The Fall of the Towers

Franz-Josef Ulm
M.I.T./CEE/1-263 – Modeling and Simulations

March 1, 2006

Abstract

This note is a contribution to the continuum part of 'Modeling and Simulation'. Through an analysis of the Fall of the Twin Towers on Sept. 11, 2001, we aim at showing how to go from a discrete system to a continuum system. At the end of this note is a problem set.

1 Introduction

We consider an event that, like no event before in our generation, continues to affect the lives of people all over the world: the Fall of the Twin Towers on September 11, 2001. The emotions may one day be forgotten, and no-doubt cooler thinking will return. In this case, we may be able to reconstruct the sequence of events, when after fire fighters and policemen, engineers were called in to explain how it had happened? Here is a personal account which exemplifies how engineers lived that very day:

> 'As in the morning hours of Sept. 11 I was anxiously watching on TV the dramatic events taking place in New York City, and saw the two World Trade Center towers engulfed in immense flames brought about by terrorists who deliberately crashed two passenger jets into them, my training in Structural Engineering instantly elicited in me visions of doom, and a feeling that the towers were in imminent danger of collapse. Still, knowing that half a decade earlier the towers had resisted massive damage in a terrorist attack, and being unaware of similar cases of skyscraper

collapse, I hoped against reason that they might survive yet again. To my horror, I then witnessed the unthinkable unfolding in front of my eyes. In retrospect, I should have been 100% sure that they would fail, but the idea was so disgusting that I allowed my wishful thinking to prevail instead. Soon after the tragedy occurred, cooler thoughts and the engineer in me returned, and I began to ponder about the mechanics that led to the catastrophe.'[1]

The North Tower was hit at 8:46 am above the 96th floor and remained erect until 10:28 am; the South Tower which was hit above the 80th floor at 9:03 am collapsed less than an hour later at 9:59 am. Both towers imploded on themselves, and did not tilt; thus avoiding an even greater catastrophe in lower Manhatten. Several mechanisms may have been at play in this failure, but the most stunning information that surfaced was that the towers should have collapsed almost in free fall down to the ground. It's worthwhile to check this conjecture.

2 Free Fall Assumption

Provided constant mass, the conservation of momentum informs us that $m_0 \overrightarrow{a} = \overrightarrow{F}_{ext}$. In the absence of drag forces (caused e.g. by air), the external force is produced by the earth acceleration, $\overrightarrow{F}_{ext} = m_0 \overrightarrow{g}$, where $\overrightarrow{g} = +g \overrightarrow{e}_z$ is the earth gravity vector (positive in the downward direction $\overrightarrow{e}_z$). In free fall, the acceleration of m_0 remains $a = g$ from the top to the ground. Integrating once w.r.t. time yields the velocity, $V(t) - V_0 = gt$; and integrating twice the free fall distance, $z - z_0 = V_0 t + \frac{1}{2} g t^2$, where $V_0 = \sqrt{2gh}$ is the initial velocity of the floors above the crash elevation z_0, which fall onto the structure below over a height h that had been destroyed by the plane crash. Eliminating time t in the previous relations yields the free fall velocity and the corresponding time as:

$$V = V_0 \sqrt{1 + \frac{2g(z - z_0)}{V_0^2}}; \; t = \frac{V_0}{g} \left(\sqrt{1 + \frac{2g(z - z_0)}{V_0^2}} - 1 \right) \tag{1}$$

[1]Kausel, E., Inferno at the World Trade Center, NY, reported in *Tech Talk*, M.I.T., Cambridge, MA, September 23, 2001.

The maximum velocity is reached at ground zero, where $z - z_0 = Mh$, where M is the floor number where the plane collided. Hence, from (1) we obtain:

$$V_{\max} = V_0\sqrt{1+M};\ \tau = \sqrt{\frac{2h}{g}}\left(\sqrt{1+M}-1\right) \tag{2}$$

For an interstory height of $h = 3.75\,\mathrm{m}$, we obtain $\tau\,(M = 96) \approx 7.7\,\mathrm{s}$ ($V_{\max} = 9.85V_0 = 84.5\,\mathrm{m}\,/\,\mathrm{s}$) for the North tower, and $\tau\,(M = 80) \approx 7.0\,\mathrm{s}$ ($V_{\max} = 9V_0 = 77.2\,\mathrm{m}\,/\,\mathrm{s}$) for the South tower. The values are somewhat smaller than the time we were anxiously replaying over and over again on TV, which was on the order of $10\,\mathrm{s}$.

3 Kausel's Discrete Mass Formulation

In the days following the events of 9/11, Structural Engineers were having a closer look into this failure, refining the engineering mechanics analysis. The following analysis is based on calculations made by Eduardo Kausel, Professor in Civil Engineering (Course 1) at MIT. The following assumptions are made: The towers are considered as a system of N discrete masses, where $N = 110$ is the number of floors. All floors have the same mass m and interstory height h. As the upper block falls down through one floor, it collides plastically with the floor underneath, and instantly imparts onto it a velocity whose value may be computed by means of the principle of conservation of momentum.

We start with the top mass m_0 falling in free fall onto the floor below. The velocity right before the collision with the floor $M - 1$ below is:

$$V_1^- = V_0 = \sqrt{2gh} \tag{3}$$

Now, if we neglect any resisting forces when the mass falls onto the floor below, there are no external forces that change the momentum during the collision. What changes is the mass $m_0 \to m_1 = m_0 + m$. This change of mass is balanced by a change of velocity so that the momentum be conserved:

$$\delta\overrightarrow{\wp} = \overrightarrow{\wp}^+ - \overrightarrow{\wp}^- = 0 \Leftrightarrow m_1 V_1^+ = m_0 V_1^- \tag{4}$$

Herein, superscripts '−' and '+' indicate 'before' and 'after' the collision. Since we assume that the collision occurs instantaneously, neither time nor time derivatives intervene in the problem formulation; we write $\delta\overrightarrow{\wp}$ instead of $d\overrightarrow{\wp}/dt$. The velocity V_1^+ is the initial condition for the free fall of the mass m_1 over the next interstory height h. The velocity right before the collision with the floor $M-2$ is obtained by letting $V_1^+ \to V_0$ and $z - z_0 = h$ into (1):

$$V_2^- = \sqrt{\left(V_1^+\right)^2 + (V_0)^2} = V_0\sqrt{\left(\frac{m_0}{m_1}\right)^2 + 1} \tag{5}$$

The momentum balance before and after collision yields:

$$\delta\overrightarrow{\wp} = 0 : V_2^+ = \frac{m_1}{m_2}V_2^- = V_0\sqrt{\left(\frac{m_0}{m_2}\right)^2 + \left(\frac{m_1}{m_2}\right)^2} \tag{6}$$

where $m_2 = m_0 + 2m$. We can continue repeating this sequence down to the ground. The algorithm developed by E. Kausel is given in Box 1.

As an alternative to a numerical procedure, one can continue analytically. In fact, from (4) to (6), it is recognized that the velocity at floor i counted from the floor of impact is

$$V_i^- = \sqrt{\left(V_{i-1}^+\right)^2 + (V_0)^2} = V_0\sqrt{\sum_{k=1}^{k=i-1}\left(\frac{m_{k-1}}{m_{i-1}}\right)^2 + 1};\ i = 2, M \tag{7}$$

before the collision, and

$$V_i^+ = \frac{m_{i-1}}{m_i}V_i^- = V_0\sqrt{\sum_{k=1}^{k=i}\left(\frac{m_{k-1}}{m_i}\right)^2};\ i = 1, M \tag{8}$$

after collision. Substituting $m_J = m\,(N - M + J)$ in (7) (where N is the total number of floors, and M the level of impact) allows us to determine the maximum velocity at ground zero as:

$$\frac{V_{\max}}{\sqrt{2gh}} = \frac{V_{i=M}^-}{V_0} = \sqrt{\sum_{k=1}^{k=M-1}\left(1 - \frac{M-k}{N-1}\right)^2 + 1} \tag{9}$$

In contrast to the free fall model (2), the maximum velocity now depends on both the impact level M and the total number of floors N. In particular, for the WTC towers where $N = 110$, we obtain:

$$\frac{V_{\max}\,(N = 110)}{V_0} = \frac{1}{654}\sqrt{(431\,646M - 3942M^2 + 12M^3)} \tag{10}$$

Algorithm	Comment
Basic Data:	
$N = 110$	Number of Floors
$M = 80$	Floor number where plane collided
$g = 9.8$	Gravity $[\mathrm{m}/\mathrm{s}^2]$
$h = 3.75$	Floor height $[\mathrm{m}]$
$V_0^2 = 2gh$	Square of change of velocity per floor
Initialisation:	
$N1 = N + 1$	
$t = zeros(N1, 1)$	Time vector
$v1 = zeros(N1, 1)$	Velocity before falling through one floor
$v2 = zeros(N1, 1)$	Velocity after falling through one floor
Calculation:	
$for\ j = N - M : N$	Loop over floors from impact level (j=N-M)
$j1 = j + 1$	to ground zero (j=N)
$v2\,(j1) = \sqrt{(v1(j))^2 + V_0^2}$	Velocity after falling through one floor
$dv = v2(j1) - v1(j)$	Change in velocity
$t(j1) = t(j) + dv/g$	Accumulated time
$v1(j1) = j * v2(j1)/j1$	Conservation of momemtum
end	

Table 1: MATLAB algorithm for WTC collapse (by E.Kausel, MIT, 2001).

The velocities are $V_{\max}\,(M = 96) = 6.06V_0 = 52.0\,\mathrm{m}/\mathrm{s}$ for the North tower, and $V_{\max}\,(M = 80) = 6.01V_0 = 51.5\,\mathrm{m}/\mathrm{s}$ for the South tower. These velocities represent roughly 2/3rd of the calculated free fall velocities. Hence, the conservation of momentum at each floor level slows the collapse down. This is shown in figure 1 which compares the velocity profiles over the story numbers of the South tower for the free fall assumption and the momentum conservation assumption. There is some significant difference!

Finally, since the floor-to-floor collision is assumed to occur instantaneously, the time elapsed between two collisons is defined by the free fall in between floors. If we note that $V\,(t_{i-1}) = V_{i-1}^+$ and $V\,(t_i) = V_i^-$, the time of the free fall over each interstory height h is due to the change of velocity:

$$\Delta t_i = t_i - t_{i-1} = \frac{V_i^- - V_{i-1}^+}{g} \tag{11}$$

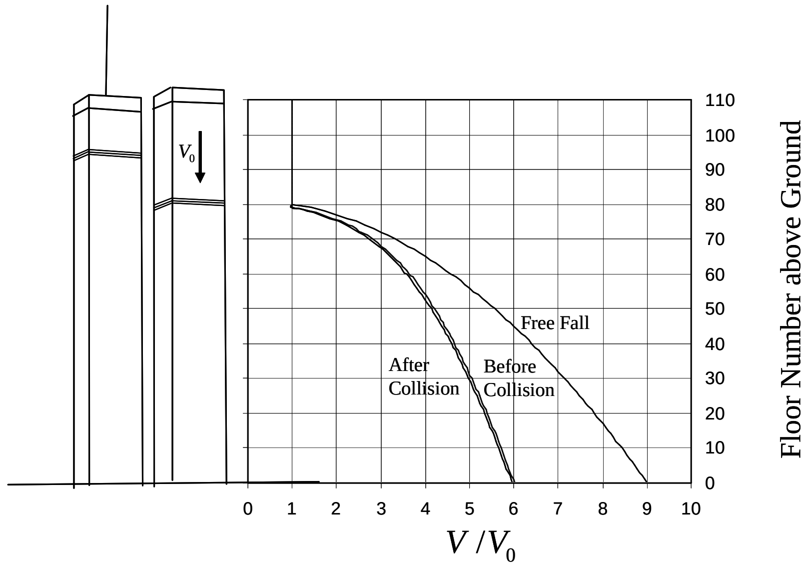

Figure 1: Fall of the WTC South tower: velocity profiles based on the conservation of momentum at each floor level and free fall.

The duration of collapse is the sum of the interstory time intervals Δt_i:

$$\tau = \sqrt{\frac{2h}{g}} \left[1 + \sum_{i=2}^{M} \left(\sqrt{\sum_{k=1}^{k=i-1} \left(\frac{N-M+k-1}{N-M+i-1} \right)^2 + 1} - \sqrt{\sum_{k=1}^{k=i-1} \left(\frac{N-M+k-1}{N-M+i-1} \right)^2} \right) \right] \tag{12}$$

The evaluation of this (monster) sum yields $\tau\,(M=96) = 11.7\,\mathrm{s}$ for the North tower, and $\tau\,(M=80) = 9.7\,\mathrm{s}$ for the South tower, which is much closer to the observed collapse duration of $\sim 10\,\mathrm{s}$.

3.1 From Discrete to Continuum Description

In the discrete description, each floor is considered a discrete mass, and the momentum balance is written at each floor level (see Eqn. (4), (6)). Since the interstory height is small compared to the size of the towers, *ie.* $h \ll H$, we could have alternatively adopted a continous description of the mass involved in the collapse; for instance in the form:

$$m_i = m_0 + i\,m \rightarrow m\,(z) = \frac{m}{h} z\,(t) \tag{13}$$

where z is initialized at the top of the building. Imagine now an observer attached to the collapse front $z = z\,(t)$. What she or he will record is a continuous change of the momentum (chain rule):

$$\frac{d\overrightarrow{\wp}}{dt} = \frac{d}{dt}\left(m\overrightarrow{V}\right) = \frac{dm}{dt}\overrightarrow{V} + m\,(z)\,\frac{d\overrightarrow{V}}{dt} \tag{14}$$

due to a continuous change of the gravity force of the mass above her/him:

$$\overrightarrow{F}_{ext} = m\,(z)\,g\,\overrightarrow{e}_z \tag{15}$$

Setting (15) equal to (14) yields the momentum balance at the collapse front:

$$\frac{dz}{dt}\overrightarrow{V} + z\,(t)\,\frac{d\overrightarrow{V}}{dt} = z\,(t)\,g\,\overrightarrow{e}_z \tag{16}$$

In a cartesian coordinate system, in which $\overrightarrow{e}_z$ points downwards, we readily recognize from (16) that the external force vector imposes its direction onto the velocity, $\overrightarrow{V} = V\,(z)\;\overrightarrow{e}_z$; as expected from Newton's 1st Law. Furthermore, in the continuum description, the downward velocity V of the collapse front z is equal to $V\,(z) = dz/dt = \dot{z}$, and the change of velocity is $dV/dt = \ddot{z}$. These considerations allow us to rewrite the momentum balance (16) in the form of a differential equation:

$$\dot{z}^2 + z\,\ddot{z} = z\,g \tag{17}$$

This differential equation needs to be solved for the intial values:

$$z\,(t=0) \;=\; z_0 = (N-M)\,h \tag{18a}$$
$$\dot{z}\,(t=0) \;=\; V_0 = \sqrt{2gh} \tag{18b}$$

The solution of this differential equation is far from trivial! But, fortunately, mathematics software packages allow us to solve this type of initial value problem numerically (for instance using the Runge-Kutta method, see lecture notes). The solution of (17) with (18) yields the sought position of the collapse front $z\,(t)$. Evaluating this function for the building height

$z(t=\tau) = Nh$ yields the collapse duration: $\tau(M=96) = 10.9\,\mathrm{s}$ for the North tower, and $\tau(M=80) = 8.9\,\mathrm{s}$ for the South tower. The values predicted by the continuum model compare quite favorably with the ones predicted by the discrete model. Compared to the discrete model, the continuum model turns out to be more compact. It involves the following assumptions and procedures:

1. Scale separability condition: The characteristic length of the elementary unit involved in the problem is much smaller than the structural scale; here $h/H = 1/110 \ll 1$. In this case, the physical properties that govern the problem can be assumed to continuously vary from one point to another; here the mass $m = m(z)$).

2. The physics laws (here the momentum balance) are written for the elementary unit, here an infinitessimal floor height. The problem is solved as a boundary/initial value problem.

We keep this in mind when using a continuum modeling approach.

4 Problem Set

Consider the continuum model of the Fall of the Twin Towers, defined by the (somewhat messy) differential equation (17) and the initial conditions (18).

1. By accurately discretizing the problem (see lecture notes, Chapter 3), develop an algorithm that allows you to solve this problem. Implement it into matlab (and hand it in with your homework).

2. Display (1) the velocity profiles (similar to fig. 1), (2) the position of the colllapse front vs. time, and (3) the time to failure for both the North and the South Tower. Compare your results with the ones developed in this note.

3. How does the numerical implementation of the continuum solution compare to the direct discrete solution presented in this note?