Quiz 1: Continuum Modeling

Introduction to Modeling and Simulation

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1. **Interface Conditions**

   Describe in words and equations the conditions that need to be applied at a fluid-solid interface in a coupled fluid-structure interaction problem. Consider the cases in which the fluid may be considered inviscid and compressible vs. viscous incompressible.

2. **1-D heat conduction stiffness matrix**

   The steady-state heat conduction equation is given by:

$$\nabla \cdot (k \nabla T) + f = 0$$

   where $f = s/\rho c$ is the source term due to chemical reactions, Joule heating by electric currents, etc. The stiffness matrix K_{ij} for heat conduction in a material of conductivity k is defined as integrals of products of gradients of the shapefunctions $N_i(\vec{x})$:

$$K_{ij} = \int k \nabla N_i \cdot \nabla N_j d\vec{x},$$

   and the unknown temperatures T_j which comprise an approximate solution to the steady-state heat conduction problem are computed by solving the matrix equation:

$$K_{ij} T_j = \int N_i f d\vec{x}.$$

   Heat conduction in 0.1 meter thick slab of material with uniform conductivity $k = 10 \frac{\text{W}}{\text{m}\cdot\text{K}}$ is simulated in one dimension using finite elements. Its 0.1 m thickness is discretized into five linear finite elements of equal size. There are thus six evenly-spaced nodes $x_0, ..., x_5$, where x_0 and x_5 are at the two ends.

   Calculate the value of the non-zero members of row 2 of the stiffness matrix ($i = 2$).

3. Simple Mass-Flow Relation for Channels

In the lectures we argued that:

1. Volume is conserved for the water flowing down a river.

2. The "density" of volume (volume per unit length, in this case) is just the filled cross-sectional area of the river $A = A(x,\, t)$, where x is the length coordinate along the river.

Hence, if $Q = Q(x,\, t)$ is the volume flow along the river, the following equation follows:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0\,. \tag{1}$$

In order to "close" the system, we then argued that for rivers in plains — and when the water level is not changing too rapidly — the flow speed down the river is basically a function of how much water the river carries. That is: $u = u(A)$. Hence $Q = u\, A$ is also a function of A. Once a form $Q = Q(A)$ is plugged into equation (1) above, the system is closed, as we end up with one equation for the single un-known A.

What should we take for the function $Q = Q(A)$? For real rivers this must be measured. However, here we will give a simple argument for what $Q = Q(A)$ should look like — which is not too bad for man-made, very regular, channels. The argument goes as follows:

A. The flow velocity u, when the flow is steady, must result from the balance of the force of gravity down the river slope, with the frictional forces along the river bottom.

B. The force of gravity (downstream) on any section dx of the river is proportional to the total mass in the cross-section $\rho\, A\, dx$ times the component of the acceleration of gravity along the river: $g\, \sin\theta$ — where ρ is the density of water and θ the slope of the river bed.

C. The frictional forces per unit length can be obtained from the (empirical) law $F_f = C_f\, u\, P\, dx$, where P is the contact perimeter of the water, and C_f is a friction coefficient.

The Problem: Complete the argument started above: Assuming that the river bed has some fixed given shape, postulate a (generic) relationship[1] giving the contact perimeter P as a function of A. Then balance the two forces to obtain a relationship giving the flow velocity u in terms of A. From this obtain a formula for $Q = Q(A)$ — what you should obtain is that $Q = \alpha\, A^{1.5}$, where α is some constant.

IMPORTANT: Give a real and complete argument. Do not simply "back-engineer" the given answer using an incomplete or shaky argument.

[1] Think of the special example when the river bed has a triangular cross-section. In this case the relationship is exact. For other cases it will be only roughly true.