

# Quiz 1: Continuum Modeling

## Introduction to Modeling and Simulation

### Solutions

1. The problem is to find suitable expressions satisfying conservation of mass and linear momentum at the fluid structure interface for the specific type of flow.

In both cases, mass conservation leads to imposing that the relative normal component of the velocity vanishes at the interface and this is regardless of the compressibility of the fluid. This guarantees that the fluid fill not interpenetrate the structure and viceversa and also that vacuum bubbles will not appear at the interface.

In both cases, a necessary condition for conservation of linear momentum is that the pressure exerted by the fluid on the structure is equal to the normal component of the stress at the interface.

In the case in which the fluid may be considered inviscid and compressible, the shear stress at the interface (and everywhere) must be zero. This also implies that the tangential component of the fluid velocity is not constrained.

In the case in which the fluid may be considered viscous and incompressible, the flow is governed by the Navier-Stokes equations, which requires the satisfaction of the non-slip boundary condition at the interface. In the case of a moving boundary, such as happens in fluid-structure interaction, the relative tangential component of the velocity between the flow and the solid needs to vanish at the interface. Also, the shear stress at the interface must be the same for the fluid and the solid.

Summarizing this in equations:

- Inviscid compressible

$$[\mathbf{v} \cdot \mathbf{n}] = 0, \quad [p] = 0$$

- Viscous incompressible

$$[\mathbf{v}] = 0, \quad [p] = 0 \text{ and } [\tau] = 0$$

In these expressions,  $[\ ]$  represent relative values or jumps,  $\mathbf{v}$  is the velocity field,  $\mathbf{n}$  is the normal to the solid fluid interface,  $p$  is the pressure (fluid) or normal stress component (solid) and  $\tau$  is the shear stress.

2. For five elements in a slab 0.1 m thick, the nodes  $x_i$  are at  $x = 0.02i$ , e.g.  $x_2 = 0.04$  m.

The linear shapefunction for node  $i$ , called  $N_i(x)$ , is a “triangle” function which is one at  $x_i$  and ramps down to zero at both  $x_{i-1}$  to the left and  $x_{i+1}$  to the right. It can be written:

$$N_i(x) = \begin{cases} 0, & x \leq x_{i-1} \\ 50(x - x_{i-1}), & x_{i-1} \leq x \leq x_i \\ 50(x_{i+1} - x), & x_i \leq x \leq x_{i+1} \\ 0, & x \geq x_{i+1} \end{cases}$$

Its derivative is therefore:

$$\frac{dN_i}{dx} = \begin{cases} 0, & x \leq x_{i-1} \\ 50, & x_{i-1} \leq x \leq x_i \\ -50, & x_i \leq x \leq x_{i+1} \\ 0, & x \geq x_{i+1} \end{cases}$$

Obviously, the product of shapefunction derivatives is zero everywhere for nodes two or more elements apart:

$$\frac{dN_i}{dx} \frac{dN_{i+2}}{dx} = 0.$$

That means that for row 2,  $K_{2,0} = K_{2,4} = K_{2,5} = 0$ , and it remains only to calculate  $K_{2,1}$ ,  $K_{2,2}$  and  $K_{2,3}$ .

$$K_{2,1} = \int_{x_1}^{x_2} k \frac{dN_1}{dx} \frac{dN_2}{dx} dx = \int_{0.02}^{0.04} (10)(-50)(50) dx = -25,000 \times 0.02 = -500,$$

$$K_{2,2} = \int_{x_1}^{x_3} k \frac{dN_2}{dx} \frac{dN_2}{dx} dx = \int_{0.02}^{0.04} (10)(50)(50) dx + \int_{0.04}^{0.06} (10)(50)(50) dx = 500 + 500 = 1000,$$

$$K_{2,3} = \int_{x_2}^{x_3} k \frac{dN_2}{dx} \frac{dN_3}{dx} dx = \int_{0.04}^{0.06} (10)(-50)(50) dx = -500.$$

The second row of the stiffness matrix is thus: (0, -500, 1000, -500, 0, 0).

3. It should be clear that the contact perimeter, in general, behaves (approximately) like the square root of the cross-sectional area. Thus we can write  $P \approx C_g \sqrt{A}$ , where  $C_g$  is a form factor. The two forces acting on the water are then:

$$F_g = \rho g \sin \theta A dx \dots\dots\dots \text{gravitational force per unit length.}$$

$$F_f = C_f C_g \sqrt{A} u dx \dots\dots\dots \text{friction force per unit length.}$$

Then  $F_g = F_f$  yields  $u = \alpha \sqrt{A}$ , where  $\alpha = \rho g \sin \theta / (C_f C_g)$  — hence  $Q = u A = \alpha A^{1.5}$ .