1. The 1-D heat conduction equation is written:

$$\frac{\partial T}{\partial t} = \nu \frac{\partial^2 T}{\partial x^2} + f,$$

where ν is the thermal diffusivity ($=k/\rho c$) and f is temperature change due to heat generation ($=\dot{q}/\rho c$, $\dot{q}$ is the volumetric heat generation rate).

   (a) Write this as a difference equation with implicit (*a.k.a.* backward Euler) timestepping on a Cartesian grid with uniform spacing Δx, and rearrange to put all new temperatures (timestep $n+1$) on the left side and old temperatures (timesteps n) on the right.

   (b) For the infinitely periodic array of temperatures below, and assuming $f = \dot{q} = 0$, demonstrate the stability of the algorithm by computing the next timestep for a mesh Fourier number ($\text{Fo}_M = \nu \Delta t / \Delta x^2$) of 10.

...	8°	10°	8°	10°	...

Initial temperatures at $t = t_0$, an infinite array with repeating temperatures.

Solution

(a) For $T_{i,n} = T|_{x=x_i,t=t_n}$, replacing the PDE with the corresponding difference equation with implicit timestepping gives:

$$\frac{T_{i,n+1} - T_{i,n}}{\Delta t} = \nu \frac{T_{i-1,n+1} - 2T_{i,n+1} + T_{i+1,n+1}}{(\Delta x)^2} + f_{i,n+1}.$$

Then the rearrangement is straightforward:

$$\left(1 + 2\frac{\nu \Delta t}{(\Delta x)^2}\right) T_{i,n+1} - \frac{\nu \Delta t}{(\Delta x)^2}(T_{i-1,n+1} + T_{i+1,n+1}) = T_{i,n} + f_{i,n+1}\Delta t.$$

(b) If we label the temperature in the cells starting at 8° at timestep n as $T_{A,n}$, and those in the 10° cells $T_{B,n}$, then we need to solve two simultaneous equations for timestep $n+1$:

$$(1 + 2\text{Fo}_M)T_{A,n+1} - \text{Fo}_M(T_{B,n+1} + T_{B,n+1}) = T_{A,n},$$

$$(1 + 2\text{Fo}_M)T_{B,n+1} - \text{Fo}_M(T_{A,n+1} + T_{A,n+1}) = T_{B,n}.$$

For Fo_M=10, this becomes:

$$21T_{A,n+1} - 20T_{B,n+1} = T_{A,n}$$

$$21T_{B,n+1} - 20T_{A,n+1} = T_{B,n}$$

Adding 21 times the first to 20 times the second gives:

$$441T_{A,n+1} - 420T_{B,n+1} + 420T_{B,n+1} - 400T_{A,n+1} = 21T_{A,n} + 20T_{B,n}$$

$$T_{A,n+1} = \frac{21 \cdot 8^\circ + 20 \cdot 10^\circ}{41} = 8.976^\circ$$

The opposite gives:

$$T_{B,n+1} = \frac{20 \cdot 8^\circ + 21 \cdot 10^\circ}{41} = 9.024^\circ$$

So even for this very long timestep, the algorithm is stable, and the temperatures draw close to their long-term steady-state value of 9°.

1. Legal flow rate of cars on a one-lane road

   In class, we discussed the problem of continuum traffic flow, which was characterized by car density ρ (cars per unit length) and flux Q (cars passing a point per unit time), where $Q = \rho u$, u being the speed of the cars. The conservation equation was:

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0.$$

   Here you will derive the legal flow rate Q as a function of ρ using the following two traffic laws:

   L1 The legal speed limit is c so $u \leq c$.

   L2 The minimum legal distance between cars $d = \frac{1}{\rho} - L$ (L is the length of a car) is proportional to the speed, $d \geq \tau u$, where τ is a constant measured in legally required feet of separation per (mile per hour) of speed.

   Problem tasks:

   (a) Write expressions for Q as a function of ρ for each of these laws.

   (b) Sketch the allowed values of Q as a function of ρ, showing the overlap between the two regions as the legal window of $Q(\rho)$.

   (c) What is the jamming density of cars, where $Q = 0$?

   (d) Discuss the relative utility of continuum and particle approaches to modeling traffic flow, particularly considering drivers with varying degrees of obnoxiousness and respect for the law on a multi-lane highway in Boston. (Okay, we can leave out this part.)

1. Legal flow rate of cars on a one-lane road: solution

   (a) For L1, we have $Q = \rho u$ and $u \leq c$, so the expression for Q goes:

$$Q \leq \rho c.$$

   For L2, we again have $Q = \rho u$. This time, $d \geq \tau u \Rightarrow u \leq d/\tau$, so substitute the expression for d:

$$u \leq \frac{\frac{1}{\rho} - L}{\tau},$$

$$Q = \rho u \leq \frac{1 - L\rho}{\tau}.$$

   (b) The legal window of $Q(\rho)$ looks like:

   Insert sketch of triangle with vertices at (0,0), $(1/(\tau c + L)$, $c/(\tau c + L))$, $(1/L$,0).

   (c) L1 has $Q = 0$ at $\rho = 0$, and that's obviously not "jamming". L2 has $Q = 0$ when $\rho = 1/L$, so that's the jamming density: one car per distance L, so the distance between cars is zero.

   (d) (Insert essay here)