

Problems for Modeling and Simulations.

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Abstract

Problems for the Conservation Laws section of the course. **Do problems 1.1 and 2.1.**

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1 Traffic Flow Problems.

In this section we present problems involving traffic flow.

1.1 Shock structure.

As explained in the lectures, when using conservation laws to derive continuum approximation mathematical models, it is necessary to supplement the conservation laws with constitutive equations (also known as equations of state). The reason for this is that, generally, systems of conservation laws involve more unknowns (densities and fluxes) than equations. Thus, extra relationships between the densities and the fluxes are needed.

Typically, in a first level of approximation, the constitutive relations are obtained from thermodynamic-like, quasi-equilibrium arguments.¹ This then lead to situations where the fluxes are functions of the densities only, so that the system of conservation laws obtained has the mathematical form

$$\mathbf{u}_t + \mathbf{F}_x = 0, \tag{1.1}$$

¹Which neglect transport of densities by discrete-level processes — precisely because of the quasi-equilibrium approximation.

if there are no sources or sinks, where \mathbf{u} is the vector of conserved densities and $\mathbf{F} = \mathbf{F}(\mathbf{u})$ is the vector of corresponding fluxes.

Systems of equations such as (1.1) have the problem that, quite often, their solutions develop infinite slopes in a finite time (even if the initial conditions are smooth) — a phenomena known as **wave breaking**. When wave breaking occurs, the system of equations ceases to be valid as a mathematical model for whatever phenomena is of interest — since quasi-equilibrium hypothesis break down when large derivatives in the solutions arise.

How to model what happens beyond wave breaking requires a re-examination of the physics. It is NOT contained within the framework of the mathematical model in 1.1. In this subsection we will concentrate on a fairly common phenomena that determines what happens after wave-breaking: shocks form. However, it is **IMPORTANT to point out that shock-formation is NOT THE ONLY POSSIBILITY after wave-breaking. The nature of the physics in the problem decides what happens, mathematics alone is NOT enough!**

In the **particular case**² where the transport of densities by discrete level processes are dominated by dissipation,³ what generally happens is that the **wave steepening** (derivatives growing) that leads to the **wave breaking** phenomena is **stopped** by the discrete level transport processes when the derivatives reach some (large) value. Even more important is the fact that the regions where the derivatives are large are restricted to very small regions in space. This gives rise to **shock waves** which are (in 3-D space) very thin (moving) sheets across which the densities vary rapidly. In 1-D they are thin intervals $|x - s(t)| \leq \ell$ across which the densities vary rapidly from one value on the left, to another on the right — here ℓ is some length scale which is much smaller than the length scales of interest in the continuum approximation, but still large compared with the discrete level length scales.

We illustrate the point in the prior paragraph with the case of Traffic Flow. In this case, if ρ is the car density, and q is the car flow rate, the corresponding conservation law is

$$\rho_t + q_x = 0, \quad (1.2)$$

where a **quasi-equilibrium** approximation leads to the equation $q = Q(\rho)$ — for some appropriate function Q . However, drivers (generally) drive “preventibly”, and do not look just at the car ahead of them, but further down the road. In other words, they do not just use the local car density, but its derivatives. A simple model that incorporates this idea is the following: take

$$q = Q(\rho) - \nu \rho_x, \quad (1.3)$$

where $\nu > 0$ is some “small” constant. Then the conservation law in (1.2) yields the equation

$$\rho_t + c(\rho)\rho_x = \nu \rho_{xx}, \quad (1.4)$$

²Though quite common.

³A precise definition of what this means in “general” is not easy. In the context of physical processes it means that energy is transferred from the macro to the micro scales, so that the entropy increases.

where $c = dQ/d\rho$. When the derivatives are not too large, the term on the right in this last equation can be neglected, and we thus obtain the model that was studied in the lectures. Namely:

$$\rho_t + c(\rho)\rho_x = 0. \quad (1.5)$$

However, this last equation is **non-linear** (c is not a constant) and, as shown in class, it leads to solutions that develop very steep gradients. When this happens, the term on the right in equation (1.4) can no longer be neglected. This term then stops the steepening, and leads to the formation of “traffic flow shock waves”.

Problem 1.1 *This problem has several parts, as follows:*

P.1 *What are the dimensions of ν in equation (1.3)?*

P.2 *Consider a simple version of equation (1.4), where c is linear in ρ . That is:*

$$\rho_t + \rho\rho_x = \nu\rho_{xx}. \quad (1.6)$$

Then (explicitly) show that this equation has steady⁴ solutions with one value of ρ as $x \rightarrow -\infty$ (say ρ_L) and a different one as $x \rightarrow +\infty$ (say ρ_R).

P.3 *Show that, for fixed values of ρ_L and ρ_R , the solutions in item **P.2** approach a piece wise discontinuous function as $\nu \rightarrow 0$. Namely: $\rho = \rho_L$ for $x < s(t)$, and $\rho = \rho_R$ for $x > s(t)$ — for some $s = s(t)$.*

P.4 *The solutions in **P.2** are the shock waves, with position $s = s(t)$. How does the **shock width**⁵ scale as a function of ν and as a function of the **shock strength** ($\rho_L - \rho_R$)?*

Hint 1.1 *Reduce the pde to an ode for the steady solution profile $\rho = f(x - Ut)$. This ode will be second order, but it is easy to integrate it once, to obtain a separable first order ode which you can integrate explicitly. Write the constant of integration, as well as the propagation velocity U , in terms of the values for the solution as $x \rightarrow \pm\infty$, namely ρ_L and ρ_R . Notice that the derivatives of ρ must vanish in the limit $x \rightarrow \pm\infty$.*

2 Incompressible fluid flow in thin elastic pipes.

Here we will consider the problem of an incompressible, inviscid, fluid flowing down a narrow cylindrical pipe with elastic thin walls. This set up constitutes a (very) crude model for the flow of blood in arteries.

⁴That is, solutions of the form $\rho = f(x - Ut)$, where U is a constant velocity.

⁵The shock width is the thickness of the transition region connecting the two extreme values on each side of the shock.

2.1 Simplest model for incompressible fluid flow in thin elastic pipes.

The objective in this subsection is to derive, using a conservation laws approach, the “simplest” model for fluid flow down an elastic pipe. We make the approximations listed below. Notice that many of these approximations are quite drastic, and of very dubious validity for blood vessels. The resulting model gives some interesting insights, though it should not be taken too seriously.

We consider a flexible pipe, filled with a **fluid under a pressure** p — of course, p is a function of time and position along the fluid. For the tube not to collapse, and for flow to be possible, it must be that $p > p_0 =$ **outside pressure.** Furthermore, let x be the **length coordinate along the tube axis.**

- A.01 Neglect transport effects,**⁶ such as viscosity, heat conductivity, etc. Thus we neglect the dissipation which occurs inside the fluid, at/near the walls of the pipe (fluid-solid interface), and inside the pipe walls (as they move and flex). Heat conduction (by both the fluid and the pipe walls) is neglected as well.
- A.02** Ignore gravity and any other (possible) body forces. Therefore we assume that **the only force acting on the fluid is the pressure.** Furthermore, assume that **the pressure p_0 outside the pipe is constant** — notice that this is the **only external force acting on the system.**
- A.03** The fluid is incompressible, with **constant density** ρ .
- A.04** The **pipe walls are homogeneous:** they have the same properties everywhere.
- A.05** The **pipe walls are thin and flexible, with no bending strength.** Thus we ignore their thickness, and **treat them as a mathematical surface** bounding the liquid. To be precise, “thin” here has the following meaning: *Let L be the scale over which the motion occurs (see **A.08** below), and let h be the tube walls’ thickness. Then $L \gg h$, so that the amount of bending in the longitudinal direction is small enough to have negligible effects on the force balances. This also has the consequence of keeping the pipe cross-section circular (see **A.09** below), so that there is no bending in the transversal direction either.*
- A.06** The **repose perimeter of the pipe is constant** — i.e.: it does not depend on the coordinate x along the tube. It then follows from **A.04** that, under steady state conditions — so that the fluid pressure is the same everywhere — the pipe has cylindrical shape. Let $2\pi r_0$ be the repose (no forces) perimeter of the tube, corresponding to a **repose radius r_0 and repose cross-sectional area $S_0 = \pi r_0^2$.**
- A.07** There is **no longitudinal tension** applied along the pipe.

⁶Generic name given to effects that are due to transport of various quantities by molecular level processes.

- A.08 Long wave assumption.** The situation we have in mind is that of fluid flowing down the pipe, with some kind of disturbances (waves) propagating down the length of the tube. Let L be the scale (wave-length) over which these disturbances occur. Then we assume that L is much bigger than the pipe diameter — or, for that matter, the tube's walls thickness. *This has the important consequences listed below in A.09, A.10, and A.11.*
- A.09 Cylindrical straight geometry.** The long wave assumption implies that, at any given point along the tube, the pressure is (essentially) constant on the cross-section. Hence the **cross-section of the tube is circular everywhere**. In addition, because of the absence of any external forces other than the outside constant pressure (and the homogeneity of the tube walls) the **pipe must remain straight**.⁷ Hence the pipe geometry is completely specified by giving its **radius** $r = r(x, t)$ — as a function of time t and the length x along the pipe axis. Alternatively, one can specify the **fluid filled cross sectional area** $S = S(x, t) = \pi r^2$.
- A.10 One dimensional approximation.** The long wave assumption implies that we can neglect any fluid motions in directions transversal to the fluid axis. Thus the **fluid dynamics can be described by the two scalar functions: $p = p(x, t)$ (pressure) and $u = u(x, t)$ (axial fluid velocity)**.
- A.11 Radial wall forces only.** The long wave assumption implies that the strain on the tube's walls is mainly along the perimeter, with very little stretching (or bending) in the longitudinal direction. Thus we **assume that the only force by the walls is that caused by a tension $T = T(x, t)$ per unit length along the perimeter, trying to pull it back to its repose length $2\pi r_0$** . Of course, the tension results from the stretching of the walls beyond their equilibrium radius by the liquid pressure excess over p_0 .
- A.12 Elastic regime.** The amount of wall stretching produced by the motion is small enough (and happens slowly enough) that the walls respond elastically. That is to say: **the tube walls oppose stretching with a force that depends only on the amount of stretching**. *It is important to notice that this assumption involves not just space scales, but time scales as well. If the deformations are too large, the walls will not respond elastically — permanent deformations will occur, etc. Furthermore, when the deformations occur too fast (even if they are small) dissipation can become important. Of course: what "too large" or "too fast" means depends on the physical properties of the tube walls.*

Thus, we can write, for the tension introduced in **A.11**, a formula of the form:

$$T = f\left(\frac{\Delta r}{r_0}\right), \quad (2.1)$$

where $\Delta r = r - r_0$ measures the amount of stretching (the tube perimeter changes by $2\pi\Delta r$), f is a function characterizing the elastic response of the walls, and Tdx is the **tension force along the perimeter of a transversal "slice" of the tube of length dx** .

⁷Actually, motion where the axis of the pipe wanders is possible, but we will ignore it here.

Remark 2.1 Notice that, at least for now, we are not making the assumption of small deviations from a steady state⁸. In particular, this would imply that r is nearly constant: $r \approx r_s$ — where $r_s - r_0$ is the stretching needed to balance the equilibrium steady state pressure. In this case Hooke's law applies, and equation (2.1) can be linearized to

$$\delta T = Eh \frac{\delta r}{r_s}, \quad (2.2)$$

where $\delta r = r - r_s$, E is the Young's modulus for the wall material, $\delta T = T - T_s$ is the deviation of the tension from its equilibrium value T_s , and h is the wall thickness — which can be assumed constant in this approximation.

Of course, if we do not make the small deviation from a steady state assumption, we cannot assume that the wall thickness h is a constant. The variations in h do, of course, affect the forces produced by the walls. We can, however, assume that the wall thickness is a function of the stretching, and thus we can incorporate the effect of these variations into the force law in equation (2.1) — without being forced to track an extra variable $h = h(x, t)$.

The small deviation from a steady state leads to a great simplification of the equations (linearization).

A.13 Neglect wall inertia. Thus, when considering the balance of forces in the radial direction, the inertial forces (mass times acceleration) due to the walls are neglected.

Problem 2.1 Use the conservation of mass and momentum to derive a system of two coupled pde's for the functions $S = S(x, t)$ and $u = u(x, t)$. You should be able to write the pressure p as a function of the cross sectional area S .

The large number of assumptions listed earlier can seem a bit frightening. Their net effect, however, is to simplify the problem massively. The **hints below should lead you to the answer without excessive trouble.**

H.1 Because the density is constant, the liquid volume is conserved. The volume density (volume per unit length) is S . What is the volume flux? Once you have the volume flux, the conservation of volume will give you one of the (two) equations that you need.

H.2 Linear momentum in the axial direction should be conserved. The momentum density (momentum per unit length) is $\rho u S$. To calculate the momentum flux, notice that (in the absence of transport effects — see **A.01**) momentum can be transferred by either advection (fluid moves and carries momentum with it) or macroscopic mechanical forces. Because of the earlier assumptions, the only forces acting on any parcel of liquid (say, the liquid in some

⁸Flow at constant speed and constant pressure.

section of the pipe $a < x < b$) are: (1) the pressure force from the liquid in $x < a$ across the cross-section at $x = a$, (2) the pressure force from the liquid in $x > b$ across the cross-section at $x = b$, and (3) the longitudinal component of the force by the walls on the liquid along the section $a < x < b$. Thus

- Write an expression for the momentum flux caused by advection: \mathcal{F}^A .
- Calculate the longitudinal force per unit length by the walls on the liquid \mathcal{F}^W . Notice that the force per unit area by the walls on the liquid is normal to the walls (no friction — see **A.01**) and equal to the pressure.
- Consider the momentum balance on an arbitrary section $a < x < b$ of the pipe:

$$\frac{d}{dt} \int_a^b (\rho u S) dx = (\mathcal{F}^A + pS) \Big|_b^a + \int_a^b \mathcal{F}^W dx \quad (2.3)$$

- As in the lectures, use the fact that (2.3) is valid for all $a < b$ to write a pde. This is the second equation you need — though, you are not done yet!

H.3 At this point you will find that you have two partial differential equations, but that they involve three unknowns: S , u , and p . You now **need an equation of state**. Below follows a strategy to write the pressure p as a function of the cross-sectional area.

Because of **A.13** the forces on the wall by the liquid (caused by the pressure difference with the outside) in the radial direction must be balanced by the radial wall forces — see **A.11** and **A.12**. Because the tension in the wall is a function of the wall perimeter, and you can write the perimeter in terms of the cross-sectional area S , this balance will provide the desired relation.

H.4 Step **H.3** requires that you calculate the radial forces by the wall as a function of the tension along their perimeter. Below a hint for to how to do this.

Consider a circular string of radius r that is under tension⁹ T_s . In order to keep the string in place, a (constant) radial force F_r per unit length must be applied along the string. The dependence of F_r on T_s can be found as follows: Consider the work needed to increase the radius of the circle by dr . This work can be calculated in two ways: (i) The work done by the radial force as the radius changes. (ii) The work done by the tension as the string stretches. Both answers must be the same, hence equating (i) and (ii) yields a relationship between F_r and T_s .

THE END

⁹In the calculation that you need to do in step **H.3**, the string is replaced by a slice of length dx of the tube wall, and T_s is replaced by Tdx .