

**A Two-Method Solution
to the Investment Timing Option**

by

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Abstract

Within the realm of derivative asset valuation, two types of methods are available for solving the investment timing option, each with a serious limitation for practical projects. Methods that use Monte Carlo simulation of risk-adjusted probability measures allow consideration of the complicated cash flow models typical of real projects, in the face of pre-specified operating policies, but they do not provide an adequate way to determine what the optimal policy is. Formulation of the problem as an American option in the vein of Black-Scholes and Merton permits calculation of an optimal start policy, but only in situations with drastically simplified cash flow models.

The solution to this dilemma is the development of an approach which applies the two methods in tandem. The rights to explore and develop an oil field are used as an example, and Monte Carlo simulation is used to calculate the value of these rights as a function of start time and contemporaneous oil price. This payoff function is then input to a Black-Scholes-Merton option calculation. The resulting optimal start policy is then reinserted to the Monte Carlo model for further analysis of project and individual cash-flow magnitudes and risks. Also, possible bias because of numerical-analysis errors are checked by direct search of start policies in the vicinity of the calculated optimum.

A TWO-METHOD SOLUTION TO THE INVESTMENT TIMING OPTION

David G. Laughton and Henry D. Jacoby*

I. THE INVESTMENT TIMING PROBLEM

Often an investor has a choice to start a project immediately or to wait and make decisions at some time in the future. An assessment that values the project on a "now or never" basis may be seriously in error in this circumstance, because it does not take account of the opportunity cost of the option, lost when the project is started, to wait and initiate the project at a later time.¹ In this paper we develop a practical method for analyzing the investment timing problem. Our focus is on investments with complex project cash flows, but which have simple contingent control possibilities, and which occur in situations that can be defined by simple underlying information models.

Fortunately many important applications fall within this limited class of problems. One example is the valuation of the rights to explore and develop an oil field. A common form of such rights specifies a fixed length of time after which the rights, if not already exercised, are relinquished. Timing choices arise at several stages in the process of exploration and development under arrangements of this type. We will use an example where the rights-holder has the opportunity to develop an oil reserve that has already been discovered, and for which the delineation drilling has been completed (establishing reservoir size) and costs estimated. At this stage the developer may want to value the rights (for possible sale, for

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example) or to know under what circumstances to initiate development. We examine this problem for an example where, in any state at any time, project cash flows are uncertain only because of uncertainty in the then future oil prices. Moreover, we restrict ourselves to models of oil price uncertainty where, in any state at any time, the contemporaneous oil price is a sufficient state variable for determining the conditional distribution of the then future oil prices. These conditions are the basis of our simple information model. We also restrict ourselves to projects that, once begun, are operated according to a contingent policy that is already specified. The only aspect of project management policy that is not known to the investor at the outset is the rule that determines when to begin development. This is the simple control structure.

We allow unlimited complexity in the way in which cash flows depend on the pattern of oil prices and the commencement date of the project. Our specific example is an oil field in the UK North Sea. The system of petroleum rent collection in the UK introduces considerable complexity to our cash-flow model. In particular it includes a field-level profits tax, called the petroleum revenue tax (PRT), which has important nonlinear, intertemporal features such as tax loss carryforwards and a complex oil allowance deduction.

Several methods are available for analyzing this timing problem, but each by itself has serious limitations in application to investments with complex cash flows. All the methods we consider use decision tree analysis to solve for the optimal investment rule and the associated value of the investment opportunity.

In one form of this analysis the decision tree is solved with dynamic programming methods, over a tree where the chance nodes reflect possible oil

price movements and where the objective function is determined by a DCF evaluation of the cash flows of the project.² Unfortunately, this approach requires the exogenous determination of a discount rate for project net cash flows. For some investments it may be possible to estimate such a rate, for instance by comparison with the market valuation of another asset of equivalent risk. But most traded assets, such as corporate equities, represent bundles of dissimilar projects. If a project has complex cash flows, such as those produced by complicated tax or contract provisions, or if it offers the type of operating flexibility considered here, guesses based on these rough analogies can lead to large errors.

An alternative is to apply derivative asset valuation (DAV) methods to determine the objective function. These methods have the virtue of being economically rigorous and of carrying out the discounting at the simple level of the underlying uncertainties, which in our example are oil prices (Jacoby and Laughton [7]). The discounting of the objective function is then endogenous to the analysis. This is the approach we take.

One DAV method solves the valuation problem by directly determining the time-discounted, risk-adjusted expectation of the net cash flows.³ This expectation is estimated by the sample mean of a sample generated by Monte Carlo simulation of the risk-adjusted measure (Jacoby and Laughton [7]). Such a model can be used to compute the value of an investment on the assumption it is initiated according to some pre-specified (yet possibly contingent) decision rule.

The problem is how to compute the best policy for startup. In our example, a "policy" takes the form of a critical price for each of various start times: at any possible start time the project (if it has not already been started) is begun provided the oil price is above the critical level.

If the valuation for any one policy were very cheap and the number of possible policies not too large, then it might be possible to search over the space of possible policies to determine the optimal one. Unfortunately, when the set of possible policies is large (as it is for the time series of critical prices in our example) and each valuation requires a significant amount of computer time (as the Monte Carlo calculations do in our case), this brute-force direct-search approach is not feasible, at least for practical analysis using the current generation of computers.

Another formulation of the problem sets up the timing choice as an American options valuation, with the measure of oil prices modelled to follow a diffusion process. A long-term oil bond can serve as the relevant underlying asset,⁴ and the value is determined by solving the Black-Scholes-Merton free boundary value problem. Very restrictive assumptions are required for an "analytical" solution (McDonald and Siegel [10]). Even if numerical methods are used to implement a dynamic programming solution of what is effectively a continuous decision tree, there is a fundamental problem in modelling projects with complex, path-dependent cash flows such as those influenced by tax loss carryforwards.

The difficulties arise because, for the project in our example, the value at exercise (the option payoff) cannot be determined analytically, or by numerical dynamic programming methods, as a function of the contemporaneous price (the underlying source of uncertainty). Once the project is begun, its remaining value depends in a highly nonlinear way on the path of past prices. The intertemporal (and thus multidimensional) nonlinearities rule out an analytic solution, and the multidimensional state space defined by the price paths is too large to be examined by numerical dynamic programming. If path dependence is ignored and cash flows are

simplified so that valuation of the project once begun is easy, important aspects of project structure may be lost and the particular results called into question. This should be avoided if possible.

Thus each of the two implementations of the DAV method has a strength and a weakness. A solution by Monte Carlo simulation and numerical integration can handle cash-flow details, but may require prohibitive amounts of computer time to find the optimal operating policy if there are many possible alternatives. An optimal policy can be calculated by application of Black-Scholes-Merton, but only for a highly simplified model of the project.

Our solution to this dilemma is to apply the two methods in tandem.⁵ The Monte Carlo approach is used to compute the value of the asset as a function of start time and the state of the world (in our example, contemporaneous oil price) at that time. This is done by performing the calculation over a grid in this two-dimensional space of start time and price, and interpolating. The resulting payoff function is then used as an input to a Black-Scholes-Merton option calculation. The optimal policy can in turn be re-inserted into the cash-flow model to be analyzed using Monte Carlo simulation and numerical integration. Because of possible numerical errors in the steps leading to the dynamic programming solution, the results may be checked by a limited direct search of policies around the calculated optimum.

The presentation of the method proceeds as follows: Section II lays out the dynamic programming solution for the American timing option and discusses the assumptions made in this application. The calculation uses a payoff function which is created as described in Section III. Section IV describes the procedure for the final direct search. Section V then

presents the application to the North Sea development project. Some possible extensions of these ideas to other types of operational flexibility are discussed in Section VI.

II. THE AMERICAN OPTION

We treat a case where oil price is the only source of uncertainty. Oil price P is assumed to follow a one-dimensional geometric Brownian motion,

$$dP_s = \alpha P_s ds + \sigma P_s dZ_s, \quad (1)$$

where s is a variable that represents the progress of time.⁶ The parameter α is the rate of oil price drift. Equivalently, in terms that will be used below, α is the rate of capital gain to the holding of oil. The term σ is the instantaneous volatility of oil prices, and dZ_s is a unit normal random deviate with expectation 0 and variance ds . The random deviates at different times are uncorrelated.

The risk-adjusted rate of return on oil is denoted μ , and we define

$$\delta = \mu - \alpha, \quad (2)$$

where δ is interpreted as the convenience yield on oil (Brennan & Schwartz [2]). Later, for ease of exposition, we specify that σ and δ be constant over time, but at this point we only need to specify that they be non-stochastic. Also needed at this time is the riskless interest rate, r , also assumed to be non-stochastic.⁷

The Black-Scholes-Merton formulation for the American option is set up in a conventional way (Brennan and Schwartz [2]) with one change. In our application the free boundary must be consistent with the initial and final Monte Carlo calculations. These calculations consider cash flows on an annual basis and apply a beginning-of-year convention for specifying when

cash flows occur. One implication is that the American option must be set up to allow exercise only at the turn of the year. This constraint on exercise is introduced as follows. Ownership of the development rights conveys the right to receive once, at a time s , the payoff $V(P, s)$ ⁸ where s is in the set S ,

$$S = \{ 0, 1, 2, \dots, R \},$$

and where R is the relinquishment date.

The value of the unexercised rights is a function of P and s (Merton [11]). We denote this value as $H(P, s)$ where H satisfies

$$H_s + 1/2\sigma^2 P^2 H_{PP} + (r - \delta)PH_P - rH = 0 \quad (4)$$

subject to the following boundary conditions:

$$H(P, R) = \max\{V(P, R), 0\} \quad (5a)$$

$$H(0, s) = 0 \quad (5b)$$

$$H_{PP}(P, s) \rightarrow 0 \quad \text{as } P \rightarrow \infty \quad (5c)$$

$$H(P, s) = V(P, s) \quad P > P^*(s) \quad s \in S \quad (5d)$$

$$H(P, s) > V(P, s) \quad P < P^*(s) \quad s \in S. \quad (5e)$$

Condition 5a states that at the point of relinquishment the project is begun if its value is positive, otherwise the rights lapse without cost. Condition 5b says that if the oil price drops to zero at time s (which would mean in our information model that it would remain at 0) the option is worthless at that time. Condition 5c is the equivalent in this problem of a circumstance, familiar in the original Black-Scholes stock call option problem, where the option value is approximately linear in stock price when this price is very high (Black and Scholes [1]). The details of Condition 5c are elaborated in Appendix A.

Condition 5d requires that the rights be exercised at any time s if the oil price at that time exceeds the exercise price $P^*(s)$. Condition 5e

determines $P^*(s)$ by requiring that the unexercised rights be more valuable than the exercised rights if the price at that time is less than the critical level.

The model of Equations 4 and 5a to 5e is solved using standard implicit finite difference methods which are outlined in Appendix B. We should note that the cost of this type of computation increases geometrically in the dimension of the state space. This underlines the importance of simplicity in the formulation of the information model.

III. THE PROJECT VALUE FUNCTION

For simple projects, the project value function $V(P, s)$ might be calculated by properly implemented DCF methods. For many real-life investments, however, the assumptions required for DCF analysis do not hold and significant errors can result (Jacoby and Laughton [7]). An example of this problem is the effect of tax loss carryforwards, other tax and contract terms, and operating policies which render the cash flows non-linear in oil price. For these circumstances we have developed a method of derivative asset valuation which estimates V based on the approach outlined in Cox, Ingersoll and Ross [3], with the restriction that the underlying assets be valued with nonstochastic discounting structures under conditions set forth by Fama [4].⁹

In our model, the price process is stated in terms of the expectation of price rather than the price itself. This formulation was chosen for its consistency with the mechanics of the oil bond valuation method, which is a continuous time analog of the method outlined by Fama [4]. We find it has the additional advantage of facilitating estimation of the parameters of the price process when expert judgement is a key input (Laughton [9]).¹⁰

Any expectation of a variable will reflect all the information available up to time s , and thus s may be thought of as "information time." It is distinguished from time index t which we use to describe when various events occur. In this notation, the equivalent process to that stated in Equation 1 is

$$d_s E(P_t) = E_s(P_t) \sigma dZ_s, \quad (6)$$

where for ease of exposition we limit ourselves to term structures of initial expectations of the form

$$E_0(P_t) = P_0 \exp(\alpha t) \quad (7)$$

where α is a constant, and to patterns of volatility σ that are constant in both information time s and across oil price maturity times t .¹¹

We imagine a set of oil bonds, one for each oil price, which mature at times t . Each bond is a claim to the oil price at that time. The bonds are the underlying assets in the derivative asset valuation, and their current value can be stated as

$$V_0(P_t, t) = E_0(P_t) \exp(-\mu t), \quad (8)$$

where $V_0(P_t, t)$ is read as "the value at (information) time zero of a bond which pays an amount P_t in year t ." The expected return on the bond is the sum of the risk-free rate r (taken to be constant for ease in presentation) and a risk premium, $\sigma\phi$,

$$\mu = r + \sigma\phi. \quad (9)$$

The term ϕ is the market price of oil risk, also taken to be constant for convenience in presentation.¹² More generally, at any time s and in any state where the price at time s is P , these bond values are given by

$$V_s(P_t, t \mid P_s = P) = E_s(P_t \mid P_s = P) \exp(-\mu(t - s)), \quad (10)$$

where $E_s(P_t \mid P_s = P)$ is the expectation at time s of the oil price P_t conditional on the price at time s being P .

Following Cox, Ingersoll and Ross [3], these bond values are used as the starting point for the construction by Monte Carlo simulation of a sample which is distributed according to the risk-adjusted measure for oil prices in any state (P, s) .¹³ We denote this measure by $dm(P_{\geq s} | P_s = P)$. It is used in the computation of the value at time t of any asset with cash flows (denoted by the index CF and occurring at time t_{CF}) which are contingent only on the price path of oil,¹⁴

$$V_s(\text{Asset} | P_s = P) = \sum_{CF} V_s(\text{CF} | P_s = P), \quad (11)$$

where

$$V_s(\text{CF} | P_s = P) = \exp(-r(t_{CF} - s)) \int dm_s(P_{\geq s} | P_s = P) X_{CF}(P_{\geq s}), \quad (12)$$

and where X_{CF} specifies the functional dependence of the cash-flow amount on oil price.

This type of calculation is used to determine the payoff function $V(P, s)$, which represents the value at time s of a project started at time s if the state at that time is an oil price with magnitude P . The function is estimated by interpolation using a set of data points spanning the relevant time periods $S = 1, \dots, R$, and an appropriate range at each time of conditioning prices P .

Here again the importance of a simple information model becomes clear. If the state space of the information model were too large, the cost of determining this payoff function would be too great, for it increases geometrically with the dimension of the state space.

IV. THE FINAL DIRECT SEARCH

If the calculation of the project value function and the solution of the resulting Black-Scholes-Merton American option were both exact, our description of the method would be finished. However, the option problem

is solved by numerical methods that have some residual error, and the payoff function is represented by an inexact interpolation among points which are computed by numerical integration (also imprecise), that is based on a Monte Carlo sample determined by pseudo-random numerical procedures. Therefore both the "optimal" policy and its associated value are imprecise.

Nevertheless we have found that the policy determined by this procedure does provide a good starting policy for a search for a closer approximation to the optimal policy. This search is carried out by inserting a series of (possibly contingent) policies into the cash-flow model of the project and computing the value of each using the Monte Carlo method of Section III. Some errors of Monte Carlo sampling may remain, but these can be reduced as needed by increasing the sample size.

V. THE VALUE OF AN UNDEVELOPED OIL RESERVE

A. $V(P, s)$ for a Sample Project

We can illustrate the method in application to an oil development project in the U.K. sector of the North Sea. The U.K. tax system includes a Petroleum Revenue Tax (PRT) and a Corporate Income Tax (CIT), and this regime has complicated deductions and carryforward provisions.¹⁵ Two sample projects are studied, one with a 195 million barrel reserve, which will be produced according to a specified output profile over its life, and another which is otherwise identical but with a 150 million barrel oil reserve and a lower variable cost of production.

The oil price process assumes that the volatility of oil price expectations σ is 0.1 in annual terms. The initial oil price P_0 is \$18 per barrel and the expected rate of real price increase α (which is the expected real capital gain for holding oil) is 0.035 per year. It is further assumed

that the price of oil risk is 0.4. The economic assumptions include a real riskless interest rate r of 0.03 per year, and thus by Equation 9 the risk-adjusted rate of return on oil bonds μ is 0.07.¹⁶ The rate of inflation is constant at 0.05 per year.

It is assumed that the project may be started immediately, at $s = 0$, or in any year up to the relinquishment date, $R = 5$. The function $V(P, s)$ for the project is thus calculated for the exercise dates $s = 0, . . . , 5$ and for several prices spanning the relevant range for exercise. A sample of the results of these calculations for the 195 million barrel project are shown in Figure 1. Each individual value was determined from a 1000 iteration Monte Carlo simulated sample of risk-adjusted oil prices.

For any exercise year s , V is non-linear in the range of \$15 to \$30, with increases in the price at exercise bringing less than a proportional increase in project value because of the various tax shields. At prices above the mid-\$30 range these shields are quickly used up and V more closely approximates a linear function of price. For a given P , V decreases with start date s , because the costs increase with inflation while the revenues have the same value at each start time for a given conditioning price. If inflation were zero, $V(P, s)$ would be the same for all exercise dates s under the assumptions made here.¹⁷

For future reference, we may note that the value of the 195 million barrel project if exercised immediately is $V(P_0, 0) = \$53$ million. It is "in the money."

B. Value of the American Option

The function $V(P, s)$ is calculated at each time by linear interpolation from the results illustrated in Figure 1, and it is then used as the payoff in the option model of Equations 4 and 5a to 5e. The optimal strike

prices $P^*(s)$ for the 195 million barrel project are given in nominal and real terms in the upper part of Table 1. In real terms, the value of the option to wait is less valuable the shorter the waiting time. This effect decreases the real exercise price at times closer to relinquishment.

Also shown is the probability of project start in each year. It will not be started at $s=0$ even though it has positive value if developed immediately. The oil price is known to be \$18 at that time and the project should not be started below \$20.70. The probability that the project will be started at some time before relinquishment is the sum over all the periods, which is 0.83. The reason for waiting can be seen in the value of the unexercised right. If the Monte Carlo DAV calculation is performed using P^* as the optimal start rule, the value of H is \$132 million.

Table 2 presents the components of the value of the American option, and compares this case with three others: immediate exercise, a commitment now to exercise in year 5 no matter what, and a European option in year 5. The 195 million barrel project is shown in the upper part of the table.

The story of this project is told most clearly by tracing the value of the revenue and cost components across the table. In the shift from immediate exercise to mandatory exercise at $s=5$, the costs decrease in value because they are subject to additional real time discounting over the five years, while the revenues decrease because of both time discounting and the loss of convenience yield from not producing and selling the oil. Thus the revenues decrease more in value than does cost, and the result is a decrease in the pre-tax and net-of-tax values.

In the shift from mandatory exercise at $s=5$ to the European option policy with optional exercise at that time, the project is not undertaken in the 21% of states with the lowest contemporaneous oil price, which would

otherwise contribute negative value to the project. The revenues are lower in these low price states, when the project is not undertaken, than in the rest of the states when the project is begun, while the cost is the same in all states. Therefore the value of costs drop more than the value of revenues, raising pre-tax and net-of-tax value for the optimal policy.

Finally, in the shift to the American option, the value of the revenues and costs go up both because the probability of exercise is higher and because in many states exercise takes place earlier, resulting in lower time and risk discounting. The American option has higher net value because it allows the most freedom to the project managers as they seek to maximize value.¹⁸

The analysis can also be used to study the risk characteristics of the project. For this purpose we calculate the real Equivalent Constant Discount Rate (ECDR) for each stream of cash flows. The ECDR is defined as that constant rate that will discount the real expected cash flow series to its calculated value.¹⁹ (The same procedure can be applied to an individual cash flow.) Table 2 shows the ECDR results for the net cash flow and cash-flow components of the 195 million barrel project for the four different exercise rules.

As an example of what these discount rates reveal, consider once again the revenue and cost streams. In the shift from immediate exercise to mandatory exercise at $s=5$, there is no change in the discount rate of the revenues and costs: the costs are still risk-free and the revenues still have the same per-period risk discounting. The combination results, however, in more risk discounting of pre-tax, and net-of-tax cash flows. The introduction of the European option makes the costs risky and also increases the risk of the revenues: the high price states in which cash

flows occur have greater real than risk-adjusted probability. For the reasons given in the analysis of values, the effect is greater on the costs, and overall operating leverage decreases, so that the discounting of the pre-tax and net-of-tax cash flows decreases. The extra options inherent in the American option increase the revenue and cost discounting more, but in such a way as to decrease the overall discounting.

Also presented in Tables 1 and 2 is the smaller, 150 million barrel project. As Table 1 shows, higher oil prices are required to make it worthwhile to start this project, and so the probability of it ever being developed is lower than for a 195 million barrel reserve. Table 2 shows a shortened version of the cash-flow analysis. Note that the project is out of the money at $s=0$ (the value for immediate exercise is \$-57 million). As required, the rights to develop have a positive value for either option.

A number of trade-offs are made between accuracy and the cost of computation and data handling in performing the analysis summarized in Tables 1 and 2. The most important choices are the step grid size in the interpolation of V and the size of the Monte Carlo sample used in computing the V function. As noted above, because of the associated numerical errors P^* may not be the optimum exercise rule, and improvement may be gained by direct search in the vicinity of P^* .²⁰ Because of these numerical errors, the value of H from the two methods may differ. For the 195 million barrel project the H from the American options formulation is \$132.7, which may be compared with \$132.1 from the Monte Carlo solution using P^* and a sample of 10,000 points.²¹

C. Other Applications

Calculations of the type shown here can be used in several ways in the process of contract negotiation and investment decision. For example, they can be used to develop rules of thumb about conditions under which the option value of particular types of projects may be important. In the example above, the development rights have value but should not be exercised now. If the reserves were larger, for a similar development cost, immediate exercise might be indicated (i.e., $P^*(0)$ would be less than the current oil price). On the other hand, a less economic project might be so far "out of the money" that the value of the rights, including the options to wait, would be very small. Moreover, it is easy to investigate now sensitive all these decisions--to exercise or wait, to consider the option value or ignore it--are to potential errors in the calculation of P^* . This is done by exploring the region of exercise prices in the vicinity of the optimal exercise.

In negotiations over lease terms and the analysis of fiscal systems and joint venture contracts, it may be very useful to know how much a longer lease period is worth (which can be calculated by performing the analysis with different assumptions about the relinquishment time R), how various work requirements affect the value of the rights, or how the option to wait affects taxes and vice versa.

Finally, it may be possible to use the two-method approach to analyze the timing choice even when the assumptions applied in the two models are not precisely the same. There are restrictions on the price process that hold for the boundary value problem but need not hold for the Monte Carlo calculations. To use a process for which contemporaneous price is not a sufficient state variable, we may look at an approximate problem for which

the price is sufficient, and then conduct the final direct search with the full process, possibly in a larger state space (e.g., one including the value of a longer-term oil bond as well as the contemporaneous oil price). In this situation the direct search could yield a substantive improvement.

VI. POSSIBLE EXTENSIONS

In this application we apply the two methods to a start-up option where the decision is irreversible. Other applications appear worth investigating, the most obvious being the two-stage exploration-development decision. In this case a developer faces a choice about exploration drilling, which if successful presents a choice about development, all within some set of lease terms. The procedure would use calculations of the type used here (expanded to cover projects of different size that exploration might reveal) to compute a new value function $V^e(P, s)$ which is the value of exercise at the exploration stage. The V^e would then be input to a calculation of H^e , the value of the option to explore.

Another possibility is analysis of the abandonment option. Abandonment often involves complex tax arrangements which are hard to incorporate in a dynamic programming boundary value formulation. In a full-blown treatment of the problem, all the tax balances as well as the price variables and the start times would be state variables. One might use simulation to determine the important dependencies, solve a reduced problem for the optimal policy, and then bring the rest back in the final Monte Carlo direct search.

APPENDIX A

Condition 5c indicates that, at any time in a region of states with a high enough contemporaneous oil price, the second derivative of the option

value function with respect to that price is close to zero. This condition may be derived as follows. In a state with a high oil price there is very little probability, risk-adjusted or real, that there will be any low oil prices in the realized oil price path. In this high oil price regime there is a very small probability that the project will not be started at the earliest opportunity. Therefore the value of the option is very nearly the value of the net cash-flows of the project begun with certainty at this earliest time. Each of these net cash-flow values is itself the time discounted risk-adjusted expectation of the relevant net cash-flow amount, where the expectation is with respect to the conditional distribution of the conditioning state being considered.

Because at very high oil prices the probability is very low that there will be any low oil prices in the realized oil price path, we need only consider the form of the net cash-flow amount function at high oil prices when forming the relevant conditional expectation. At high oil prices the net cash-flow for the type of project we consider is nearly a linear function of its contemporaneous oil price, with no dependence on non-contemporaneous oil prices. In our model of oil price valuation, the risk-adjusted conditional expectation of that price is the product of a non-stochastic risk-discount factor and the true conditional expectation, which is itself proportional to the conditioning price.²² Therefore the value of each net cash-flow is nearly linear in the conditioning price. This means that the value of the payoff and thus of the option is nearly linear in the conditioning price. This nearly-linear function has a second derivative approaching zero, which is Condition 5c.

APPENDIX B

To solve the free boundary problem set up in Equations 4 and 5, the following steps are taken. A discrete uniformly spaced lattice is created in the time interval $[0, R]$. At each time, a lattice is set up on the state space where, in this situation, the state space is the positive real line of prices. We cut off this lattice at some price P_{Max} and approximate the problem by setting Condition 5c to hold exactly at this boundary. On each time slice a finite difference approximation of the differential equation and the boundary conditions is established. The horizon boundary Condition 5a fills in the lattice on the horizon slice. This begins a backward induction, with the approximation of the time derivative in the differential equation connecting two adjacent time slices. The other terms in the approximation of the differential equation and the boundary Conditions 5b and 5c comprise a system of linear equations in the value on each new time slice to be solved.²³

The system of equations is tridiagonal and can be solved by row reduction and back substitution. On slices where exercise is possible, row reduction is performed going from the lower boundary at zero price up to the upper boundary at P_{Max} , and then back-substitution is carried out against a check whether the payoff is greater than the back substitution solution. If the payoff is greater, it is made the solution and the back substitution continues down in price with the payoff in place as the solution. The boundary is set when the back substitution is greater than the payoff. At prices below this level back substitution occurs with no check.

Table 1. Exercise Boundary P^* and Start Probabilities,
American Option Solution

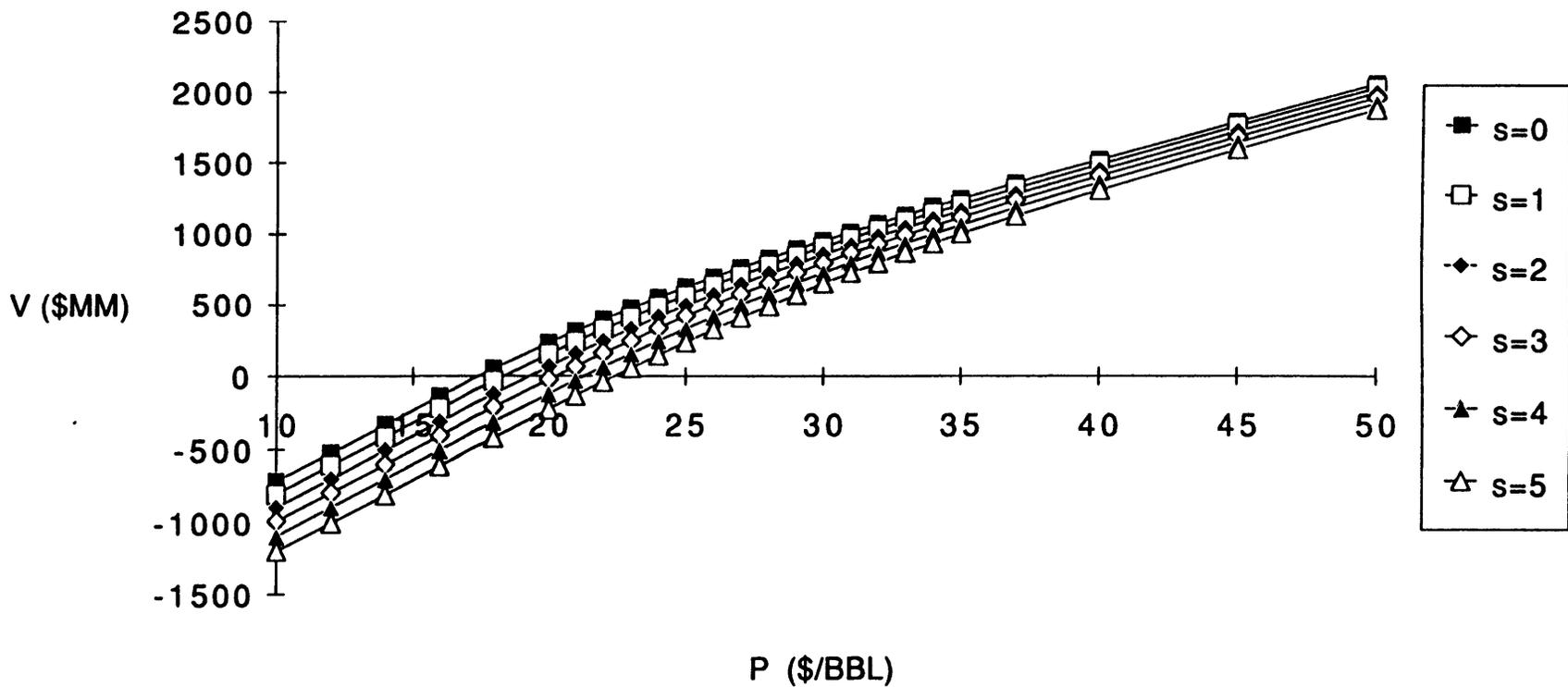
	Time of Exercise, t					
	0	1	2	3	4	5
195 MMBL Project						
P^* Nominal	20.70	21.50	22.35	23.05	23.35	22.40
P^* Real	20.70	20.45	20.22	19.94	19.12	17.45
Start Prob.	0.00	0.16	0.22	0.17	0.13	0.15
150 MMBL Project						
P^* Nominal	22.45	23.40	24.30	25.00	25.25	24.10
P^* Real	22.45	22.26	21.99	21.52	20.67	18.77
Start Prob.	0.00	0.03	0.13	0.17	0.15	0.20

Table 2. Project and Cash-Flow Value and Risk

Project Results	Exercise Permitted			
	Mandatory t=0	Mandatory t=5	European t=5	American 0≤t≤5
<u>195 MMBBL Project</u>				
Value (\$ millions)				
Net-of-tax CF	53	-4	117	132
Pre-tax CF	219	130	261	290
Revenue	2710	2274	1266	1474
Cost	2490	2143	1005	1184
PRT	25	36	34	31
CIT	146	99	110	127
ECDR				
Net-of-tax CF	.111	.162	.117	.112
Pre-tax CF	.126	.164	.135	.131
Revenue	.070	.070	.106	.110
Cost	.030	.030	.084	.093
PRT	.272	.206	.209	.232
CIT	.137	.146	.133	.137
Cum. Start Probability	1.0	1.0	.79	.83
<u>150 MMBBL Project</u>				
Value (\$ millions)				
Net-of-tax CF	-57	-82	67	72
ECDR				
Net-of-tax CF	.112	a	.119	.114
Cum. Start Probability	1.0	1.0	.68	.70

a. No real root exists.

Figure 1. Project Value Function $V(P,s)$



NOTES

1. See Pindyck [12] for an introduction to the investment timing problem. McDonald and Siegel [10] present an important early analysis using some of the methods applied here.
2. For an application of this method to an oil project, and its comparison with an options approach, see Pindyck [12].
3. For the theory behind the use of risk-adjusted probability measures, see Cox, Ingersoll and Ross [3], and Huang [6].
4. This type of information model follows Brennan and Schwartz [2] who analyze a simple mine that can be opened and closed repeatedly before it is abandoned. They use a futures price as device for creating the Black-Scholes hedge. In a model with one source of uncertainty an oil bond provides an equivalent hedge.
5. An earlier implementation of this approach was carried out by Hong [5] who solved the American options problem using a binary algorithm.
6. We reserve the index t , conventionally used in this setting, for the specification of particular events in time rather than the progress of an observer through time. See Section III.
7. Actually, to set up our American options problem δ , σ and r need not be non-stochastic; they need only be known functions of the state variable, the contemporaneous price of oil (Merton [11]). If this were not so we would need more state variables to determine them, which would add unwanted complexity to the information model.
8. The term $V(P, s)$ is an abbreviation for a more complete notation which we use in Section III. There we denote the value of an asset in a state with contemporaneous oil price P at time s as $V_s(\text{Asset} \mid P_s=P)$. Our asset is an oil field beginning development at time s , which we denote (OF, s) , so the abbreviation is $V(P, s) = V_s((\text{OF}, s) \mid P_s=P)$.
9. For an extensive treatment of the method, see Laughton [9]. A shorter description is available in Jacoby and Laughton [7].
10. This framework can be used to generate commodity price processes for which the contemporaneous price is the only state variable, and that result in simple commodity bond valuation, yet show reversion of the price to some central tendency. This property, which is missing from the standard risk-adjusted lognormal process (which is used here to facilitate exposition), is required by most notions of long-run equilibrium in economics.
11. The σ can be allowed to depend on information time s . However, dependence on maturity time t is restricted if contemporaneous oil price is to be the only state variable: σ must be either independent of t or exponentially dependent on the time to maturity at time s , which is $t-s$.

The time scale of this exponential decay is a measure of how long it takes for information to become stale. When a model with exponential decay is expressed as a price process, one gets the reversion mentioned in endnote 10. The time scale of the reversion is given by the time scale of the decay in volatility.

12. In an iterated simple CAPM, ϕ is the ratio of the expected risk premium of the portfolio of diversification and its volatility, all multiplied by the correlation between the uncertainty in that return and the uncertainty in the oil price (Jacoby and Laughton [7]). While it is for ease of exposition that r and ϕ are constant in time, it is fundamental to DCF methods that they, along with σ , be non-stochastic (Fama [4]).

13. Because there is no interest rate risk in our model (i.e., time discounting is non-stochastic) we can use forward oil prices as the starting point for constructing the risk-adjusted measure (Jacoby and Laughton [7]). Thus we do not need a special notation for handling a price which may be used in the model of a cash flow at one time, and also for cash flows at later times, as occurs with tax-loss carryforwards.

Also note that we use more than one risky asset even though our diffusion process is one-dimensional. To construct the hedge, we actually need only a single oil bond with a maturity greater than the project horizon. It is for ease of construction and exposition that we use multiple instruments.

14. This is true only in a model such as ours with of nonstochastic interest rates and prices and quantities of oil price risk.

15. The PRT tax base is the project operating income less a deduction for capital recovery and a complicated "oil allowance". There are restrictions on the PRT deduction, but carryforward is allowed. It is a project-level tax and typically there are substantial losses to carry forward from the investment phase. This gives the PRT the features of an iterated call option on oil, where the exercise price is the tax loss balance. The tax base for the CIT is corporate-level income, and the contribution of the project is operating income less allowances for capital cost and PRT paid. Corporate income is assumed to be very large, so CIT losses attributable to the project are used immediately to shield other income.

16. See Jacoby and Laughton [7] for a discussion of the setting of these parameters in practice.

17. The critical features are the time stationarity of costs in real terms, and the constant discount rate, constant inflation, and the constant expected increase in real oil prices.

18. Explanations of the same type can be given for the pattern of tax values.

19. This summary indicator can only be calculated after the fact. In this example the actual discount rates are stochastic. Note the equivalent constant discount rate may not exist for all projects because there may be no real root to the IRR equation used to calculate it.

20. For this example such a search was conducted by raising and lowering elements of the exercise boundary by amounts from \$0.10 to several dollars. Such calculations convey an impression of the shape of H in the region of exercise rules around $P^*(s)$. In this case no significant improvement was found; the best rule, which was a \$0.10 shift, yielded $H = \$132.16$ million as compared to \$132.10 million yielded by P^* .

21. The exact numerical results also depend on the seed used for the random number generator underlying the Monte Carlo analysis. If the analysis is conducted with a seed of -2 (as compared to -1 for the results in the text) the American options result is \$132.9 million whereas the value by Monte Carlo is \$134.7 million. The random numbers were generated using the algorithm suggested in Press, et al [13].

22. In our single-factor model of oil price expectations, in any period the volatility of the expectations does not depend on which oil price is being considered. Thus the expectation of all oil prices changes by the same proportional amount in any period in any given scenario. The proportional dependence of conditional expectations on the conditioning price is one result of this type of specification.

23. For details of the construction of this implicit finite difference approximation of the boundary problem, see a numerical analysis text such as Lapidus and Pinder [8].

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