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Wave propagation in a cylindrical waveguide loaded by equally spaced irises with circular holes is discussed in two steps, for the case of infinitely thin irises. (1) An equivalent circuit consisting of equally spaced capacities shunting a continuous transmission line is shown to have the correct behavior of modes, and depends on one disposable constant, the capacity, which can be evaluated if the wavelength of the π-mode is known. (2) An approximate solution of Maxwell's equations is constructed for the π-mode, by matching solutions valid in different parts of the region. This process leads to an infinite series for the solution, in which the first coefficient is determined by numerical methods, the rest by an asymptotic formula. The result is a calculation of phase velocity as a function of dimensions, presented in forms of table and graph, which is quite accurate over the whole range of variables, and also a method of numerically calculating the field. In a final section, a perturbation method is applied to finding the correction for finite iris thickness.

ELECTROMAGNETIC WAVES IN IRIS-LOADED WAVEGUIDES

In a report entitled "The Design of Linear Accelerators,"[1] the writer has considered the general properties of waveguides loaded with periodically spaced irises. In particular, a cylindrical guide, loaded with disks with circular holes, such as is being used in the M. I. T. linear accelerator, was taken up in some detail. It is the purpose of the present paper to go into detailed calculations of the electromagnetic field in such a guide. Somewhat similar discussions have been given by several other writers;[2] but the present methods are somewhat different from any of those presented by others, and seem to the writer to have some advantages. The discussion will be divided into two parts. First we give a simple equivalent circuit which gives quite an accurate picture of the relation between frequency and guide wave length, the pass bands, and so on, in terms of a single disposable constant. Secondly, we give a detailed solution of Maxwell's equations in the guide for the π-mode, that in which the successive irises differ in phase by π, for arbitrary values of all the parameters of the problem. From this solution we can determine the disposable constant of the other solution, and hence can get frequencies of all the modes. All of this discussion is for infinitely thin irises; in a final section we give some treatment of the effect of finite thickness of irises. We shall assume throughout that the reader is familar with Reference 1, particularly Section 1.

1. <u>An Equivalent Circuit for the Loaded Guide</u>. An iris of arbitrary shape in a waveguide can be represented as a shunt reactance,[3] provided we are far enough away from the iris so that we can neglect the attenuated modes in the guide. Thus the loaded guide can be represented as a continuous transmission line, representing the cylindrical cavity, shunted by equally spaced shunt reactances, representing the irises. This description is accurate if the spacing between irises is great enough. For small spacing, though it is not quantitatively accurate, it is still qualitatively correct. Furthermore, the type of iris we are considering, in the frequency range of importance, has a reactance whose frequency dependence is like that of a lumped capacity. Thus in particular our model is a continuous transmission line shunted by capacities. Such a periodic structure shows properties common to all such problems, discussed to some extent in Ref. 1, Sec. 1, and more extensively in L. Brillouin, "Wave Propagation in Periodic Structures", McGraw-Hill Book Co., Inc., 1946. Equivalent circuits similar to that of the present section are taken up by Brillouin in Secs. 59 and 60 of his book. However, since his method of treatment is somewhat involved, we give an elementary derivation here. The value of such an equivalent circuit was pointed out in the Research Laboratory of Electronics by Mr. S. J. Mason and Dr. W. H. Bostick, and the derivation we shall give leads to the same results as a derivation by Mr. Mason.

We start by considering a single section of the line, as shown in Fig. 1. This consists of a length L of line, and a single condenser of capacity C. Let i_n, V_n, i_{n+1}, V_{n+1} be the current and voltage at the respective terminals of this four-terminal network.

Figure 1. Equivalent circuit for capacity-loaded line.

It is also useful to introduce i_n', the current in the line immediately to the right of the condenser. Since the line has distributed inductance and capacity, i_n' and V_n are not the same as i_{n+1}, V_{n+1}. The relations between them may be written, by well-known methods,

$$i_n' = V_n \, jY_o \cot(2\pi L/\lambda_1) - V_{n+1} \, jY_o \csc(2\pi L/\lambda_1)$$

$$i_{n+1} = V_n \, jY_o \csc(2\pi L/\lambda_1) - V_{n+1} \, jY_o \cot(2\pi L/\lambda_1).$$

Here λ_1 is the guide wave length of the wave in the unloaded transmission line. To make connections with a waveguide, this must be given by the relation

$$\frac{1}{\lambda_o^2} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_c^2},$$

where λ_o is the wave length in free space, λ_c is the cutoff wave length, given for the lowest TM-mode of a circular guide by the equation $2\pi R/\lambda_c = 2.405$, where R is the radius of the guide, 2.405 the first value of w for which $J_o(w) = 0$. Y_o is the characteristic admittance of the line, which for a TM-mode may be written as

$$Y_o = \sqrt{\frac{\epsilon_o}{\mu_o}} \, \frac{\lambda_1}{\lambda_o}.$$

We now note that i_n plus the current in the capacity equals i_n'; that is, $i_n = i_n' - V_n jC\omega$. Thus

$$i_n = V_n(jY_o \cot(2\pi L/\lambda_1) - jC\omega) - V_{n+1} \, jY_o \csc(2\pi L/\lambda_1)$$

$$i_{n+1} = V_n \, jY_o \csc(2\pi L/\lambda_1) - V_{n+1} \, jY_o \cot(2\pi L/\lambda_1).$$

We replace n by n-1 in the second equation above, and combine with the first equation to eliminate i_n, obtaining

$$(V_{n-1} + V_{n+1})\ jY_o \csc (2\pi L/\lambda_1) = V_n (2jY_o \cot (2\pi L/\lambda_1) - jC\omega).$$

We may solve this equation by the assumption $V_n = e^{2\pi jnL/\lambda_g}$, where λ_g will be the guide wave length in the loaded line, and is to be determined. Substituting, and rearranging terms, using the relation between λ_o, λ_1, and λ_c, we have

$$\cos (2\pi L/\lambda_g) = \cos \left(2\pi L \sqrt{\frac{1}{\lambda_o^2} - \frac{1}{\lambda_c^2}}\right) - \frac{\pi C}{\epsilon_o \sqrt{\frac{1}{\lambda_o^2} - \frac{1}{\lambda_c^2}}} \sin \left(2\pi L \sqrt{\frac{1}{\lambda_o^2} - \frac{1}{\lambda_c^2}}\right). \quad (1)$$

In Eq. (1) we have a relation between the free-space wave length λ_o and the guide wave length λ_g. It is this relation that was used to plot Figs. (2), (4), and (6) of Ref. 1. It shows all the correct qualitative features of propagation in a loaded line, and in addition, tests made in the laboratory show that it is quite accurate quantitatively in the range of variables concerned with the M. I. T. accelerator. In addition to the cut-off wave length λ_c, Eq. (1) has just one arbitrary constant, C. This may be determined in terms of λ_π, the wave length of the π-mode. We note in the first place that the wavelength of the O-mode (that for which $2\pi L/\lambda_g = 0$) is λ_c, the cutoff wave length. To find the relation for the π-mode, we set $2\pi L/\lambda_g = \pi$, and solve (1) for C. We find

$$\frac{\pi C}{\epsilon_o} = \frac{\cot \pi L \sqrt{\frac{1}{\lambda_\pi^2} - \frac{1}{\lambda_c^2}}}{\sqrt{\frac{1}{\lambda_\pi^2} - \frac{1}{\lambda_c^2}}}. \quad (2)$$

If we know the wave length of the π-mode, either by experiment or by the type of theory to be discussed in the next section, and know λ_c, either from measurement of R or from observation of the O-mode, we can then determine C from (2), and use it in (1) to get a uniquely determined equation.

2. Accurate Solution for the π-Mode of the Loaded Guide. We assume a perfectly conducting circular cylinder of radius R, extending indefinitely along the z axis. At z = 0, L, 2L, - - - we have thin perfectly conducting irises, with holes of radius a in each. We wish to solve Maxwell's equations within this loaded guide. Maxwell's equations in cylindrical coordinates, for a TM-mode in which there is no dependence on Θ, lead to

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rH_\Theta)\right) + \frac{\partial^2 H_\Theta}{\partial z^2} + k^2 H_\Theta = 0, \quad k = \frac{\omega}{c} = \frac{2\pi}{\lambda_o}$$

$$rE_r = j\sqrt{\frac{\mu_o}{\epsilon_o}}\,\frac{1}{k}\,\frac{\partial}{\partial z}(rH_\Theta); \qquad rE_z = -j\sqrt{\frac{\mu_o}{\epsilon_o}}\,\frac{1}{k}\,\frac{\partial}{\partial r}(rH_\Theta). \quad (3)$$

From the second and third of Eqs. (3) we see that E_r and E_z can be determined from rH_θ. In fact, we see that the direction of E is at right angles to the gradient of rH_θ, so that the lines rH_θ = constant in the r-z plane are also the lines of electric force. The boundary condition at a perfectly conducting surface is the vanishing of the tangential component of E, or of the normal derivative of rH_θ, so that the lines of constant rH_θ meet conducting surfaces normally. In a two-dimensional r-z plane, electric lines of force computed with equal intervals between rH_θ will measure the value of rE by their closeness of spacing; for the condition for determining field strength by spacing of lines of force is that the divergence of the field vanish, and the two-dimensional divergence of rE vanishes by (3), which gives $\partial/\partial r(rE_r) + \partial/\partial z(rE_z) = 0$. Thus we see that almost all our essential information about the field can be found from a computation of rH_θ.

Equations (3) have a familiar solution

$$H_\theta = e^{j(\omega t \mp \beta_z)} \frac{J_1(\sqrt{k^2 - \beta^2}\, r)}{\sqrt{k^2 - \beta^2}}$$

$$E_r = \pm e^{j(\omega t \mp \beta_z)} \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{\beta}{k} \frac{J_1(\sqrt{k^2 - \beta^2}\, r)}{\sqrt{k^2 - \beta^2}}$$

$$E_z = -j\, e^{j(\omega t \mp \beta_z)} \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{1}{k} J_o(\sqrt{k^2 - \beta^2}\, r). \qquad (4)$$

Here the upper sign refers to a wave propagated to the right, the lower sign to a wave propagated to the left, along the z axis, and if only one such solution is used, β equals $2\pi/\lambda_g$, where λ_g is the guide wavelength. Solution (4) forms the whole solution of the problem for the unloaded cylindrical guide; there the boundary condition is that E_z must be zero when r = R, leading, for the lowest mode, to $\sqrt{k^2 - \beta^2}\, R = 2.405$, equivalent to the condition we have stated earlier.

In Ref. 1, Section 1, it was shown that for a periodically loaded guide, we can set up the solution by superposing functions of the nature of (4), with an infinite set of β's, equal to $\beta_o + 2\pi n/L$, where β_o is a constant, each function having its appropriate coefficient. For the π-mode, β_o in particular is given by π/L, so that β can take on the values $(2n+1)\pi/L$, or $q\pi/L$, where n is a positive or negative integer, and q is an odd integer. Now the π-mode is of necessity a standing wave, so that there must be equal coefficients for equal positive and negative values of q. This means that the exponentials $e^{\mp jq\pi/L}$ combine to form either sines or cosines. We readily find that the combination we wish for the mode we are considering is of the form

$$H_\theta = e^{j\omega t} \sum_q A_q \sin \frac{q\pi z}{L} \frac{J_1(\sqrt{k^2 - (q\pi/L)^2}\, r)}{\sqrt{k^2 - (q\pi/L)^2}}$$

$$E_r = je^{j\omega t} \sum_q A_q \cos \frac{q\pi z}{L} \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{q\pi}{kL} \frac{J_1(\sqrt{k^2 - (q\pi/L)^2}\, r)}{\sqrt{k^2 - (q\pi/L)^2}}$$

$$E_z = -je^{j\omega t} \sum_q A_q \sin \frac{q\pi z}{L} \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{1}{k} J_o(\sqrt{k^2 - (q\pi/L)^2}\, r), \qquad (5)$$

where the A_q's are arbitrary constants, and q takes on all positive odd integral values. The other solution, in which we interchange sines and cosines in (5), is also a legitimate solution, but not the one we want; in Fig. (4) of Ref. 1, it represents the π-mode of the second, rather than the first, pass band, and for it we have only one term of (5), just as if the irises were absent, for the corresponding solution automatically satisfies the boundary conditions on the surfaces of the irises.

Our problem is to determine the coefficients A_q in (5), so that it will represent a field which not only satisfies Maxwell's equations (which it certainly will do in any case), but also will satisfy the correct boundary conditions. We may now conclude something about the nature of the series. The correct function, as we shall see later, has a singularity at the edge of the iris; that is, for r = a, and z = 0, L, etc. Thus, regarding (5) as series in r, we may expect that these series will diverge, and fail to represent the correct function for r greater than a, just as a power series representing a function of a complex variable diverges outside its circle of convergence. We must therefore use another series representation for r greater than a, analogous to the method of analytic continuation in function theory. This series must likewise be made up out of the functions (4), but now the condition on β is different. We wish to satisfy the boundary condition of E being normal to the surface, at z = 0, z = L, and r = R, but we do not care about the periodic behavior from one section of the guide to the next, for we can use one function for z between 0 and L, the negative of this solution for z between L and 2L (the negative because we are dealing with the π-mode), and so on, without requiring any continuity from one section to the next, since for r > a the different sections are separated from each other by metallic barriers, the irises. To satisfy the boundary conditions at r = R, we use, not the Bessel function J_o and J_1, as in (4), which was chosen to remain finite when r = 0, but instead a combination of the Bessel function and a Neumann or Hankel function, which we may call Z_o and Z_1, so determined that the corresponding Z_o will reduce to zero when r = R. To satisfy the boundary conditions at z = 0 and z = L, we must interchange sines and cosines in (5). Thus we find

$$H_\theta = e^{j\omega t} \sum_m B_m \cos \frac{m\pi z}{L} \frac{Z_1(\sqrt{k^2 - (m\pi/L)^2}\, r)}{\sqrt{k^2 - (m\pi/L)^2}}$$

$$E_r = -je^{j\omega t} \sum_m B_m \sin \frac{m\pi z}{L} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{m\pi}{kL} \frac{Z_1(\sqrt{k^2 - (m\pi/L)^2}\, r)}{\sqrt{k^2 - (m\pi/L)^2}}$$

$$E_z = -j\, e^{j\omega t} \sum_m B_m \cos \frac{m\pi z}{L} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{k}\, Z_0(\sqrt{k^2 - (m\pi/L)^2}\, r), \qquad (6)$$

where Z_0 is a solution of Bessel's equation of order zero, made up of Bessel and Neumann functions, satisfying the condition $Z_0(\sqrt{k^2 - (m\pi/L)^2}\, R) = 0$, and Z_1 is a corresponding solution of Bessel's equation of order unity, related to Z_0 by the equation $dZ_0(w)/dw = -Z_1(w)$. Clearly, so long as m is an integer, (6) satisfies the boundary conditions on the irises as well as on the cylindrical surface. On the other hand, the π-mode has a type of symmetry about the plane $z = L/2$, as a result of which only the terms for which m is an even integer have coefficients B_m different from zero.

We now have two expansions, (5) and (6), for the field. The expansion (5) should hold for $r < a$, and (6) for $r > a$. To determine the coefficients, we must equate the two expansions along the cylindrical surface $r = a$. For $r = a$, the two solutions are expressed in quite different forms as far as z is concerned. We shall, however, express the sines and cosines which appear in (5) in terms of the sines and cosines of (6), and this will allow us to find the necessary relations. By straightforward Fourier analysis, we have

$$\sin \frac{q\pi z}{L} = \frac{2}{q\pi} + \frac{4}{\pi} \sum_m{}' \frac{q}{q^2 - m^2} \cos \frac{m\pi z}{L}$$

$$\cos \frac{q\pi z}{L} = \frac{4}{\pi} \sum_m{}' \frac{m}{m^2 - q^2} \sin \frac{m\pi z}{L},$$

where q is an odd integer, m an even integer, the prime over the summation indicates that it is not to include the term $m = 0$, which is handled separately, and where the expansions hold for z between 0 and L. We substitute these expressions in (5), converting these formulas into summations over m. Thus in both (6) and the revised form of (5) we have the field expressed as a Fourier series in the sines and cosines of $m\pi z/L$. Setting $r = a$, we are then allowed to equate the coefficients of each such term. Proceeding in this way, we find the equations from H_θ: $$\sum_q A_q \frac{2}{q\pi} \frac{J_1(\sqrt{k^2 - (q\pi/L)^2}\ a)}{\sqrt{k^2 - (q\pi/L)^2}} = B_o \frac{Z_1(ka)}{k}$$

$$\sum_q A_q \frac{4}{\pi} \frac{q}{q^2 - m^2} \frac{J_1(\sqrt{k^2 - (q\pi/L)^2}\ a)}{\sqrt{k^2 - (q\pi/L)^2}} = B_m \frac{Z_1(\sqrt{k^2 - (m\pi/L)^2}\ a)}{\sqrt{k^2 - (m\pi/L)^2}}$$

from E_z: $$\sum_q A_q \frac{2}{q\pi} J_o(\sqrt{k^2 - (q\pi/L)^2}\ a) = B_o Z_o(ka)$$

$$\sum_q A_q \frac{4}{\pi} \frac{q}{q^2 - m^2} J_o(\sqrt{k^2 - (q\pi/L)^2}\ a) = B_m Z_o(\sqrt{k^2 - (m\pi/L)^2}\ a), \qquad (7)$$

where $m = 2, 4, 6, \ldots$

The equations resulting from E_r are identical with those coming from H_θ.

We may now eliminate the B_m's from (7), and derive an infinite set of simultaneous equations for the coefficients A_q. These equations may be written

$$\sum_q A_q J_o(\sqrt{k^2 - (q\pi/L)^2}\ a) \frac{q}{q^2 - m^2}$$

$$\times\left\{\frac{Z_1(\sqrt{k^2 - (m\pi/L)^2}\ a)}{\sqrt{k^2 - (m\pi/L)^2}\ Z_o(\sqrt{k^2 - (m\pi/L)^2}\ a)} - \frac{J_1(\sqrt{k^2 - (q\pi/L)^2}\ a)}{\sqrt{k^2 - (q\pi/L)^2}\ J_o(\sqrt{k^2 - (q\pi/L)^2}\ a)}\right\} = 0 \qquad (8)$$

These equations are to be solved simultaneously for the A_q's, the integer m taking on all even values including zero (which can be combined in the general equation (8). Since they form an infinite set of simultaneous homogeneous equations for the unknowns A_q, these equations have no solution unless the determinant of their coefficients vanishes. Setting the determinant equal to zero thus furnishes a secular equation for the problem, which can be satisfied only for certain definite values of k, or for certain definite frequencies. The desired π-mode is the lowest of these frequencies. When we have determined the frequency, we can use (8) to find the coefficients A_q, and can then express the field by means of (5) for r less than a, and by means of (6) for r greater than a, using (7) to determine the B's in terms of the A's. This outline of procedure is in principle correct; but in practice, on account of the infinite range of q, it cannot be carried out, and we must look for approximate ways of solving the problem.

The idea behind our approximation is simple. We shall try to find asymptotic values of the A_q's for large values of q, and asymptotic forms of the equations of the set (8) for large values of m, so that the relations, for large q and m, will be automatically fulfilled. Then we shall set up equations specifically only for small q and small m, leading to a finite number of unknowns, which we can solve numerically. We shall in fact use asymptotic values of all the A_q's for q greater than 1, and asymptotic forms of Eq. (8) for m = 4, 6, . . . , solving only the two simultaneous equations connected with m = 0, 2, for the coefficient A_1. We shall then check our method by showing that, even though we have not specifically solved the equations for m = 4 and greater, nevertheless our assumptions regarding the A's in fact lead to very accurate agreement with these equations.

First we consider the asymptotic behavior of the A_q's. We derive this by considering the expression (5) for E_z, for r = a. According to (5), E_z is proportional to $\sum_q A_q J_o(\sqrt{k^2 - (q\pi/L)^2}\ a) \sin \frac{q\pi z}{L}$ for r = a. That is, the quantities $A_q J_o(\sqrt{k^2 - (q\pi/L)^2}\,a)$, which we may abbreviate by a_q, and which appear in (8), are proportional to the Fourier coefficients of the expansion of E_z along the line r = a, in sine series. Now along this line, E_z will have a singularity as z approaches z = 0 or z = L. We can see this, since as we approach very close to the edge of the iris, the actual field will approach the electrostatic field, and as we approach the edge of a conducting plate held at a potential different from ground, there is a well-known solution, proved by conjugate function methods, which says that E_z goes infinite as $1/\sqrt{z}$, where z is the distance from the edge. (See, for instance, J.H. Jeans, "Electricity and Magnetism," Cambridge University Press, for some discussion of this problem). Thus in our case E_z must go infinite as $1/\sqrt{z}$ when z approaches zero, and as $1/\sqrt{L-z}$ when z approaches L. Furthermore, by the symmetry of the π-mode, E_z must be symmetrical about the point z = L/2. A very simple function which shows this behavior is $1/\sqrt{z(L-z)}$. In a moment we shall find the Fourier coefficients for the expansion of this function in sine series. (This function for E_z, and its Fourier coefficients, are used in the TRE report listed in Ref. 2; our handling of the rest of the problem, however, is different from that used by TRE, and should be more accurate). We cannot assume that the function given above is an accurate representation of the real value of E_z. However, its singularities must be like the real singularities, and these singularities will be produced primarily by the terms of high q value in the Fourier series. Thus we may assume that the A_q's for high q's asymptotically approach the values found from expanding our function. We shall make this assumption, but shall keep A_1 as a disposable constant.

We wish, then, to find the coefficients a_q in the expansion

$$\frac{1}{\sqrt{z(L - z)}} = \sum_q a_q \sin \frac{q\pi z}{L}.$$

To find these coefficients, in the usual way, we multiply by sin qπz/L, integrate from z = 0 to z = L, and find

$$a_q = \frac{2}{L}\int_0^L \frac{\sin q\pi z/L}{\sqrt{z(L-z)}}dz.$$

To evaluate this integral, we introduce the change of variables $w = \frac{2z}{L} - 1$. We then have

$$a_q = \pm \frac{4}{L}\int_0^1 \frac{\cos \frac{q\pi}{2} w \, dw}{\sqrt{1-w^2}},$$

where the + sign is to be used for $q = 1, 5, 9, \ldots$, and the - sign for $q = 3, 7, 11, \ldots$ We may now use the formula, which can be found in Jahnke-Emde, "Tables of Functions,"

$$\int_0^1 \frac{\cos aw \, dw}{\sqrt{1-w^2}} = \frac{\pi}{2} J_o(a).$$

Thus we have

$$a_q = \pm \frac{2\pi}{L} \quad J_o(q\pi/2).$$

We are interested in this only for q equal to or greater than 3 (since we do not propose to use our expansion for $x = 1$). For large values of z, $J_o(z)$ approaches $\sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right)$. Using this approximation, we have finally

$$a_q = \frac{2}{L}\sqrt{\frac{2}{q}} \quad ,$$

a formula accurate for large q's, and good enough to use for $q = 3$, though not for $q = 1$. Since we do not care about the absolute value of the a_q's, on account of the homogeneous nature of Eqs. (8), we shall then assume $a_q = 1/\sqrt{q}$ for $q = 3, 5, \ldots$, and a_1 is to be separately determined. This gives us the set of coefficients we desire.

Next we must consider the asymptotic nature of Eqs. (8). The key to this question is provided if we consider the behavior of the functions Z_1, J_1, etc., for large values of the integers q and m. We note that when q and m are large the quantities $\sqrt{k^2 - (q\pi/L)^2}$ or $\sqrt{k^2 - (m\pi/L)^2}$ becomes imaginary, and if q and m are large enough compared to kL/π, they approach $jq\pi/L$ and $jm\pi/L$ respectively. We are thus led to examine the solutions of Bessel's equation for a large imaginary argument. For a large imaginary argument, a solution of Bessel's equation with the argument jw approaches $e^{\pm w}/\sqrt{w}$. For large positive w's, the positive exponontial is the leading term. We thus may write

$$\lim_{q\to\infty} J_0\left(\sqrt{k^2 - (q\pi/L)^2}\ a\right) = \frac{\text{const. } e^{q\pi a/L}}{\sqrt{q\pi a/L}}$$

$$\lim_{q\to\infty} J_1\left(\sqrt{k^2 - (q\pi/L)^2}\ a\right) = -\frac{d \lim J_0\left(\sqrt{k^2 - (q\pi/L)^2}\ a\right)}{d\left(\sqrt{k^2 - (q\pi/L)^2}\ a\right)}$$

$$= j \lim_{q\to\infty} J_0\left(\sqrt{k^2 - (q\pi/L)^2}\ a\right),$$

where in the last formula we have neglected a term in $1/a$. Thus we have

$$\lim_{q\to\infty} \frac{J_1\left(\sqrt{k^2 - (q\pi/L)^2}\ a\right)}{\sqrt{k^2 - (q\pi/L)^2}\ J_0\left(\sqrt{k^2 - (q\pi/L)^2}\ a\right)} = \frac{L}{q\pi}.$$

Similarly, remembering that Z_0 was defined so as to go to zero when $r = R$, we have

$$\lim_{m\to\infty} Z_0\left(\sqrt{k^2 - (m\pi/L)^2}\ a\right) = \frac{\text{const. } \sinh m\pi(R - a)/L}{\sqrt{m\pi a/L}}$$

$$\sim \frac{\text{const. } e^{m\pi(R - a)/L}}{2\sqrt{m\pi a/L}}$$

$$\lim_{m\to\infty} Z_1\left(\sqrt{k^2 - (m\pi/L)^2}\ a\right) = -\frac{j \text{ const. } \cosh m\pi(R - a)/L}{\sqrt{m\pi a/L}}$$

$$\sim -j \lim_{m\to\infty} Z_0\left(\sqrt{k^2 - (m\pi/L)^2}\ a\right)$$

so that

$$\lim_{m\to\infty} \frac{Z_1\left(\sqrt{k^2 - (m\pi/L)^2}\ a\right)}{\sqrt{k^2 - (m\pi/L)^2}\ Z_0\left(\sqrt{k^2 - (m\pi/L)^2}\ a\right)} = -\frac{L}{m\pi}. \quad (10)$$

To indicate the accuracy of these limiting values, we quote results from a practical case, in which $k = \pi/L$ (the phase velocity equals the velocity of light), and $ka = 1.822$. In this case, the value on the left side of (9), for $q = 3$, proves to be $0.318\ L/\pi$, compared with the approximation 0.333 L/π. The approximation rapidly gets better as q increases. Similarly for the same case, with kR = 3.00, the left side of (10) for m = 4 is $-0.276\ L/\pi$, compared to $-0.250\ L/\pi$ for the right side. We shall use the approximate values given by (9) for $q \geq 3$, and by (10) for $m \geq 4$. For q = 1, and m = 0 and 2, the approximations are too poor to use, and we use the exact values.

We now use the approximations (9) and (10) in Eq. (8), and in addition use our assumption $A_q\ J_o(\sqrt{k^2 - (q\pi/L)^2}\ a) = a_q = 1/\sqrt{q}$, for q = 3,5,. . . In making this substitution, we encounter summations, which we symbolize as follows:

$$\Sigma_1(m) = \sum_{q=3,5,\ldots} \frac{\sqrt{q}}{q^2 - m^2}$$

$$\Sigma_2(m) = \sum_{q=3,5,\ldots} \frac{1}{\sqrt{q}(q^2 - m^2)} .$$

For small values of m, it is not hard to compute these sums numerically; for larger values, we can approximately replace the summation over q by an integration, which can be performed, and thus obtain an asymptotic value for large m. In this way we find the values given in Table 1:

Table 1

m	$\Sigma_1(m)$	$\Sigma_2(m)$
0	.6888	.1043
2	.871	.1586
4	.456	-.0048
6	.346	-.0131
8	.293	-.0120
	- - - - - - -	
$\lim_{m\to\infty}$	$\frac{\pi/4}{\sqrt{m}} + \frac{1}{m^2}$	$-\frac{\pi/4}{m^{3/2}} + \frac{1}{m^2}$.

With these values, and our assumptions, (8) then becomes

$$m = 0:\quad a_1\left\{\frac{Z_1(ka)}{kZ_o(ka)} - \frac{J_1(\sqrt{k^2 - (\pi/L)^2}\ a)}{\sqrt{k^2 - (\pi/L)^2}\ J_o(\sqrt{k^2 - (\pi/L)^2}\ a)}\right\} + .6888\,\frac{Z_1(ka)}{kZ_o(ka)} - .1043\,\frac{L}{\pi} = 0. \tag{11}$$

$$m = 2: \quad -\frac{a_1}{3}\left\{\frac{Z_1(\sqrt{k^2-(2\pi/L)^2}\,a)}{\sqrt{k^2-(2\pi/L)^2}\,Z_0(\sqrt{k^2-(2\pi/L)^2}\,a)} - \frac{J_1(\sqrt{k^2-(\pi/L)^2}\,a)}{\sqrt{k^2-(\pi/L)^2}\,J_0(\sqrt{k^2-(\pi/L)^2}\,a)}\right\}$$

$$+ \frac{.871 \quad Z_1(\sqrt{k^2-(2\pi/L)^2}\,a)}{\sqrt{k^2-(2\pi/L)^2}\,Z_0(\sqrt{k^2-(2\pi/L)^2}\,a)} - .1586\,\frac{L}{\pi} = 0. \qquad (12)$$

$$m \geqslant 4: \quad \frac{a_1}{1-m^2}\left\{-\frac{L}{m\pi} - \frac{J_1(\sqrt{k^2-(\pi/L)^2}\,a)}{\sqrt{k^2-(\pi/L)^2}\,J_0(\sqrt{k^2-(\pi/L)^2}\,a)}\right\} - \sum_1(m)\,\frac{L}{m\pi} - \sum_2(m)\,\frac{L}{\pi} = 0. \qquad (13)$$

We note that (13) is automatically satisfied for large m. For then, using the limiting values from Table 1, the leading terms become those in $m^{-3/2}$ from the terms involving the two summations, which are seen to cancel. We may then with a certain plausibility satisfy (11) and (12) exactly, by proper choice of a_1, and of the frequency, which is proportional to k, and hope that (13) will be satisfied with sufficient accuracy. This procedure proves in fact to be remarkably satisfactory, as we shall show from an example. If we proceed as we have just indicated, we then eliminate a_1 from (11) and (12), to obtain a condition on the frequency. This can conveniently be handled in practice by solving each equation for a_1, and computing this quantity as a function of ka, keeping kR and kL fixed. In Fig. 2 we show the resulting curves for a practical case (kR = 3, kL = π, the case actually

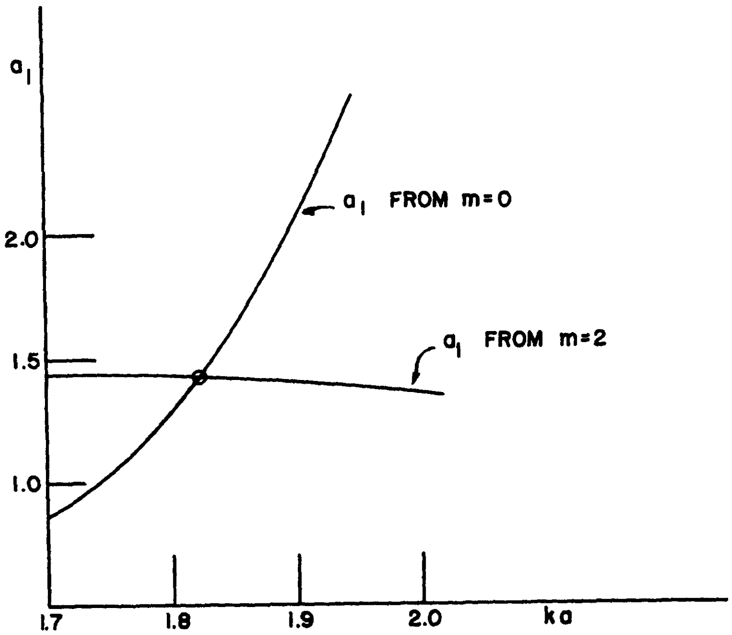

Figure 2. Calculation of a_1.

chosen for the M. I. T. accelerator). The two curves intersect for ka = 1.822, a_1 = 1.41, thus giving the iris diameter in this particular case. To investigate the accuracy of our assumption regarding the a's, we may use these values of the constants, and compute the value of a_1 from Eqs. (13), for m = 4,6, . . . , to see whether these equations are consistent with the two, (11) and (12), which we have satisfied exactly. When we substitute from Table 1, we have the values given in Table 2,

Table 2

m	a_1
4	1.41
6	1.44
8	1.49

These values are in extraordinarily good agreement with the value 1.41 resulting from the first two equations. In other words, our assumed values for the a's are really very accurate, and come very close to satisfying Eq. (8), our fundamental relation.

From the accuracy with which our Eq. (8) is satisfied, we may expect that our procedure will result in very accurate values for the relation between kR, ka, and kL. By simultaneously solving (11) and (12), as in the special case described above, we have calculated kL as a function of ka, for various values of kR, and these values are plotted in Fig. 3, which thus contains the main numerical results of the present theory. In Fig. 3, instead of plotting kL, we plot kL/π, which equals the ratio of the phase velocity of the wave in the guide, to the velocity of light. The abscissa, being ka, measures the size of the hole in the iris; the parameter held constant is kR, proportional to the diameter of the guide. Thus Fig. 3 is a plot, in suitable scale, of phase velocity as a function of the size of the hole in the iris. This is the convenient functional relation of comparison with experiment, or for practical use, and a careful inspection of Fig. 3 will show many relations to the general conclusions of Ref. 1, Sec. 1. In Table 3 we give the numerical values from which Fig. 3 was constructed, and at the same time give the corresponding values of a_1 for each point. The writer wishes to acknowledge the valuable work of Miss Patricia J. Boland, who carried out the calculations of these points.

In solving for the frequencies, we have at the same time found the a_1's, and this, together with our assumption regarding the other a's, gives us values for the coefficients A_q in the expansion (5) of the field for $r < a$. We may then use Eqs. (7) to find the B's, the coefficients in the expansion (6) of the field for $r > a$. Thus in principle we have determined the field, and in particular H_0, from which we have already seen that we can determine most useful properties of the field. In practice, this gives the field only as an infinite series; but it is more convenient than one would at first think to make actual calculations. The series (5) converges rapidly for r considerably less than a, and for r nearly a, near the irises, we know the asymptotic form of the field, from our electrostatic analogy, and can conveniently fit the series onto this asymptotic function which correctly describes the singularity of the field. The relations (7) between the A's and B's can be simplified, using the asymptotic values of the Bessel's functions already discussed, and

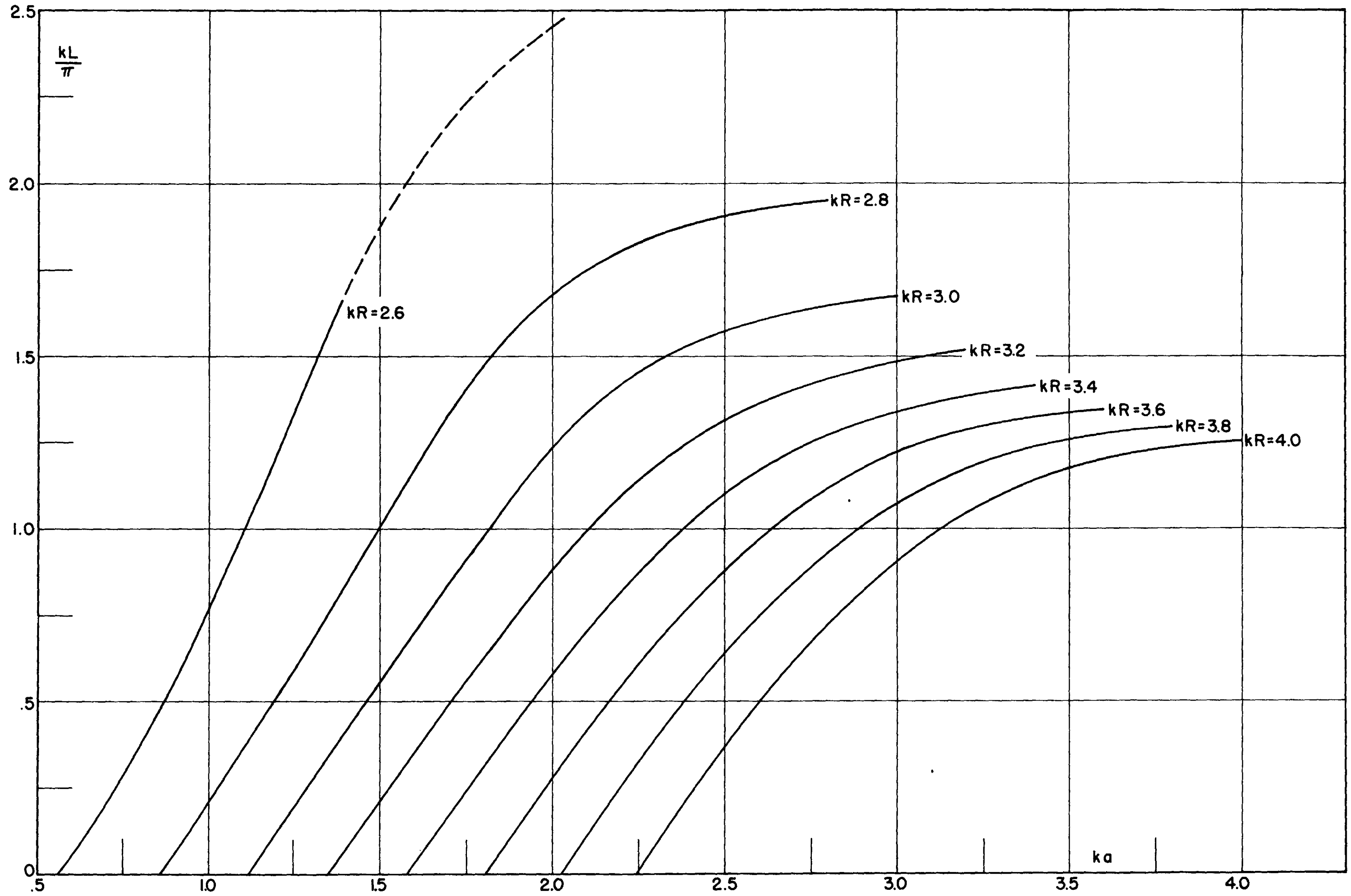

Figure 3. kL/π (equal to ratio of phase velocity of wave in guide to velocity of light) as function of ka ($k = 2\pi$/free-space wavelength, a = radius of hole in iris) for fixed values of kR (R = radius of cylindrical pipe), for π-mode of loaded guide.

Table 3

kR = 2.6			kR = 2.8			kR = 3.0		
ka	kL/π	a_1	ka	kL/π	a_1	ka	kL/π	a_1
0.560	0		0.856	0		1.1125	0	
0.816	.4	1.50	1.126	.4	1.38	1.388	.4	1.305
0.9245	.6	1.63	1.2565	.6	1.45	1.529	.6	1.360
1.0175	.8	1.75	1.380	.8	1.52	1.677	.8	1.385
1.061	.9	1.80	1.440	.9	1.55	1.753	.9	1.405
1.106	1.0	1.84	1.50	1.0	1.58	1.822	1.0	1.410
1.365	1.6	2.18	1.907	1.6	1.71	2.065	1.6	1.595
2.6	2.6318		2.8	1.9528		3.0	1.6729	

kR = 3.2			kR = 3.4			kR = 3.6		
ka	kL/π	a_1	ka	kL/π	a_1	ka	kL/π	a_1
1.3525	0		1.5825	0		1.8075	0	
1.637	.4	1.280	1.8695	.4	1.26	2.089	.4	1.24
1.7825	.6	1.30	2.0205	.6	1.26	2.2475	.6	1.23
1.9385	.8	1.30	2.1885	.8	1.24	2.4230	.8	1.19
2.021	.9	1.29	2.279	.9	1.22	2.523	.9	1.16
2.113	1.0	1.30	2.380	1.0	1.20	2.6385	1.0	1.125
2.691	1.4	1.37	2.6475	1.2	1.16	2.954	1.2	1.015
3.2	1.5159		3.4	1.4147		3.6	1.3439	

kR = 3.8			kR = 4.0		
ka	kL/π	a_1	ka	kL/π	a_1
2.0260	0		2.2425	0	
2.309	.4	1.23	2.525	.4	1.22
2.4685	.6	1.21	2.6895	.6	1.19
2.653	.8	1.15	2.8775	.8	1.125
2.762	.9	1.115	2.990	.9	1.075
2.885	1.0	1.06	3.128	1.0	1.010
3.2685	1.2	.895	3.3035	1.1	.915
3.8	1.2916		4.0	1.2515	

we find that the series (6) for $r > a$ converges rapidly, so long as r is considerably larger than a. It is not hard, then to find satisfactory values of field. For calculating various quantities of interest in Ref. 1, in particular the quantities Q_0 and α, we may use the values of field so found numerically. For Q_0, we must know the stored energy, or integral of H_0^2 over the volume, and the integral of H_0^2 over the surface. This can be found by computing H_0 at discrete points of the cavity, and substituting a summation over these points for an integration. For α, we must know the strength of field along the axis, in its relation to the stored energy. We get the stored energy as above; the field strength along the axis comes at once from the coefficient a_1 of the first term in the Fourier expansion. Since these numerical calculations are straightforward, we shall not describe them further.

Measurements of resonant frequency of iris cavities, with thin but not infinitely thin irises, have been made in this laboratory for various values of the parameters. Measurements have been made for various iris thicknesses, and extrapolated to zero thickness. The evidence indicates that the experimental values so obtained agree within a small fraction of a percent with the values predicted by the calculations of Fig. 3 or Table 3. The measurements have been made by Mr. P. Demos, and the writer is much indebted to him for his careful experimental work.

3. The Effect of Finite Iris Thickness. In the M. I. T. accelerator, the irises, though of finite thickness, are thin in comparison to their spacing L, and we can treat the effect of the finite thickness as a perturbation on the solution we have already found, for zero thickness. From general electromagnetic theory it can be shown[5] that if the wall of a cavity is pushed in by a small amount, the original angular frequency ω will be changed to a perturbed frequency ω', given by

$$\omega'^2 = \omega^2(1 + \int(H_0^2 - E_0^2)\,dv),$$

where the integration is over the small volume removed from the cavity when the wall is pushed in, and where E_0, H_0 are real functions of position independent of time, proportional to the E and H in the mode in question, but normalized so that the integral of E_0^2 or H_0^2 over the cavity is unity. Let us apply this theorem first to the change of frequency of the 0-mode, when an iris with a hole of radius a is increased in thickness from zero (in which case the frequency is the cut-off frequency) to a thickness d. Let us make a cavity as shown in Fig. 4, bounded by two planes midway between irises, with an iris in the middle.

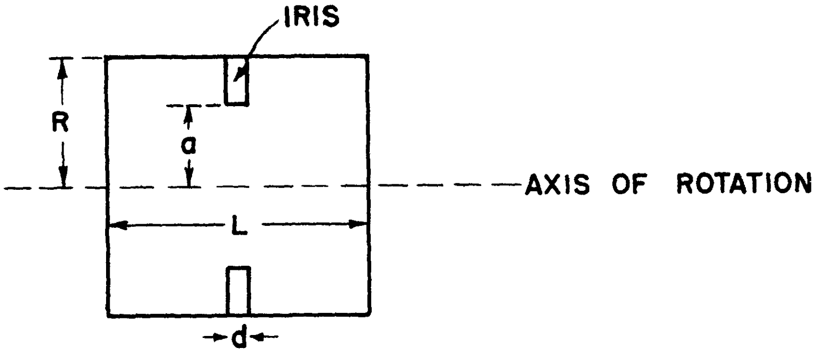

Figure 4. Cavity to illustrate effect of iris thickness on 0-mode.

Such a cavity will be equivalent to a section of the complete tube, for the 0-mode, even for irises of finite thickness. We then find that

$$H_{o\Theta} = \frac{k\, J_1(kr)}{\sqrt{\pi L}\,(2.405)\, J_1(2.405)}$$

$$E_{oz} = \frac{k\, J_o(kr)}{\sqrt{\pi L}\,(2.405)\; J_1(2.405)}$$

Thus the integral $\int (H_o^{\,2} - E_o^{\,2})\, dv$ is

$$\int (H_o^{\,2} - E_o^{\,2})\, dv = \frac{k^2}{\pi (2.405)^2 J_1^{\,2}(2.405)} \; \frac{d}{L} \; 2\pi \int_a^R r(J_1^{\,2}(kr) - J_o^{\,2}(kr))\, dr$$

$$= \frac{2ka\; J_o(ka)\; J_1(ka)}{(2.405)^2 J_1^{\,2}(2.405)} \; \frac{d}{L} ,$$

in which we have used familiar theorems regarding the integrals of squares of Bessel functions. We note that when a = 0 the result reduces to zero, as it must, for we know that a surface with no aperture, perpendicular to z, can be pushed into the cavity without affecting the frequency; the increase of frequency caused by compressing the magnetic field, which is strong for large r values, is compensated by the decrease caused by compressing the electric field, which is strong for small r. Also for a = R, or no iris at all, our formula indicates no change of frequency, on account of the function $J_o(ka)$, as of course it must. In between, however, there is a positive value for the integral, indicating an increase of frequency. For thin irises, this formula has been checked, and is satisfactorily accurate. We have only investigated rather thin irises, but an easy interpolation would give approximate values for any value of d/L, for we have the initial slope of the curve of ω vs. d/L from the results of the present paragraph, and the case of d/L = 1 is the case of cutoff of a guide of radius a. An interpolation between these values proves not to be difficult.

For the effect of iris thickness on the frequency of the π-mode, we may use the same theorem we have just been discussing, but of course we must use the field distribution in the π-mode, as found in Sec. 2. Since we know this field only numerically, the integrations required must be carried out numerically, much as the integrations described in Sec. 2 which are needed for finding Q_o and α. This numerical integration has been carried out in some cases, and the results again prove to be in very satisfactory agreement with experiment.

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