Problems for Modeling and Simulations.

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Abstract

Problems for the Conservation Laws section of the course. **Problems 1.1 and 2.1 — with their ANSWERS.**

Contents

1 Traffic Flow Problems.

In this section we present problems involving traffic flow.

1.1 Shock structure.

As explained in the lectures, when using conservation laws to derive continuum approximation mathematical models, it is necessary to supplement the conservation laws with constitutive equations (also known as equations of state). The reason for this is that, generally, systems of conservations laws involve more unknowns (densities and fluxes) than equations. Thus, extra relationships between the densities and the fluxes are needed.

In a first level of approximation, the constitutive relations are obtained (in most cases) from thermodynamic-like, quasi-equilibrium arguments.[1] This then lead to situations where the fluxes

[1] Which neglect transport of densities by discrete-level processes — precisely because of the quasi-equilibrium approximation.

are functions of the densities only, so that the system of conservation laws obtained has the mathematical form

$$\mathbf{u}_t + \mathbf{F}_x = 0, \tag{1.1}$$

if there are no sources or sinks, where $\mathbf{u}$ is the vector of conserved densities and $\mathbf{F} = \mathbf{F}(\mathbf{u})$ is the vector of corresponding fluxes.

Systems of equations such as (1.1) have the problem that, quite often, their solutions develop infinite slopes in a finite time (even if the initial conditions are smooth) — a phenomena known as **wave breaking**. When wave breaking occurs, the system of equations ceases to be valid as a mathematical model for whatever phenomena is of interest — since quasi-equilibrium hypothesis break down when large derivatives in the solutions arise.

How to model what happens beyond wave breaking requires a re-examination of the physics. It is NOT contained within the framework of the mathematical model in 1.1. In this sub-section we will concentrate on a fairly common phenomena that determines what happens after wave-breaking: shocks form. However, it is **IMPORTANT to point out that shock-formation is NOT THE ONLY POSSIBILITY after wave-breaking. The nature of the physics in the problem decides what happens, mathematics alone is NOT enough!**

In the **particular case**[2] where the transport of densities by discrete level processes are dominated by dissipation,[3] what generally happens is that the **wave steepening** (derivatives growing) that leads to the **wave breaking** phenomena is **stopped** by the discrete level transport processes when the derivatives reach some (large) value. Even more important is the fact that the regions where the derivatives are large are restricted to very small regions in space. This gives rise to **shock waves** which are (in 3-D space) very thin (moving) sheets across which the densities vary rapidly. In 1-D they are thin intervals $|x - s(t)| \leq \ell$ across which the densities vary rapidly from one value on the left, to another on the right — here ℓ is some length scale which is much smaller than the length scales of interest in the continuum approximation, but still large compared with the discrete level length scales.

We illustrate the point in the prior paragraph with the case of Traffic Flow. In this case, if ρ is the car density, and q is the car flow rate, the corresponding conservation law is

$$\rho_t + q_x = 0, \tag{1.2}$$

where a **quasi-equilibrium** approximation leads to the equation $\boxed{q = Q(\rho)}$ — for some appropriate function Q. However, drivers (generally) drive "preventibly", and do not look just at the car ahead of them, but further down the road. In other words, they do not just use the local car density, but its derivatives. A simple model that incorporates this idea is the following: take

$$q = Q(\rho) - \nu \rho_x, \tag{1.3}$$

[2] Though quite common.

[3] A precise definition of what this means in "general" is not easy. In the context of physical processes it means that energy is transferred from the macro to the micro scales, so that the entropy increases.

where $\nu > 0$ is some "small" constant. Then the conservation law in (1.2) yields the equation

$$\rho_t + c(\rho)\rho_x = \nu \rho_{xx}, \tag{1.4}$$

where $c = dQ/d\rho$. When the derivatives are not too large, the term on the right in this last equation can be neglected, and we thus obtain the model that was studied in the lectures. Namely:

$$\rho_t + c(\rho)\rho_x = 0. \tag{1.5}$$

However, this last equation is **non-linear** (c is not a constant) and, as shown in class, it leads to solutions that develop very steep gradients. When this happens, the term on the right in equation (1.4) can no longer be neglected. This term then stops the steepening, and leads to the formation of "traffic flow shock waves".

Problem 1.1 *This problem has several parts, as follows:*

P.1 *What are the dimensions of ν in equation (1.3)?*

P.2 *Consider a simple version of equation (1.4), where c is linear in ρ. That is:*

$$\rho_t + \rho\rho_x = \nu \rho_{xx}. \tag{1.6}$$

Then (explicitly) show that this equation has steady[4] *solutions with one value of ρ as $x \to -\infty$ (say ρ_L) and a different one as $x \to +\infty$ (say ρ_R).*

P.3 *Show that, for fixed values of ρ_L and ρ_R, the solutions in item* **P.2** *approach a piece wise discontinuous function as $\nu \to 0$. Namely: $\rho = \rho_L$ for $x < s(t)$, and $\rho = \rho_R$ for $x > s(t)$ — for some $s = s(t)$.*

P.4 *The solutions in* **P.2** *are the* shock waves, *with position $s = s(t)$. How does the* **shock width**[5] *scale as a function of ν and as a function of the* **shock strength** *$(\rho_L - \rho_R)$?*

Hint 1.1 *Reduce the pde to an ode for the steady solution profile $\rho = f(x - Ut)$. This ode will be second order, but it is easy to integrate it once, to obtain a separable first order ode which you can integrate explicitly. Write the constant of integration, as well as the propagation velocity U, in terms of the values for the solution as $x \to \pm\infty$, namely ρ_L and ρ_R. Notice that the derivatives of ρ must vanish in the limit $x \to \pm\infty$.*

[4] That is, solutions of the form $\rho = f(x - Ut)$, where U is a constant velocity.

[5] The shock width is the thickness of the transition region connecting the two extreme values on each side of the shock.

1.2 Answer to the shock structure problem.

Answer to problem 1.1:
We proceed as follows:

P.1 The flux q must have dimensions stuff/time, while ρ has dimensions stuff/length. It follows that ν **must have dimensions** length^2/time. Hence ν is a **diffusivity.**

P.2 Substituting $\rho = f(x - Ut)$, where U is a constant, in equation (1.6), yields the ode

$$\nu \frac{d^2 f}{d\zeta} = -U \frac{df}{d\zeta} + f \frac{df}{d\zeta}, \tag{1.7}$$

where $\zeta = x - U\,t$. This can be integrated once, to yield

$$\nu \frac{df}{d\zeta} = \kappa - U\,f + \frac{1}{2} f^2, \tag{1.8}$$

where κ is an integration constant. Since $f \to \rho_L$ as $\zeta \to -\infty$, and $f \to \rho_R$ as $\zeta \to +\infty$, we can write

$$\kappa = \frac{1}{2} \rho_L \, \rho_R \qquad \text{and} \qquad U = \frac{1}{2} (\rho_L + \rho_R). \tag{1.9}$$

The equation then takes the form:

$$\frac{df}{dz} = \frac{1}{2} (f - \rho_L)\,(f - \rho_R) \tag{1.10}$$

where $z = \dfrac{\zeta}{\nu} = \dfrac{x - U\,t}{\nu}$. This is easily solved, and yields, finally:

$$\rho = \frac{\rho_L + \rho_R}{2} - \frac{\rho_L - \rho_R}{2} \tanh\left(\frac{\rho_L - \rho_R}{4\,\nu} (x - x_0 - U\,t)\right) \tag{1.11}$$

where x_0 is an arbitrary constant. **Notice the important restriction: for this solution to be consistent with the earlier assumptions that $\rho \to \rho_L$ as $x \to -\infty$, and that $\rho \to \rho_R$ as $x \to +\infty$, it MUST be that $\rho_L > \rho_R$. There are NO shock solutions with $\rho_L < \rho_R$.**

P.3 From equation (1.11), and the properties of the hyperbolic tangent, it should be clear that for $\rho_L > \rho_R$ fixed:

$$\rho \to \rho_L \ \text{ for } \ x < x_0 + U\,t, \ \text{ as } \ \nu \to 0.$$
$$\rho \to \rho_R \ \text{ for } \ x > x_0 + U\,t, \ \text{ as } \ \nu \to 0.$$

P.4 From equation (1.11), and the properties of the hyperbolic tangent, it should be clear that **the shock width scales like**

$$\frac{\nu}{\rho_L - \rho_R}.$$

Hence **shocks get "thinner" as they get stronger, or as ν gets smaller.**

2 Incompressible fluid flow in thin elastic pipes.

Here we will consider the problem of an incompressible, inviscid, fluid flowing down a narrow cylindrical pipe with elastic thin walls. This set up constitutes a (very) crude model for the flow of blood in arteries.

2.1 Simplest model for incompressible fluid flow in thin elastic pipes.

The objective in this subsection is to derive, using a conservation laws approach, the "simplest" model for fluid flow down an elastic pipe. We make the approximations listed below. Notice that many of these approximations are quite drastic, and of very dubious validity for blood vessels. The resulting model gives some interesting insights, though it should not be taken too seriously.

We consider a flexible pipe, filled with a **fluid under a pressure** p — of course, p is a function of time and position along the fluid. For the tube not to collapse, and for flow to be possible, it must be that $p > p_0 =$ **outside pressure.** Furthermore, let x be the **length coordinate along the tube axis.**

A.01 **Neglect transport effects,**[6] such as viscosity, heat conductivity, etc. Thus we neglect the dissipation which occurs inside the fluid, at/near the walls of the pipe (fluid-solid interface), and inside the pipe walls (as they move and flex). Heat conduction (by both the fluid and the pipe walls) is neglected as well.

A.02 Ignore gravity and any other (possible) body forces. Therefore we assume that **the only force acting on the fluid is the pressure.** Furthermore, assume that **the pressure** p_0 **outside the pipe is constant** — notice that this is the **only external force acting on the system.**

A.03 The fluid is incompressible, with **constant density** ρ.

A.04 The **pipe walls are homogeneous:** they have the same properties everywhere.

A.05 The **pipe walls are thin and flexible, with no bending strength.** Thus we ignore their thickness, and **treat them as a mathematical surface** bounding the liquid. To be precise, "thin" here has the following meaning: *Let L be the scale over which the motion occurs (see* **A.08** *below), and let h be the tube walls' thickness. Then $L \gg h$, so that the amount of bending in the longitudinal direction is small enough to have negligible effects on the force balances. This also has the consequence of keeping the pipe cross-section circular (see* **A.09** *below), so that there is no bending in the transversal direction either.*

A.06 The **repose perimeter of the pipe is constant** — i.e.: it does not depend on the coordinate x along the tube. It then follows from **A.04** that, under steady state conditions — so that

[6] Generic name given to effects that are due to transport of various quantities by molecular level processes.

the fluid pressure is the same everywhere — the pipe has cylindrical shape. Let $2\pi r_0$ be the repose (no forces) perimeter of the tube, corresponding to a **repose radius** r_0 **and repose cross-sectional area** $S_0 = \pi r_0^2$**.**

A.07 There is **no longitudinal tension** applied along the pipe.

A.08 **Long wave assumption.** The situation we have in mind is that of fluid flowing down the pipe, with some kind of disturbances (waves) propagating down the length of the tube. Let L **be the scale (wave-length) over which these disturbances occur. Then we assume that** L **is much bigger than the pipe diameter** — or, for that matter, the tube's walls thickness. *This has the important consequences listed below in* **A.09**, **A.10**, *and* **A.11**.

A.09 **Cylindrical straight geometry.** The long wave assumption implies that, at any given point along the tube, the pressure is (essentially) constant on the cross-section. Hence the **cross-section of the tube is circular everywhere.** In addition, because of the absence of any external forces other than the outside constant pressure (and the homogeneity of the tube walls) the **pipe must remain straight.**[7] Hence the pipe geometry is completely specified by giving its **radius** $r = r(x, t)$ — as a function of time t and the length x along the pipe axis. Alternatively, one can specify the **fluid filled cross sectional area** $S = S(x, t) = \pi r^2$.

A.10 **One dimensional approximation.** The long wave assumption implies that we can neglect any fluid motions in directions transversal to the fluid axis. Thus the **fluid dynamics can be described by the two scalar functions:** $p = p(x, t)$ **(pressure) and** $u = u(x, t)$ **(axial fluid velocity).**

A.11 **Radial wall forces only.** The long wave assumption implies that the strain on the tube's walls is mainly along the perimeter, with very little stretching (or bending) in the longitudinal direction. Thus we **assume that the only force by the walls is that caused by a tension** $T = T(x, t)$ **per unit length along the perimeter, trying to pull it back to its repose length** $2\pi r_0$**.** Of course, the tension results from the stretching of the walls beyond their equilibrium radius by the liquid pressure excess over p_0.

A.12 **Elastic regime.** The amount of wall stretching produced by the motion is small enough (and happens slowly enough) that the walls respond elastically. That is to say: **the tube walls oppose stretching with a force that depends only on the amount of stretching.** *It is important to notice that this assumption involves not just space scales, but time scales as well. If the deformations are too large, the walls will not respond elastically — permanent deformations will occur, etc. Furthermore, when the deformations occur too fast (even if they are small) dissipation can become important. Of course: what "too large" or "too fast" means depends on the physical properties of the tube walls.*

[7] Actually, motion where the axis of the pipe wanders is possible, but we will ignore it here.

Thus, we can write, for the tension introduced in **A.11**, a formula of the form:

$$T = f\left(\frac{\Delta r}{r_0}\right), \tag{2.1}$$

where $\Delta r = r - r_0$ measures the amount of stretching (the tube perimeter changes by $2\pi\Delta r$), f is a function characterizing the elastic response of the walls, and $T\,dx$ **is the tension force along the perimeter of a transversal "slice" of the tube of length** dx**.**

Remark 2.1 *Notice that, at least for now, we are not making the assumption of* small deviations from a steady state[8]*. In particular, this would imply that r is nearly constant: $r \approx r_s$ — where $r_s - r_0$ is the stretching needed to balance the equilibrium steady state pressure. In this case Hooke's law applies, and equation (2.1) can be linearized to*

$$\delta T = Eh\frac{\delta r}{r_s}, \tag{2.2}$$

where $\delta r = r - r_s$, E is the Young's modulus for the wall material, $\delta T = T - T_s$ is the deviation of the tension from its equilibrium value T_s, and h is the wall thickness — which can be assumed constant in this approximation.

Of course, if we do not make the small deviation from a steady state assumption, we cannot assume that the wall thickness h is a constant. *The variations in h do, of course, affect the forces produced by the walls. We can, however, assume that the wall thickness is a function of the stretching, and thus we can incorporate the effect of these variations into the force law in equation (2.1) — without being forced to track an extra variable $h = h(x,t)$.*

The small deviation from a steady state leads to a great simplification of the equations (linearization).

A.13 Neglect wall inertia. Thus, when considering the balance of forces in the radial direction, the inertial forces (mass times acceleration) due to the walls are neglected.

Problem 2.1 *Use the conservation of mass and momentum to derive a system of two coupled pde's for the functions $S = S(x,t)$ and $u = u(x,t)$. You should be able to write the pressure p as a function of the cross sectional area S.*

The large number of assumptions listed earlier can seem a bit frightening. Their net effect, however, is to simplify the problem massively. The **hints below should lead you to the answer without excessive trouble.**

[8] Flow at constant speed and constant pressure.

H.1 Because the density is constant, the liquid volume is conserved. The volume density (volume per unit length) is S. What is the volume flux? Once you have the volume flux, the conservation of volume will give you one of the (two) equations that you need.

H.2 Linear momentum in the axial direction should be conserved. The momentum density (momentum per unit length) is $\rho\, u\, S$. To calculate the momentum flux, notice that (in the absence of transport effects — see **A.01**) momentum can be transferred by either advection (fluid moves and carries momentum with it) or macroscopic mechanical forces. Because of the earlier assumptions, the only forces acting on any parcel of liquid (say, the liquid in some section of the pipe $a < x < b$) are: (1) the pressure force from the liquid in $x < a$ across the cross-section at $x = a$, (2) the pressure force from the liquid in $x > b$ across the cross-section at $x = b$, and (3) the longitudinal component of the force by the walls on the liquid along the section $a < x < b$. Thus

- Write an expression for the momentum flux caused by advection: $\mathcal{F}^A$.
- Calculate the longitudinal force per unit length by the walls on the liquid $\mathcal{F}^W$. Notice that the force per unit area by the walls on the liquid is normal to the walls (no friction — see **A.01**) and equal to the pressure.
- Consider the momentum balance on an arbitrary section $a < x < b$ of the pipe:

$$\frac{d}{dt}\int_a^b (\rho u S)dx = \left(\mathcal{F}^A + pS\right)\Big|_b^a + \int_a^b \mathcal{F}^W\, dx \tag{2.3}$$

- As in the lectures, use the fact that (2.3) is valid for all $a < b$ to write a pde. This is the second equation you need — though, you are not done yet!

H.3 At this point you will find that you have two partial differential equations, but that they involve three unknowns: S, u, and p. You now **need an equation of state.** Below follows a strategy to write the pressure p as a function of the cross-sectional area.

Because of **A.13** the forces on the wall by the liquid (caused by the pressure difference with the outside) in the radial direction must be balanced by the radial wall forces — see **A.11** and **A.12**. Because the tension in the wall is a function of the wall perimeter, and you can write the perimeter in terms of the cross-sectional area S, this balance will provide the desired relation.

H.4 Step **H.3** requires that you calculate the radial forces by the wall as a function of the tension along their perimeter. Below a hint for to how to do this.

Consider a circular string of radius r that is under tension[9] T_s. In order to keep the string in place, a (constant) radial force F_r per unit length must be applied along the string. The

[9] In the calculation that you need to do in step **H.3**, the string is replaced by a slice of length dx of the tube wall, and T_s is replaced by $T dx$.

dependence of F_r on T_s can be found as follows: Consider the work needed to increase the radius of the circle by dr. This work can be calculated in two ways: (i) The work done by the radial force as the radius changes. (ii) The work done by the tension as the string stretches. Both answers must be the same, hence equating (i) and (ii) yields a relationship between F_r and T_s.

2.2 Answer to problem 2.1: simplest model for . . . flow in . . . pipes.

As requested in problem 2.1, here we derive the governing equations for the **simplest model for incompressible fluid flow in a thin elastic pipe.** We do this using the conservation of mass and momentum, and follow the hints in subsection 2.1.

Conservation of Mass. The fluid mass must be conserved. However, the fluid density is a constant, hence the **fluid volume must be conserved.** Since $S = \pi r^2$ is the cross-sectional area of the tube, it follows that the **volume density (per unit length** dx**) is** S. On the other hand, it should be clear that the **volume flow (per unit time) is** uS**.** Since there are no mass/volume sources, we must have:

$$(S)_t + (uS)_x = 0\,. \tag{2.4}$$

By the way: notice that the mass in the tube walls does not "flow". Thus we do not have to worry about it when considering the equation for the conservation of mass (or, for that matter, the equation for the conservation of momentum).

Conservation of Momentum. The fluid linear (along the pipe axis) momentum must be conserved. The linear **momentum density per unit length** dx **is** $\rho u S$**.** On the other hand, the linear **momentum flux has two components:** the **advective component** $\rho u^2 S$ (momentum carried by the flow), and the **momentum flux due to the pressure force** pS **across the tube section**. In addition, there is a **momentum source per unit length,** caused by the forces (in the flow direction) on the fluid by the tube walls. It should be clear that this momentum source is given by[10]

$$M_s = 2\pi r p \frac{r_x}{\sqrt{1 + r_x^2}} \approx 2\pi r p r_x = p S_x, \tag{2.5}$$

where (long wave assumption **A.08**) we have used that $1 + r_x^2 \approx 1$. The conservation of momentum for any section $a < x < b$ of the pipe then yields:

$$\frac{d}{dt}\int_a^b (\rho u S)dx = \left(\rho u^2 S + pS\right)\Big|_b^a + \int_a^b pS_x\,dx = \int_a^b \left(pS_x - \left(\rho u^2 S + pS\right)_x\right)\,dx$$

Since this should be valid for *any* $a < b$, we conclude that:

$$(\rho u S)_t + \left(\rho u^2 S + pS\right)_x = pS_x \qquad \Longleftrightarrow \qquad (uS)_t + \left(u^2 S\right)_x + \frac{S}{\rho}p_x = 0\,. \tag{2.6}$$

[10]See **P.1** below, with the calculation of the equation for the pressure, and then use it in the expression: "wall longitudinal force per unit area" $\times$ "tube perimeter" $=$ "momentum source per unit length".

Using (2.4), this last equation can also be written in the form:

$$u_t + u u_x + \frac{1}{\rho} p_x = 0\,. \tag{2.7}$$

Equation for the pressure. The pressure in the fluid $p = p(x,t)$ must be balanced by both the pressure outside the tube p_0, and the elastic forces exerted by the tube walls. At the tube wall, the pressure force by the fluid on the wall can be decomposed into two components:

P.1 Longitudinal force (per unit area) .... $p\,\dfrac{r_x}{\sqrt{1+r_x^2}} \approx p\,r_x,$

P.2 Radial force (per unit area) .... $p\,\dfrac{1}{\sqrt{1+r_x^2}} \approx p,$

where the long wave approximation **A.08** has been used to replace $1 + r_x^2$ by 1. We note that **an entirely similar decomposition applies to the force by the outside pressure.**

As a consequence of the long wave approximation, the longitudinal force is much smaller than the radial one — this is consistent with **A.11:** the wall (elastic) forces are transversal only, corresponding to stretching mainly only along the perimeter.

The tension T per unit length on the tube walls (see equation (2.1)) results in a radial force — per unit area — by the tube walls (see remark 2.2) of magnitude

$$\text{Force} = \frac{1}{r}\,T = \frac{1}{r}\,f\left(\frac{\Delta r}{r_0}\right). \tag{2.8}$$

Because of assumption **A.13**, this force must balance the radial forces caused by the pressures (internal and external) on the pipe's walls — see **P.2** above. Thus, we end up with the following formula for the pressure $p = p(x,t)$:

$$p - p_0 = \frac{1}{r}\,T = \frac{1}{r}\,f\left(\frac{\Delta r}{r_0}\right)\,. \tag{2.9}$$

Of course, since $r = \sqrt{S/\pi}$, **this last equation gives the pressure as a function of the cross-sectional area** S**.**

Remark 2.2 **Calculation of the radial force from the tension.** *Consider a section of the tube's walls, of length dx and radius r. There is a tension $T dx$ along the section, which must be balanced by a radial force f_r per unit length along the perimeter — else the section would shrink. Consider now the work dW needed to change the radius from r to $r + dr$. If we calculate it using the work done by the radial force, we obtain $dW = 2\pi r f_r dr$ (work is force time distance, and we have to add the work done all along the perimeter by the radial force). On the other hand, we can also calculate this work in terms of the work done against the tension $T dx$, as the perimeter changes by $2\pi dr$. This yields $dW = 2\pi dr T dx$. Since the two expressions must be equal, we conclude that $f_r = (T/r)dx$. It follows that* **the radial force per unit area caused by the transversal tension in the pipe's walls is given by** T/r**.**

Final Equations. We now have a complete system of equations for $S = S(x,t)$ and $u = u(x,t)$

$$\left.\begin{array}{rcl} 0 & = & S_t + (u\,S)_x\,, \\ 0 & = & u_t + u\,u_x + \dfrac{1}{\rho}\,p_x\,, \end{array}\right\} \tag{2.10}$$

where $p = p(S)$ is given by equation (2.9), via the relationship $S = \pi\,r^2$, and ρ is a constant.

THE END