IM/S Problem Set 2

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Solutions

Part I: The Freezing Lake

1. This was a straightforward application of finite difference principles:

$$\begin{aligned} \frac{H_{i,n+1} - H_{i,n}}{\Delta t} &= \frac{\left[k\frac{\partial T}{\partial x}\right]_{i+\frac{1}{2},n} - \left[k\frac{\partial T}{\partial x}\right]_{i-\frac{1}{2},n}}{\Delta x} \\ &= \frac{k_{i+\frac{1}{2},n}(T_{i+1,n} - T_{i,n}) - k_{i-\frac{1}{2},n}(T_{i,n} - T_{i-1,n})}{(\Delta x)^2} \end{aligned} \quad (1)$$

2. (Not covered this year) At the bottom of the lake, the flux at $x_{i+\frac{1}{2}}$ is zero, so we remove it from equation 1:

$$\frac{H_{i,n+1} - H_{i,n}}{\Delta t} = \frac{k_{i+\frac{1}{2},n}(T_{i+1,n} - T_{i,n})}{(\Delta x)^2}. \quad (2)$$

   At the top of the lake, the flux is given by $h(T_{air} - T)$, so we replace the $i + \frac{1}{2}$ part of the equation. But we have to be careful with the Δxes, distinguishing between the one used for the divergence of the flux, and that used to calculate the flux itself at $x_{i-\frac{1}{2}}$:

$$\frac{H_{i,n+1} - H_{i,n}}{\Delta t} = \frac{h(T_{air} - T_{i,n}) - \frac{k_{i-\frac{1}{2},n}(T_{i,n} - T_{i-1,n})}{\Delta x}}{\Delta x}. \quad (3)$$

3. First, extend the properties table in the problem set to include thermal diffusivity and stable timestep (for $\Delta x = 0.01$m):

Material	Liq. water	Sol. water
Conductivity k, $\frac{\text{W}}{\text{m}\cdot\text{K}}$	0.56	2.3
Heat capacity c_p, $\frac{\text{J}}{\text{kg}\cdot\text{K}}$	4200	2100
Density ρ, $\frac{\text{kg}}{\text{m}^3}$	1000	920
Thermal diffusivity α, $\frac{\text{m}^2}{\text{s}}$	1.33×10^{-7}	1.19×10^{-6}
Stability criterion, sec	$\Delta t \leq 375$	$\Delta t \leq 42$

   If we use the liquid water criterion, the liquid is stable but the solid clearly is not. So we use the smaller timestep criterion (solid water, 42 seconds), which corresponds to the larger thermal diffusivity.

4. The three timescales:

   (a) For $L = 0.1$m, the time required to reach steady-state in liquid water is given by $t \sim L^2/\alpha = 75,000$ seconds (about 20 hours).

(b) First we solve the differential equation:

$$\rho \Delta H_f \frac{dX}{dt} = k\frac{\Delta T}{X} \tag{4}$$

$$X dX = \frac{k\Delta T}{\rho \Delta H_f} dt$$

$$\frac{X^2}{2} = \frac{k\Delta T}{\rho \Delta H_f} t + const$$

For $X = 0$ at $t = 0$, the integration constant is zero, so the time is given by

$$t = \frac{\rho \Delta H_f X^2}{2k\Delta T} = \frac{920\frac{\text{kg}}{\text{m}^3} \cdot 334,000\frac{\text{J}}{\text{kg}} \cdot (0.1\text{m})^2}{2 \cdot 2.3\frac{\text{J}}{\text{s}\cdot\text{m}\cdot\text{K}} \cdot 10\text{K}} = 66,800\text{s}. \tag{5}$$

(c) The differential equation for convection-limited freezing is even simpler:

$$\rho \Delta H_f \frac{dX}{dt} = h\Delta T \tag{6}$$

$$\rho \Delta H_f X = h\Delta T t + const.$$

Again the integration constant is zero, so using $h = 10\frac{W}{m^2 \cdot K}$ (from problem 5), required time is given by:

$$t = \frac{\rho \Delta H_f X}{h\Delta T} = \frac{920\frac{\text{kg}}{\text{m}^3} \cdot 334,000\frac{\text{J}}{\text{kg}} \cdot 0.1\text{m}}{10\frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 10\text{K}} = 307,280\text{s}. \tag{7}$$

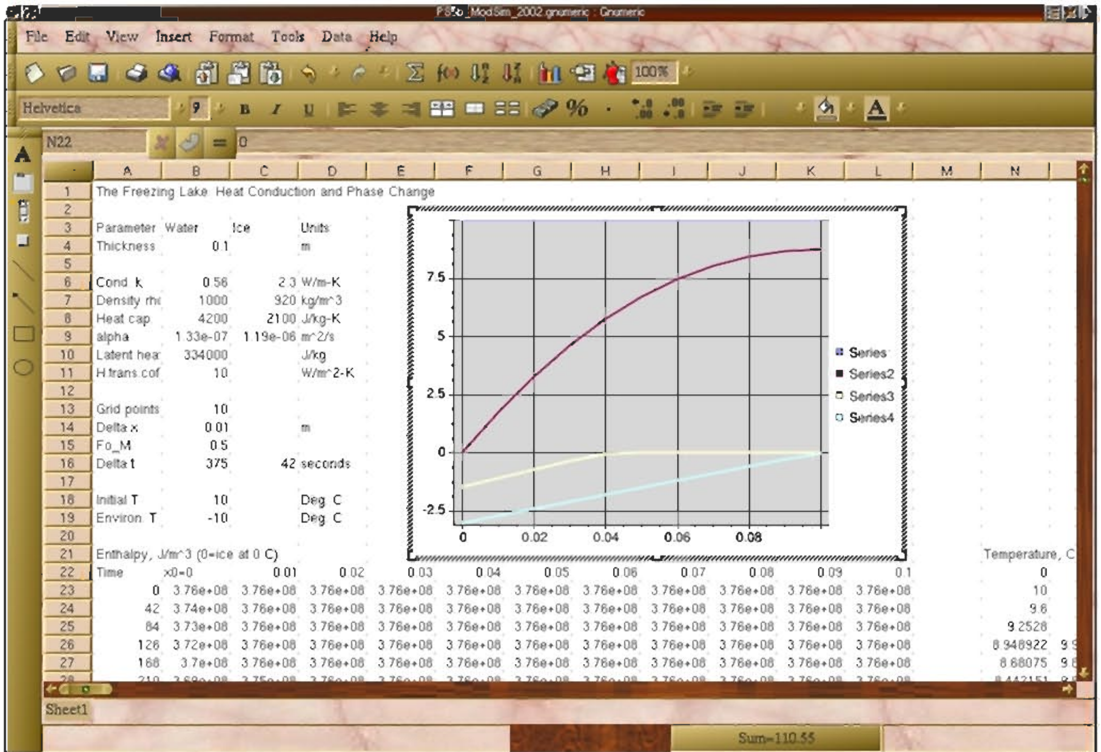
This is clearly the dominant timescale in this problem.

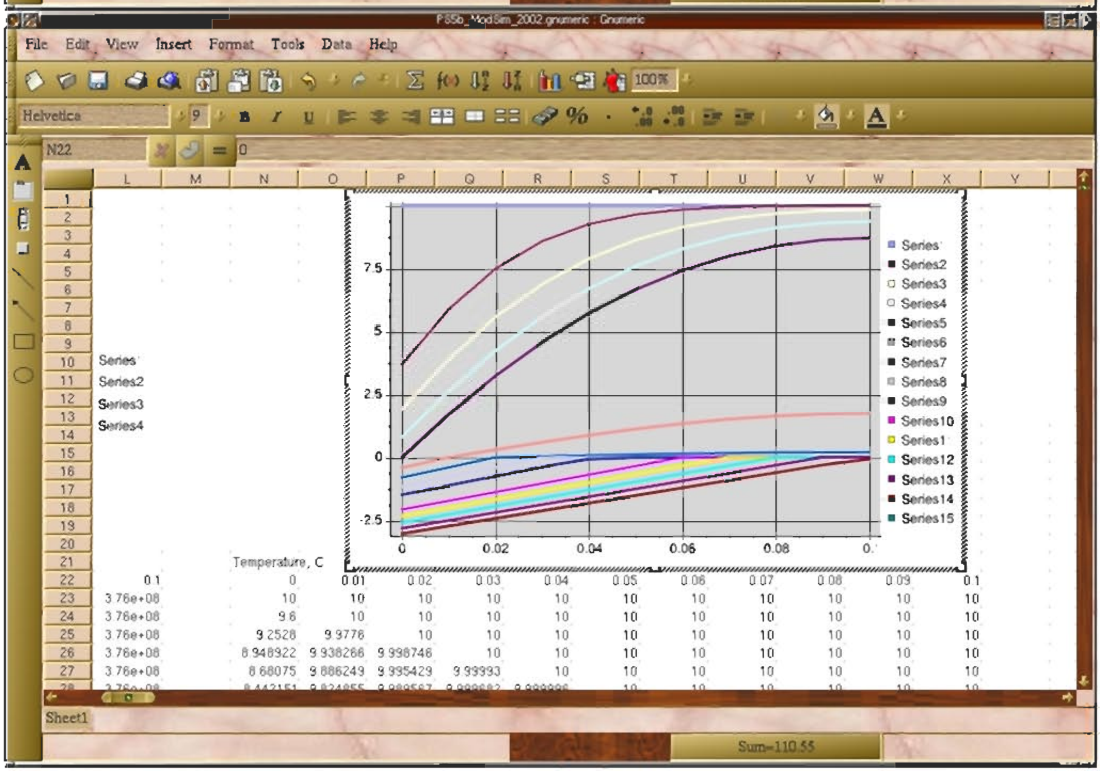
We can add the above results to give a rough estimate of the total time required to freeze the lake, which gives us 449,000 seconds, or about 5 days. At 42 seconds/timestep (from problem 3), this will require about ten thousand timesteps (!).

5. The spreadsheet provided implements the above equations and boundary conditions. Two plots are provided: one giving the information requested in the asignment, and the other with plots at a few more times in between. Times for each set of plots are as follows:

Time	Plot 1 series #	Plot 2 series #	Time	Plot 1 series #	Plot 2 series #
0	1	1	158634		8
3360		2	204414	3	9
6846		3	251034		10
10080		4	301434		11
13760	2	5	351834		12
62034		6	402234		13
110334		7	448560	4	14

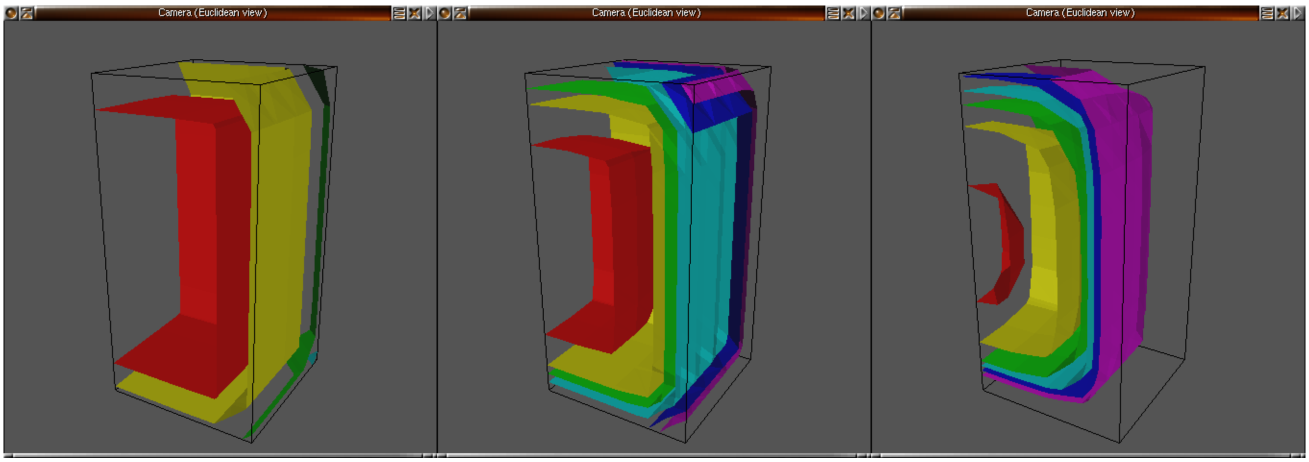
Note that the spreadsheet-calculated total time of freezing is within 0.2% of the estimate in problem 4.


PS5b_ModSim_2002.gnumeric : Gnumeric
File Edit View Insert Format Tools Data Help
The Freezing Lake: Heat Conduction and Phase Change
Parameter Water Ice Units
Thickness 0.1 m
Cond. k 0.56 2.3 W/m-K
Density rho 1000 920 kg/m^3
Heat cap 4200 2100 J/kg-K
alpha 1.33e-07 1.19e-06 m^2/s
Latent heat 334000 J/kg
H.trans.coef 10 W/m^2-K
Grid points 10
Delta x 0.01 m
Fo_M 0.5
Delta t 375 42 seconds
Initial T 10 Deg. C
Environ. T -10 Deg. C
Enthalpy, J/m^3 (0=ice at 0 C)
Temperature, C
Time x0=0 0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.1
0 3.76e+08 3.76e+08 ...
42 3.74e+08 3.76e+08 ...
84 3.73e+08 3.76e+08 ...
126 3.72e+08 3.76e+08 ...
168 3.7e+08 3.76e+08 ...
Series Series2 Series3 Series4
Sheet1
Sum=110.55


PS5b_ModSim_2002.gnumeric : Gnumeric
File Edit View Insert Format Tools Data Help
Series Series2 Series3 Series4
Temperature, C
0.1 0 0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.1
3.76e+08 10 10 10 10 10 10 10 10 10 10 10
3.76e+08 9.6 10 10 10 10 10 10 10 10 10 10
3.76e+08 9.2528 9.9776 10 10 10 10 10 10 10 10 10
3.76e+08 8.948922 9.938266 9.998746 10 10 10 10 10 10 10 10
3.76e+08 8.68075 9.886249 9.995429 9.99993 10 10 10 10 10 10 10
Series Series2 Series3 Series4 Series5 Series6 Series7 Series8 Series9 Series10 Series11 Series12 Series13 Series14 Series15
Sheet1
Sum=110.55

Part II: Cast-A-Box

1. The following images were grabbed at $t = 0.005$, 0.025, and 0.065 seconds. The default temperatures are shown as contour surfaces, from inside to outside these are: 1195, 1150, 1100, 1050, 1000 and 950 degrees. In the plot on the right, the dark blue contour covers the top of the box, indicating that the top has frozen over and a considerable amount of liquid is trapped inside, which will almost certainly lead to a defect.



2. The next images were captured at times $t = 0.04$, 0.15, and 0.4 during a simulation run with option `-top_tenv 1000`, which might correspond to heating the top surface to that temperature using a flame. The same temperature contours are shown, and it is evident that the last liquid to solidify will be on the top surface of the casting, preventing such a defect. The last bit of liquid solidifies at $t = 0.515$.