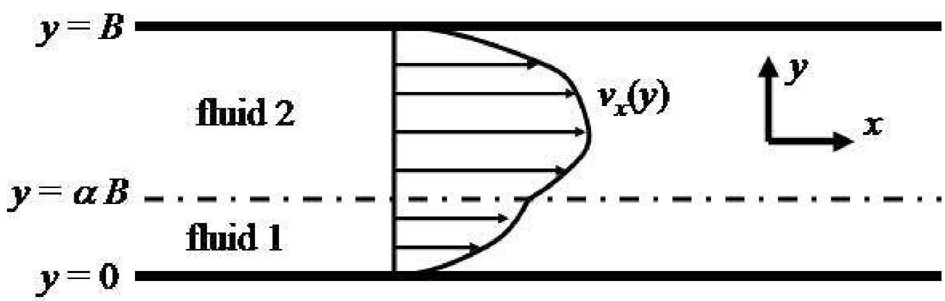
Introduction to Modeling and Simulation

Fluid Dynamics HW problems Spring 2006

Problem 1. Analytical solution of a 1-D laminar flow problem

Consider the laminar flow of two immiscible fluids between two stationary, horizontal parallel plates, separated by a distance $B = 0.01\ m$. The first fluid, of higher density $\rho_1 = 1{,}050\ Kg/m^3$ and viscosity $\mu_1 = 0.0012\ Pa*s$, is found on the bottom of the channel in the region $0 \le y \le \alpha B$, $0 < \alpha < 1$, and the second fluid, of lower density $\rho_2 = 980\ Kg/m^3$ and viscosity $\mu_2 = 0.0009\ Pa*s$, is found on the top of the channel in the region $\alpha B < y \le B$. The flow is driven from left to right by a constant dynamic pressure gradient $\Delta P/\Delta x < 0$.



(1.a) *Compute analytically the steady-state velocity profile $v_x(y)$, as a function of $\Delta P/\Delta x$ and α,* by solving the Navier-Stokes equation of motion and the equation of continuity,

$$\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{\nabla} P + \mu\, \nabla^2 \underline{v} \qquad\qquad \underline{\nabla} \cdot \underline{v} = 0$$

The x-direction component of the equation of motion is written in component form as

$$\rho \frac{\partial v_x}{\partial t} + \rho \left[v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right] = -\frac{\partial P}{\partial x} + \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right]$$

At the bottom and upper plate walls, use no-slip boundary conditions,

$$v_x(y=0) = 0$$
$$v_x(y=B) = 0$$

At the interface, use a force balance that states that the shear stresses on either side of the interface match one another. For a Newtonian fluid, this yields the condition

$$\mu_1 \frac{\partial v_x}{\partial y}\Big|_{\alpha B^-} = \mu_2 \frac{\partial v_x}{\partial y}\Big|_{\alpha B^+}$$

αB^- is the y-value just below the interface and αB^+ is the y-value just above the interface. As $\mu_1 > \mu_2$, the velocity gradient will be smaller on the bottom (fluid 1) side of the interface than on the top (fluid 2) side.

(1.b) For $\alpha = 0.5$, make a plot of the average velocities in each region of two-fluid flow,

$$\langle v_x \rangle_1 = \frac{1}{\alpha B} \int_0^{\alpha B} v_x(y)\, dy \qquad \langle v_x \rangle_2 = \frac{1}{(1-\alpha) B} \int_{\alpha B}^{B} v_x(y)\, dy$$

as a function of the imposed pressure gradient $\Delta P/\Delta x < 0$, expressed in units of *Pa/m*. Plot as well the dimensionless Reynolds' numbers for each region,

$$\mathrm{Re}_1 = \frac{\rho_1 \langle v_x \rangle_1 [\alpha B]}{\mu_1} \qquad \mathrm{Re}_2 = \frac{\rho_2 \langle v_x \rangle_2 [(1-\alpha) B]}{\mu_2}$$

At what value of the pressure gradient is $\mathrm{Re}_{\max} = \max\{\mathrm{Re}_1, \mathrm{Re}_2\}$ *equal to one?*

Problem 2. Numerical solution of time dependent flow problem

Let us simplify the problem above by making it a flow involving only one fluid with the properties of fluid 2, such that the time-dependent velocity field $v_x(y,t)$ satisfies

$$\rho_2 \frac{\partial \underline{v}}{\partial t} + \rho_2 \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{\nabla} P + \mu_2 \nabla^2 \underline{v} \qquad\qquad \underline{\nabla} \cdot \underline{v} = 0$$

That is, $\alpha = 0$. Let us use the same boundary conditions as for the steady-state flow problem above and use the initial value condition that at time $t = 0$, the fluid is at rest,

$$v_x(y, t=0) = 0$$

Write a computer problem (MATLAB, C/C++, FORTRAN, etc.) that solves this start-up flow problem numerically using the following procedure:

(2.a) Use the assumed functional form of the flow $v_x(y,t)$ to find a single partial differential equation relating $\partial v_x / \partial t$ to the driving pressure gradient and to the spatial derivatives of $v_x(y,t)$,

$$\frac{\partial v_x}{\partial t} = \mathit{function}\left\{ \frac{\Delta P}{\Delta x}, \frac{\partial^2 v_x}{\partial y^2}, \ldots \right\}$$

Use the value of the pressure gradient that you found in problem 1 made $\mathrm{Re}_{\max} = 1$.

(2.b) Form a grid of points (say at least 100) that spans $0 < y < B$, labeled $0 < y_1 < y_2 < \cdots < y_N < B$.

(2.c) For each velocity field value at each grid point, you have an ordinary differential equation in time,

$$\frac{\partial v_x(y_j)}{\partial t} = function\left\{ \frac{\Delta P}{\Delta x}, \frac{\partial^2 v_x}{\partial y^2}\Big|_{y_j}, \ldots \right\}$$

(2.d) Use finite difference approximations to write each spatial derivative as an algebraic difference of neighboring grid point values.

(2.e) Use the fourth-order Runge-Kutta method to integrate this set of ODEs forward in time until the flow reaches steady state. This should occur on the time span of several values of the characteristic time, $t_{flow} = \rho_2 B^2 / \mu_2$.

(2.f) Make a plot of the average velocity,

$$\langle v_x \rangle = \frac{1}{B} \int_0^B v_x(y)\, dy$$

as a function of time. Plot the final steady-state value, and show by comparison that it agrees with the analytical solution of problem 1 in this limit.