma_z} \frac{e^{Xz + z^2}}{z} dz \qquad \text{coinciding pole integral} \qquad 16(I.3)$$

The integral 16(I.3) can be expressed in terms of error functions of complex argument. We have

$$\int_0^X e^{zx} dx = \frac{1}{z} e^{zx} \Big|_0^X = \frac{1}{z} e^{zX} - \frac{1}{z} \qquad 17(I.3)$$

Hence

$$I_k = \frac{1}{2\pi i} \int_{-iR}^{+iR} \int_0^X e^{z^2 + zx} \, dx \, dz + \frac{1}{2\pi i} \int_{-iR}^{+iR} \frac{e^{z^2}}{z} dz \qquad 18(I.3)$$

Along the contour shown in Fig. 4(I.3) one gets for the second integral (pole at $z = 0$)

$$\frac{1}{2\pi i} \int_{-iR}^{+iR} \frac{e^{z^2}}{z} dz = \frac{1}{2}, \quad \text{as } R \rightarrow \infty \qquad 19(I.3)$$

In the first integral one can reverse the order of integration, so that one gets, by letting $z = ir$,

$$\frac{1}{2\pi} \int_{-r}^{+r} \int_0^X e^{-r^2} e^{irx} \, dr \, dx = \frac{1}{2\pi} \int_0^X dx \int_{-r}^{+r} e^{-r^2 + irx} dr \qquad 20(I.3)$$

$$\approx \frac{1}{2\pi} \int_0^X dx \int_{-\infty}^{+\infty} e^{-r^2 + irx} dr \qquad 21(I.3)$$

since for very large values of r, e^{-r^2} is negligible and

$$\left| \int_r^\infty e^{-r^2 + irx} dr \right| \leqslant \int_r^\infty e^{-r^2} dr \rightarrow \text{zero for large } r$$

Consider now the integral

$$\int_{-\infty}^{+\infty} e^{-r^2 + irx}\, dr$$

and introduce the new transformation

$$r^2 - irx = u^2 + \frac{1}{4}x^2 \qquad 22(I.3)$$

therefore

$$\left(r - i\frac{x}{2}\right)^2 = u^2$$

$$u = r - \frac{ix}{2} \qquad\qquad du = dr$$

Then

$$\int_{-\infty}^{+\infty} e^{-r^2 + irx}\, dr = e^{-\frac{1}{4}x^2} \int_{-\infty - \frac{ix}{2}}^{+\infty - \frac{ix}{2}} e^{-u^2}\, du = e^{-\frac{1}{4}x^2} \sqrt{\pi} \qquad 23(I.3)$$

substituting 23(I. 3) and 19(I. 3) in 16(I. 3) one gets

$$I_k = \left\{ \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \int_0^X e^{-\frac{1}{4}x^2}\, dx \right\} \qquad 24(I.3)$$

Now let

$$\frac{1}{2}x = V \qquad 25(I.3)$$

$$\frac{1}{\sqrt{\pi}} \int_0^X e^{-\frac{1}{4}x^2}\, dx = \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{2}X} e^{-V^2}\, dV = \operatorname{erf}\frac{X}{2}$$

By using 15(I. 3) and the values of a and b in 13(I. 3), one gets

$$I_k = \frac{1}{2}\left(1 + \operatorname{erf}\frac{X}{2}\right) \qquad 26(I.3)$$

Then the first integral of 13(I. 3), which gives the effect of the pole, is given by

$$\left.\begin{aligned} \frac{R_k}{2\pi i} \int_{\gamma_s} \frac{1}{s - s_k} e^{W(s,t)}\, ds &\approx R_k\, e^{W(s_k,t)}\, \frac{1}{2}\left(1 + \operatorname{erf}\frac{X}{2}\right) \\ \frac{1}{2}X &= \frac{W^{I}(s_k,t)}{\sqrt{2W^{II}(s_k,t)}} \end{aligned}\right\} \qquad 27(I.3)$$

I. 34 THE ERROR-FRESNEL SOLID. FIRST NOTION OF GENERATING FUNCTIONS. The function X is, in general, a complex quantity. Consequently, the function defined by

$$I_k = G(X) = \frac{1}{2}\left(1 + \text{erf}\,\frac{X}{2}\right) \qquad 28(I.\,3)$$

is an extension of the so-called error function, which is classically defined for real values of X. In general, G(X) is a complex quantity and is called, as far as this investigation is concerned, the Error-Fresnel "Generating Function." The name of Fresnel is given, since, for $X = |X|\ e^{i\pi/4}$, the G(X) coincides with the so-called Fresnel function. If we construct the surfaces defined by Real G(X), Imag G(X), |G(X)|, and the phase angle of G(X), and these functions are measured along the perpendicular line to the X plane at the point X, then we obtain the so-called Error-Fresnel solids. These solids are important because they contain all possible transients (envelopes, etc.) corresponding to the transient family, which can be approximated in the way already shown. Figure 5(I. 3) shows some typical cross sections of the amplitude solid

$$\left|\frac{1}{2}\left(1 + \text{erf}\,\frac{X}{2}\right)\right| \text{for } X = \left|X\right| e^{i\phi} \qquad \phi = 0°,\ 10°,\ 20°,\ 30°,\ 40°,\ 45°,\ 50°$$

It is important to note the strong effect of the value of ϕ on the overshoot of the wave. The 45° and 225° cross section produces the so-called Fresnel behavior. This behavior is associated with a nondissipative system.

Figure 6(I. 3) shows the 45°-225°cross section of

$$\left|\frac{1}{2}\left(1 + \text{erf}\,\frac{X}{2}\right)\right| = \psi(X)$$

Figure 7(I. 3) shows the corresponding argument function (cross section 45°-225°).

$$\arg\left[\frac{1}{2}\left(1 + \text{erf}\,\frac{X}{2}\right)\right] = \phi(X)$$

$$G(X) = \psi(X)\ e^{i\phi(X)} \qquad 29(I.\,3)$$

The function $\phi(X)$ is called the phase deviation function.

Figure 8(I. 3) shows the polar plot of G(X) for the 45°-225° cross section, that is, Real G(X) versus Imag G(X). The result is the so-called Cornu's spiral. The values of the |X| are plotted along the spiral. (The spirals are important because they contain a visual relationship of the amplitude and phase deviation function.) Finally, from the amplitude solid, it can be noted that for the cross section beyond 45° (and less than 135°, not shown) the overshoot wiggles start increasing very rapidly (regions of instability).

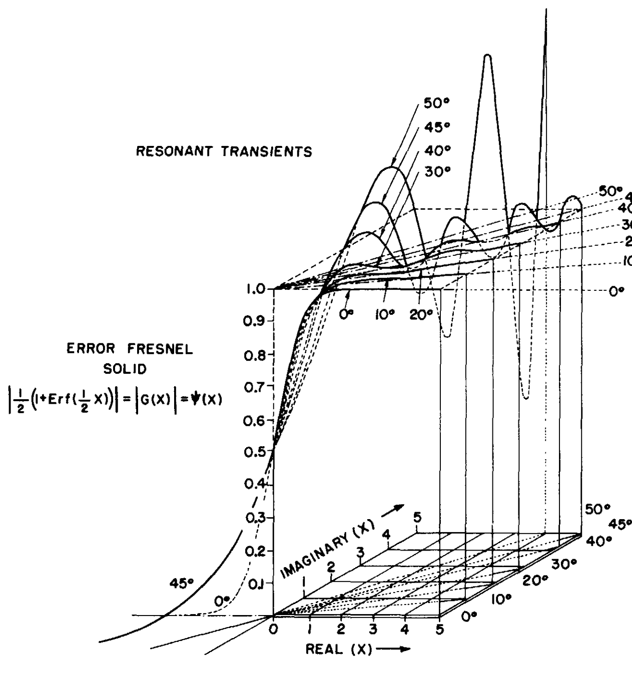

Fig. 5(I. 3)

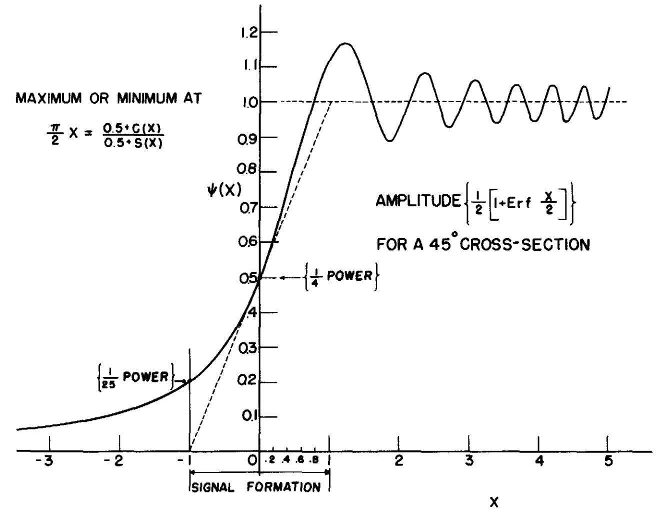

Fig. 6(I. 3)

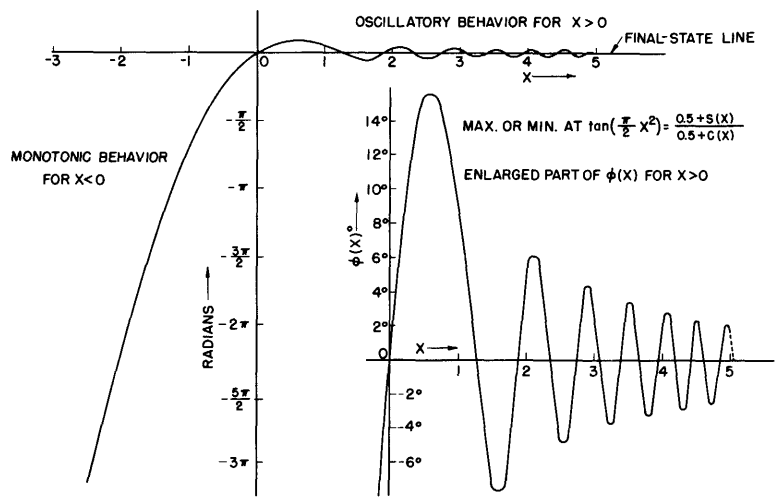

Fig. 7(I. 3)

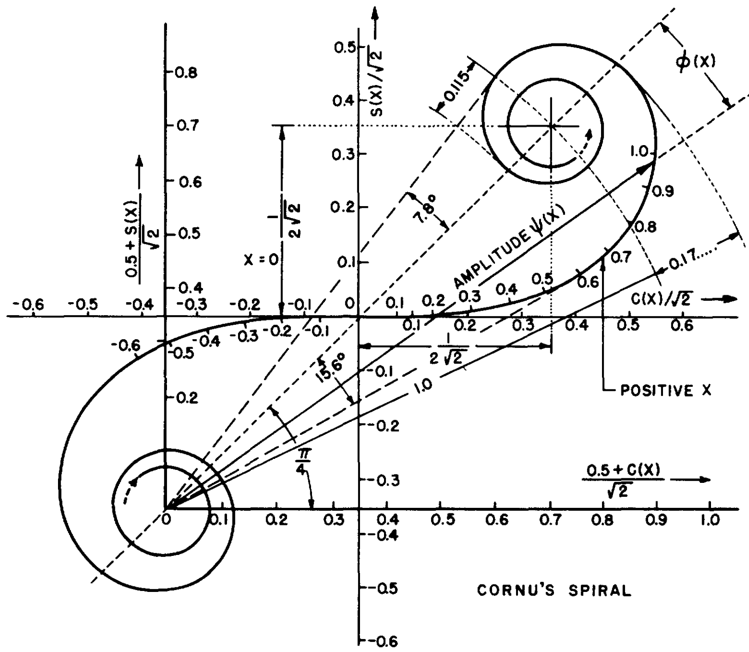

Fig. 8(I. 3)

I. 35 PARTICULAR TRANSIENTS. TRANSIENT TRAJECTORY. LONGITUDINAL DEFORMATION. COMPLETE TRANSIENT WAVE. A BRIEF OUTLINE OF THE MECHANISM OF THE GENERATING FUNCTIONS.

Suppose we have at hand a transient of the family which is characterized by the passage of a saddlepoint over a certain pole at some time in the interval $t_a \leq t \leq t_b$, and that W(s, t) can be approximated up to the second-degree term of the Taylor expansion.

A. A particular transient is now completely defined by X = X(t).
The computation of the function X can be made immediately, without a new process of integration, since

$$X = \left[\sqrt{2}\ W^{I}(s_k, t)\right] / \sqrt{W^{II}(s_k, t)} \qquad 30(I. 3)$$

B. In general, X is a complex quantity. Equation 30(I. 3) defines the parametric equation of a line (trajectory) in the X plane in terms of the variable time. This trajectory is, of course, different from the saddlepoint orbit. (That is why we prefer to use different names for these lines: the transient trajectory and the saddlepoint orbit.)

C. Once the trajectory X(t) is determined for a particular transient, the expression in brackets in 27(I. 3) is then completely determined as a function of t. The complete particular transient is now obtained by the multiplication of the factor

$$R_k\ e^{W(s_k, t)} \qquad 31(I. 3)$$

as required by 27(I. 3). This represents the contribution of a primary saddlepoint. If there is more than one saddlepoint, the final expression can be obtained by the sum of the terms of the form 27(I. 3).

D. Once the trajectory X(t) is determined in the X plane, for a particular transient, the function G(X(t)) = 1/2(1 + erf X(t)/2), now in terms of time, can be extracted from the generating solid (or corresponding numerical tables). What we have to do is to plot in the X plane of the solid the corresponding X(t) trajectory, marking along the trajectory the values of time (as indicated in Fig. 9(I.3)). Then we draw the cylindrical surface generated by the perpendicular lines to the X plane, at every point of the trajectory. The intersection of this cylindrical surface and the surface of the generating solid gives a line along which the particular transient is contained in the corresponding solid. By picking up the amplitude, phase, etc., in the solid, we can obtain the corresponding function of G(X(t)) as a function of the time. The trajectory is the same in every solid.

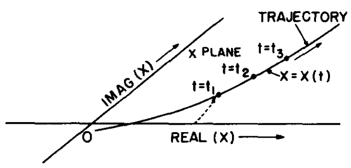

Fig. 9(I. 3)

E. We can regard the function X(t) as a transformation. In fact it defines a continuous transformation with respect to time. In a particular transient we continuously transform the generating function G(X) by means of X = X(t). The effect of the transformation X(t) upon G(X) is simply one of stretching, or compressing, along the direction of the independent variable. In general, this stretching effect is linear. Now, since X(t) defines a line on the X plane, what we really stretch or compress is only the line of intersection of the generating solid which corresponds to the particular transient.

Since X(t) acts only upon the independent variable of G(X), the vertical scales remain invariant. THE MAXIMA AND MINIMA OF G(X) ARE THUS RETAINED; they are only displaced from their original positions. This property is of basic importance in connection with design problems.

The longitudinal stretching or contraction of G(X) is called longitudinal deformation. By extension of these ideas, G(X) is called the "Longitudinal Generating Function" of the Error-Fresnel type. (In a later discussion, we shall introduce the theory of "orthogonal deformations.")

F. Graphical methods of transformation are very suggestive and they simplify considerably the amount of labor needed in the computations of a particular transient, when the required accuracy is more than, say, two percent. The methods for obtaining this graphical transformation are well known and they will not be discussed at the present

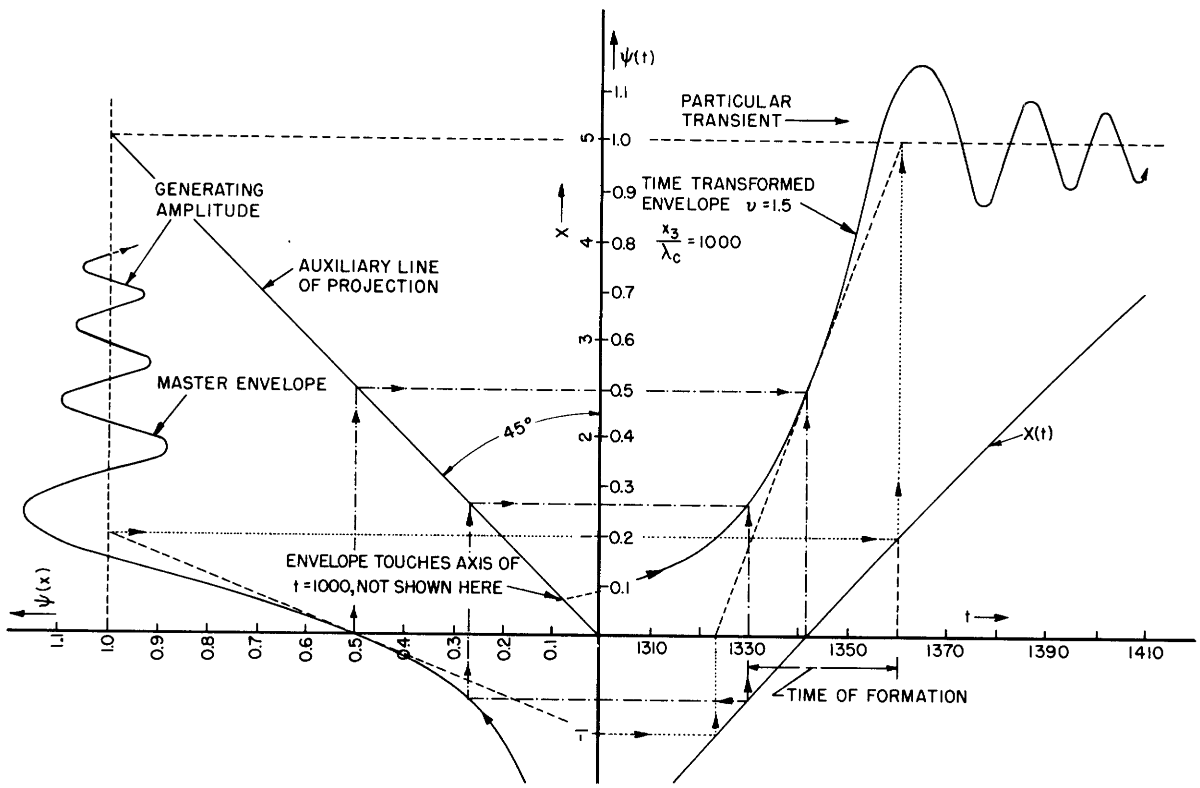

Fig. 10(I.3)

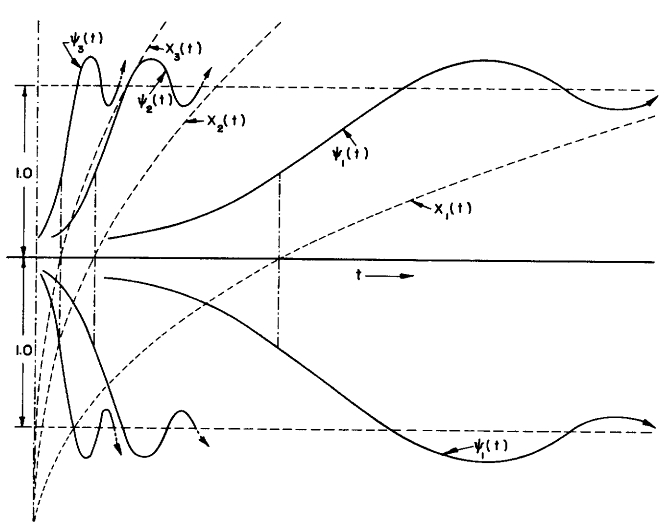

Fig. 11(I.3)

moment. Illustrative examples are given in Fig. 10(I. 3), which is drawn for a 45° straight line trajectory through the origin and in the solid of Fig. 5(I. 3). Figure 11(I. 3) shows the effect of different laws of transformation, say $X_1(t)$, $X_2(t)$, $X_3(t)$, indicated by the dotted lines. Since Figs. 10 and 11(I. 3) are almost self-explanatory, no more details are given.

SECTION I. 4 TRANSIENT COMPOSITION. SATELLITE SADDLEPOINTS.

I. 40 FIRST INTRODUCTION TO THE IDEAS OF TRANSIENT COMPOSITION. THE RESONANT FAMILY. BASIC MEMBERS. Except in very simple and unimportant cases, it is almost impossible to obtain for a given transient a single approximate solution which is valid for all values of t in the whole range $0 < t < \infty$. Approximate solutions are valid only in a specific interval of time. It is not possible, at least in the beginning of the transient theory, to preassociate time intervals and the corresponding solutions in a one-to-one correspondence. Although this basic problem has been solved in our theory of transients, we omit a discussion on this matter because the proofs are involved and may lead to some confusion at this point. We shall assume that this correspondence exists. Following a step-by-step method we shall be able to develop the ideas which are needed in the solution of this problem.

In what follows, when we speak of a transient solution, or simply of a transient, we refer implicitly to the solution, or to the part of the transient, which is valid in a specific interval of time, say $t_a \leq t \leq t_b$.

I. 41 THE RESONANCE FAMILY. The resonance family appears as a sort of generalization of the above results. This generalization is not yet complete. The asymptotic character of the solution is still assumed.

The resonance family is defined as those transients which are expressed (in a given interval of time) by the integral

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} F(s)\, e^{A_o + A_1(s - s_c) + A_2(s - s_c)^2}\, ds \qquad 1(I. 4)$$

in which F(s) is a rational function of s; A_o, A_1, A_2 are parametric functions of time and independent of s; γ_s is a contour of integration which is topologically equivalent to the line joining $c_o - i\infty$ to $c_o + i\infty$; and c_o is the abscissa of uniform convergence (Laplace transforms). Finally, s_c is a fixed or time-variable point in the s plane. In the cases already discussed, s_c was either the saddlepoint or a pole at s_k which lay in the vicinity of the orbit of the saddlepoint.

I. 42 TRANSIENT REDUCIBLE TO THE RESONANCE TYPE. SATELLITE SADDLE-POINT. If a transient given by

$$\frac{1}{2\pi i} \int_{\gamma_s} F(s)\, e^{W(s,t)}\, ds \qquad 1(I. 1)$$

can be approximated in a given interval of time by one or more integrals of the type 1(I.4), then one says that the transient is resonant-reducible (in the same interval of time).

From the previous discussion of the saddlepoint integration, we can observe that the point s_c must be a sort of saddlepoint of $W(s,t)$ or some point in the vicinity of its orbit.

Let s_s be a primary saddlepoint of second order of $W(s,t)$. By definition, s_s is a solution of

$$\frac{d}{ds} W(s,t) = W^{I}(s,t) = 0 \qquad 2(I.4)$$

Suppose that $W(s,t)$ is expanded in Taylor series around the point s_c. One gets

$$W(s,t) = W(s_c,t) + W^{I}(s_c,t)(s - s_c) + \frac{W^{II}(s_c,t)}{2!}(s - s_c)^2 + \ldots \qquad 3(I.4)$$

It is then clear that

$$\left.\begin{aligned} A_o &\approx W(s_c,t) \\ A_1 &\approx W^{I}(s_c,t) \\ A_2 &\approx \frac{1}{2!} W^{II}(s_c,t) \end{aligned}\right\} \qquad 4(I.4)$$

The following notation will be used:

By $W^*(s,t)$ we shall denote a function which approximates the function $W(s,t)$ inside a given interval of time.

DEFINITION OF SATELLITE SADDLEPOINT. A solution of

$$\frac{d}{ds} W^*(s,t) = W^{*I}(s,t) = 0 \qquad 5(I.4)$$

is called a satellite saddlepoint of $W(s,t)$ in the given interval of time. A satellite saddlepoint will be denoted by $s_s^* = s_s^*(t)$. The name satellite was chosen because the points s_s^* are attached to, and somehow follow, within the given interval of time, the corresponding saddlepoints s_s of $W(s,t)$.†

In the resonance family, $s_s^*(t)$ are given by

$$s_s^* = s_c - \frac{A_1}{2A_2}$$

or also

$$s_s^* = s_c - \frac{W^{I}(s_c,t)}{W^{II}(s_c,t)} \qquad 6(I.4)$$

---

†Satellite saddlepoints follow only the primary saddlepoints of $W(s,t)$ which control the integral in a given interval of time. The satellite saddlepoint plays an important role in the theory of approximate integration.

If one choses $s_c = s_s$, then $W^I(s_s, t) = 0$, by definition of the second-order saddlepoint of $W(s,t)$, and hence we have

$$s_s^* = s_s \qquad \text{for all } t \text{ in the interval}$$

Otherwise $s_s^* \neq s_s$ in general. Note from 6(I.4) that for the points for which $W^{II}(s_c, t) = 0$, the satellite saddlepoints are at infinity. But we are under the assumption that $W^{II} \neq 0$ and $\left|W^{II}(s_c, t)\right|$ are rather large, so s_s^* is close to s_c.

The satellite saddlepoints will be used in a future discussion.

I. 43 BASIC MEMBER OF THE RESONANCE FAMILY. In the study of the resonance transients it is convenient to introduce the transformation

$$z = (s - s_c)\sqrt{A_2} \qquad |A_2| \neq 0 \qquad 7(I.4)$$

Hence, the integral which defines the family becomes

$$f(t) = \frac{e^{A_o}}{2\pi i} \frac{1}{\sqrt{A_2}} \int_{\gamma_z} F(z)\, e^{Vz + z^2}\, dz \qquad 8(I.4)$$

where $V = A_1/\sqrt{A_2}$, and γ_z is the corresponding transformed contour of integration in the z plane. The transformation 7(I. 4) transforms the point s_c into the origin of the z plane.

Instead of using the complete integral 8(I. 4), we shall study the integral

$$\frac{1}{2\pi i} \int_{\gamma_z} F(z)\, e^{Vz + z^2}\, dz \qquad 9(I.4)$$

We shall consider three cases which have a basic importance:

$$\left.\begin{array}{lll} \text{A.} & F(z) = \dfrac{\sqrt{A_2}}{z - z_b} & \text{Pole resonance transient}^{\dagger} \\ \text{B.} & F(z) = \dfrac{z - z_a}{\sqrt{A_2}} & \text{Zero resonance transient} \\ \text{C.} & F(z) = \dfrac{z - z_a}{z - z_b} & \text{Dipole resonance transient} \end{array}\right\} \qquad 10(I.4)$$

---

†The factor $\sqrt{A_2}$ appears in the expression for $F(z)$ because a pole transient in the s plane will look like $F(s) = 1/(s - s_b)$. Now using the transformation 7(I. 4) one gets

$$F(z) = \frac{\sqrt{A_2}}{s - s_c - (s_b - s_c)} = \frac{\sqrt{A_2}}{z - z_b}$$

with the assumption that $|z_b|$ and $|z_a|$ are simultaneously small.

I.431 THE POLE TRANSIENT. The pole transient is given by

$$I_p = \frac{\sqrt{A_2}}{2\pi i} \int_{\gamma_z} \frac{1}{z - z_b} e^{Vz + z^2} dz \qquad 11(I.4)$$

Let $z - z_b = u$. Then we have

$$I_p = e^{Vz_b + z_b^2} \frac{\sqrt{A_2}}{2\pi i} \int_{\gamma_u} \frac{e^{Xu + u^2}}{u} du \qquad 12(I.4)$$

where $X = V + 2z_b$.

From 16(I.3) and 26(I.3), one gets

$$I_p = e^{Vz_b + z_b^2} \frac{\sqrt{A_2}}{2} \left(1 + \operatorname{erf} \frac{X}{2}\right) \qquad 13(I.4)$$

Then, for a pole transient, the complete integral 8(I.4) becomes

$$\left.\begin{aligned} f(t) &= e^{A_o + Vz_b + z_b^2} \frac{1}{2} \left(1 + \operatorname{erf} \frac{X}{2}\right) \\ X &= V + 2z_b = \frac{A_1}{\sqrt{A_2}} + 2z_b \end{aligned}\right\} \qquad 14(I.4)$$

Using 7(I.4), one gets

$$A_o + Vz_b + z_b^2 = A_o + A_1(s_b - s_c) + A_2(s_b - s_c)^2 = W^*(s_b, t) \qquad 15(I.4)$$

So, finally, we have

$$\left.\begin{aligned} f(t) &= e^{A_o + A_1(s_b - s_c) + A_2(s_b - s_c)^2} \frac{1}{2} \left(1 + \operatorname{erf} \frac{X}{2}\right) \\ &= e^{W^*(s_b, t)} \frac{1}{2} \left(1 + \operatorname{erf} \frac{X}{2}\right) \\ \frac{1}{2}X &= \frac{A_1}{2\sqrt{A_2}} + z_b = \frac{W^I(s_c, t)}{\sqrt{2\, W^{II}(s_c, t)}} + \frac{(s_b - s_c)}{\sqrt{2}} \sqrt{W^{II}(s_c, t)} \end{aligned}\right\} \qquad 16(I.4)$$

If $z_b = 0$, then we say that we have the coinciding pole transient. This means that the pole is now at $z = 0$ or, by using 7(I.4), the pole is at s_c.

It can be observed that if $z_b = 0$, 16(I.4) is equivalent to 27(I.3) if one sets

$s_c = s_k$.† Then, 27(I. 3) is the solution corresponding to the coinciding (with s_c) pole.

By means of 16(I. 4) we can establish a basic property of pole resonant transients. We shall discuss the effect of the pole shift (from the coinciding position) only in the factor

$$\frac{1}{2}\left(1 + \text{erf}\,\frac{X}{2}\right)$$

The effect on $W^*(s_c, t)$ is simply a change in frequency and attenuation due to the position of the shifted pole. Momentarily, let us call

$$X_o = \frac{W^I(s_c, t)}{\sqrt{2W^{II}(s_c, t)}}$$

Then

$$\Delta X = X - X_o = (s_b - s_c)\frac{\sqrt{W^{II}(s_c, t)}}{\sqrt{2}} \qquad 17(I. 4)$$

17(I. 4) shows that the shift of a pole is equivalent to a displacement of the wave, which is produced by the generating function. The displacement is proportional to the pole shift.††

Figure 1(I. 4) shows, for example, the effect of a pole shift in the 45° cross section of the generating solid in Fig. 5(I. 3) when the pole deviation is also directed in the 45° direction. Note that in this case the overshoot of the wave remains the same. The whole effect is simply a shift of the whole wave. When the pole deviation is made in other directions than the 45° line, slight modification of the overshoot may result, as can be observed by looking at the solid in Fig. 5(I. 3).

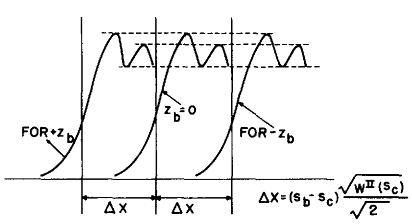

Fig. 1(I. 4)

Similar shift effect is observed in the corresponding phase deviation function, see Fig. 7(I. 3), as well as in the other associated functions of G(X).

The wave shapes indicated in Fig. 1(I. 4) are not the final transient waves. In accordance with 16(I. 4) it is still necessary to multiply the envelopes by the factor $e^{W^*(s_b, t)}$. Consequently, a damping factor and a constant change of the instantaneous oscillation will appear. Since the effect of this exponential function is well known, it is omitted in Fig. 1(I. 4) in order to show clearly the principal effect of the pole deviation on the envelope function.

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†Remember that s_c represents an appropriate point of expansion.

††This effect does not necessarily take place in other families of transients; the reader must not conclude that this property is true for all possible transients.

The corresponding transients for poles of higher order can be studied in a similar way. Solutions of these cases are of no importance for the following sections.

I.432 THE ZERO TRANSIENT. FIRST EXTENSION OF THE INTEGRAL PROCEDURE. The integral which corresponds to this case is given (see 8 and 10(I. 4)) by

$$f(t) = \frac{e^{A_o}}{2\pi i A_2} \int_{\gamma_z} (z - z_a)\, e^{Vz + z^2}\, dz \qquad 18(I.\,4)$$

Let us introduce the transformation

$$z - z_a = u \qquad 19(I.\,4)$$

and consider only the integral factor of 18(I. 4)

$$\frac{1}{2\pi i} \int_{\gamma_z} (z - z_a)\, e^{Vz + z^2}\, dz = \frac{1}{2\pi i}\, e^{Vz_a + z_a^2} \int_{\gamma_u} u\, e^{uX + u^2}\, du \qquad 20(I.\,4)$$

where $X = V + 2z_a$.

The integrals 18 and 20(I. 4) are alternative forms of the zero transient expression.

We can now proceed with the integration by using the saddlepoint method already described. In this case one has to use the satellite saddlepoint for the exponent. Let, for instance,

$$I = \frac{1}{2\pi i} \int_{\gamma_u} u\, e^{Xu + u^2}\, du \qquad 21(I.\,4)$$

Since in the saddlepoint the first derivative is zero, the saddlepoint can be obtained, if the transformation

$$Xu + u^2 + \frac{X^2}{4} = -p^2 \qquad 22(I.\,4)$$

is introduced; that is

$$u + \frac{X}{2} = ip \qquad 23(I.\,4)$$

or

$$I = \frac{1}{2\pi i} \int_{\gamma_u} u\, e^{Xu + u^2}\, du = \frac{1}{2\pi}\, e^{-X^2/4} \int_{\gamma_p} \left(ip - \frac{X}{2}\right) e^{-p^2}\, dp$$

$$= \frac{e^{-X^2/4}}{2\pi}\, i \int_{\gamma_p} p\, e^{-p^2}\, dp - \frac{e^{-X^2/4}}{2\pi}\, \frac{X}{2} \int_{\gamma_p} e^{-p^2}\, dp \qquad 24(I.\,4)$$

Now it can be seen that γ_p is equivalent to the integration from $-\infty$ to $+\infty$, so that finally we get

$$I = -\frac{e^{-X^2/4}}{2\pi} X \int_0^\infty e^{-p^2} dp = -\frac{X}{4} \frac{e^{-X^2/4}}{\sqrt{\pi}} \qquad 25(I.4)$$

Since

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} pe^{-p^2} dp \equiv 0 \qquad 26(I.4)$$

By using 20(I. 4) the zero-deviated transient becomes

$$f(t) = -\frac{e^{\left(A_o + Vz_a + z_a^2\right)}}{4\sqrt{\pi}} \frac{V + 2z_a}{A_2} e^{-(V + 2z_a)^2/4}$$

$$= -\frac{e^{W^*(s_a, t)}}{2A_2\sqrt{\pi}} \left(\frac{W^I(s_c, t)}{\sqrt{2W^{II}(s_c, t)}} + \frac{(s_a - s_c)\sqrt{W^{II}(s_c, t)}}{\sqrt{2}} \right) e^{-(X/2)^2} \qquad 27(I.4)$$

where

$$\frac{X}{2} = \left(\frac{W^I(s_c, t)}{\sqrt{2W^{II}(s_c, t)}} + \frac{(s_a - s_c)\sqrt{W^{II}(s_c, t)}}{\sqrt{2}} \right) \qquad 28(I.4)$$

If $s_a = s_c$ the solution 27(I. 4) vanishes identically. If s_c is the saddlepoint of $W(s_c, t)$, then $W^I(s_c, t) \equiv 0$. This means that:

"If a primary saddlepoint of the second order of $W(s,t)$ passes exactly over a zero of first† order of $F(s)$, then the corresponding transient is zero (or almost zero, since the integration is an approximate one) in the interval of time in which the approximation is valid."

It is convenient to refer the deviation of the zero of $F(s)$ with respect to the saddlepoint of $W(s,t)$. In that case the zero-deviated transient is given simply by

$$f(t) = -\frac{e^{W^*(s_a, t)}}{\sqrt{2\pi}} \frac{s_a - s_s}{\sqrt{W^{II}(s_s, t)}} e^{-\frac{(s_a - s_s)^2 W^{II}(s_s, t)}{2}} \qquad 29(I.4)$$

This equation shows that the disappearance of the zero transient becomes very

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†The cancellation of the transient is also true for zeros of odd order, but is not so for zeros of even order. This is a consequence of the cancellation of 8(I. 3) for zeros of odd order.

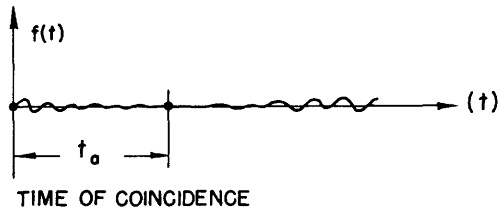

Fig. 2(I. 4)

pronounced, when the exponent in the equation becomes real. A typical wave configuration of a zero transient is shown in Fig. 2(I. 4). When W(s, t) has the typical form (Laplace transform)

$$W(s, t) = st - \phi(s) \tag{30(I. 4)}$$

then the time of coincidence is simply given by

$$W^I(s, t) = 0 \quad s = s_a \tag{31(I. 4)}$$

or

$$t_a = \phi^I(s_a) \tag{32(I. 4)}$$

This expression shows that if s_a is moved along the orbit of the saddlepoint, the time of coincidence in Fig. 2(I. 4) becomes correspondingly displaced. Since all expansions are usually made around the primary saddlepoint, 29(I. 4) is commonly used. If, for convenience, the expansion is made around some other point in the neighborhood of the saddlepoint, say at s_c, then 27(I. 4) must be used. The main shape of the transient, however, remains the same.

A comparison of Figs. 1 and 2(I. 4) shows the different behavior of the pole and zero transient waves.

I. 433 THE DIPOLE TRANSIENT.

A dipole transient is given (see 8 and 10(I. 4)) by the integral

$$f(t) = \frac{e^{A_o}}{2\pi i} \frac{1}{\sqrt{A_2}} \int_{\gamma_z} \frac{z - z_a}{z - z_b} e^{Vz + z^2} dz \tag{33(I. 4)}$$

We can write

$$\frac{z - z_a}{z - z_b} = \frac{z - z_b + z_b - z_a}{z - z_b} = 1 + \frac{z_a - z_b}{z - z_b} \tag{34(I. 4)}$$

Let $\delta = z_a - z_b$. Then we have

$$f(t) = \frac{e^{A_o}}{A_2}\left\{\frac{\sqrt{A_2}}{2\pi i}\int_{\gamma_z} e^{Vz + z^2}\, dz + \frac{\delta\sqrt{A_2}}{2\pi i}\int_{\gamma_z} \frac{e^{Vz + z^2}}{z - z_b}\, dz\right\} \qquad 35(I.4)$$

The first integral represents the envelope wave of a pure resonant transient. The second integral represents a pole resonance transient. If $z_a - z_b = 0$, the pole transient disappears, leaving only the pure resonant wave. For this reason we consider a pure transient as a limiting case of the dipole transient when $z_a \to z_b$.

The solution of the second integral in 35(I.4) is already known. It is given by 11(I.4) to 16(I.4) or

$$f_2(t) = e^{W^*(s_b, t)}\, \frac{1}{2}\left(1 + \text{erf}\, \frac{X}{2}\right) \frac{\delta}{\sqrt{A_2}} \qquad 36(I.4)$$

$$\frac{X}{2} = \left(\frac{W^I(s_c, t)}{\sqrt{2W^{II}(s_c, t)}} + \frac{(s_b - s_c)\sqrt{W^{II}(s_c, t)}}{\sqrt{2}}\right) \qquad 37(I.4)$$

The factor $(\delta/\sqrt{A_2})$ changes the scale of the envelope magnitude.

To obtain a solution of the first integral in 35(I.4) one sets

$$Vz + z^2 + \frac{V^2}{4} = -p^2 \qquad 38(I.4)$$

or

$$\left(z + \frac{V}{2}\right) = ip \qquad 39(I.4)$$

so that

$$\frac{1}{2\pi i}\int_{\gamma_z} e^{Vz + z^2}\, dz = \frac{e^{-V^2/4}}{2\sqrt{\pi}} \qquad 40(I.4)$$

Hence, finally,

$$f(t) = \frac{e^{W^*(s_b, t)}}{\sqrt{2\pi W^{II}(s_c, t)}}\, e^{-W^I(s_c, t)(s_b - s_c) - \frac{W^{II}(s_c, t)}{2}(s_b - s_c)^2}$$

$$\times\, e^{-\frac{\left[W^I(s_c, t)\right]^2}{2W^{II}(s_c, t)}} + e^{W^*(s_b, t)}\, (s_b - s_a)\, \frac{1}{2}\left(1 + \text{erf}\, \frac{X}{2}\right) \qquad 41(I.4)$$

where

$$\frac{X}{2} = \frac{W^{I}(s_c, t)}{\sqrt{2W^{II}(s_c, t)}} + (s_b - s_c)\sqrt{\frac{W^{II}(s_c, t)}{2}} \qquad 37(I. 4)$$

If the expansion is made around the saddlepoint, then $s_c = s_s$ and $W^{I}(s_s, t) \equiv 0$, so that

$$f(t) = e^{W^{*}(s_b, t)} \left[e^{-\frac{W^{II}(s_s, t)}{2}(s_b - s_s)^2} + (s_b - s_a)\frac{1}{2}\left(1 + \text{erf}\,\frac{X}{2}\right) \right] \qquad 42(I. 4)$$

where

$$\frac{X}{2} = (s_b - s_s)\sqrt{\frac{W^{II}(s_s, t)}{2}} \qquad 43(I. 4)$$

INTEGRATION THROUGH SADDLEPOINTS OF HIGHER ORDER. THE THEORY OF TRANSITION. TRANSITIONAL TRANSIENTS.

SECTION II. 1 DEFINITIONS. DOMINANT TERMS. TRANSITION.

II. 10 INTRODUCTORY REMARKS. The theory of integration through saddlepoints of the second order has been outlined in the previous notes. Now we shall give a basic and brief discussion of the theory when saddlepoints are of an order greater than two.

The second-order theory certainly considers and includes the main curvature of the function W(s, t) or, let us say, of the phase function. The reason is that we consider the terms up to the second derivative in the Taylor expansion. Besides, it was assumed that the magnitude of this derivative was large enough to guarantee a good approximation in the asymptotic range of the variables. The second-order theory is meaningless when $(W^{II}(s_s, t))$ vanishes.

The integration through saddlepoints of higher order permits us to consider rapid changes in the curvature of the function W(s, t). We shall also be able to investigate what happens when $W^{II}(s_s, t)$ vanishes or becomes small.

We shall still assume that the function W(s, t) is analytic in a certain domain, inside of which is found the main contribution to the integral in question. Poles, branch points, and essential singularities of W(s, t) are in the outside of the needed domain in the s plane. In other words, W(s, t) can be still expanded in Taylor series.

Suppose that the required domain contains one or several saddlepoints. Let s_c be a point of the s plane which is inside the given domain.† The Taylor expansion of W(s, t) around s_c is

$$W(s,t) = W(s_c,t) + W^{I}(s_c,t)(s - s_c) + \ldots + \frac{W^{(n)}(s_c,t)}{n!}(s - s_c)^n + \ldots \qquad 1(II.\,1)$$

In the approximation process, one has to consider only a certain number of terms, say n, and neglect the terms after n + 1. We shall designate by $W^*(s, t)$ the polynomial of n degrees in $(s - s_c)$ which is obtained by this process of syncopation.

A question naturally arises. "What is the criterion for knowing how many terms of 1(II. 1) must be taken?"

This question is of fundamental importance in the theory of approximate integration. The answer leads to the production of the theorems of existence of the approximate solution inside of definite frames of tolerances.

Unfortunately, the production of these theorems of existence is a very involved and difficult task. Only through a step-by-step procedure along this method of integration will the reader be in a position to grasp the needed basic criterion on which we base the proof of the existence theorems.

†The point s_c may, or may not, be a saddlepoint. For the following explanation the unique determination of s_c is of no consequence.

The integration through saddlepoints of higher order cannot be properly conducted without setting some basic principles and ideas regarding the way in which we must approximate the function W(s, t). The ideas are found in these notes.

II. 11 PRIMARY AND DOMINANT TERMS.

Let

$$W^{*}(s,t) = W(s_c, t) + W^{I}(s_c, t)(s - s_c) + \ldots + \frac{W^{(n)}(s_c, t)}{n!}(s - s_c)^n \qquad 2(II.\ 1)$$

be the approximate function $W^{*}(s,t)$ which is valid inside a domain, say G, of the s plane. We shall assume that the approximation is valid inside the interval $t_a \leq t \leq t_b$, where t_a and t_b are now unspecified.

The domain G is, by hypothesis, one that yields a predominant part of the contribution of the integral, for $t_a \leq t \leq t_b$,

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} F(s)\, e^{W(s,t)}\, ds \qquad 3(II.\ 1)$$

If there is more than one G region, the following discussion may be extended to the rest.

The original contour of integration must be deformed, in these methods of integration, so that it will run inside of G. Figure 1(II. 1) shows one such region G and the contour

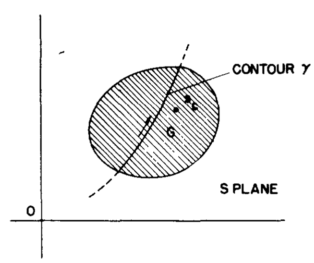

Fig. 1(II. 1)

of integration running through it. The exact position of the contour γ_s inside G is of no consequence in this explanation.

In order to simplify the following discussion we shall assume that the function F(s) plays an unimportant role in the integration inside G. We shall concentrate our discussion on $W^{*}(s,t)$, which approximates the function W(s, t), as we stated above.†

We now are confronted with a two-fold condition problem:

A. To approximate W by W^{*} in such a way that the last function represents the first, within certain tolerances, inside of G and for $t_a \leq t \leq t_b$.

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†This restriction imposed on F(s) is, in fact, not a serious one, for if F(s) would become important, then similar conclusions drawn for W(s, t) may be applied to a function, say $F^{*}(s)$, which approximates F(s) inside G and in the same interval $t_a \leq t \leq t_b$.

B. Let

$$f_G^*(t) = \frac{1}{2\pi i} \int_G F(s)\, e^{W^*(s,t)}\, ds \qquad 4(\text{II. 1})$$

and $W^*(s,t)$ be such as to comply with the requirement: $f_G^*(t)$ must approximate $f_G(t)$ inside the interval $t_a \leq t \leq t_b$.†

It can be seen at once that the fulfillment of A does not necessarily imply B. In fact, prior experience with integral approximation tells us that in most of the cases, even when W is numerically well represented by W^*, the function $f_G^*(t)$ has a poor resemblance, or none at all, to the function $f_G(t)$.

The question of the requirements placed on $W^*(s,t)$ in order to satisfy B is a very difficult and delicate one. In fact, a general solution constitutes the cornerstone on which the theory of approximate integration rests. The following discussion will touch on the first steps in this direction.

Let us assume that W^* satisfies condition A. In regard to B, W^* must contain three principal types of terms:

a. Dominant terms

b. Fast variation or primary variation terms

c. Slow variation or secondary variation terms.

DOMINANT TERMS. If the exact integral 1(II. 1) is such that a large bulk of the contributions comes from the inside of G and is negligible in the outside, then $W^*(s,t)$ must possess certain terms which guarantee that the contribution of the approximate integral 4(II. 1) must also be negligible outside of G. The terms in $W^*(s,t)$ which control and guarantee this requirement are called dominant terms. The proof that $W^*(s,t)$ must contain these dominant terms is difficult and will be omitted here. However, the reader can easily visualize the necessity for this requirement. It can be translated into analytical terms as a sufficient condition, but not a necessary one, by saying that $W^*(s,t)$ must be such that

$$\text{Real } W^*(s,t) < 0 \qquad 5(\text{II. 1})$$

for values of s along the points of the line γ (contour of integration) which lie in the outside of G†† and $t_a \leq t \leq t_b$.

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†The general condition includes W^* as well as F^*. Then, condition B must refer to

$$f_G^*(t) = \frac{1}{2\pi i} \int F^*(s)\, e^{W^*(s,t)}\, ds$$

We shall discuss this situation in the theory of transitional-mixed transients.

††Under this condition $e^{W^*(s,t)}$ is very small outside G and the integral is negligible. This condition is, of course, not a necessary one.

FAST VARIATION TERMS. The original function W(s, t) may suffer sudden changes in value inside G when we travel along the contour line (of integration), or when t passes from t_a to t_b. If W(s, t) possesses singularities inside of G (poles, branch points, essential singularities), then it is very likely that such a sudden change will take place. But these singularities are not necessarily the only cause of fast changes. Rapid oscillation or fast changes in curvature may also occur.

It can be shown that these terms of fast variation play a fundamental role in the genesis of the integral process of summation inside of G. These changes need not be drastically large in order to have a profound influence on the integration process. Relatively small changes count as well when the rate of change is fast, either with respect to s or with respect to t. Since W(s, t) is the exponent of e, then the effect of the changes of W(s, t) are largely magnified and so their effect on the whole integrand may be large enough to be considered.

One basic theorem, which guarantees the fulfillment of condition B is that:

$W^*(s,t)$ must contain all terms of fast variation. The singularities of W(s, t) must then appear in $W^*(s,t)$. By using the material of these notes, we cannot offer a definite proof of this fact. In the first place, we have not yet defined what we mean by "fast variation." In the course of these notes, we shall have the opportunity to dig more precisely into this matter.

The so-called terms of fast variation do not necessarily coincide, at all points of γ or for all values of t, with the so-called dominant terms. Take, for example, 2(II. 1). The terms of fast variation are associated with some of the derivatives $W^{(n)}(s_c, t)$. At some instant of time, some derivative, for example, increases very fast, while at some other value of time some other derivative may be the important source of changes in $W^*(s,t)$.

It is important to notice that $W^*(s,t)$ is in fact a polynomial with variable coefficients (with respect to the time). Under this circumstance, the number of terms in 2(II. 1) may change as it moves in the interval $t_a \leqslant t \leqslant t_b$.

It may be helpful to give a simple illustrative example. Assume that W(s, t) has the form

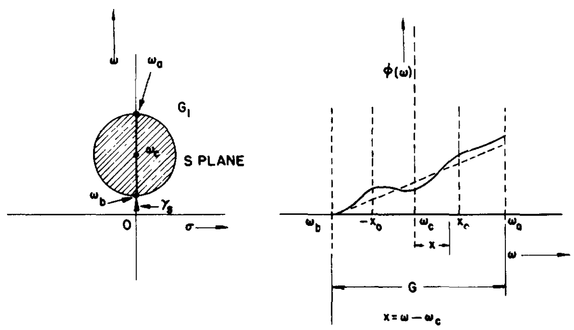
$$W(s,t) = st - \phi(s) \qquad 6(\text{II. }1)$$

and suppose, just as an example, that the contour of integration is taken along the imaginary axis of the s plane ($s = \sigma + i\omega$).

In order to introduce terms of fast variation in W(s, t), we will suppose that $\phi(s)$, along $s = i\omega$, oscillates around the linear phase in the manner indicated in Fig. 2b(II. 1). In accordance with the previous ideas, $W^*(s,t)$ must show the same behavior for $s = i\omega$ inside G.

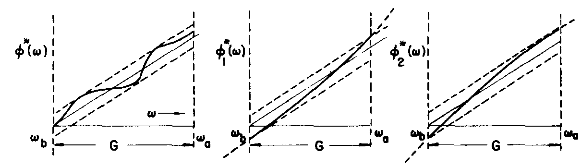
Let

$$\phi(\omega) = \phi(\omega_c) + x\phi^{I}(\omega_c) + \frac{x^2}{2!}\phi^{II}(\omega_c) + \frac{x^3}{3!}\phi^{III}(\omega_c) + \ldots + \frac{x^5}{5!}\phi^{V}(\omega_c) + \ldots$$



(a) (b)

Fig. 2(II. 1)



(a) (b) (c)

Fig. 3(II. 1)

We shall show that W(s, t) will have the form, at least

$$W^{*}(s,t) = \omega_c t + x\left[t - \phi^{I}(\omega_c)\right] + \frac{x^3}{3!}\,\phi^{III}(\omega_c) + \frac{x^5}{5!}\,\phi^{V}(\omega_c) \qquad 7(\text{II. }1)$$

PROOF:

1. Let us write

$$\phi^{*}(\omega) = \phi(\omega_c) + x\phi^{I}(\omega_c) + \frac{x^2}{2!}\,\phi^{II}(\omega_c) + \frac{x^3}{3!}\,\phi^{III}(\omega_c) + \frac{x^4}{4!}\,\phi^{IV}(\omega_c) + \frac{x^5}{5!}\,\phi^{V}(\omega_c) \qquad 8(\text{II. }1)$$

Inside G, the function $\phi^{I}(\omega)$ shows three points of maxima at $x = 0$, $x = +x_o$; $x = -x_o$. Then: $\phi^{II}(\omega)$ must vanish at these points

$$\phi^{II} = \phi^{II}(\omega_c) + x\phi^{III}(\omega_c) + \frac{x^2}{2!}\,\phi^{IV}(\omega_c) + \frac{x^3}{3!}\,\phi^{V}(\omega_c) = 0$$

and $\phi^{II}(\omega_c) \equiv 0$, since the expression above must vanish at $x = 0$ in accordance with Fig. 2b(II. 1). Now the symmetry of the two other points of maxima requires that $\phi^{IV}(\omega_c) = 0$. For in this case, the equation

$$\phi^{III}(\omega_c) + \frac{x^2}{3!}\,\phi^{V}(\omega_c) = 0$$

has two symmetric roots at

$$x = \pm i\sqrt{\frac{6W^{III}(\omega_c)}{W^{V}(\omega_c)}} \qquad 9(\text{II. }1)$$

2. Now, if we should take $\phi^{*}(\omega)$ in 8(II. 1) of a degree smaller than 5(II. 1), it would be impossible to retain in 8(II. 1) the same number of oscillations that are shown in the original function $\phi(\omega)$ inside G.

3. By substituting the above results in 8(II. 1) and by expanding 6(II. 1) around ω_c, then 7 (II. 1) follows immediately.

The following remark is important.

In Fig. 2a(II. 1) the amplitude of the oscillation of $\phi(\omega)$ around the dotted lines may be relatively small. Nevertheless, in order to comply with condition B, the oscillatory character has to be kept in $W^*(s, t)$. In other words, if the approximation of $\phi(\omega)$ is made in three different fashions, as in Fig. 3(II. 1) (in which the lateral tolerance was kept equal), then the result of the integral 4(II. 1) will certainly be different, and can be made strongly different, for each of the three suggested ways of approximation. We are not prepared to give a proof of this fact with the elements presented at this point. That this is a fact, can be shown, step by step, in the future course of these notes. For a reader familiar with transient behavior in filters, the difference in the transient behavior associated with phase functions of type a, b, and c is almost obvious.

We shall come back to the discussion of terms of fast variation in subsequent chapters. In order to show the importance of the dominant and fast variation terms, we shall introduce a basic theorem without proof.

Let

$$W^*(s, t) \text{ approximate } W(s, t)$$

and

$$F^*(s, t) \text{ approximate } F(s).$$

Then we have

$$f_G(t) = \frac{1}{2\pi i} \int_G F(s)\, e^{W(s,t)}\, ds; \qquad f_G^*(t) = \frac{1}{2\pi i} \int_G F^*(s)\, e^{W^*(s,t)}\, ds \qquad 10(\text{II. } 1)$$

"The necessary and sufficient condition that $f_G^*(t)$ approximate $f_G(t)$, within certain tolerances, is that $F^*(s, t)$ and $W^*(s, t)$ contain, respectively, the dominant and fast variation terms of $F(s)$ and $W(s, t)$."

(The theorem is not sharply stated. It will be in the future. It was introduced in this way just to show the direction of our steps in the development of the theory of approximate integration. Here, the theorem has a rather qualitative character.)

SLOW VARIATION TERMS. It may happen, and is often the case, that $W(s, t)$ contains terms which suffer very small changes when s belongs to G and is taken along the contour γ, for $t_a \leq t \leq t_b$. Some of these terms may also appear in $W^*(s, t)$. Terms of slow variation play a secondary role in the theory of approximate integration. They are handled with ease and they produce small changes, or perturbation, in the solution obtained with the fast variation terms. Slow variation terms, however, may suffer fast changes outside G. It may then happen that they contribute definitely as dominant terms in the outside of G. The idea of slow variation terms is mentioned here rather as a matter of definition of the possible components of $W(s, t)$ or $W^*(s, t)$.

II. 12 TRANSITION. PURE TRANSITIONAL TRANSIENTS. A brief explanation will be given of the meaning of "transition" in connection with our theories of approximate integration.

The polynomial 2(II. 1), which defines $W^*(s,t)$, has a time-variable coefficient. The relative value of the coefficient may change with t, so that it may happen that at a certain time, say $t = t_1$, a certain term (or terms) of this polynomial has a predominant effect on the s behavior of $W^*(s,t)$, while at other values of time, say $t = t_2$, this predominant effect is switched to another term, or to other terms, of the polynomial. This time variation of the predominant terms of the polynomial is called "time transition" or simple "transition" of $W^*(s,t)$. It is to be noted that the term applies to $W^*(s,t)$ and not to F(s) or $F^*(s)$, unless otherwise specified.

As an example, let us take the expression 7(II. 1). The point ω_c of expansion is fixed. Therefore, the coefficients of the third and fifth powers are constant, but the corresponding coefficients of the first and zero powers are functions of the time. At $t = \phi'(\omega_a)$, for example, the first-power term disappears and has no effect on the integral. But at some other time, this term may, or may not, control the main behavior of expression 7(II. 1).

DEFINITION. If at a certain interval of time, say at $t_a \leq t \leq t_b$, the contribution of the integral 3(II. 1), coming from the region G, is controlled by this transitional process of W^*, then the transient (between t_a and t_b), is called "transitional transient." We are under the assumption that W(s, t) suffers rapid changes in G but that they are finite. If W(s, t) has poles or essential singularities, this definition does not hold.

If the function F(s) is of slow variation in G, then the corresponding transient is called "pure transitional."

If F(s) has terms of fast variation (pole, zero, dipoles, etc.) in G, then the transient is called "mixed transitional."

The "theory of transition" in connection with our method of integration is a sort of generalization of the integration through saddlepoints of higher order.

SECTION II. 2 SADDLEPOINTS OF HIGHER ORDER.

II. 20 DEFINITIONS. Let H(s) be an analytic function inside G. Let s_c be a point in G. The Taylor expansion around s_c gives

$$H(s) = H(s_c) + (s - s_c)H'(s_c) + \ldots + (s - s_c)^n \frac{H^{(n)}(s_c)}{n!} + (s - s_c)^{n+1} \frac{H^{(n+1)}(s_c)}{(n+1)!} + \ldots \qquad 1(II.2)$$

DEFINITION 1(II. 2). "If s_c is such that the first n derivatives of $H(s_c)$ for $s = s_c$ vanish simultaneously

$$\left.\begin{array}{ll} H^{(\nu)}(s_c) \equiv 0 & \text{for } \nu = 1, 2, \ldots, n \\ \text{but} & \\ H^{(n+1)}(s_c) \not\equiv 0 & \end{array}\right\} \qquad 2(II.2)$$

then we say that s_c is a saddlepoint of G(s) of the order n + 1."

This definition does not require that $H(s_c)$ must be zero. If $H(s_c)$ happens to be zero, then we say that s_c is simultaneously a saddlepoint of (n + 1) order and a zero of n order.

If $s = s_c$ is a saddlepoint of (n + 1) order, then 1(II. 2) can be written as

$$H(s) = H(s_c) + \frac{H^{(n+1)}(s_c)}{(n+1)!}(s - s_c)^{n+1} + \ldots \qquad 3(\text{II. } 2)$$

We must now extend the definition of the (n + 1) order to the function W(s, t). The Taylor expansion is

$$W(s,t) = W(s_c,t) + (s - s_c)\, W^{I}(s_c,t) + \ldots + (s - s_c)^n \frac{W^{(n)}(s_c,t)}{n!}$$

$$+ (s - s_c)^{n+1} \frac{W^{(n+1)}}{(n+1)!}(s - s_c)^{n+1} + \ldots \qquad 4(\text{II. } 2)$$

DEFINITION 2(II. 2). "If the point s_c has the property that at a certain time, say $t = t_o$, the first n derivatives of $W(s_c, t)$ with respect to s vanish simultaneously

$$\left.\begin{aligned} &W^{(\nu)}(s_c, t_o) = 0 \qquad \text{for } \nu = 1, 2, \ldots, n \\ \text{but} \quad & \\ &W^{(n+1)}(s_c, t_o) \neq 0 \end{aligned}\right\} \qquad 5(\text{II. } 2)$$

then we say that s_c is a transitional saddlepoint of (n + 1) order at the time $t = t_o$."

This definition does not require that $W(s_c, t)$ must be zero.

At t_o the function $W(s_c, t)$ can be expanded around s_c as

$$W(s,t_o) = W(s_c,t_o) + \frac{W^{(n+1)}(s_c,t_o)}{(n+1)!}(s - s_c)^{n+1} + \ldots \qquad 6(\text{II. } 2)$$

REMARKS. I. This definition coincides with the corresponding definition regarding H(s), if the derivatives $W^{(\nu)}(s_c, t_o) = 0$ for $\nu = 1, \ldots, n$ for all values of time. II. A saddlepoint of W(s, t) has a transitional character. That is, a point s_c may be a saddlepoint of (n + 1) order at $t = t_o$, but in general it is not necessarily a saddlepoint at $t = t + \Delta t$. From this remark, the reader may immediately understand why the saddlepoint method of integration is generally inadequate for obtaining a solution valid for $t = t_o$. The transitional theory was developed to overcome this difficulty. In connection with transitional transients, the saddlepoint method of higher order provides solutions that are accurate at $t = t_o$ only. The solution holds good in a very narrow interval around t_o, which makes it of no practical importance. Outside this narrow interval, the solution differs very rapidly from the correct answer.

II. 21 BEHAVIOR OF W(s, t) IN THE VICINITY OF A SADDLEPOINT. LEVEL LINES. LINES OF STEEPEST DESCENT. ASYMPTOTIC STAR. In the immediate vicinity of s_c, and for $t = t_o$, the first two terms of 6(II. 2) control the behavior of W(s, t).

Let us study the behavior of

$$W(s,t) - W(s_c, t) = M(s, t_o) = \frac{W^{(n+1)}}{(n+1)!} (s - s_c)^{n+1} = P + iQ \qquad 7(II.2)$$

By setting $s - s_c = z = re^{i\phi}$ one gets

$$\frac{W^{(n+1)}(s_c, t)}{(n+1)!} = \left| \frac{W^{(n+1)}}{(n+1)!} \right| e^{i\theta_{n+1}} = h_{n+1} e^{i\theta_{n+1}} \qquad 8(II.2)$$

and one gets

$$\left. \begin{aligned} P &= h_m r^m \cos (m\phi + \theta_m) \\ Q &= h_m r^m \sin (m\phi + \theta_m) \end{aligned} \quad m = n + 1 \right\} \qquad 9(II.2)$$

The lines defined by $P = 0$ and $Q = 0$ are called lines of steepest descent. The lines $P = 0$ are solutions of

$$\cos (m\phi + \theta_m) = 0$$

or

$$m\phi + \theta_m = \frac{\pi}{2} + K\pi \qquad K = \text{positive, zero, or negative integer} \qquad 10(II.2)$$

or

$$\phi_p = \frac{\pi}{2m} + \frac{K\pi}{m} - \frac{\theta_m}{m}$$

If one gives to K successive integer values, starting with $K = 0$, then the angle ϕ_p repeats itself when

$$K = 2m = 2(n+1)$$

Then we have 2m different ($P = 0$) lines in the direction given by

$$\phi_p = \frac{\pi}{2m} - \frac{\theta_m}{m} + \frac{K\pi}{m} \qquad K = 0, 1, 2, \ldots, 2m-1 \qquad 11(II.2)$$

Figure 1(II.2) shows the uniform distribution of the lines $P = 0$ in the vicinity of the saddlepoint of (n+1) order. The lines $Q = 0$ are solutions of

$$\sin (m\phi + \theta_m) = 0$$

or

$$m\phi + \theta_m = K\pi \qquad 12(II.2)$$

or

$$\phi_Q = -\frac{\theta_m}{m} + \frac{K\pi}{m}$$

We also have, therefore, $2m = 2(n+1)$, $Q = 0$ for $K = 0, 1, 2, \ldots, 2m - 1$, which are distributed as

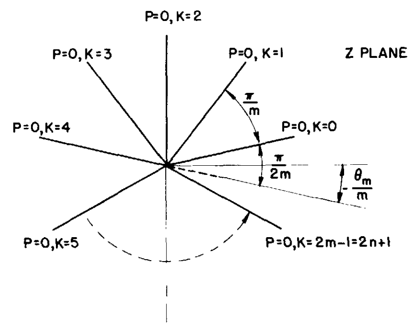


Fig. 1(II. 2)

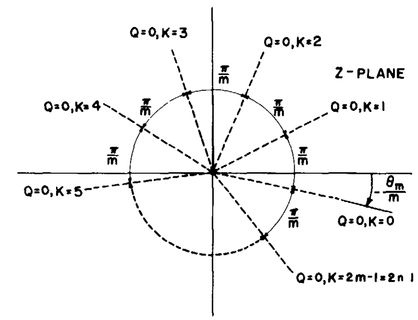


Fig. 2(II. 2)

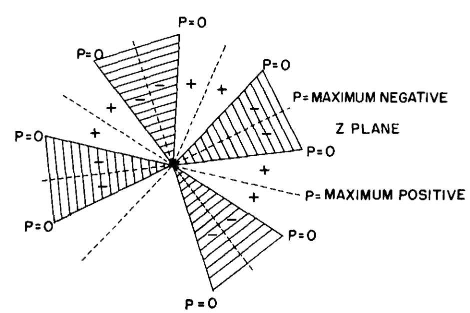
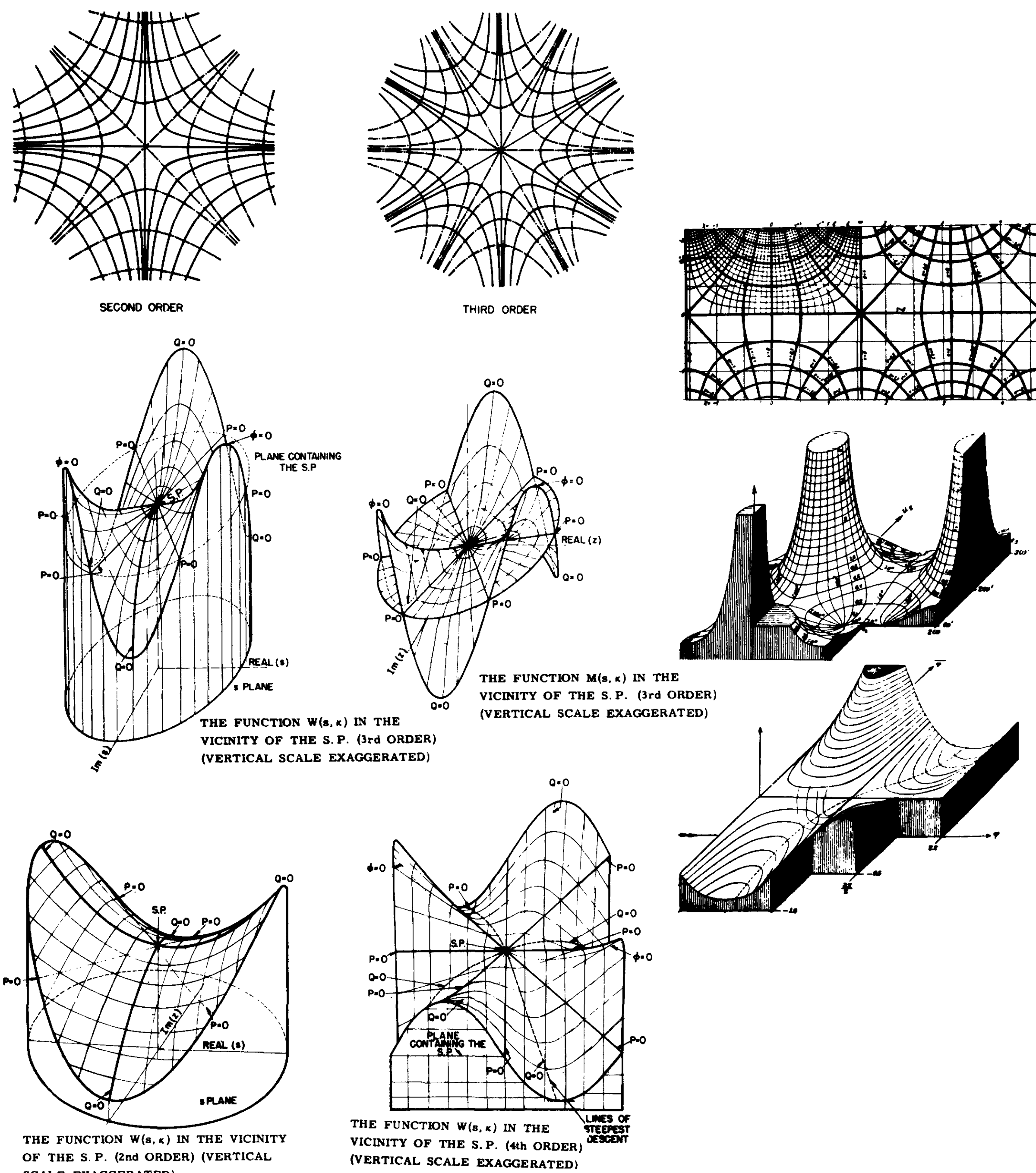


Fig. 3(II. 2)

LEVEL LINES IN THE VICINITY OF A SADDLEPOINT (TOP VIEW)



THE FUNCTION W(s, κ) IN THE VICINITY OF THE S.P. (3rd ORDER) (VERTICAL SCALE EXAGGERATED)

THE FUNCTION M(s, κ) IN THE VICINITY OF THE S.P. (3rd ORDER) (VERTICAL SCALE EXAGGERATED)

THE FUNCTION W(s, κ) IN THE VICINITY OF THE S.P. (2nd ORDER) (VERTICAL SCALE EXAGGERATED)

THE FUNCTION W(s, κ) IN THE VICINITY OF THE S.P. (4th ORDER) (VERTICAL SCALE EXAGGERATED)

Fig. 4(II. 2)

$$\phi_Q = -\frac{\theta_m}{m} + \frac{K\pi}{m} \qquad K = 0, \ldots, 2m-1 \qquad 13(\text{II. }2)$$

in Fig. 2(II. 2). The lines $Q = 0$ are uniformly distributed around the saddlepoint. The lines $P = 0$ and $Q = 0$ alternate with each other.

NEGATIVE AND POSITIVE AREAS OF THE REAL PART P. A comparison of 9, 11, and 13(II.2) shows that the real part P attains an extremum, for a given r, along the $Q=0$ lines, where

$$M = P = \text{pure real} \qquad 14(\text{II. }2)$$

The values of the function along $Q = 0$ are then given by

$$M = P = h_m r^m \cos K\pi = (-1)^K h_m r^m \qquad 15(\text{II. }2)$$

so that the sign of P alternates from one $Q = 0$ to the adjacent one.

Figure 3(II. 2) shows the region of the z plane, in which real $M = P$ are positive and negative.

ASYMPTOTIC STAR. The configuration of Fig. 3(II.2) is frequently used in the following discussion. For this reason, we prefer to name it. We have chosen the name "asymptotic star" but perhaps this is not the final designation.

Level lines, and isometric plots of several saddlepoints of the second, third, and fourth order are shown in the different plots of Fig. 4(II. 2).

SECTION II. 3 INTEGRATION THROUGH A SADDLEPOINT OF (n + 1) ORDER.

II. 30 THE (n + 1) n/2 SOLUTIONS. INDEPENDENT BASIC SOLUTIONS. The contour γ_s must be deformed along the lines of steepest descent with maximum negative P. The number of these lines is equal to $(n + 1)$. The values of K which follow P maximum negative lines are: $K = 1, 3, 5, \ldots, 2n + 1$. The even values correspond to P maximum positive. The direction of these lines is given by

$$\phi_{Q=0} = -\frac{\theta_{n+1}}{n+1} + \frac{K\pi}{n+1} \qquad K = 1, 3, 5, \ldots, 2n+1 \qquad 1(\text{II. }3)$$

We have to pick up the lines of steepest descent which are topologically equivalent to the contour of integration γ_s.

Quite often we may have more than one primary saddlepoint. In this case, one has to follow the lines of the steepest negative descent of one saddlepoint and then change the corresponding negative line of steepest descent of the other primary saddlepoint in such a way that the final contour, obtained in this way, is equivalent to the given contour of integration.

Figure 1(II. 3) illustrates this procedure for the particular case of two saddlepoints of the third order. Two possible deformed contours of integration are shown; both of them lead to the same solution. For the deformed contour a, one has to switch from one star to the other at a finite point in the s plane. For case b, the corresponding switching takes place at the point of infinity.

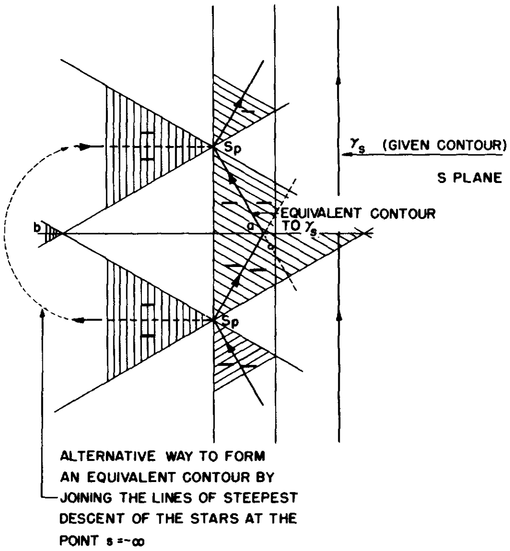


Fig. 1(II. 3)

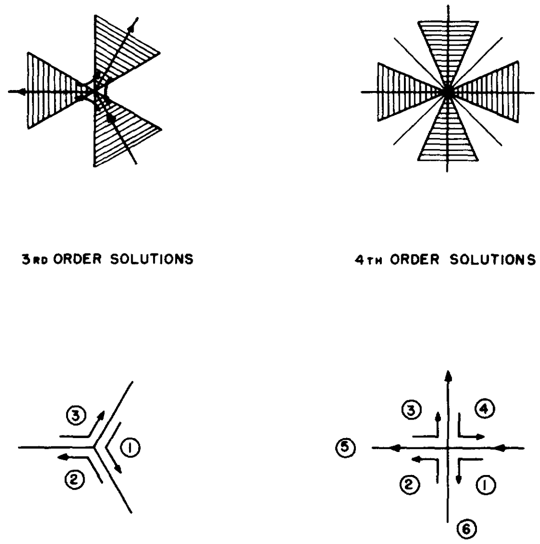


Fig. 2(II. 3)

We shall consider only the process of integration through a single saddlepoint of (n + 1) order. To complete the integral we have to add the contribution of each saddlepoint.

We enter to the saddlepoint along a line of steepest descent, say for K = M, and leave along another line, say for K = N. We shall label such a solution with the subindices (M, N).

Since K = 1, 3, 5, 7, . . . , 2n + 1, we can easily show that

$$\text{the number of possible solutions} = \frac{(n + 1)n}{2} \qquad 2(\text{II. }3)$$

For n + 1 = 3 we have three solutions and for n + 1 = 4 we have six possible solutions. This is schematically shown in Fig. 2(II. 3). A solution obtained by changing the direction of integration is not a new one.

All the (n + 1)n/2 are not independent. (The term independent is not quite correct here.) Some of them can be expressed as linear combinations of the others. For example, for n + 1 = 3, solution 3 can be expressed in terms of 1 and 2. For n + 1 = 4, solution 5 can be expressed as a sum of 1 and 2, and so forth. The solutions which cannot be expressed in terms of the others are called the basic integral solutions of the saddlepoint of (n + 1) order.

It can be shown that the number of basic integrals is equal to n (for an (n + 1) order saddlepoint).

Therefore

$$\text{the number of basic solutions} = n \qquad 3(\text{II. }3)$$

II.31 ANALYTICAL EXPRESSION OF A BASIC INTEGRAL SOLUTION. Let us consider the integral

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} F(s)\, e^{W(s,t)}\, ds \qquad 4(\text{II. }3)$$

and assume that $W(s,t)$ possesses a primary saddlepoint of $(n+1)$ order, say at s_c, at a certain time $t = t_o$.

The Taylor series expansion of $W(s,t)$, around s_c is given by 6(II. 2). We shall assume that $\left|W^{(n+1)}(s_c, t_o)\right|$ is large enough when s_c is taken along the lines of steepest descent corresponding to $K = M$ and $K = N$, which are taken as the deformed contour of integration. This assumption guarantees the fact that the expression

$$e^{-\left|\frac{W^{(n+1)}(s_c, t_o)}{(n+1)!}\right| r^{n+1}} \qquad 5(\text{II. }3)$$

suffers a sudden change in the vicinity of the saddlepoint and becomes negligible outside the vicinity of the saddlepoint.

We shall suppose, first, that $F(s)$ is a constant whose normalized value is equal to one.

Let $\gamma_{M,N}$ denote the contour of integration along the lines of steepest descent $K = M$ and $K = N$.

The integral 4(II. 3) becomes

$$f(t_o) = \frac{e^{W(s_c, t_o)}}{2\pi i} \int_{\gamma_{M,N}} e^{+\frac{W^{(n+1)}(s_c, t_o)}{(n+1)!}(s - s_c)^{n+1}}\, ds \qquad 6(\text{II. }3)$$

in which the terms of $(s - s_c)^{n+2}$ etc. are assumed to be negligible, since in the vicinity of the $(n + 1)$ order saddlepoint, the term

$$\frac{W^{(n+1)}(s_c, t_o)(s - s_c)^{n+1}}{(n+1)!}$$

controls the behavior of $W(s, t_o)$.

Let us introduce the transformation

$$\sqrt[(n+1)]{\frac{W^{(n+1)}(s_c, t_o)}{(n+1)!}}\,(s - s_c) = z = \rho e^{i\psi} \qquad 7(\text{II. }3)$$

Then, one gets

$$f(t_o) = \frac{e^{W(s_c, t_o)}}{\sqrt[n+1]{\frac{1}{(n+1)!}\, W^{(n+1)}(s_c, t_o)}}\, \frac{1}{2\pi i} \int_{\gamma_{M,N}} e^{z^{n+1}}\, dz \qquad 8(\text{II. }3)$$

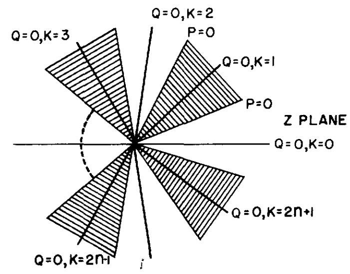


Fig. 3(II. 3)

The set of $(n + 1)n/2$ expressions

$$h_{M,N} = \frac{1}{2\pi i}\int_{\gamma_{M,N}} e^{z^{(n+1)}}\,dz \qquad 9(\text{II. }3)$$

is called the integral solution of the $(n + 1)$ order saddlepoint.

Considering the transformation under 7(II. 2) and 13(II. 2), 14(II. 2), 1(II. 3), and 8(II. 3), it can be seen that the star of Fig. 3(II. 2) rotates in the positive direction an angle $\theta_{(n+1)}/n+1$ when the transformation 7(II. 3) is introduced. The star has the position in the z plane indicated by Fig. 3(II. 3). The lines of steepest descent for $Q = 0$ are given, in the z plane, by

$$\psi_Q = \frac{K\pi}{n+1} \qquad K = 0, 1, \ldots, 2n + 1 \qquad 10(\text{II. }3)$$

The lines of steepest descent in the shaded areas (P negative) are given by 10(II. 3) for K odd.

We are now ready to evaluate the integral 9(II. 3).

$$h_{M,N} = \frac{1}{2\pi i}\int_0^\infty e^{-\rho^{n+1}} e^{i\frac{N\pi}{n+1}}\,d\rho - \frac{1}{2\pi i}\int_0^\infty e^{-\rho^{n+1}} e^{+i\frac{M\pi}{n+1}}\,d\rho$$

By using

$$\int_0^\infty e^{-x^P}\,dx = \frac{\Gamma\left(\frac{1}{P}\right)}{P}$$

we have

$$h_{M,N} = \frac{1}{2\pi i}\left(e^{i\frac{N\pi}{n+1}} - e^{i\frac{M\pi}{n+1}}\right)\frac{\Gamma\left(\frac{1}{n+1}\right)}{(n+1)}$$

$$h_{M,N} = \frac{1}{\pi}\,\frac{\Gamma\left(\frac{1}{n+1}\right)}{(n+1)}\,e^{i\frac{\pi}{n+1}\frac{N+M}{2}}\sin\left[\frac{\pi}{n+1}\,\frac{N-M}{2}\right] \qquad 11(\text{II. }3)$$

In terms of these quantities, the explicit solution of 6(II. 3) can be written as

$$f(t_o) = e^{W(s_c,t_o)} \frac{1}{\pi} \frac{\Gamma\left(\frac{1}{n+1}\right)}{(n+1)} e^{i\frac{\pi}{n+1}\frac{N+M}{2}} \sin\left[\frac{\pi}{n+1}\frac{N-M}{2}\right] \frac{1}{\left(\frac{1}{(n+1)!} W^{(n+1)}(s_c,t_o)\right)^{\frac{1}{n+1}}}$$

12(II. 3)

Solution 12(II. 3) presupposes that F(s) = 1 in the integral 4(II. 3). Now we shall obtain a solution of 4(II. 3) when F(s) is present. We shall assume that F(s) is analytic in the vicinity of the saddlepoint s_c of (n + 1) order.

The Taylor series expansion of F(s) is

$$F(s) = F(s_c) + F'(s_c)(s - s_c) + \ldots + \frac{F^{(\nu)}}{\nu!}(s - s_c)^\nu + \ldots \qquad 13(II.3)$$

In accordance with 7(II. 3) we have

$$F(z) = F(s_c) + \frac{F'(s_c)z}{\left[\frac{1}{(n+1)!} W^{(n+1)}(s_c,t_o)\right]^{1/n+1}} + \ldots + \frac{F^{(\nu)}z^\nu}{\nu!\left[\frac{1}{(n+1)!} W^{(n+1)}(s_c,t_o)\right]^{\nu/n+1}} + \ldots$$

$$= \sum_0^\infty a_\nu z^\nu \qquad 14(II.3)$$

Hence, the integral 4(II. 3) becomes

$$f(t_o) = \frac{e^{W(s_c,t_o)}}{\left[\frac{1}{(n+1)!} W^{(n+1)}(s_c,t_o)\right]^{1/n+1}} \sum_0^\infty a_\nu \frac{1}{2\pi i}\int_{\gamma_{M,N}} z^\nu e^{-z^{(n+1)}} dz \qquad 15(II.3)$$

The integral in 15(II. 3) can be performed, along the lines K = M and K = N, as

$$\frac{1}{2\pi i}\int_0^\infty \rho^\nu e^{i\nu\frac{N\pi}{n+1}} e^{-\rho^{n+1}} e^{i\frac{N\pi}{n+1}} d\rho - \frac{1}{2\pi i}\int_0^\infty \rho^\nu e^{i\nu\frac{M\pi}{n+1}} e^{-\rho^{n+1}} e^{i\frac{M\pi}{n+1}} d\rho$$

$$= \frac{1}{2\pi i}\left\{e^{i\frac{\nu+1}{n+1}N\pi} - e^{i\frac{\nu+1}{n+1}M\pi}\right\}\int_0^\infty \rho^\nu e^{-\rho^{n+1}} d\rho$$

$$= \frac{1}{\pi} e^{i\frac{\nu+1}{n+1}\pi\frac{N+M}{2}} \sin\left(\frac{\nu+1}{n+1}\pi\frac{N-M}{2}\right)\frac{\Gamma\left(\frac{\nu+1}{n+1}\right)}{n+1}$$

and the solution for 15(II. 3) becomes

$$f(t_o) = \frac{e^{W(s_c, t_o)}}{\pi(n+1)} \sum_{\nu=0}^{\infty} \left\{ \frac{F^{(\nu)}(s_c)\,\Gamma\left(\frac{\nu+1}{n+1}\right)}{\left[\frac{1}{(n+1)!} W^{(n+1)}(s_c, t_o)\right]^{(\nu+1)/(n+1)}} e^{i\frac{\nu+1}{n+1}\pi\frac{N+M}{2}} \sin\left[\frac{\nu+1}{n+1}\pi\frac{N-M}{2}\right] \right\}$$

16(II. 3)

The series above converges extremely fast when $\left|W^{(n+1)}(s_c, t_o)\right|$ is large. Only the first few terms are needed for obtaining a good degree of approximation for $f(t_o)$.

II. 32 REMARKS ON SUBSECTION II. 31. In the last subsection, II. 31, we gave the expression for computing the integrals in connection with the (n + 1) order saddlepoint. These solutions have a formal character; they are given merely as a matter of illustration for certain basic steps and for the terminology which has to be followed in the development of the theory of transition. Confusion may result if we go ahead with the transition theory without first showing some of the basic ideas involved.

Due to the presence of time, a saddlepoint of higher order can be found only at a certain time, say $t = t_o$, and therefore the solutions in 12(II.3) and 16(II.3) can be applied only at $t = t_o$. This type of solution is practically useless. However, it has a theoretical meaning that will permit us to construct a more adequate theory from which we can derive solutions valid in a rather broad interval of time around t_o.

After $t = t_o$, the (n + 1) saddlepoint dissolves. What happens is that the saddlepoint breaks up into two or more saddlepoints, each of lower order, with the sum of their order equal to (n + 1). Each of these new saddlepoints moves away along different orbits. Before $t = t_o$ the individual saddlepoints are moving toward s_c and they meet at $t = t_o$. In other words, a saddlepoint of (n + 1) order can be considered as the time confluence of two or more saddlepoints whose orders add up to (n + 1).

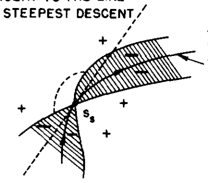


Fig. 4(II. 3)

The asymptotic star configuration of Fig. 3(II. 2) changes considerably as the colliding saddlepoints move apart. There is only a single point at which the lines P = 0 and Q = 0 meet. In the vicinity of s_c the lines of steepest descent are no longer straight lines. They are sharply curved.

These sharp curvatures in the vicinity of s_c are the cause of failure of the ordinary saddlepoint methods of integration in the case of saddlepoints of higher orders.

For, suppose that the lines of steepest descent and the negative shaded areas have a configuration of Fig. 4(II. 3). When we apply the saddlepoint method we are dealing with two main ideas:

1. We approximate the integrand in the vicinity of the saddlepoint.

2. We express the integral along the small segment of the contour contained in the vicinity of s_s. This segment coincides in direction with the tangent to the line of steepest

descent at s_s. Once we have settled the integral along this segment, with end points, say, at $r = r_o$ and $r = -r_o$, then we replace

$$\int_{-r_o}^{+r_o} \text{ by } \int_{-\infty}^{+\infty}$$

What we are really doing in this last step is to extend the integral along the tangent to the line of steepest descent, and the tangent may leave, as in Fig. 4(II. 3), the shaded negative area. An integral along this line does not necessarily satisfy condition B given in subsection II. 11.

These considerations show us how we must direct our next steps: Develop a method of integration in which we can follow the curved lines of steepest descent, not only in the vicinity of s_s but also for a sufficient distance from the saddlepoint, so that the function

$$e^{\frac{W^{(n+1)}}{(n+1)!}(s - s_c)^{n+1}}$$

becomes negligible, even if

$$\left| \frac{W^{n+1}(s_c, t)}{(n+1)!} \right|$$

is not very large. The method of integration which accomplishes these possibilities is called "the extended saddlepoint method of integration."

The theory of the extended saddlepoint method of integration is involved and cannot be given in its full and final form here. We shall start with the so-called third-order transition as a convenient introductory approach to the advanced theory. There is a rather large class of transients which belong to the third-order transitional family, and therefore the results of this first approach find a large number of applications.

CHAPTER III

PURE TRANSITION OF THE THIRD ORDER

SECTION III. 1 DEFINITIONS. EXAMPLE.

III. 10 DEFINITION OF PURE TRANSITION AND TRANSITIONAL TRANSIENTS.

Let us consider the integral

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} F(s) e^{W(s,t)} ds \qquad 1(III. 1)$$

together with a domain, say G, of the s plane, from which a prominent contribution to the integral takes place at the interval $t_a \leq t \leq t_b$. In this domain, for a certain value of the time, say $t_a \leq t = t_o \leq t_b$, there is a point, say $s_c \in G$, that becomes a saddle-point of the order (n + 1). Let W(s, t) be analytic in G. We shall study the integration of 1(III. 1) in two separate cases:

I. When F(s) is constant (or practically constant) in G, we have

$$f_G(t) = \frac{1}{2\pi i} \int_{\gamma_{s_G}} e^{W(s,t)} ds \qquad 2(III. 1)$$

which is called a "pure transitional integral" of the (n + 1) order. The transient it represents is called a "pure transitional transient of the (n + 1) order."

II. When F(s) is meromorphic in G (having, therefore, no exponential behavior), we have

$$f_G(t) = \frac{1}{2\pi i} \int_{\gamma_{s_G}} F(s) e^{W(s,t)} ds \qquad 3(III. 1)$$

which is called a "mixed transitional integral of the (n + 1) order." The corresponding transient is called "mixed transitional transient of the (n + 1) order."

The case (n + 1) = 3 will be fully discussed as an introduction to the general case.

Under section III. 30 we shall consider the pure "transitional case." In Chapter IV we shall introduce mixed transients and the mechanism of the transitional transient formation.

III. 11 INTRODUCTORY EXAMPLE. Let

$$W(s,t) = st - \log\sqrt{s^2 + 1} \qquad 4(III. 1)$$

The saddlepoints are given by

$$W^{I}(s, t) = 0$$

or

$$t - \frac{s}{1 + s^2} = 0 \qquad 5(\text{III. } 1)$$

from which we obtain

$$s_s = \frac{1}{2t} \pm \sqrt{\left(\frac{1}{2t}\right)^2 - 1} = \begin{cases} s_{s_1} \\ s_{s_2} \end{cases} \qquad \text{two saddlepoints} \qquad 6(\text{III. } 1)$$

The orbits of the saddlepoints are given in Fig. 1(III. 1). The saddlepoints meet at $t = 1/2$. At this point the orbits suffer a 90° change in direction. The saddlepoints move toward the points $\pm i$ when $t \to \infty$.

Let us now consider the second derivative of 4(III. 1).

$$W^{II}(s, t) = -\frac{1 - s^2}{(1 + s^2)^2}$$

The second derivative vanishes at $s = 1$, which is the point in which the two saddlepoints meet and therefore at $s = 1$ we have

$$\left.\begin{array}{l} W^{I}(s, t) = 0 \\ W^{II}(s, t) = 0 \end{array}\right\} \quad \text{for } t = \frac{1}{2}$$

Therefore, $s = 1$ is, at $t = 1/2$, a saddlepoint of the third order.

The third derivative is

$$-\frac{2s(s^2 - 3)}{(1 + s^2)^3}$$

and for $s = 1$ its value is

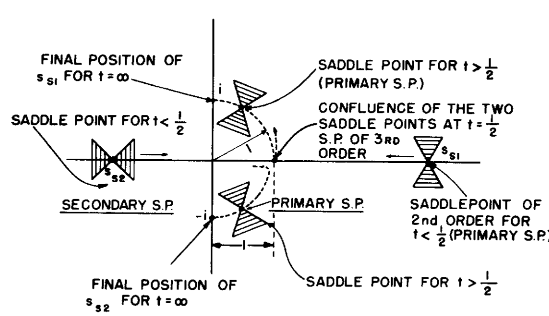


Fig. 1(III. 1)

$$W^{III}(1,t) = \frac{1}{2}$$

Then, the expansion of W(s, t) around s = 1 is given by

$$W(s,t) = \left(-\frac{1}{2}\log 2 + t\right) + \left(t - \frac{1}{2}\right)(s-1) + \frac{1}{2 \times 3!}(s-1)^3 + 0.0625\,(s-1)^4 + \ldots \qquad 7(\text{III. }1)$$

III. 12 MORE ABOUT TERMS OF FAST VARIATION IN $W = st - \log\sqrt{s^2+1}$. A simple mathematical procedure will allow us to have a visual picture of the terms of fast variation.

Let us compute the successive derivatives of $st - \log\sqrt{s^2+1}$ and plot, in the s plane, the positions of their zeros and poles.

	Derivatives	Poles	Zeros	$\left.\frac{W^{(n)}(s,t)}{n!}\right\|_{s=1}$
	$W^{I} = t - \frac{s}{1+s^2}$			
independent of t	$W^{II} = -\frac{1-s^2}{(1+s^2)^2}$	$s = \pm i$, 2^{nd} order	$s = \pm 1$	0
	$W^{III} = -\frac{2s(s^2-3)}{(1+s^2)^3}$	$s = \pm i$, 3^{rd} order	$s = 0$; $s = \pm\sqrt{3}$	+0.0833
	$W^{IV} = +6\,\frac{s^4 - 6s^2 + 1}{(1+s^2)^4}$	$s = \pm i$, 4^{th} order	$s = {+2.42 \atop -2.42}$; $s = {+0.414 \atop -0.414}$	+0.0625
	etc.	etc.	etc.	etc.

The values of the first four derivatives, their poles, and zeros are tabulated above. The behavior of the derivatives after the first is important in this discussion. In Fig. 2(III. 1) the positions of the poles and zeros are indicated and the saddlepoints and saddlepoint orbits are shown.

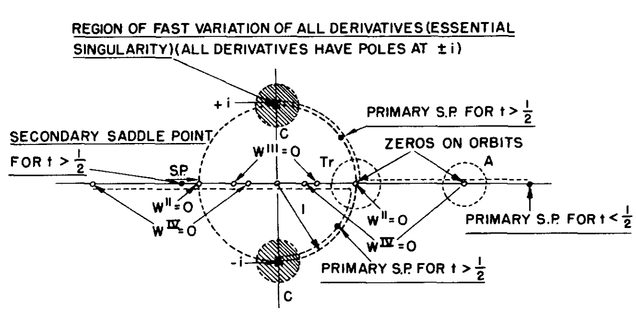


Fig. 2(III. 1)

Now, following the motion of the primary saddlepoints one can have a simple notion of the number of terms we need to keep in the expansion of W(s, t) for different values of time.

For $t < 1/2$, the saddlepoint s_{s1} is the primary one. At $t = 0$, the saddlepoint is at infinity; and as t increases, s_{s1} moves along the positive real axis toward the point $s = +1$. In its motion, s_{s1} will encounter the points at which some derivatives may be zero. In other problems, the saddlepoint may encounter some points at which some derivatives have poles.

In the region A, Fig. 2(III. 1), the saddlepoint will encounter the point in which the four derivatives vanish (at $s = +2.42\ldots$). If, at this point, we compute the values of the successive derivatives, we find

$$\frac{W^{II}}{2!} = -0.05175 \qquad \frac{W^{IV}}{4!} = 0$$

$$\frac{W^{III}}{3!} = +0.0075 \qquad \frac{W^{V}}{5!} = -0.0000$$

This situation clearly indicates that in A one can cut the expansion of W(s, t) up to the third power, since the higher derivatives have a small influence. Now, it can be observed that the third derivative has the nearest zero (at $s = +\sqrt{3}$) and the pole at $\pm i$, which are both far from A. There, the third derivative term has a slow variation in A and the integral is mostly controlled by the second derivative. The first derivative term is constantly zero along the orbit, and the constant term can be taken out of the integral sign.

The situation changes when the saddlepoint moves in the region Tr.

In the first place, the second derivative is going to vanish and therefore is going to lose control of the integral.

In the second place, the other saddlepoint s_{s2} will become active, and both will change in direction after their confluence at $s = +1$.

If one computes the values of the second, third, fourth, etc., derivatives at $s = +1$, one finds the values

$$\frac{W^{II}}{2!} = 0 \qquad \frac{W^{IV}}{4!} = +0.0625$$

$$\frac{W^{III}}{3!} = +0.0833 \qquad \frac{W^{V}}{5!} = \text{negligible}$$

It is clear that we have to keep the first terms in the expansion of W(s, t). Now let us give attention to the confluence of the asymptotic stars of s_{s1} and s_{s2} at $s = +1$. They clearly will form at $t = 1/2$, $s = 1$, a sort of mixed star between the third and fourth orders, whose actual shape is, for the present, immaterial. The actual behavior of the confluent asymptotic stars will be discussed later.

When the saddlepoints s_{s1} and s_{s2} abandon the region Tr, the derivatives start increasing, and more and more terms of the expansion of W are needed. Inside the region C, all derivatives tend to infinity, and finally, at $s = i$, W has an essential singularity. In the region C, the whole idea of approximation becomes meaningless. In the particular case of $W = st - \log \sqrt{s^2 + 1}$, the essential singularity is removable.

$$e^{st - \log\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}} e^{st}$$

Therefore, in the vicinity of $s = i$, the saddlepoint method becomes meaningless, but we can use some other method of integration which may be applicable to $e^{st}/\sqrt{s^2+1}$.

SECTION III.2 PURE TRANSITIONAL TRANSIENTS OF THE THIRD ORDER. AIRY-HARDY FUNCTIONS.

III.20 DEFINITION OF A PURE TRANSITIONAL TRANSIENT OF THE THIRD ORDER. Let $a = a(t)$, $b = b(t)$, $c = c(t)$, and $d = d(t)$ be time-variable coefficients independent of s. Let $s = s_c$ be a fixed point of the s plane.

An expression of the form

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} e^{a + b(s - s_c) + c(s - s_c)^2 + d(s - s_c)^3} ds \qquad 1(\text{III. } 2)$$

is called, by definition, a pure transitional transient of the third order.

III.21 TRANSIENTS REDUCIBLE TO THE THIRD ORDER. Now, consider the integral 2(III. 1) which defines the general pure transitional transient.

Let D_ν be a region of the s plane such that

1. it contains the orbits of primary saddlepoints,
2. the primary saddlepoints move inside D_ν in the interval of time $t_a \leqslant t \leqslant t_b$, and
3. $s_c \in D_\nu$. If for $s \in D_\nu$ and for t, $t_a \leqslant t \leqslant t_b$, the expansion of $W(s, t)$ on the integral 2(III. 1) can be written as

$$W(s,t) \approx W(s_c, t) + (s - s_c)\frac{W^{I}(s_c, t)}{1!} + (s - s_c)^2 \frac{W^{II}(s_c, t)}{2!} + (s - s_c)^3 \frac{W^{III}(s_c, t)}{3!}$$

Then, the transient represented by

$$f(t) = \frac{1}{2\pi i} \int_{\gamma_s} e^{W(s,t)} ds \qquad 2(\text{III. } 2)$$

is said to be reducible, in $t_a \leqslant t \leqslant t_b$, to a transitional transient of the third order. In such a case

$$a \approx W(s_c, t) \qquad c \approx \frac{W^{II}(s_c, t)}{2!}$$
$$b \approx W^{I}(s_c, t) \qquad d \approx \frac{W^{III}(s_c, t)}{3!} \qquad 3(\text{III. }2)$$

III. 22 CANONICAL FORM. THE VARIABLE B. The integral 1(III. 2) which defines the third-order transitional transient, is a function of t through the variables a, b, c, d. The large number of variables makes the problem very difficult to discuss.

The following transformation reduces the number of variables and puts the integral in a more suitable form.

Let

$$s - s_c = \frac{1}{\sqrt[3]{d}} z - \frac{c}{3d} \qquad 4(\text{III. }2)$$

After performing algebraic operations one gets

$$f(t) = \frac{e^{A(t)}}{\sqrt[3]{d}} \frac{1}{2\pi i} \int_{\gamma_z} e^{Bz + z^3} dz \qquad 5(\text{III. }2)$$

where

$$\left.\begin{aligned} A &= a - \frac{bc}{3d} + \frac{2}{27}\frac{c^3}{d^2} \\ B &= \frac{1}{\sqrt[3]{d}}\left(b - \frac{c^2}{3d}\right) \\ &\gamma_z \text{ is the transformed contour of cut in the } z \text{ plane} \end{aligned}\right\} \qquad 6(\text{III. }2)$$

Since in the integral 5(III. 2) the functions $e^{A(t)}/\sqrt[3]{d}$ are simple factors of the integral, we usually disregard them momentarily and consider only the integral

$$Ah(B) = \frac{1}{2\pi i} \int_{\gamma_z} e^{Bz + z^3} dz \qquad 7(\text{III. }2)$$

which is called the canonical integral. (The notation Ah(B) comes from the names Airy and Hardy. In fact, in the future we shall use a more complete notation

$$Ah_{\nu,3}(B)$$

For the moment, we shall dispense with the indices.)

A complete discussion of the integral 7(III. 2) follows in the next sections.

III. 23 LINES OF STEEPEST DESCENT. THE FUNCTION M(B, z). SATELLITE SADDLEPOINT. STAR CONFIGURATIONS. LAGOONS. Let us introduce the notation

$$\left.\begin{aligned} M(B,z) &= Bz + z^3 \\ B &= |B|\, e^{i\beta} \\ z &= re^{i\phi} \end{aligned}\right\} \qquad 8(\text{III. }2)$$

I. Zeros of M(B, z).

There are three zeros at

$$\left\{\begin{aligned} z_o &= 0 \\ z_{o_1} &= i\sqrt{B} = \sqrt{|B|}\, e^{i\left(\frac{\pi}{2}+\frac{\beta}{2}\right)} \\ z_{o_2} &= -i\sqrt{B} = \sqrt{|B|}\, e^{i\left(-\frac{\pi}{2}+\frac{\beta}{2}\right)} \end{aligned}\right\} \qquad 9(\text{III. }2)$$

See Fig. 1(III. 2).

II. P = 0 and Q = 0 lines.

Let

$$M = P + iQ \qquad P \text{ and } Q \text{ real} \qquad 10(\text{III. }2)$$

We can immediately write

$$\left.\begin{aligned} P &= r\left\{|B| \cos(\phi+\beta) + r^2 \cos 3\phi\right\} \\ Q &= r\left\{|B| \sin(\phi+\beta) + r^2 \sin 3\phi\right\} \end{aligned}\right\} \qquad 11(\text{III. }2)$$

The lines of steepest descent of M(B, z) are given by β

$$\left.\begin{aligned} P = 0 \qquad r = 0; \qquad r &= \sqrt{|B|}\sqrt{-\frac{\cos(\phi+\beta)}{\cos 3\phi}} \\ Q = 0 \qquad r = 0; \qquad r &= \sqrt{|B|}\sqrt{-\frac{\sin(\phi+\beta)}{\sin 3\phi}} \end{aligned}\right\} \qquad 12(\text{III. }2)$$

where r must be real and positive, since it is a magnitude.

The computation of the lines is a rather tedious process of algebraic computation. We are not going to follow all of the mathematical steps required for obtaining the lines of steepest descent from 12(III. 2). While all of these steps have been worked out in notes, the limitations of space prevent their detailed presentation here. Illustrative examples of final results will be given by means of convenient figures.

It can be shown that Eqs. 12(III. 2) admit three sets of P = 0 lines and three sets

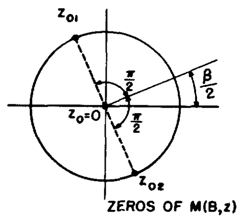


Fig. 1(III. 2)

of Q = 0 lines. The form and position of the lines are strongly influenced by the angle β. See 8(III. 2). The effect of $|B|$, for a fixed β, is a sort of scale factor.

Figure 2(III. 2) shows the lines of steepest descent for $\beta = -33\ 1/3°$. The shaded areas correspond to the parts of the z plane in which P attains a negative value. These shaded areas are called lagoons. Along the lines Q = 0, the function M(B, z) is pure real.

Figure 3(III. 2) shows the corresponding configuration for $\beta = -60°$; Fig. 4(III. 2) is plotted for $\beta = 0$.

Finally, the satellite saddlepoints of the z plane can be computed by

$$\left.\begin{aligned} &\frac{dM(B,z)}{dz} = 0 = B + 3z^2 \\ &\therefore z^*_{s_1} = i\sqrt{\frac{B}{3}};\ z^*_{s_2} = -i\sqrt{\frac{B}{3}} \end{aligned}\right\} \qquad 13(\text{III. }2)$$

In Fig. 2(III. 2), the lines of steepest descent of M do not run over the satellite saddlepoints. In Figs. 3(III. 2) and 4(III. 2), which are plotted for the so-called critical values of β, lines of steepest descent run over the satellite saddlepoints.

III. 24 THE THREE POSSIBLE CONTOURS. DEFINITION OF THE FUNCTIONS $Ah_\nu(B)$. Let us consider the integrals 1(III. 2) and 2(III. 2), and let us study the transformation of the contour γ_s into the γ_z contour, in the z plane, under the transformation 4(III. 2). This transformation shows that γ_z is obtained from γ_s by a shift and a rotation.

There are three possible positions of γ_z in which the integral 7(III. 2) converges. They are indicated by γ_1, γ_2, γ_3. Each contour corresponds to an integral. See Fig. 5(III. 2).

Then, the integral 7(III. 2) breaks into three functions, which are given by

$$Ah_\nu(B) = \frac{1}{2\pi i}\int_{\gamma_\nu} e^{Bz + z^3}\, dz \qquad \nu = 1, 2, 3 \qquad 14(\text{III. }2)$$

Often, as we shall see in the case of transitions of n order, two indices are attached to Ah(B). One corresponds to the contour from which it is generated and the other indicates the order of transition. In our case, the complete notation would be

$$Ah_{1,3}(B) \quad \text{for} \quad Ah_1(B)$$

$$Ah_{2,3}(B) \quad \text{for} \quad Ah_2(B)$$

$$Ah_{3,3}(B) \quad \text{for} \quad Ah_3(B)$$

In the present chapter we shall dispense with the functions of the order index.

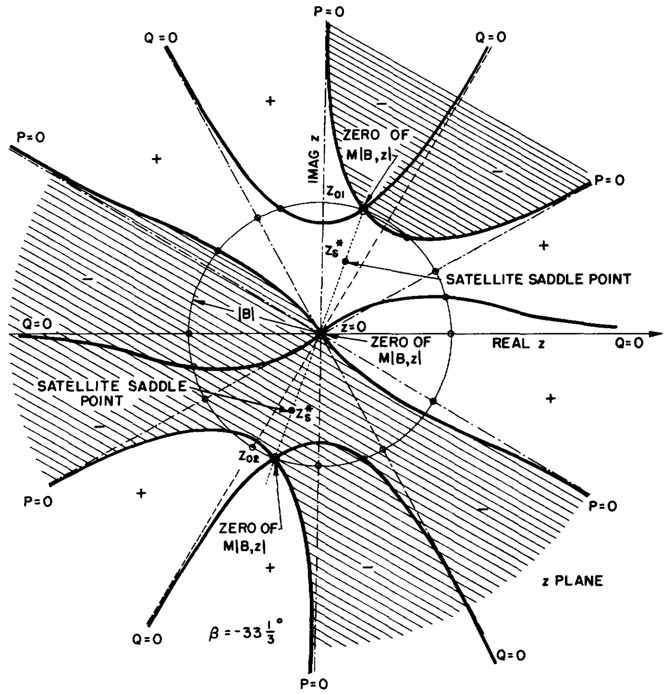


Fig. 2(III. 2)

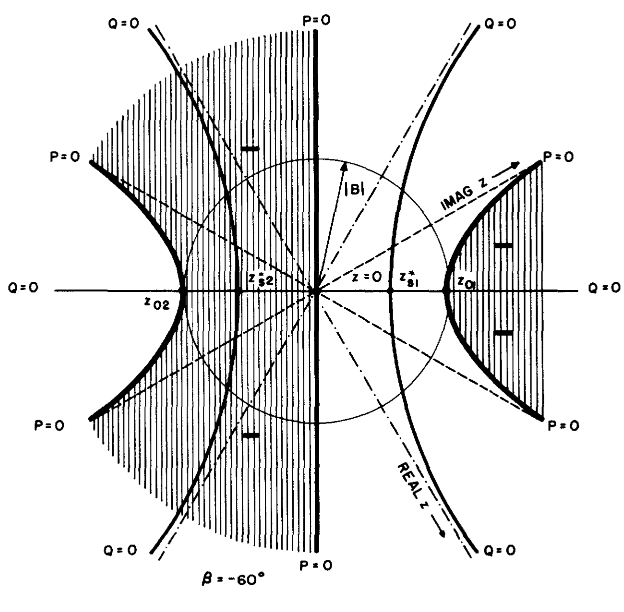


Fig. 3(III. 2)

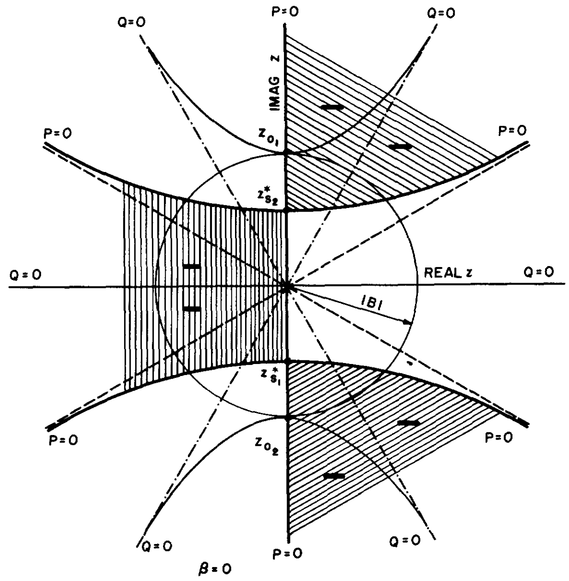


Fig. 4(III.2)

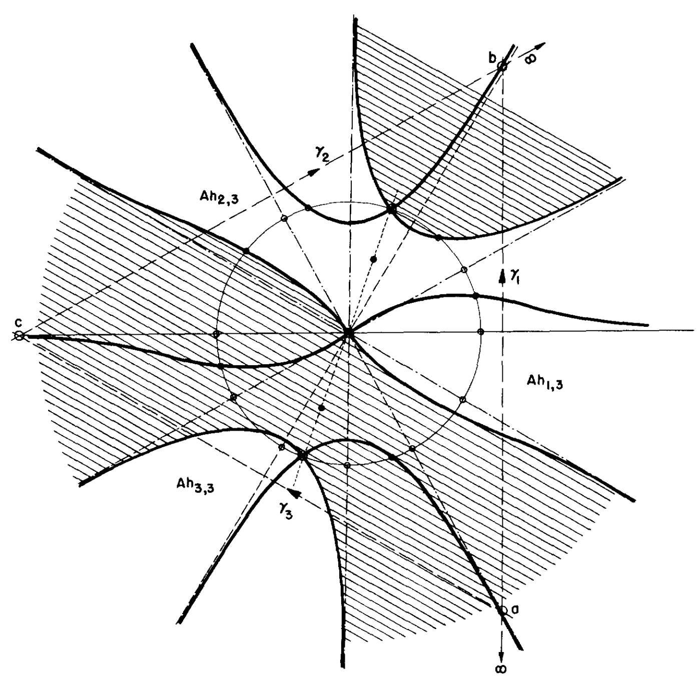


Fig. 5(III.2)

Only two functions are independent, since we can deduce from Fig. 5(III. 2) that

$$Ah_3(B) = Ah_1(B) - Ah_2(B) \qquad 15(III.2)$$

III. 25 EVALUATION OF THE FUNCTION $Ah_1(B)$.

$$Ah_1(B) = \frac{1}{2\pi i} \int_{\gamma_1} e^{Bz + z^3} dz \qquad 16(III.2)$$

Figure 5(III. 2) shows that the integration along γ_1 is equivalent to integrating along the straight lines a to 0 and 0 to b when a and b $\rightarrow \infty$. It can be noted that there is not a continuous passage from a to b along the lines of steepest descent. We shall use the lines of steepest descent only at great distances from the origin of the z plane.

1. Integration along the line $z = \tau e^{i\pi/3}$

$$z^3 = \tau^3 e^{i\pi} = -\tau^3 = \text{real}$$

$$dz = e^{i\frac{\pi}{3}} d\tau$$

Then

$$\frac{1}{2\pi i}\int_0^\infty e^{B\tau e^{i\frac{\pi}{3}} - \tau^3} e^{i\frac{\pi}{3}} d\tau = \frac{e^{i\frac{\pi}{3}}}{2\pi i}\int_0^\infty e^{B\tau e^{i\frac{\pi}{3}}} e^{-\tau^3} d\tau$$

$$= \frac{e^{i\frac{\pi}{3}}}{2\pi i}\sum_0^\infty \frac{B^\nu e^{i\frac{\pi}{3}\nu}}{\nu!}\int_0^\infty e^{-\tau^3}\tau^\nu d\tau$$

$$= \frac{1}{6\pi i}\sum_0^\infty \frac{\Gamma\left(\frac{\nu+1}{3}\right)}{\nu!} B^\nu e^{i(\nu+1)\frac{\pi}{3}} \qquad 17(III.2)$$

since

$$e^{B\tau e^{i\frac{\pi}{3}}} = \sum_0^\infty \frac{B^\nu \tau^\nu e^{i\frac{\pi}{3}\nu}}{\nu!}$$

and

$$\int_0^\infty e^{-\tau^3}\tau^\nu d\tau = \frac{1}{3}\Gamma\left(\frac{\nu+1}{3}\right) \qquad 18(III.2)$$

2. Integration along the line $z = \tau e^{-i\frac{\pi}{3}}$.

In a similar way one gets

$$\frac{-e^{-i\frac{\pi}{3}}}{2\pi i}\int_0^\infty e^{-\tau^3} e^{B\tau e^{-i\frac{\pi}{3}}} d\tau = -\frac{1}{6\pi i}\sum_0^\infty \frac{\Gamma\left(\frac{\nu+1}{3}\right)}{\nu!} B^\nu e^{-i(\nu+1)\frac{\pi}{3}} \qquad 19(\text{III. }2)$$

3. The complete integral is then given by

$$Ah_1(B) = \frac{1}{2\pi i}\int_{\gamma_1} e^{Bz + z^3} dz = \frac{1}{3\pi}\sum_0^\infty \frac{\Gamma\left(\frac{\nu+1}{3}\right)}{\nu!} B^\nu \sin(\nu+1)\frac{\pi}{3}$$

$$= \frac{1}{2\sqrt{3}\pi}\left\{\Gamma\left(\frac{1}{3}\right) + \frac{\Gamma\left(\frac{2}{3}\right)}{1!} B - \frac{\Gamma\left(\frac{4}{3}\right)}{3!} B^3 - \frac{\Gamma\left(\frac{5}{3}\right)}{4!} B^4 + \frac{\Gamma\left(\frac{7}{3}\right)}{6!} B^6 + \ldots\right\} \qquad 20(\text{III. }2)$$

III. 26 EVALUATION OF THE FUNCTIONS $Ah_2(B)$ AND $Ah_3(B)$. In a similar way, it can be shown that

$$\left.\begin{aligned} Ah_2(B) &= \frac{1}{3\pi}\sum_0^\infty \frac{\Gamma\left(\frac{\nu+1}{3}\right)}{\nu!} B^\nu e^{i(2\nu-1)\frac{\pi}{3}} \sin\frac{\pi}{3}(\nu+1) \\ &= \frac{1}{2\sqrt{3}\pi}\left\{\Gamma\left(\frac{1}{3}\right) e^{-\frac{\pi}{3}i} + \Gamma\left(\frac{2}{3}\right) B e^{i\frac{\pi}{3}} - \frac{\Gamma\left(\frac{4}{3}\right)}{4!} B^3 e^{5\frac{\pi}{3}i} \right. \\ &\quad \left. - \frac{\Gamma\left(\frac{5}{3}\right)}{4!} B^4 e^{7i\frac{\pi}{3}} + \ldots\right\} \end{aligned}\right\} \qquad 21(\text{III. }2)$$

$$\left.\begin{aligned} Ah_3(B) &= \frac{1}{3\pi}\sum_0^\infty \frac{\Gamma\left(\frac{\nu+1}{3}\right)}{\nu!} B^\nu e^{i\frac{\nu+1}{3}\pi} \sin\frac{2}{3}(\nu+1)\pi \\ &= \frac{1}{2\sqrt{3}\pi}\left\{\Gamma\left(\frac{1}{3}\right) e^{i\frac{\pi}{3}} - \Gamma\left(\frac{2}{3}\right) B e^{\frac{2}{3}\pi i} + \frac{\Gamma\left(\frac{4}{3}\right)}{3!} B^3 e^{i\frac{4}{3}\pi} - \frac{\Gamma\left(\frac{5}{3}\right)}{4!} B^4 e^{i\frac{5\pi}{3}} + \ldots\right\} \end{aligned}\right\} 22(\text{III. }2)$$

III. 27 RELATION BETWEEN Ah_1, Ah_2, AND Ah_3. We have shown, Eq. 15(III. 2), that

$$Ah_1(B) = Ah_2(B) + Ah_3(B)$$

Besides, from 20, 21, and 22(III. 2) we can easily get

$$\begin{aligned} Ah_2(B) &= e^{-i\frac{\pi}{3}} Ah_1\left(B e^{i\frac{2\pi}{3}}\right) \\ Ah_3(B) &= e^{i\frac{\pi}{3}} Ah_1\left(B e^{-i\frac{2\pi}{3}}\right) \end{aligned} \qquad 23(\text{III. }2)$$

Relations 23(III. 2) show that we need only to compute and stick to $Ah_1(B)$.

SECTION III. 3 GENERATING FUNCTIONS.

III. 30 GENERAL EXPRESSIONS FOR THE PURE TRANSITIONAL TRANSIENT OF THE THIRD ORDER.

From 5, 6, and 7(III. 2) one gets

$$\left.\begin{aligned} f(t) &= \left\{\frac{e^{A}}{\sqrt[3]{d}}\right\}\left\{Ah_{\nu}(B)\right\} \\ A &= a - \frac{bc}{3d} + \frac{2}{27}\frac{c^{3}}{d^{2}} \\ B &= \frac{1}{\sqrt[3]{d}}\left(b - \frac{c^{2}}{3d}\right) \end{aligned}\right\} \qquad 1(\text{III. }3)$$

The values of a, b, c in terms of the derivatives of W(s, t) are given by 3(III. 2).

III. 31 TRANSVERSAL AND LONGITUDINAL FUNCTIONS. In accordance with 1(III. 3) the transient is given by the product of two factors. The first factor depends on A and d. The second depends on B.

The first factor is called the transversal function.

$$\left\{\frac{e^{A}}{\sqrt[3]{d}}\right\} = \text{transversal generating function} \qquad 2(\text{III. }3)$$

The second factor is called the longitudinal function.

$$\left\{Ah_{\nu}(B)\right\} = \text{longitudinal generating function} \qquad 3(\text{III. }3)$$

The transversal generating function (GF) has a simple mathematical expression and needs no special discussion. The longitudinal generating function is given by the newly introduced Ah_{ν} function which we shall study in some detail.

Relation 23(III. 2) shows that it is sufficient to consider only $Ah_1(B)$.

III. 32 B PLANE TRANSIENT TRAJECTORY. The argument B is, in general, a complex quantity. Let us introduce the B plane as indicated by Fig. 1(III. 3).

For a particular transient, the quantities a(t), b(t), c(t), and d(t) are given. See 1 and 3(III. 2).

From 1(III. 3), we have

$$B = B(t) = \frac{1}{\sqrt[3]{d}}\left(b - \frac{c^{2}}{3d}\right) \qquad 4(\text{III. }3)$$

Then, as time passes, point B will describe in the B plane a line which is called the

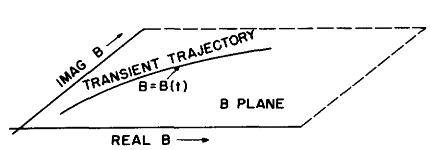


Fig. 1(III. 3)

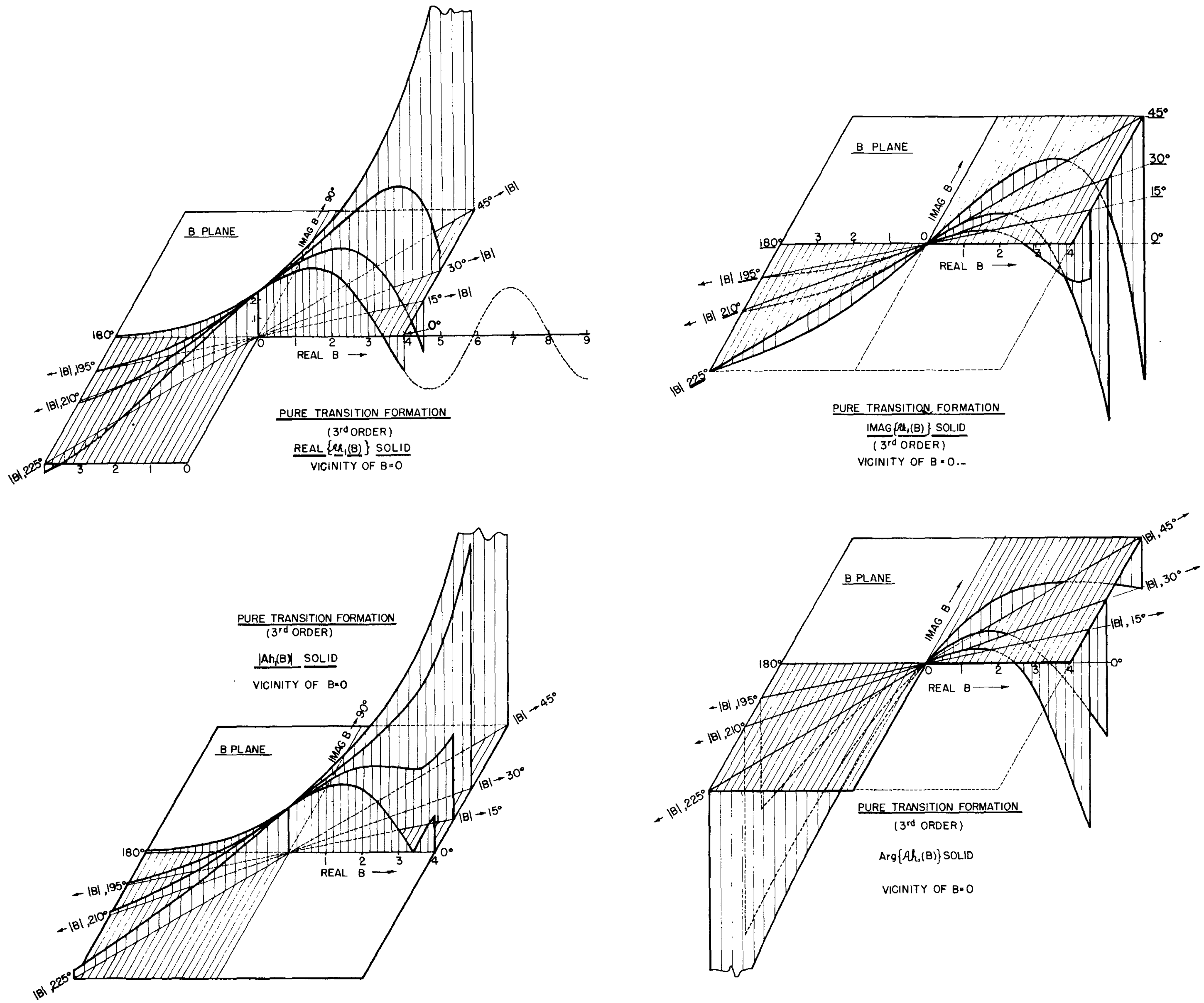

B PLANE
IMAG B → 90°
REAL B →
PURE TRANSITION FORMATION
(3rd ORDER)
REAL {Ah₁(B)} SOLID
VICINITY OF B=0
B PLANE
IMAG B
REAL B →
PURE TRANSITION FORMATION
IMAG{Ah₁(B)} SOLID
(3rd ORDER)
VICINITY OF B=0
PURE TRANSITION FORMATION
(3rd ORDER)
|Ah₁(B)| SOLID
VICINITY OF B=0
B PLANE
IMAG B → 90°
REAL B →
B PLANE
IMAG B
REAL B →
PURE TRANSITION FORMATION
(3rd ORDER)
Arg{Ah₁(B)} SOLID
VICINITY OF B=0

Fig. 2(III. 3)

trajectory of the given transient. In the future discussion we shall refer to a particular transient as its trajectory in the B plane.

III. 33 THE TRANSITIONAL SOLIDS OF THE THIRD ORDER. Let us consider the following functions:

$$\text{Real } Ah_1(B); \qquad \text{Imag } Ah_1(B)$$

$$|Ah_1(B)|; \qquad \text{Arg } (Ah_1(B))$$

The transitional solids are formed as follows. Pick up a value of B. For this selected value of B, compute, for example, Real $Ah_1(B)$. Now measure Real $Ah_1(B)$ along a line perpendicular to the B plane, passing through point B. If we proceed in a similar manner for every point of the B plane, we shall generate a surface which, together with the B plane, will define the Real $Ah_1(B)$ solid.

Solids for Imag $Ah_1(B)$, $|Ah_1(B)|$, Arg $(Ah_1(B))$ may be constructed in a similar way. These solids are indicated as isometric plots for certain strategic regions of the B plane, in Fig. 2(III. 3).

These solids provide a visual means for seeing directly the behavior of the longitudinal part of a particular transient. We first plot the corresponding transient trajectory. The intersection of a given solid with the cylindrical surface perpendicular to the plane and generated by the trajectory gives us visually the corresponding part of the longitudinal factor.

Figure 2(III. 3) shows a typical straight-line cross section of the transitional solid. The cross section along the real B axis appears very frequently in several problems. Along this cross section the function $Ah_1(B)$ is pure real. This cross section coincides with the so-called Airy-Hardy functions. Integral 7(III. 2) is a generalization of the above-mentioned functions.

CHAPTER IV
MIXED TRANSITIONAL TRANSIENTS
MECHANISM OF TRANSIENT FORMATION

SECTION IV. 1 DEFINITIONS.

IV. 10 DEFINITIONS OF PURE TRANSITIONAL TRANSIENTS. Pure transitional transients are given, by definition, by

$$\frac{1}{2\pi i}\int_{\gamma} e^{W(s,t)}\, ds$$

We shall start with the discussion of mixed transients, in which the effect of F(s) is considered. We shall first study simple typical cases which will help us in the attack of the general problem.

IV. 11 THE COINCIDING POLE MIXED TRANSIENTS OF THE THIRD ORDER. Let F(s) be reduced to

$$F(s) = \frac{1}{s - s_c} \qquad 1(IV.1)$$

The denomination "coinciding" simply indicates that the pole is at $s = s_c$, which is the point of confluence.

The relation 23(III. 2) allows us to treat only the case for the contour γ_1 which generates the $Ah_1(B)$ functions.

The corresponding integral for the coinciding pole transient of the third order is given by

$$f(t) = \frac{1}{2\pi i}\int \frac{e^{a+b(s-s_c)+c(s-s_c)^2+d(s-s_c)^3}}{(s - s_c)}\, ds = e^A \frac{1}{2\pi i}\int_{\gamma_1} \frac{e^{Bz + z^3}}{z}\, dz \qquad 2(IV.1)$$

when the transformation 4(III. 2) is introduced.

We shall study in some detail the canonical integral, which reads

$$\phi_p(B) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{e^{Bz + z^3}}{z}\, dz \qquad 3(IV.1)$$

In order to integrate 3(IV. 1) one uses the auxiliary integral

$$\int_0^B e^{Bz}\, dB = \frac{e^{Bz}}{z} - \frac{1}{z} \qquad 4(IV.1)$$

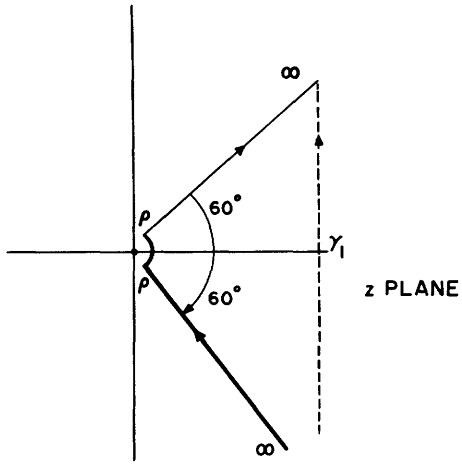


Fig. 1(IV. 1)

Then 3(IV. 1) becomes

$$\phi_p(B) = \frac{-1}{2\pi i}\int_{\gamma_1} \frac{e^{z^3}dz}{z} + \frac{1}{2\pi i}\int_{\gamma_1} e^{z^3}\,dz \int_0^B e^{Bz}\,dB = \frac{-1}{2\pi i}\int \frac{e^{z^3}dz}{z} + \int_0^B Ah_1(B)\,dB \qquad 5(IV.1)$$

since

$$Ah_1(B) = \frac{1}{2\pi i}\int_{\gamma_1} e^{Bz + z^3}\,dz$$

The first integral of 5(IV. 1) has a pole at z = 0. Figure 1(IV. 1) shows the contour which is needed

$$\frac{1}{2\pi i}\int \frac{e^{z^3}}{z}\,dz = \frac{1}{2\pi i}\int_\rho^\infty \frac{e^{\tau^3}\,d\tau}{\tau} + \frac{1}{2\pi i}\int_{-\pi/3}^{+\pi/3} \frac{e^{\rho e^{i\phi}}}{\rho}\,d\phi - \frac{1}{2\pi i}\int_\rho^\infty \frac{e^{\tau^3}\,d\tau}{\tau} = \frac{1}{3} \qquad 6(IV.1)$$

Then, expression 5(IV. 1) becomes, by using 6(IV. 1) and 20(III. 2),

$$\left.\begin{aligned} \phi_{1,p}(B) &= \frac{1}{3} + \sum_0^\infty \frac{\Gamma\left(\frac{\nu+1}{3}\right)}{(\nu+1)!} B^{\nu+1} \sin\frac{\pi}{3}(\nu+1) \\ &= \frac{1}{3}\left\{1 + \frac{\sqrt{3}}{2\pi}\left[\Gamma\left(\frac{1}{3}\right)B + \frac{\Gamma\left(\frac{2}{3}\right)}{2!}B^2 - \frac{\Gamma\left(\frac{4}{3}\right)}{4!}B^4 + \ldots\right]\right\} \end{aligned}\right\} \qquad 7(IV.1)$$

which is the corresponding solution of the coinciding pole transient.

Curve 2 in Fig. 2a(IV. 1) shows the function $\phi_p(B)$ for B real. (In Figs. 2a, 2b, and 2c(IV. 1) the point s_c was selected as the point s_d of confluence of the 2 saddlepoints which correspond to the third-order case.)

The complete pole transient can then be written as

$$f_p(t) = e^A \phi_{1,p}(B) \qquad 8(IV.1)$$

IV. 12 THE COINCIDING ZERO TRANSIENT. Let F(s) be reduced to

$$F(s) = s - s_c \qquad 9(IV.1)$$

The coinciding zero transient is written as

$$f_o(t) = \frac{1}{2\pi i} \int (s-s_c)\, e^{a+b(s-s_c)+c(s-s_c)^2+d(s-s_c)^3} ds = \frac{e^A}{(\sqrt[3]{d})^2} \frac{1}{2\pi i} \int_{\gamma_1} z e^{Bz + z^3} dz$$

10(IV.1)

with canonical form

$$\phi_{1,0}(B) = \frac{1}{2\pi i} \int_{\gamma_1} z e^{Bz + z^3} dz \qquad 11(IV.1)$$

This integral can be computed immediately.

It can be seen that

$$\frac{1}{2\pi i} \int_{\gamma_1} z e^{Bz + z^3} dz = \frac{d}{dB} \frac{1}{2\pi i} \int_{\gamma_1} e^{Bz + z^3} dz = \frac{d}{dB} Ah_1(B)$$

since the function converges uniformly with respect to B.

Hence in accordance with 20(III. 2) one gets

$$\left.\begin{aligned} \phi_{1,0}(B) &= \frac{1}{3\pi} \sum_{0}^{\infty} \frac{\Gamma\left(\frac{\nu+1}{3}\right)}{(\nu-1)!} B^{(\nu-1)} \sin\frac{\pi}{3}(\nu+1) \\ &= \frac{1}{2\pi\sqrt{3}} \left\{ \Gamma\left(\tfrac{2}{3}\right) - \frac{\Gamma\left(\frac{4}{3}\right)}{2!} B^2 - \frac{\Gamma\left(\frac{5}{3}\right)}{3!} B^3 + \dots \right\} \end{aligned}\right\} \qquad 12(IV.1)$$

Curve 3 in Fig. 2a(IV. 1) shows the corresponding function 12(IV. 1) for B real.

The complete zero transient is given by

$$f_o(t) = \frac{e^A}{(\sqrt[3]{d})^2} \phi_{1,0}(B) \qquad 13(IV.1)$$

IV. 13 COMPARISON OF PURE, COINCIDING POLE, AND COINCIDING ZERO TRANSIENT (third order). As a matter of illustration of the waveforms corresponding to the pure, coinciding pole, and coinciding zero transient, we offer Fig. 2a(IV. 1) which is computed for only B real. Curve 1 shows the pure transitional transient.†

IV. 14 SHIFTED POLE TRANSIENTS. (Small shift case). This is the case when

$$F = \frac{1}{s - s_b} \qquad s_b \neq s_c \qquad 14(IV.1)$$

Here we shall study the case of small shift, that is, s_b is close to s_c.

The corresponding integral reads

$$f_{s_b}(t) = \frac{1}{2\pi i}\int_{\gamma_s} \frac{e^{a+\ldots+d(s-s_c)^3}}{(s-s_c)-(s_b-s_c)}\, ds = e^A \frac{1}{2\pi i}\int_{\gamma_1} \frac{e^{Bz+z^3}}{z-z_b}\, dz \qquad 15(IV.1)$$

where $z_b = s_b - s_c$; z_b is small.

The corresponding canonical integral now is given by

$$\phi_{1,p_b} = \frac{1}{2\pi i}\int_{\gamma_1} \frac{e^{Bz+z^3}}{z-z_b}\, dz \qquad 16(IV.1)$$

Now let us introduce the transformation

$$z - z_b = u \qquad 17(IV.1)$$

We obtain

$$Bz + z^3 = u^3 + 3u^2 z_b + 3uz_b^2 + z_b^3 + Bu + Bz_b$$

or

$$\phi_{1,p_b}(B) = \frac{e^{z_b^3+Bz_b}}{2\pi i}\int \frac{e^{u^3+u\left(B+3z_b^2\right)}}{u}\, e^{3u^2 z_b}\, du \qquad 18(IV.1)$$

Let us introduce the new variable parameter

$$X = B + 3z_b^2 \qquad 19(IV.1)$$

(for small values of z_b, $X \approx B$).

Before performing the integration 18(IV. 1) it is convenient, for reasons of further simplification, to substitute this value in 15(IV. 1) so that we shall first consider the integral

$$\phi_{1,p_a}(X) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{e^{u^3+ux}}{u}\, e^{3u^2 z_b}\, du \qquad 20(IV.1)$$

---

†In Fig. 2(IV. 1) the variable v is used instead of z. This change in notation does not change the results.

for small shifts.

$$e^{3u^2 z_b} \approx 1 + 3u^2 z_b \qquad 21(IV.1)$$

so that 20(IV. 1) becomes

$$\phi_{1,p_a}(X) \approx \frac{1}{2\pi i}\int \frac{e^{ux+u^3}}{u}\,du + 3z_b \frac{1}{2\pi i}\int u\,e^{ux+u^2}\,du \qquad 22(IV.1)$$

The first integral is a coinciding pole transient. The second is a coinciding zero transient. Then, the first theorem of composition is given by

"The shifted pole transient is equivalent to the sum of a coinciding pole transient and $3z_b$ times a coinciding zero transient." Since

$$X = B + 3z_b^2$$

there is practically no shift if z_b is small. The function 22(IV. 1) is plotted in Fig. 2b(IV. 1). The curve 0 is the coinciding pole transient. The lower curve is the coinciding zero transient. Figure 2b(IV. 1) was computed for X real, $z_b = \pm 0.1$. Curve 1 is the final shifted pole transient for $z_b = -0.1$, curve 2 is the corresponding curve for $z_b = +0.1$. It can be noted immediately that

"In transitional transients, shown here for the third order, a small shift of the coinciding pole produces a strong effect on the wave envelope overshoot, and leaves the rising part of the wave practically unchanged."

IV. 15 SHIFTED ZERO TRANSIENTS. This is the case when

$$F(s) = s - s_a \qquad s_a \neq s_c \qquad 23(IV.1)$$

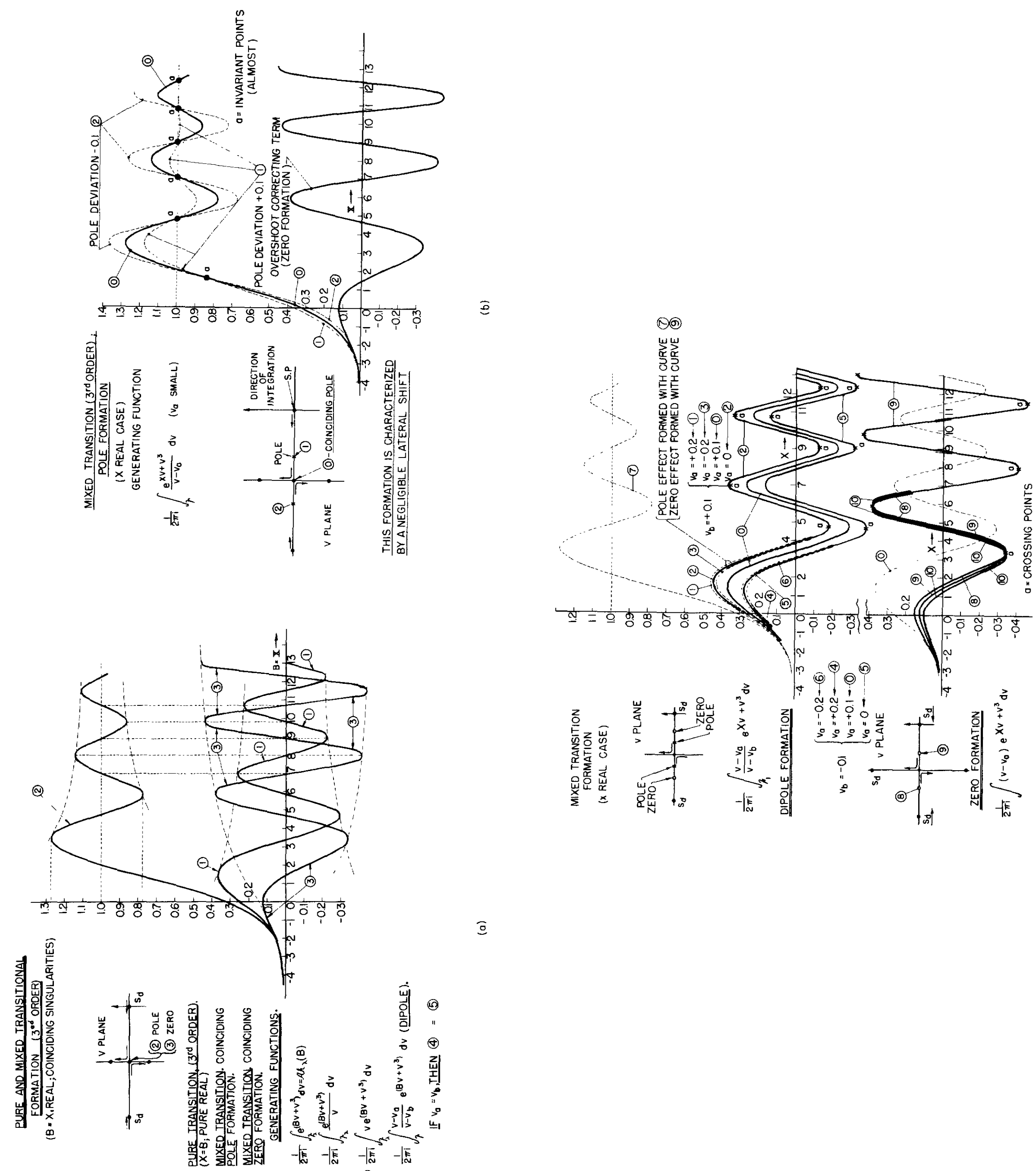
The shifted zero transient is given by

$$f_a(t) = \frac{1}{2\pi i}\int (s - s_a)\, e^{a+(s-s_c)b+\ldots+d(s-s_c)^3}\, ds$$

$$= \frac{1}{2\pi i}\int_\gamma (s - s_c)\, e^{a+(s-s_c)b+\ldots+d(s-s_c)^3}\, ds$$

$$- \frac{1}{2\pi i}(s_c - s_a)\int_\gamma e^{a+(s-s_c)b+\ldots+d(s-s_c)^3}\, ds$$

$$= \frac{e^A}{(\sqrt[3]{d})^2}\left\{\frac{1}{2\pi i}\int_\gamma z\,e^{Bz+z^3}\,dz - \frac{1}{2\pi i} z_a \int_\gamma e^{Bz+z^3}\,dz\right\} \qquad 24(IV.1)$$



PURE AND MIXED TRANSITIONAL FORMATION (3rd ORDER)
(B = X, REAL; COINCIDING SINGULARITIES)
V PLANE
② POLE
③ ZERO
① PURE TRANSITION (3rd ORDER). (X = B, PURE REAL)
② MIXED TRANSITION, COINCIDING POLE FORMATION.
③ MIXED TRANSITION, COINCIDING ZERO FORMATION.
GENERATING FUNCTIONS.
(DIPOLE).
IF Va = Vb THEN ④ = ⑤
(a)
MIXED TRANSITION (3rd ORDER). POLE FORMATION
(X REAL CASE)
GENERATING FUNCTION
(Va SMALL)
DIRECTION OF INTEGRATION
S.P
POLE
V PLANE
⓪ - COINCIDING POLE
THIS FORMATION IS CHARACTERIZED BY A NEGLIGIBLE LATERAL SHIFT
POLE DEVIATION -0.1 ②
POLE DEVIATION +0.1 ①
OVERSHOOT CORRECTING TERM (ZERO FORMATION)
a = INVARIANT POINTS (ALMOST)
(b)
MIXED TRANSITION FORMATION
(X REAL CASE)
V PLANE
POLE
ZERO
DIPOLE FORMATION
Vb = +0.1
Vb = -0.1
ZERO FORMATION
POLE EFFECT FORMED WITH CURVE ⑦
ZERO EFFECT FORMED WITH CURVE ⑨
a = CROSSING POINTS
(c)

Fig. 2(IV.1)

The first integral is a coinciding zero transient, the second is z_a times a pure transitional transient.

Figure 2c(IV. 1) shows the plot of

$$\frac{1}{2\pi i}\int z e^{Bz + z^3}\, dz - \frac{z_a}{2\pi i}\int e^{Bz + z^3}$$

for $z_a = \pm 0.01$ and B real.

In Fig. 2c(IV. 1), curve 9 is the coinciding zero transient. Curve 0 is the pure transitional transient. Curve 8 is the composite transient for a shift of $z_a = -0.01$ and curve 10 shows the effect of $z_a = 0.01$ zero displacement. It can be noted that

"In transitional transients (here proved for the third order) a small displacement of a zero from the coinciding position has a very small effect in the transient wave."

IV. 16 DIPOLE TRANSIENTS. This is the case where

$$F(s) = \frac{s - s_a}{s - s_b}$$

since

$$\left. \begin{aligned} F(s) &= \frac{s - s_a}{s - s_b} \\ F(s) &= \frac{s - s_b + s_b - s_a}{s - s_b} = 1 + \frac{\delta}{s - s_b} \end{aligned} \right\} \qquad 25(\text{IV. }1)$$

so that

$$f_d(t) = \frac{1}{2\pi i}\int e^{a+\dots+d(s-s_c)^3}\, ds + \frac{(s_b - s_a)}{2\pi i}\int_\gamma \frac{e^{a+\dots+d(s-s_c)^3}}{s - s_b}\, ds \qquad 26(\text{IV. }1)$$

The first integral leads to

$$\frac{1}{2\pi i}\int_{\gamma_1} e^{a+\dots+d(s-s_c)^3}\, ds = \frac{e^A}{\sqrt[3]{d}}\, Ah_1(B) \qquad 27(\text{IV. }1)$$

The second, in accordance with 20(IV. 1), leads to

$$\frac{(s_b - s_a)}{2\pi i}\int \frac{e^{a+\dots+d(s-s_c)^3}}{s - s_b}\, ds = \delta\, e^{\left(A+Bz_b+z_b^3\right)} \left\{ \frac{1}{2\pi i}\int \frac{e^{ux+u^3}}{u}\, du \right.$$

$$\left. + 3z_b \frac{1}{2\pi i}\int u\, e^{ux+u^3}\, du \right\} \qquad 28(\text{IV. }1)$$

which is the composition of a coinciding pole and a coinciding zero transient.

The final dipole transient then reads

$$f_a(t) = \frac{e^A}{\sqrt[3]{d}} \left\{ Ah_1(B) + e^{Bz_b + z_b^3} \left[\delta\phi_{1,p}(X) + 3\delta z_b \phi_{1,0}(X) \right] \right\} \qquad 29(IV.1)$$

when

$$X = B + 3z_b^2 \approx B \qquad 30(IV.1)$$

The functions $\phi_{1,p}(X)$ are given by 7(IV.1) and 12(IV.1), respectively.

Figure 2c(IV.1) shows the dipole transient composition that corresponds to the values

$$z_a = \pm 0.2$$

$$z_b = \pm 0.1$$

from which the following cases can be separated.

Curve

(1) } $z_b = +0.1$; $z_a = +0.2$

(3) } $z_b = +0.1$; $z_a = -0.2$

(4) } $z_b = -0.1$; $z_a = +0.2$

(6) } $z_b = -0.1$; $z_a = -0.2$

SECTION IV.2 COMPARISON OF SECOND- AND THIRD-ORDER SOLUTIONS IN THE CASE OF COINCIDING POLES.

It is of interest to compare integral solutions corresponding to the contribution of the second- and third-order saddlepoints in the case in which the saddlepoint orbit runs over a pole. Graphical comparison of the corresponding waveforms reveals at once the character of both types of transient formation.

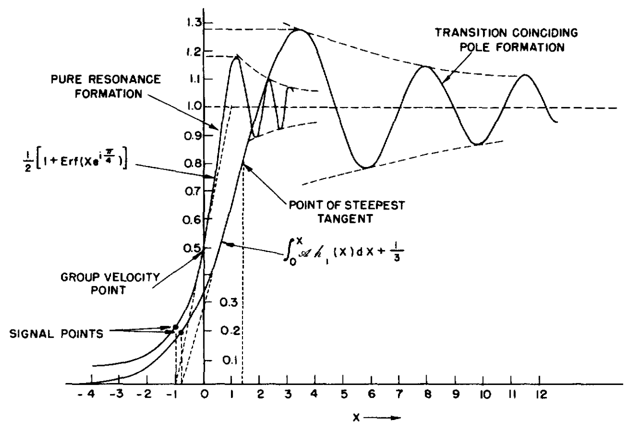


Fig. 1(IV.2)

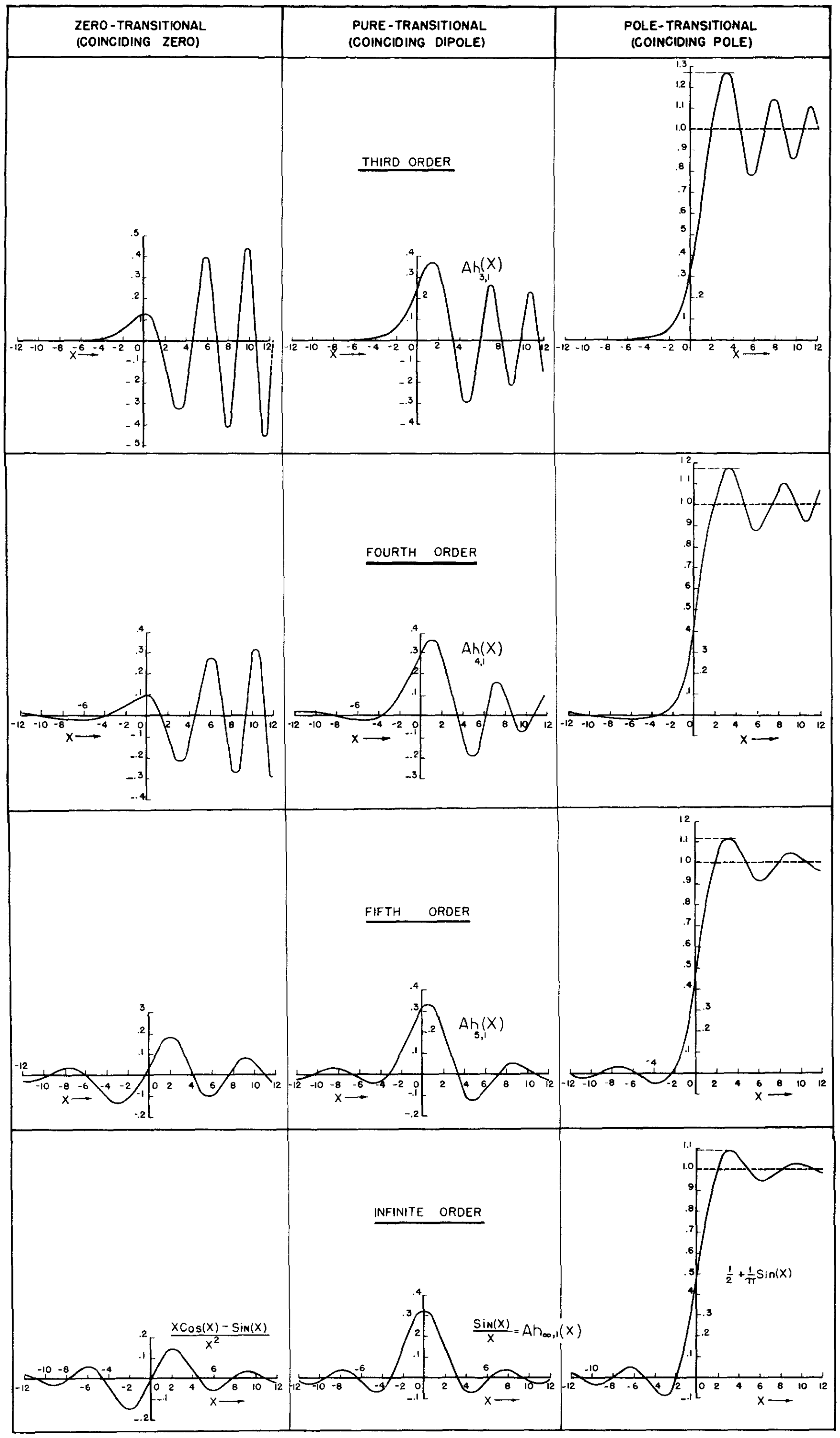

ZERO-TRANSITIONAL (COINCIDING ZERO)
PURE-TRANSITIONAL (COINCIDING DIPOLE)
POLE-TRANSITIONAL (COINCIDING POLE)
THIRD ORDER
Ah(X) 3,1
FOURTH ORDER
Ah(X) 4,1
FIFTH ORDER
Ah(X) 5,1
INFINITE ORDER
XCos(X) - Sin(X) / X²
Sin(X) / X = Ah∞,1(X)
1/2 + 1/π Si(X)

Fig. 2(IV. 2)

Figure 1(IV. 2) illustrates the graphs of the second- and third-order coinciding pole formation. Normalized units have been selected in order to give a proper basis of comparison.

It can be observed that both formations are characterized by the rapid increase of the response around the normalized point $X = 0$. For negative values of X, both cases show a slow monotonic increase reaching the signal point at practically the same value of X ($X = -1$). After this point, the second-order solution rises faster than the third-order solution. Both waves tend to the final state (1)* by oscillation. The second-order solution shows faster oscillation and smaller overshoot value, and converges faster than the wave corresponding to the third-order case.

The graphs of the third-order coinciding pole transients, illustrated in Fig. 1(IV-2), correspond to the functions associated with $Ah_{3,1}$. The cases associated with $Ah_{3,2}$ and $Ah_{3,3}$ are not shown.

SECTION IV. 3 COMPARISON OF ZERO, DIPOLE, AND POLE TRANSIENTS OF HIGHER ORDER.

At this point we should like to remind the reader that the present report, No. 55:2a, is intended only as an introduction to the theory of the saddlepoint method of integration. Since the basic ideas, procedures, and results have been very well illustrated by the previous analysis, we consider that the discussion of higher order transients is out of the scope of this introductory report. Higher order solutions are treated in other reports of this series.

We consider it of interest, however, to present some typical results which correspond to the higher order cases, particularly from the point of view of comparison of the waveforms associated with zero, dipole, and pole transients of different order.

This subsection gives the results of the analysis of transients of higher order. The waves presented here are associated with the function $Ah_{n,1}$, $n = 3, 4, 5, \infty$. The results are given in Fig. 2(IV. 2). Since this set of graphs is self-explanatory, a further description is omitted.

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*The bibliography for the entire Series 55 will be given in the final report of the series.

CHAPTER V
CONCLUDING REMARKS

This report has given, in an elementary way, the basic ideas, procedures, and mechanism of solution of the saddlepoint method of approximate integration.

This elementary and somewhat informal presentation has been advanced to a formal treatment of the problem of approximate integration because of strategic reasons. A unified and formal theory of integration is very difficult to follow if it has not been preceded by a somewhat elementary and heuristic presentation of the subject, particularly if certain mechanisms of construction of the solutions are not familiar to the reader. Several reports of Series 55 will contain a formal and rigid presentation of a unified theory of approximate integration. Some ideas of this theory are hard to accept if they have not been introduced by qualitative information on certain specific methods – the saddlepoint, cliff, and pocket methods, for example.

The spirit of report No. 55:2a is to supply the reader with some elementary knowledge on the subject of the saddlepoint method of integration. Through all of this report the reader may have noticed that all efforts were directly concentrated on showing the mechanism of construction of the solution rather than on giving a formal discussion of the subject. All details that are not relevant to this mechanism of construction were omitted. For example, proof of the asymptotic character of the solution was not given. All of this proof and analytical details were saved for the formal presentation of the theory.