The Novikov Theory for Symplectic Cohomology and Exact Lagrangian Embeddings

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Abstract

Given an exact symplectic manifold, can we find topological constraints to the existence of exact Lagrangian submanifolds?

I developed an approach using symplectic cohomology which provides such conditions for exact Lagrangians inside cotangent bundles and inside ALE hyperkähler spaces. For example, the only exact Lagrangians inside ALE hyperkähler spaces must be spheres.

The vanishing of symplectic cohomology is an obstruction to the existence of exact Lagrangians. In the above applications even though the ordinary symplectic cohomology does not vanish, one can prove that a Novikov homology analogue for symplectic cohomology does vanish.

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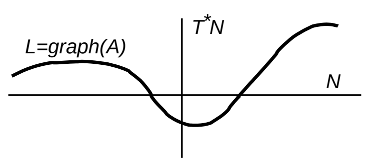
Chapter 1

Introduction

1.1 Exact Lagrangian submanifolds

Let $(M^{2n}, d\theta)$ be an exact symplectic manifold. A submanifold $L^n \subset M$ is called *Lagrangian* if $d\theta|_L = 0$ and *exact Lagrangian* if $\theta|_L$ is exact.

Example: Let $L = \text{graph}(\alpha) \subset T^*N$ be the graph of a $1-$form on N inside the cotangent bundle of N. Then L is Lagrangian if and only if α is closed, and L is exact Lagrangian if and only if α is exact. For instance, the zero section is an exact Lagrangian.



1.1.1 Exact Lagrangians inside cotangent bundles

Conjecture (Arnol'd '86) All exact Lagrangians $L \subset T^*N$ are isotopic to the zero section. (Stronger version: Hamiltonian isotopic)

Remark The Conjecture does not hold in general if L were just Lagrangian: for example, consider a Clifford torus $(S^1)^n \subset \mathbb{C}^n$ inside a small Darboux chart.

Example: $L \subset T^*S^2$**.** For homological reasons, L is either S^2, T^2, or unorientable. For $L = S^2$, it is known that L is isotopic to the zero section (Eliashberg-Polterovich '93), indeed it is Hamiltonian isotopic (R. Hind '03). It is known that L cannot be a torus (Viterbo '97) and a consequence of our thesis is that L cannot be unorientable. Thus the conjecture holds for T^*S^2.

Known results (for L, N closed orientable, $L \subset T^*N$)

1. $\pi_1 L \to \pi_1 N$ has finite cokernel (Lalonde-Sikorav '91)

2. $\pi_1 N = 1 \Rightarrow L \neq K(\pi, 1)$ (Viterbo '97)

3. If $\pi_1 N = 1$, (Maslov class of L) $= 0 \in H^1(L)$ and L, N are spin, then $H^*(N;\mathbb{R}) \cong H^*(L;\mathbb{R})$ (Nadler, Fukaya-Seidel-Smith '08)

Theorem 1. *Let $L \subset T^*N$ be an exact Lagrangian submanifold (L, N closed).*

1. *If $\pi_1 N = 1$ then $\pi_2 L \to \pi_2 N$ has finite cokernel, and $H^2(N) \hookrightarrow H^2(L)$ is injective.*

2. *If $\pi_1 N \neq 1$ and $\pi_m N$ is finitely generated for each $m \geq 2$, then $\pi_2 L \to \pi_2 N$ has finite cokernel, and $H^2(\widetilde{N}) \hookrightarrow H^2(\widetilde{L})$ is injective.*

We emphasize that there is no condition on the Maslov class of L and no assumption about the orientability of L or N.

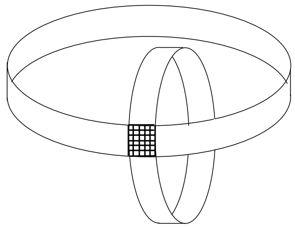
Example: Suppose $L \subset T^*S^2$ is unorientable. Then $H^2(S^2) \hookrightarrow H^2(L)$ means $\mathbb{Z} \hookrightarrow \mathbb{Z}/2\mathbb{Z}$, which is absurd. So L cannot exist.

1.1.2 Exact Lagrangians inside ALE spaces

Very little is known about exact Lagrangians inside spaces which are not cotangent bundles, and there are no known conjectures of what they ought to be. Our thesis answers this question for ALE spaces.

An *ALE space* is a hyperkähler four manifold which at infinity is asymptotic to $\mathbb{C}^2/\Gamma$ where $\Gamma \subset SL_2(\mathbb{C})$ is a finite subgroup.

Example: Take two copies of T^*S^2 and over a small patch plumb them together by



identifying the fibre directions of one with the base directions of the other. This yields an ALE space for $\Gamma = \mathbb{Z}/3\mathbb{Z}$ with Dynkin diagram $\bullet\!\!-\!\!\bullet$ (A_2). In general plumbings of copies of T^*S^2 according to a Dynkin diagram of type A_n, D_n or E_n are ALE spaces.

Theorem 2. *The only exact Lagrangians in an ALE space are spheres.*

1.2 Symplectic cohomology

For an exact symplectic manifold $(M, d\theta)$ with a sufficiently nice boundary one can associate certain invariants called *symplectic cohomology* $SH^*(M)$. Formally it is the Morse homology of the free loopspace $\mathcal{L}M = C^\infty(S^1, M)$ for the action function $A_H(\gamma) = \int(-\theta(\dot\gamma) + H(\gamma))\,dt$, where H is a Hamiltonian with sufficiently fast growth near the boundary of M.

Example: $M = T^*N$. The symplectic chain complex is generated by the $1-$periodic orbits of H (the critical points of A_H), and the chain differential is a count of cylinders connecting two generators and satisfying a PDE called *Floer's equation* (negative gradient trajectories of A_H for a suitable metric). For the cotangent bundle $M = T^*N \cong TN$, the Hamiltonian $H(v) = \frac{1}{2}|v|^2$ generates the geodesic flow and the $1-$periodic orbits of H are either the constants in N or certain closed geodesics away from N. Via a reparametrization, these closed geodesics correspond bijectively to the closed geodesics on a sphere bundle STN: the further we go away from N the longer the corresponding geodesic on STN. If we consider only the critical points of H on N (after perturbing H) then we get the Morse cohomology of N, or equivalently the cohomology of T^*N. This gives rise to a map $H^*(T^*N) \to SH^*(T^*N)$. Such a map from ordinary cohomology to the symplectic cohomology can be constructed also for more general M.

1.2.1 Viterbo functoriality

Foundational work of Viterbo (Viterbo '96) showed that an exact $L \subset (M, d\theta)$ yields a commutative diagram

$$\begin{array}{ccc} H_{n-*}(\mathcal{L}L) \cong SH^*(T^*L, d\theta) & \longleftarrow & SH^*(M, d\theta) \\ \uparrow c_* & & \uparrow c_* \\ H_{n-*}(L) \cong H^*(L) & \xleftarrow{\quad j^* \quad} & H^*(M) \end{array}$$

where the left vertical map is the inclusion of constants. Since $j^*1 = 1$, we obtain a non-zero element $c_* j^* 1$ in $H_*(\mathcal{L}L)$. Therefore $SH^*(M)$ cannot vanish if L exists.

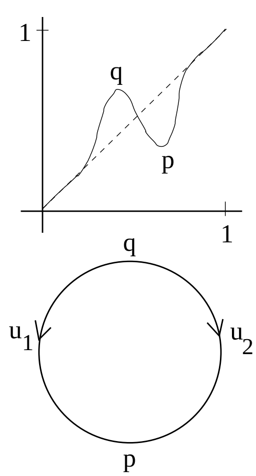
Problem. In the above applications, $SH^*(M)$ does not vanish. The key to our applications will be to tweak the setup:

Theorem 3. *The diagram holds when we use twisted coefficients in a Novikov bundle associated to a closed* $1-$*form* $[\alpha] \in H^1(\mathcal{L}M)$.

1.3 Twisted Symplectic Cohomology

Warm-up example: Novikov Homology

We will explain the twisting argument for $X = S^1$ with $\alpha = d\widetilde{\theta}$, where $\widetilde{\theta}$ is a multi-valued function on $S^1 = \mathbb{R}/\mathbb{Z}$ with two critical points $\alpha(q) = 0 = \alpha(p)$:



We now pretend to do Morse homology: instead of df for a Morse function f, we have a $1-$form α, and given a nice metric the form α defines a vector field for which we study the trajectories connecting the zeros of α.

The Morse homology differential of p counts the number of trajectories flowing into p. There are two opposite trajectories u_1, u_2 from q to p. Thus in ordinary Morse homology there are two generators q, p, generating the usual homology of S^1.

The twisted chain complex is $\mathbb{Z}((t))q \oplus \mathbb{Z}((t))p$, where $\mathbb{Z}((t))$ is the ring of formal Laurent series in t. The differential in the Morse homology picture would be $\partial p = q - q = 0$ so we would obtain the homology of S^1 over the ring $\mathbb{Z}((t))$. In the Novikov homology picture we insert weights:

$$\partial p = t^{\int_{u_1} \alpha} q - t^{\int_{u_2} \alpha} q \neq 0$$

since $\int_{S^1} \alpha \neq 0$. Therefore, q is the boundary of a multiple of p, so $NH_*(S^1)_\alpha = 0$.

If α had been exact, then $\int_{S^1} \alpha = 0$, and so we would just get back our ordinary homology: $NH_*(S^1)_{\text{exact}} = H_*(S^1) \otimes \mathbb{Z}((t))$.

Definition: The *twisted symplectic cohomology* $NSH^*(M)_\alpha$, for $[\alpha] \in H^1(\mathcal{L}M)$, is formally the Novikov homology of $\mathcal{L}M$ with respect to the action function, using weights $t^{\int_u \alpha}$.

Just as in the warm-up example, if α is exact then we just recover the ordinary symplectic cohomology (over the ring $\mathbb{Z}((t))$). When studying unorientable L, we assume that $\mathbb{Z}$ is replaced by $\mathbb{Z}/2$, otherwise nothing changes.

1.3.1 Transgressed forms

We now explain what choice of one-form $[\alpha] \in H^1(\mathcal{L}M)$ is appropriate. The transgression is defined by

$$\tau = \pi \circ ev^* : H^2(M; \mathbb{R}) \xrightarrow{ev^*} H^2(\mathcal{L}M \times S^1; \mathbb{R}) \xrightarrow{\pi} H^1(\mathcal{L}M; \mathbb{R}) ,$$

where ev is the evaluation map and π is the projection to the Künneth summand. Explicitly, $\tau\beta$ evaluated on a smooth path u in $\mathcal{L}M$ is given by integrating β over the corresponding cylinder in M. We will need some basic facts:

1. τ is an isomorphism if $\pi_1 M = 1$.

2. $\tau\beta|_M = 0$ vanishes when restricted to constant loops.

3. $\tau\beta$ can be identified with the induced map $\pi_2 M \to \mathbb{R}$ (identify $\pi_2 M \subset \pi_1(\mathcal{L}M)$).

1.3.2 Proofs of Theorem 1 and 2

Theorem 3 for this α states that there is a commutative diagram:

$$\begin{array}{ccc}
NH_{n-*}(\mathcal{L}L)_{\tau(\beta|L)} \cong NSH^*(T^*L, d\theta)_{\tau(\beta|L)} & \longleftarrow & NSH^*(M)_{\tau\beta} \\
\uparrow{\scriptstyle c_*} & & \uparrow{\scriptstyle c_*} \\
H_{n-*}(L) \otimes \mathbb{Z}(\!(t)\!) \cong H^*(L) \otimes \mathbb{Z}(\!(t)\!) & \xleftarrow{j^*\otimes 1} & H^*(M) \otimes \mathbb{Z}(\!(t)\!)
\end{array}$$

Suppose that $NSH^*(M)_{\tau\beta} = 0$ and $\tau(\beta|L) = 0$, then this simplifies to

$$\begin{array}{ccc}
H_{n-*}(\mathcal{L}L) \otimes \mathbb{Z}(\!(t)\!) & \longleftarrow & 0 \\
\uparrow{\scriptstyle c_*} & & \uparrow{\scriptstyle c_*} \\
H_{n-*}(L) \otimes \mathbb{Z}(\!(t)\!) \cong H^*(L) \otimes \mathbb{Z}(\!(t)\!) & \xleftarrow{j^*\otimes 1} & H^*(M) \otimes \mathbb{Z}(\!(t)\!)
\end{array}$$

But $c_* j^* 1$ is non-zero, so the commutativity of the digram is contradicted.

Conclusion: If $NSH^*(M)_{\tau\beta} = 0 \Rightarrow \tau(\beta|_L) \neq 0$.

Proving Theorems 1 and 2 using this Conclusion.

Case 1. $M = T^*N$**.**

Theorem 4. *Suppose* $\pi_1 N = 1$, $\beta \neq 0$. *Then* $NSH^*(T^*N)_{\tau\beta} \cong NH_{n-*}(\mathcal{L}N)_{\tau\beta} = 0$.

The proof of the vanishing of $NH_*(\mathcal{L}N)_{\tau\beta}$ (for closed simply connected N) only uses classical methods in algebraic topology. By the Conclusion, we deduce:

$\Rightarrow \tau\beta \to \tau(\beta|_L)$ is injective

$\Rightarrow \mathrm{Hom}(\pi_2 N, \mathbb{R}) \hookrightarrow \mathrm{Hom}(\pi_2 L, \mathbb{R})$

$\Rightarrow \pi_2 L \to \pi_2 N$ has finite cokernel

$\Rightarrow$ Theorem 1.

Case 2. $M =$ **ALE space.**

If L is not a sphere then:

- if L is orientable, then $\pi_2 L = 0$ so $\tau(\beta|_L) = 0$

- if L is unorientable, then $H^2(L) = \mathbb{Z}/2\mathbb{Z}$ so $\tau(\beta|_L) = 0$.

$\Rightarrow$ by the Conclusion, if $NSH^*(M)_{\tau\beta} = 0$ then L must be a sphere.

Thus Theorem 2 follows by

Theorem 5. *$NSH^*(M)_{\tau\omega} = 0$ for generic symplectic forms ω on M.*

Steps of proof:

1. There is a hyperkähler structure on M such that $\omega = g(I\cdot, \cdot)$ (Kronheimer '89).

2. For (M, ω) it is possible to prove $SH^*(M, \omega) = 0$ by exploiting an S^1-action on (M, ω) (which doesn't exist for $(M, d\theta)$).

3. By the following theorem, $SH^*(M, \omega) \cong NSH^*(M, d\theta)_{\tau\omega}$.

Theorem 6 (Deformation theorem)**.** *Let $(M, d\theta)$ be an exact symplectic manifold with nice boundary. Let β be a small closed 2-form with support $\subset$ int(M). Then*

$$SH^*(M, d\theta + \beta) \cong NSH^*(M, d\theta)_{\tau\beta}.$$

The deformation theorem is a non-trivial statement because there is no a priori control of the energy of Floer trajectories for non-exact symplectic forms. The theorem is proved by considering how the moduli space of trajectories varies as $d\theta$ is appropriately deformed into $d\theta + \beta$. Given an admissible Hamiltonian, it is possible to arrange that the generators of the two chain complexes are identical and we prove that for very small β the above isomorphism is induced by the identity map on the level of chains. This is because the moduli spaces of rigid Floer trajectories, as the symplectic form varies, form a 1$-$parameter family. The proof of this fact relies not only on a transversality result (which proves that the family is a smooth one-dimensional manifold) but also on a compactness result. It is necessary to prove that there aren't trajectories for $d\theta + \varepsilon\beta$ of arbitrarily large energy as $\varepsilon \to 0$ which cannot be detected at $\varepsilon = 0$.

The compactness result relies on a Lyapunov property of the $(d\theta)-$action $A_{d\theta}(\gamma) = \int(-\theta(\dot{\gamma}) + H(\gamma))\, dt$ with respect to $(d\theta + \beta)-$Floer trajectories. In particular, there is an a priori energy estimate for the $(d\theta+\beta)-$trajectories u connecting x to y,

$$E_{d\theta+\beta}(u) \leq 2(A_{d\theta}(x) - A_{d\theta}(y)).$$

We recall that for $(d\theta)-$Floer trajectories u there is an a priori energy estimate $E_{d\theta}(u) = A_{d\theta}(x) - A_{d\theta}(y)$, but such an estimate does not exist in general in the non-exact setup since the action functional is usually multi-valued.

1.4 Outline of the Thesis

Chapter 2 reproduces verbatim my first paper [18].

Theoretical part: we construct the twisted symplectic cohomology, which we called *Novikov-symplectic cohomology* in the paper and we denoted it $SH^*(M;\underline{\Lambda}_\alpha)$ instead of $NSH^*(M)_\alpha$. We prove the twisted functoriality property, which is Theorem 3 above. In this chapter, we always assume that the symplectic manifold is exact (*Liouville domains*).

Application: we apply these results to the study of exact Lagrangians inside cotangent bundles. We prove Theorem 1 and the vanishing Theorem 4 above.

Chapter 3 reproduces almost verbatim my second paper [19].

Theoretical part: we construct symplectic cohomology for non-exact symplectic forms and we prove the deformation theorem (Theorem 6 above).

Application: we apply these results to the study of exact Lagrangians inside ALE spaces. We prove Theorem 2 and the vanishing Theorem 5 above.

Chapter 2

Novikov-symplectic cohomology and exact Lagrangian embeddings

2.1 Summary

Let N be a closed manifold satisfying a mild homotopy assumption, then for any exact Lagrangian $L \subset T^*N$ the map $\pi_2(L) \to \pi_2(N)$ has finite index. The homotopy assumption is either that N is simply connected, or more generally that $\pi_m(N)$ is finitely generated for each $m \geq 2$. The manifolds need not be orientable, and we make no assumption on the Maslov class of L.

We construct the Novikov homology theory for symplectic cohomology, denoted $SH^*(M; \underline{\Lambda}_\alpha)$, and we show that Viterbo functoriality holds. We prove that $SH^*(T^*N; \underline{\Lambda}_\alpha)$ is isomorphic to the Novikov homology of the free loopspace. Given the homotopy assumption on N, we show that this Novikov homology vanishes when $\alpha \in H^1(\mathcal{L}_0 N)$ is the transgression of a non-zero class in $H^2(\widetilde{N})$. Combining these results yields the above obstructions to the existence of L.

2.2 Introduction

Consider a disc cotangent bundle $(DT^*N, d\theta)$ of a closed manifold N^n together with its canonical symplectic form. We want to find obstructions to the existence of embeddings $j\colon L^n \hookrightarrow DT^*N$ for which $j^*\theta$ is exact. These are called *exact Lagrangian embeddings*. For now assume that all manifolds are orientable and that we use $\mathbb{Z}-$coefficients in (co)homology.

Denote by $p\colon L \to N$ the composite of j with the projection to the base. Recall that the ordinary transfer map $p_!\colon H_*(N) \to H_*(L)$ is obtained by Poincaré duality and the pull-back p^*, by composing

$$p_!\colon H_*(N) \to H^{n-*}(N) \to H^{n-*}(L) \to H_*(L).$$

For the space $\mathcal{L}_0N$ of smooth contractible loops in N such transfer maps need not exist, as Poincaré duality no longer holds. However, using techniques from symplectic topology, Viterbo [25, 27] showed that there is a transfer homomorphism

$$\mathcal{L}p_!\colon H_*(\mathcal{L}_0N) \to H_*(\mathcal{L}_0L)$$

which commutes with the ordinary transfer map for p,

$$\begin{array}{ccc} H_*(\mathcal{L}_0L) & \xleftarrow{\mathcal{L}p_!} & H_*(\mathcal{L}_0N) \\ \uparrow c_* & & \uparrow c_* \\ H_*(L) & \xleftarrow{p_!} & H_*(N) \end{array}$$

where $c\colon N \to \mathcal{L}_0N$ denotes the inclusion of constant loops.

For any $\alpha \in H^1(\mathcal{L}_0N)$, we can define the associated Novikov homology theory, which is in fact homology with twisted coefficients in the bundle of Novikov rings $\Lambda = \mathbb{Z}((t))$ associated to a singular cocycle representing α. We denote the bundle by $\underline{\Lambda}_\alpha$ and the Novikov homology by $H_*(\mathcal{L}_0N; \underline{\Lambda}_\alpha)$.

Main Theorem. *For all exact $L \subset T^*N$ and all $\alpha \in H^1(\mathcal{L}_0N)$, there exists a* *commutative diagram*

$$\begin{array}{ccc} H_*(\mathcal{L}_0 L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & \xleftarrow{\mathcal{L}p_!} & H_*(\mathcal{L}_0 N; \underline{\Lambda}_\alpha) \\ \uparrow{\scriptstyle c_*} & & \uparrow{\scriptstyle c_*} \\ H_*(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & \xleftarrow{p_!} & H_*(N; c^*\underline{\Lambda}_\alpha) \end{array}$$

If $c^*\alpha = 0$ *then the bottom map becomes* $p_! \otimes 1 \colon H_*(L) \otimes \Lambda \leftarrow H_*(N) \otimes \Lambda$.

Suppose now that N is simply connected. Then a nonzero class $\beta \in H^2(N)$ defines a nonzero transgression $\tau(\beta) \in H^1(\mathcal{L}_0 N)$. The associated bundles $\underline{\Lambda}_{\tau(\beta)}$ on $\mathcal{L}_0 N$ and $\underline{\Lambda}_{\tau(p^*\beta)}$ on $\mathcal{L}_0 L$ restrict to trivial bundles on N and L.

Suppose $\tau(p^*\beta) = 0 \in H^1(\mathcal{L}_0 L)$. Then the above twisted diagram becomes

$$\begin{array}{ccc} H_*(\mathcal{L}_0 L) \otimes \Lambda & \xleftarrow{\mathcal{L}p_!} & H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) \\ {\scriptstyle c_*}\uparrow \; \downarrow{\scriptstyle q_*} & & \uparrow{\scriptstyle c_*} \\ H_*(L) \otimes \Lambda & \xleftarrow{p_!} & H_*(N) \otimes \Lambda \end{array}$$

where $q \colon \mathcal{L}_0 N \to N$ is the evaluation at 0 map. If N is simply connected and $\beta \neq 0$, then we will show that $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) = 0$, so the fundamental class $[N] \in H_n(N)$ maps to $c_*[N] = 0$. But $\mathcal{L}p_!(c_*[N]) = c_* p_![N] = c_*[L] \neq 0$ since c_* is injective on $H_*(L)$. Therefore $\tau(p^*\beta) = 0$ cannot be true. This shows that $\tau \circ p^* \colon H^2(N) \to H^1(\mathcal{L}_0 L)$ is injective. Now, from the commutative diagram

$$\begin{array}{ccc} H^2(N) & \xrightarrow[\sim]{\tau} & \mathrm{Hom}(\pi_2(N), \mathbb{Z}) \cong H^1(\mathcal{L}_0 N) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle (\mathcal{L}p)^*} \\ H^2(L) & \xrightarrow{\tau} & \mathrm{Hom}(\pi_2(L), \mathbb{Z}) \subset H^1(\mathcal{L}_0 L) \end{array}$$

we deduce that $p^* \colon H^2(N) \to H^2(L)$ and $\mathrm{Hom}(\pi_2(N), \mathbb{Z}) \to \mathrm{Hom}(\pi_2(L), \mathbb{Z})$ must be injective. Thus we deduce:

Main Corollary. *If* $L \subset T^*N$ *is exact and* N *is simply connected, then the image of* $p_* \colon \pi_2(L) \to \pi_2(N)$ *has finite index and* $p^* \colon H^2(N) \to H^2(L)$ *is injective.*

We emphasize that there is no assumption on the Maslov class of L in the statement – this is in contrast to the results of [17] and [6]: the vanishing of the Maslov class is crucial for their argument. Also observe that if $H^2(N) \neq 0$ then the corollary overlaps with Viterbo's result [27] that there is no exact Lagrangian $K(\pi, 1)$ embedded in a simply connected cotangent bundle.

We will prove that the corollary holds even when N and L are not assumed to be orientable. A concrete application of the Corollary is that there are no exact tori and no exact Klein bottles in T^*S^2. We will also generalize the Corollary to obtain a result in the non-simply connected setup:

Corollary. *Let N be a closed manifold with finitely generated $\pi_m(N)$ for each $m \geq 2$. If $L \subset T^*N$ is exact then the image of $p_* \colon \pi_2(L) \to \pi_2(N)$ has finite index.*

This is innovative since in [6], [17], and [27] it is crucial that N is simply connected.

The outline of the proof of the corollary required showing that the Novikov homology $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)})$ vanishes for nonzero $\beta \in H^2(\widetilde{N})$. The idea is as follows. A class $\tau(\beta) \in H^1(\mathcal{L}\widetilde{N}) = H^1(\mathcal{L}_0 N)$ gives rise to a cyclic covering $\overline{\mathcal{L}_0 N}$ of $\mathcal{L}_0 N$. Let t be a generator for the group of deck transformations. The Novikov ring $\Lambda = \mathbb{Z}((t)) = \mathbb{Z}[[t]][t^{-1}]$ is the completion in t of the group ring $\mathbb{Z}[t, t^{-1}]$ of the cover. The Novikov homology is isomorphic to $H_*(C_*(\overline{\mathcal{L}_0 N}) \otimes_{\mathbb{Z}[t,t^{-1}]} \Lambda)$.

Using the homotopy assumptions on N it is possible to prove that $H_*(\overline{\mathcal{L}_0 N})$ is finitely generated in each degree. It then easily follows from the flatness of Λ over $\mathbb{Z}[t, t^{-1}]$ and from Nakayama's lemma that

$$H_*(C_*(\overline{\mathcal{L}_0 N}) \otimes_{\mathbb{Z}[t,t^{-1}]} \Lambda) \cong H_*(\overline{\mathcal{L}_0 N}) \otimes_{\mathbb{Z}[t,t^{-1}]} \Lambda = 0.$$

The outline of the Chapter is as follows. In section 2.3 we recall the construction of symplectic cohomology and we explain how the construction works when we use twisted coefficients in the Novikov bundle of some $\alpha \in H^1(\mathcal{L}N)$, which we call Novikov-symplectic cohomology. In section 2.4 we recall Abbondandolo and Schwarz's construction [1] of the isomorphism between the symplectic cohomology of T^*N and the singular homology of the free loopspace $\mathcal{L}N$, and we adapt the isomorphism to Novikov-symplectic cohomology. In section 2.5 we review the construction of

Viterbo's commutative diagram, and we show how this carries over to the case of twisted coefficients. In section 2.6 we prove the main theorem and in section 2.7 we prove the main corollary. In section 2.8 we generalize the corollary to the case of non-simply connected cotangent bundles, and in section 2.9 we extend the results to the case when N and L are not assumed to be orientable.

2.3 Symplectic cohomology

We review the construction of symplectic cohomology, and refer to [25] for details and to [22] for a survey and for more references. We assume the reader is familiar with Floer homology for closed manifolds, for instance see [21].

2.3.1 Liouville domain setup

Let (M^{2n}, θ) be a Liouville domain, that is $(M, \omega = d\theta)$ is a compact symplectic manifold with boundary and the Liouville vector field Z, defined by $i_Z\omega = \theta$, points strictly outwards along ∂M. The second condition is equivalent to requiring that $\alpha = \theta|_{\partial M}$ is a contact form on ∂M, that is $d\alpha = \omega|_{\partial M}$ and $\alpha \wedge (d\alpha)^{n-1} > 0$ with respect to the boundary orientation on ∂M.

The Liouville flow of Z is defined for all negative time r, and it parametrizes a collar $(-\infty, 0] \times \partial M$ of ∂M inside M. So we may glue an infinite symplectic cone $([0, \infty) \times \partial M, d(e^r\alpha))$ onto M along ∂M, so that Z extends to $Z = \partial_r$ on the cone. This defines the completion $\hat{M}$ of M,

$$\hat{M} = M \cup_{\partial M} [0, \infty) \times \partial M.$$

We call $(-\infty, \infty) \times \partial M$ the collar of $\hat{M}$. We extend θ to the entire collar by $\theta = e^r\alpha$, and ω by $\omega = d\theta$. Later on, it will be convenient to change coordinates from r to $\mathrm{x} = e^r$. The collar will then be parametrized as the tubular neighbourhood $(0, \infty) \times \partial M$ of ∂M in $\hat{M}$, where ∂M corresponds to $\{\mathrm{x} = 1\}$.

Let J be an $\omega-$compatible almost complex structure on $\hat{M}$ which is of contact type on the collar, that is $J^*\theta = e^r dr$ or equivalently $J\partial_r = \mathcal{R}$ where $\mathcal{R}$ is the Reeb vector field (we only need this to hold for $e^r \gg 0$ so that a certain maximum principle applies there). Denote by $g = \omega(\cdot, J\cdot)$ the $J-$invariant metric.

2.3.2 Reeb and Hamiltonian dynamics

The Reeb vector field $\mathcal{R} \in C^\infty(T\partial M)$ on ∂M is defined by $i_{\mathcal{R}} d\alpha = 0$ and $\alpha(\mathcal{R}) = 1$. The periods of the Reeb vector field form a countable closed subset of $[0, \infty)$.

For $H \in C^\infty(\hat{M}, \mathbb{R})$ we define the Hamiltonian vector field X_H by

$$\omega(X_H, \cdot) = -dH.$$

If inside M the Hamiltonian H is a C^2-small generic perturbation of a constant, then the 1-periodic orbits of X_H inside M are constants corresponding precisely to the critical points of H.

Suppose $H = h(e^r)$ depends only on e^r on the collar. Then $X_H = h'(e^r)\mathcal{R}$. It follows that every non-constant 1-periodic orbit $x(t)$ of X_H which intersects the collar must lie in $\{e^r\} \times \partial M$ for some e^r and must correspond to a Reeb orbit $z(t) = x(t/T) \colon [0, T] \to \partial M$ with period $T = h'(e^r)$. Since the Reeb periods are countable, if we choose h to have a generic constant slope $h'(e^r)$ for $e^r \gg 0$ then there will be no 1-periodic orbits of X_H outside of a compact set of $\hat{M}$.

2.3.3 Action functional

We define the action functional for $x \in C^\infty(S^1, M)$ by

$$A_H(x) = -\int x^*\theta + \int_0^1 H(x(t))\, dt.$$

If $H = h(e^r)$ on the collar and x is a 1-periodic orbit of X_H in $\{e^r\} \times \partial M$, then

$$A_H(x) = -e^r h'(e^r) + h(e^r).$$

Let $\mathcal{L}\hat{M} = C^\infty(S^1, \hat{M})$ be the space of free loops in $\hat{M}$. The differential of A_H at $x \in \mathcal{L}\hat{M}$ in the direction $\xi \in T_x\mathcal{L}\hat{M} = C^\infty(S^1, x^*T\hat{M})$ is

$$dA_H \cdot \xi = -\int_0^1 \omega(\xi, \dot{x} - X_H)\, dt.$$

Thus the critical points $x \in \mathrm{Crit}(A_H)$ of A_H are precisely the 1-periodic Hamiltonian orbits $\dot{x}(t) = X_H(x(t))$. Moreover, we deduce that with respect to the L^2-metric $\int_0^1 g(\cdot,\cdot)\, dt$ the gradient of A_H is $\nabla A_H = J(\dot{x} - X_H)$.

2.3.4 Floer's equation

For $u\colon \mathbb{R} \times S^1 \to M$, the negative L^2-gradient flow equation $\partial_s u = -\nabla A_H(u)$ in the coordinates $(s,t) \in \mathbb{R} \times S^1$ is Floer's equation

$$\partial_s u + J(\partial_t u - X_H) = 0.$$

The action $A_H(u(s,\cdot))$ decreases in s along Floer solutions, since

$$\partial_s(A_H(u(s,\cdot))) = dA_H \cdot \partial_s u = -\int_0^1 \omega(\partial_s u, \partial_t u - X_H)\, dt = -\int_0^1 |\partial_s u|_g^2\, dt.$$

Let $\mathcal{M}'(x_-, x_+)$ denote the moduli space of solutions u to Floer's equation, which at the ends converge uniformly in t to the 1-periodic orbits $x_\pm$:

$$\lim_{s\to\pm\infty} u(s,t) = x_\pm(t).$$

These solutions u occur in $\mathbb{R}-$families because we may reparametrize the $\mathbb{R}$ coordinate by adding a constant. We denote by $\mathcal{M}(x_-, x_+) = \mathcal{M}'(x_-, x_+)/\mathbb{R}$ the space of unparametrized solutions.

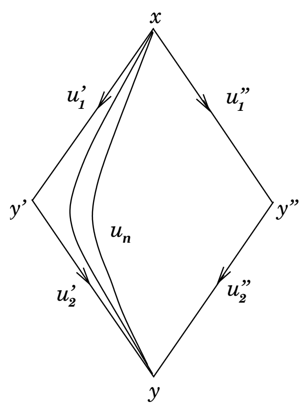


Figure 2-1: The x, y', y'', y are 1-periodic orbits of X_H, the lines are Floer solutions in $\hat{M}$. The $u_n \in \mathcal{M}_1(x, y)$ are converging to the broken trajectory $(u_1', u_2') \in \mathcal{M}_0(x, y') \times \mathcal{M}_0(y', y)$.

2.3.5 Energy

For a Floer solution u the energy is defined as

$$E(u) = \int |\partial_s u|^2 \, ds \, dt = \int \omega(\partial_s u, \partial_t u - X_H) \, ds \, dt = -\int \partial_s(A_H(u)) \, ds.$$

Thus for $u \in \mathcal{M}'(x_-, x_+)$ there is an a priori energy estimate,

$$E(u) = A_H(x_-) - A_H(x_+).$$

2.3.6 Compactness and the maximum principle

The only danger in this setup, compared to Floer theory for closed manifolds, is that there may be Floer trajectories $u \in \mathcal{M}(x_-, x_+)$ which leave any given compact set in $\hat{M}$. However, for any Floer trajectory u, a maximum principle applies to the function $e^r \circ u$ on the collar, namely: on any compact subset $\Omega \subset \mathbb{R} \times S^1$ the maximum of $e^r \circ u$ is attained on the boundary $\partial\Omega$. Therefore, if the $x_\pm$ lie inside $M \cup ([0, R] \times \partial M)$ then also all the Floer trajectories in $\mathcal{M}'(x_-, x_+)$ lie in there.

2.3.7 Transversality and compactness

Thanks to the maximum principle and the a priori energy estimates, the same analysis as for Floer theory for closed manifolds can be applied to show that for a generic time-dependent perturbation (H_t, J_t) of (H, J) the corresponding moduli spaces $\mathcal{M}(x_-, x_+)$ are smooth manifolds and have compactifications $\overline{\mathcal{M}}(x_-, x_+)$ whose boundaries are defined in terms of broken Floer trajectories (Figure 2-1). We write $\mathcal{M}_k(x_-, x_+) = \mathcal{M}'_{k+1}(x_-, x_+)/\mathbb{R}$ for the k-dimensional part of $\mathcal{M}(x_-, x_+)$.

The perturbation of (H, J) ensures that the differential $D\phi^1_{X_H}$ of the time 1 return map does not have eigenvalue 1, where $\phi^t_{X_H}$ is the flow of X_H. This non-degeneracy condition ensures that the 1-periodic orbits of X_H are isolated and it is used to prove the transversality results. In the proofs of compactness, the exactness of ω is used to exclude the possibility of bubbling-off of $J-$holomorphic spheres.

To keep the notation under control, we will continue to write (H, J) even though we are using the perturbed (H_t, J_t) throughout.

2.3.8 Floer chain complex

The Floer chain complex for a Hamiltonian $H \in C^\infty(\hat{M}, \mathbb{R})$ is the abelian group freely generated by 1-periodic orbits of X_H,

$$CF^*(H) = \bigoplus \left\{ \mathbb{Z}x : x \in \mathcal{L}\hat{M},\ \dot{x}(t) = X_H(x(t)) \right\},$$

and the differential ∂ on a generator $y \in \mathrm{Crit}(A_H)$ is defined as

$$\partial y = \sum_{u \in \mathcal{M}_0(x,y)} \epsilon(u)\, x,$$

where $\mathcal{M}_0(x, y)$ is the $0-$dimensional part of $\mathcal{M}(x, y)$ and the sign $\epsilon(u) \in \{\pm 1\}$ is determined by the choices of compatible orientations.

We may also filter the Floer complexes by action values $A, B \in \mathbb{R} \cup \{\pm\infty\}$:

$$CF^*(H; A, B) = \bigoplus \left\{ \mathbb{Z}x : x \in \mathcal{L}\hat{M},\ \dot{x}(t) = X_H(x(t)),\ A < A_H(x) < B \right\}.$$

This is a quotient complex of $CF^*(H)$ if $B \neq \infty$. Observe that increasing A gives a subcomplex, $CF^*(H; A', B) \subset CF^*(H; A, B)$ for $A < A' < B$. Moreover there are natural action-restriction maps $CF^*(H; A, B) \to CF^*(H; A, B')$ for $A < B' < B$, because the action decreases along Floer trajectories.

Standard methods show that $\partial^2 = 0$, and we denote by $HF^*(H)$ and $HF^*(H; A, B)$ the cohomologies of these complexes.

2.3.9 Continuation maps

One might hope that the continuation method of Floer homology can be used to define a homomorphism between the Floer complexes $CF^*(H_-)$ and $CF^*(H_+)$ obtained for two Hamiltonians $H_\pm$. This involves solving the parametrized version of Floer's equation

$$\partial_s u + J_s(\partial_t u - X_{H_s}) = 0,$$

where J_s are $\omega-$compatible almost complex structures of contact type and H_s is a homotopy from H_- to H_+ (i.e. an $s-$dependent Hamiltonian with $(H_s, J_s) = (H_-, J_-)$ for $s \ll 0$ and $(H_s, J_s) = (H_+, J_+)$ for $s \gg 0$). If x and y are respectively 1-periodic orbits of X_{H_-} and X_{H_+}, then we can define a moduli space $\mathcal{M}(x, y)$ of such solutions u which converge to x and y at the ends. This time there is no freedom to reparametrize u in the $s-$variable.

The action $A_{H_s}(u(s, \cdot))$ along such a solution u will vary as follows

$$\partial_s(A_{H_s}(u(s, \cdot))) = -\int_0^1 |\partial_s u|^2 \, dt + \int_0^1 (\partial_s H_s)(u) \, dt,$$

so the action decreases if H_s is monotone decreasing, $\partial_s H_s \leq 0$. The energy is

$$E(u) = \int |\partial_s u|^2_{g_s} \, ds \wedge dt = A_{H_-}(x_-) - A_{H_+}(x_+) + \int (\partial_s H_s)(u) \, ds \wedge dt,$$

so an a priori bound will hold if $\partial_s H_s \leq 0$ outside of a compact set in $\hat{M}$.

If $H_s = h_s(e^r)$ on the collar and $\partial_s h'_s \leq 0$, then a maximum principle for $e^r \circ u$ as before will hold on the collar (we refer to [22] for a very clear proof) and therefore it automatically guarantees a bound on $(\partial_s H_s)(u)$ and thus an a priori energy bound.

Thus, if outside of a compact in $\hat{M}$ we have $H_s = h_s(e^r)$ and $\partial_s h_s' \leq 0$, then (after a generic C^2-small time-dependent perturbation of (H_s, J_s)) the moduli space $\mathcal{M}(x, y)$ will be a smooth manifold with a compactification $\overline{\mathcal{M}}(x, y)$ by broken trajectories and a continuation map $\phi\colon CF^*(H_+) \to CF^*(H_-)$ can be defined: on a generator $y \in \mathrm{Crit}(A_{H_+})$,

$$\phi(y) = \sum_{v \in \mathcal{M}_0(x,y)} \epsilon(v)\, x,$$

where $\mathcal{M}_0(x, y)$ is the 0−dimensional part of $\mathcal{M}(x, y)$ and $\epsilon(v) \in \{\pm 1\}$ depends on orientations. Standard methods show that ϕ is a chain map and that these maps compose well: given homotopies from H_- to K and from K to H_+, each satisfying the condition $\partial_s h_s' \leq 0$ outside of a compact in $\hat{M}$, then the composite $CF^*(H_+) \to CF^*(K) \to CF^*(H_-)$ is chain homotopic to ϕ. So on cohomology, $\phi\colon HF^*(H_+) \to HF^*(H_-)$ equals the composite $HF^*(H_+) \to HF^*(K) \to HF^*(H_-)$.

For example, a "compactly supported homotopy" is one where H_s is independent of s outside of a compact ($\partial_s H_s = 0$ for $s \gg 0$). Continuation maps for H_s and H_{-s} can then be defined and they will be inverse to each other up to chain homotopy.

2.3.10 Symplectic cohomology using only one Hamiltonian

We change coordinates from r to $\mathrm{x} = e^r$, so the collar is now $(0, \infty) \times \partial M \subset \hat{M}$ and $\partial M = \{\mathrm{x} = 1\}$.

Take a Hamiltonian H_∞ with $H_\infty = h(\mathrm{x})$ for $\mathrm{x} \gg 0$, such that $h'(\mathrm{x}) \to \infty$ as $\mathrm{x} \to \infty$. The symplectic cohomology is defined as the cohomology of the corresponding Floer complex (after a C^2-small time-dependent perturbation of (H_∞, J)),

$$SH^*(M; H_\infty) = HF^*(H_\infty).$$

The technical difficulty lies in showing that it is independent of the choices (H_∞, J).

2.3.11 Symplectic cohomology with action bounds

Similarly one defines the groups $SH^*(M; H_\infty; A, B) = HF^*(M; H_\infty; A, B)$, but these now depend on the choice of H_∞. However, for $B = \infty$, taking the direct limit as $A \to -\infty$ yields

$$\varinjlim SH^*(M; H_\infty; A, \infty) = SH^*(M; H_\infty),$$

since $CF^*(H_\infty; A, \infty)$ are subcomplexes exhausting $CF^*(H_\infty; -\infty, \infty)$ as $A \to -\infty$.

If we use action bounds, then it is sometimes possible to vary the Hamiltonian without using continuation maps. Let $H_1 = h_1(\mathrm{x})$ for $\mathrm{x} \geq \mathrm{x}_0$, and suppose $A_{h_1}(\mathrm{x}) = -\mathrm{x}h_1'(\mathrm{x}) + h_1(\mathrm{x}) < A$ for $\mathrm{x} \geq \mathrm{x}_0$. Let $H_2 = H_1$ on $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$ and $H_2 = h_2(\mathrm{x})$ with $A_{h_2}(\mathrm{x}) < A$ for $\mathrm{x} \geq \mathrm{x}_0$ (e.g. if $h_2'' \geq 0$). Then

$$CF^*(H_1; A, B) = CF^*(H_2; A, B)$$

are equal as complexes: the orbits in $\{\mathrm{x} \geq \mathrm{x}_0\}$ get discarded by the action bounds; the orbits agree in $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$ since $H_1 = H_2$ there; and the differential on these common orbits is the same because the maximum principle forces the Floer trajectories to lie in $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$, where $H_1 = H_2$, so the Floer equations agree.

For example, let $H_1 = h_1(\mathrm{x}) = \frac{1}{2}\mathrm{x}^2$ on $\mathrm{x} > 0$, so $A_{h_1}(\mathrm{x}) = -\frac{1}{2}\mathrm{x}^2$. Take $H_2 = H_1$ on $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$ and extend H_2 linearly on $\{\mathrm{x} \geq \mathrm{x}_0\}$. Then $CF^*(H_1; -\frac{1}{2}\mathrm{x}_0^2; \infty) = CF^*(H_2; -\frac{1}{2}\mathrm{x}_0^2; \infty)$. By this trick, $SH^*(M; H_\infty; A, \infty)$ can be computed by a Hamiltonian which is linear at infinity, and so $SH^*(M; H_\infty)$ can be computed as a direct limit using Hamiltonians which are linear at infinity and whose slopes at infinity become steeper and steeper. We now make this precise.

2.3.12 Hamiltonians linear at infinity

Consider Hamiltonians H which equal

$$h_{c,C}^m(\mathrm{x}) = m(\mathrm{x} - c) + C$$

for $\mathrm{x} \gg 0$. We assume that the slope $m > 0$ does not occur as the value of the period of any Reeb orbit. If H_s is a homotopy from H_- to H_+ among such Hamiltonians, i.e. $H_s = h^{m_s}_{c_s,C_s}(\mathrm{x})$ for $x \gg 0$, then the maximum principle (and hence a priori energy bounds for continuation maps) will hold if

$$\partial_s \partial_{\mathrm{x}} h^{m_s}_{c_s,C_s} = \partial_s m_s \leq 0.$$

Suppose that $\partial_s m_s \leq 0$, satisfying $m_s = m_-$ for $s \ll 0$ and $m_s = m_+$ for $s \gg 0$, and suppose that the action values $A_{H_s}(x)$ of 1-periodic orbits x of X_{H_s} never cross the action bounds A, B. Then a continuation map can be defined,

$$\phi \colon\, CF^*(H_+; A, B) \to CF^*(H_-; A, B).$$

These maps compose well: $\phi' \circ \phi''$ is chain homotopic to ϕ (where to define ϕ', ϕ'' we use m'_s varying from m_- to some m, m''_s varying from m to m_+, and the analogous assumptions as above hold). For example if we vary only c, C, and not m, then $\partial_s m_s = 0$ outside of a compact and the continuation map ϕ for H_s can be inverted (up to chain homotopy) by using the continuation map for H_{-s}. Thus, up to isomorphism, $HF^*(H)$ is independent of the choice of the constants c, C in $h^m_{c,C}$.

2.3.13 Symplectic cohomology as a direct limit

Suppose $H_\infty = h(\mathrm{x})$ for $\mathrm{x} \gg 0$ and $h'(\mathrm{x}) \to \infty$ as $\mathrm{x} \to \infty$. Suppose also that $\mathrm{x}h''(\mathrm{x}) > \delta > 0$ for $x \gg 0$. This implies that $\partial_{\mathrm{x}} A_h = -\mathrm{x}h''(\mathrm{x}) < -\delta$ so A_h decreases to $-\infty$ as $\mathrm{x} \to \infty$.

Given $A \in \mathbb{R}$, suppose $A_h(\mathrm{x}) = -\mathrm{x}h'(\mathrm{x}) + h(\mathrm{x}) < A$ for $\mathrm{x} \geq \mathrm{x}_0$. Define $H = H_\infty$ on $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$ and extend H linearly in x for $\mathrm{x} \geq \mathrm{x}_0$. Then $CF^*(H; A, B) = CF^*(H_\infty; A, B)$, and $CF^*(H; A, B)$ is a subcomplex of $CF^*(H_\infty; -\infty, B)$.

Decreasing A to $A' < A$ defines some Hamiltonian H' which is steeper at infinity, and it induces a continuation map $CF^*(H; A, B) \to CF^*(H'; A', B)$. The direct limit over these continuation maps yields a chain isomorphism

$$\varinjlim CF^*(H; A, B) \to CF^*(H_\infty; -\infty, B),$$

which by the exactness of direct limits induces an isomorphism on cohomology

$$\varinjlim HF^*(H; A, B) \to SH^*(M; H_\infty; -\infty, B).$$

So an alternative definition is

$$SH^*(M) = \varinjlim HF^*(H),$$

where the direct limit is over the continuation maps for all the Hamiltonians which are linear at infinity, ordered by increasing slopes $m > 0$. In the above argument, we chose particular H which approximated H_∞ on larger and larger compacts. However, the direct limit can be taken over any family of H with slopes at infinity $m \to \infty$ because, up to an isomorphism induced by a continuation map, $HF^*(H)$ is independent of the choice of H for fixed m, so any two cofinal families ($m \to \infty$) will give the same limit up isomorphism.

2.3.14 Novikov bundles of coefficients

We recommend [28] as a reference on local systems. Let $\mathcal{L}N = C^\infty(S^1, N)$ denote the free loopspace of a manifold N, and let $\mathcal{L}_0 N$ be the component of contractible loops. The Novikov ring

$$\Lambda = \mathbb{Z}((t)) = \mathbb{Z}[[t]][t^{-1}]$$

is the ring of formal Laurent series. Let α be a singular cocycle representing $a \in H^1(\mathcal{L}N)$. The Novikov bundle $\underline{\Lambda}_\alpha$ is the local system of coefficients on $\mathcal{L}N$ defined by a copy Λ_γ of Λ over each loop $\gamma \in \mathcal{L}N$ and by the multiplication isomorphism $t^{\alpha[u]} \colon \Lambda_\gamma \to \Lambda_{\gamma'}$ for each path u in $\mathcal{L}N$ connecting γ to γ', where $\alpha[\cdot] \colon C_1(\mathcal{L}N) \to \mathbb{Z}$ is evaluation on singular one-chains. A different choice of representative α for a gives an isomorphic local system, so by abuse of notation we write $\underline{\Lambda}_a$ instead of $\underline{\Lambda}_\alpha$ and $a[u]$ instead of $\alpha[u]$.

We will be using the Novikov bundle $\underline{\Lambda}_{\tau(\beta)}$ on $\mathcal{L}_0 N$ corresponding to the transgression $\tau(\beta) \in H^1(\mathcal{L}_0 N)$ of some $\beta \in H^2(N)$ (see 3.5.1). This bundle pulls back to a trivial bundle under the inclusion of constant loops $c\colon N \to \mathcal{L}_0 N$, since the transgression $\tau(\beta)$ vanishes on $\pi_1(N) \subset \pi_1(\mathcal{L}_0 N)$. Therefore we just get ordinary cohomology with coefficients in the ring Λ,

$$H^*(N; c^*\underline{\Lambda}_{\tau(\beta)}) \cong H^*(N; \Lambda).$$

Moreover, for any map $j\colon L \to T^*N$ the projection $p\colon L \to T^*N \to N$ induces a map $\mathcal{L}p\colon \mathcal{L}_0 L \to \mathcal{L}_0 N$, and the pull-back of the Novikov bundle is

$$(\mathcal{L}p)^*\underline{\Lambda}_{\tau(\beta)} \cong \underline{\Lambda}_{(\mathcal{L}p)^*(\tau(\beta))} \cong \underline{\Lambda}_{\tau(p^*\beta)}.$$

If $\tau(p^*\beta) = 0 \in H^1(\mathcal{L}_0 L)$, then this is a trivial bundle and

$$H_*(\mathcal{L}_0 L; (\mathcal{L}p)^*\underline{\Lambda}_{\tau(\beta)}) \cong H_*(\mathcal{L}_0 L) \otimes \Lambda.$$

2.3.15 Novikov-Floer cohomology

Let (M^{2n}, θ) be a Liouville domain (2.3.1). Let α be a singular cocyle representing a class in $H^1(\mathcal{L}M) \cong H^1(\mathcal{L}\hat{M})$. We define the Novikov-Floer chain complex for $H \in C^\infty(\hat{M}, \mathbb{R})$ with twisted coefficients in $\underline{\Lambda}_\alpha$ to be the $\Lambda-$module freely generated by the 1-periodic orbits of X_H,

$$CF^*(H; \underline{\Lambda}_\alpha) = \bigoplus \left\{ \Lambda x : x \in \mathcal{L}\hat{M},\ \dot{x}(t) = X_H(x(t)) \right\},$$

and the differential δ on a generator $y \in \mathrm{Crit}(A_H)$ is defined as

$$\delta y = \sum_{u \in \mathcal{M}_0(x,y)} \epsilon(u)\, t^{\alpha[u]}\, x,$$

where $\mathcal{M}_0(x,y)$ and $\epsilon(u) \in \{\pm 1\}$ are the same as in (2.3.8). The new factor $t^{\alpha[u]}$ which appears in the differential is precisely the multiplication isomorphism $\Lambda_x \to \Lambda_y$ of the local system $\underline{\Lambda}_\alpha$ which identifies the $\Lambda-$fibres over x and y.

As in the untwisted case, we assume that a generic C^2-small time-dependent perturbation of (H,J) has been made so that the transversality and compactness results of (2.3.7) for the moduli spaces $\mathcal{M}(x,y)$ are achieved.

Proposition 7. $(CF^*(H;\underline{\Lambda}_\alpha);\delta)$ *is a chain complex, i.e.* $\delta\circ\delta = 0$.

Proof. We mimick the proof that $\partial^2 = 0$ in Floer homology (see [21]). Observe Figure 2-1. A sequence $u_n \in \mathcal{M}'_2(x,y)$ converges to a broken trajectory $(u'_1,u'_2) \in \mathcal{M}'_1(x,y') \times \mathcal{M}'_1(y',y)$, in the sense that there are $s_n \to -\infty$ and $S_n \to \infty$ with

$$u_n(s_n+\cdot,\cdot) \to u'_1 \text{ and } u_n(S_n+\cdot,\cdot) \to u'_2 \text{ both in } C^\infty_{\text{loc}};$$

Conversely given such (u'_1,u'_2) there is a curve $u\colon [0,1) \to \mathcal{M}'_2(x,y)$, unique up to reparametrization and up to the choice of $u(0) \in \mathcal{M}'_2(x,y)$, which approaches (u'_1,u'_2) as $r \to 1$, and the curve is orientation preserving iff $\epsilon(u'_1)\epsilon(u'_2) = 1$.

So the boundary of $\overline{\mathcal{M}}_1(x,y)$ is parametrized by $\mathcal{M}_0(x,y') \times \mathcal{M}_0(y',y)$. The value of $d\alpha = 0$ on the connected component of $\mathcal{M}_1(x,y)$ shown in Figure 2-1 is equal to the sum of the values of α over the broken trajectories,

$$\alpha[u'_1] + \alpha[u'_2] = \alpha[u''_1] + \alpha[u''_2],$$

and since $\epsilon(u'_1)\epsilon(u'_2) = -\epsilon(u''_1)\epsilon(u''_2)$, we conclude that

$$\epsilon(u'_1)\, t^{\alpha[u'_1]}\, \epsilon(u'_2)\, t^{\alpha[u'_2]} = -\epsilon(u''_1)\, t^{\alpha[u''_1]}\, \epsilon(u''_2)\, t^{\alpha[u''_2]}.$$

Thus the broken trajectories contribute opposite $\Lambda-$multiples of x to $\delta(\delta y)$ for each connected component of $\mathcal{M}_1(x,y)$. Hence, summing over x, y',

$$\delta(\delta y) = \sum_{(u'_1,u'_2)\in\mathcal{M}_0(x,y')\times\mathcal{M}_0(y',y)} \epsilon(u'_1)\, t^{\alpha[u'_1]}\, \epsilon(u'_2)\, t^{\alpha[u'_2]}\, x = 0. \quad \square$$

Denote by $HF^*(H;\underline{\Lambda}_\alpha)$ the $\Lambda-$modules corresponding to the cohomology groups of the complex $(CF^*(H;\underline{\Lambda}_\alpha);\delta)$. We call these the Novikov-Floer cohomology groups. By filtering the chain complex by action as in (2.3.8), we can define

$$HF^*(H;\underline{\Lambda}_\alpha;A,B) = H^*(CF^*(H;\underline{\Lambda}_\alpha;A,B);\delta).$$

2.3.16 Twisted continuation maps

We now show that the continuation method described in (2.3.9) can be used in the twisted case under the same assumptions that we made in the untwisted case. Recall that this involves solving

$$\partial_s v + J_s(\partial_t v - X_{H_s}) = 0,$$

and that under suitable assumptions on (H_s, J_s) the moduli spaces $\mathcal{M}(x,y)$ of solutions v joining 1-periodic orbits x, y of X_{H_-} and X_{H_+} are smooth manifolds with compactifications $\overline{\mathcal{M}}(x,y)$ whose boundaries are given by broken trajectories.

So far, using a twisted differential does not change the setup. However, to make the continuation map $\phi\colon\, CF^*(H_+;\underline{\Lambda}_\alpha) \to CF^*(H_-;\underline{\Lambda}_\alpha)$ into a chain map we need to define it on a generator $y \in \mathrm{Crit}(A_{H_+})$ by

$$\phi(y) = \sum_{v\in\mathcal{M}_0(x,y)} \epsilon(v)\, t^{\alpha[v]}\, x,$$

where $\mathcal{M}_0(x,y)$ and $\epsilon(v) \in \{\pm 1\}$ are as in (2.3.9).

Proposition 8. *$\phi\colon\, CF^*(H_+;\underline{\Lambda}_\alpha) \to CF^*(H_-;\underline{\Lambda}_\alpha)$ is a chain map.*

Proof. We mimick the proof that ϕ is a chain map in the untwisted case [21]. Denote by $\mathcal{M}^{H_\pm}(\cdot,\cdot)$ the moduli spaces of Floer trajectories for $H_\pm$. Observe Figure 2-2.

A compactness result in Floer homology shows that a sequence of solutions $v_n \in \mathcal{M}_1(x,y)$ will converge to a broken trajectory

$$(u'_-, v') \in \mathcal{M}_0^{H_-}(x,x') \times \mathcal{M}_0(x',y) \;\text{ or }\; (v', u'_+) \in \mathcal{M}_0(x,y') \times \mathcal{M}_0^{H_+}(y',y).$$

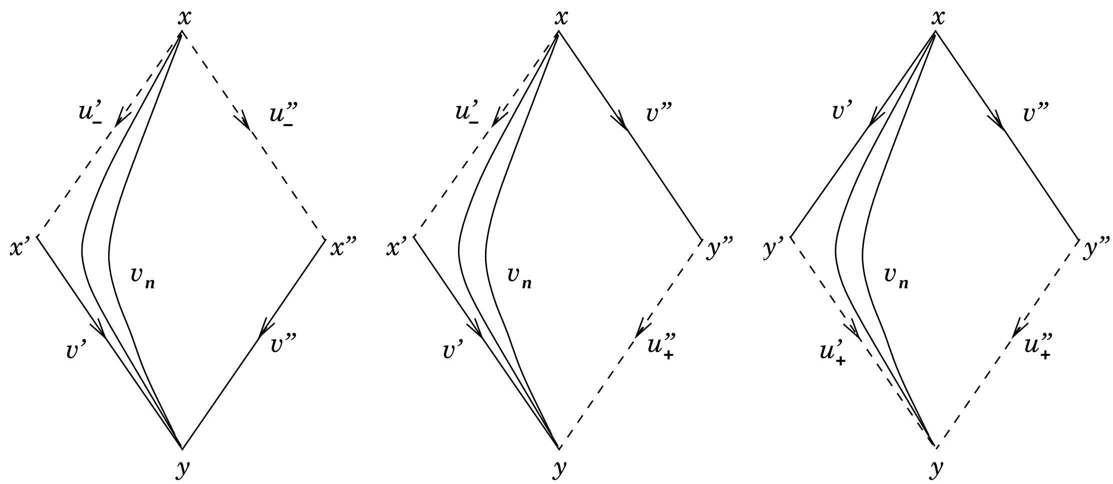


Figure 2-2: The dashed lines $u_\pm$ are Floer solutions converging to 1-periodic orbits of $X_{H_\pm}$, the solid lines are continuation map solutions, the $v_n \in \mathcal{M}_1(x,y)$ are converging to broken trajectories.

Conversely, given such (u'_-, v') or (v', u'_+) there is a smooth curve $v\colon\ [0,1) \to \mathcal{M}_1(x,y)$, unique up to reparametrization and up to the choice of $v(0)$, which approaches the given broken trajectory as $r \to 1$, and the curve is orientation preserving iff respectively $\epsilon(u'_-)\epsilon(v') = -1$ and $\epsilon(v')\epsilon(u'_+) = 1$.

Thus the boundary of $\overline{\mathcal{M}}_1(x,y)$ is parametrized by $-\mathcal{M}_0^{H_-}(x,x') \times \mathcal{M}_0(x',y)$ and by $\mathcal{M}_0(x,y') \times \mathcal{M}_0^{H_+}(y',y)$. The value of $d\alpha = 0$ on a connected component of $\mathcal{M}_1(x,y)$ as in Figure 2-2 is equal to the sum of the values of α over the broken trajectories. For instance, in the second figure

$$\alpha[u'_-] + \alpha[v'] = \alpha[v''] + \alpha[u''_+],$$

and since $\epsilon(u'_-)\epsilon(v') = \epsilon(v'')\epsilon(u''_+)$,

$$\epsilon(u'_-)\, t^{\alpha[u'_-]}\, \epsilon(v')\, t^{\alpha[v']} = \epsilon(v'')\, t^{\alpha[v'']}\, \epsilon(u''_+)\, t^{\alpha[u''_+]}.$$

Thus the broken trajectories contribute equal $\Lambda-$multiples of x to $\delta(\phi(y))$ and $\phi(\delta y)$ for that component of $\mathcal{M}_1(x,y)$. A similar computation shows that in the first or third figures, the two broken trajectories contribute opposite $\Lambda-$multiples of x and so in total give no contribution to $\delta(\phi(y))$ or $\phi(\delta y)$. We deduce that

$$\begin{aligned}\delta(\phi(y)) &= \sum_{(u'_-,v')\in\mathcal{M}_0^{H_-}(x,x')\times\mathcal{M}_0(x',y)} \epsilon(u'_-)\, t^{\alpha[u'_-]}\, \epsilon(v')\, t^{\alpha[v']}\, x \;= \\ &= \sum_{(v',u'_+)\in\mathcal{M}_0(x,y')\times\mathcal{M}_0^{H_+}(y',y)} \epsilon(v')\, t^{\alpha[v']}\, \epsilon(u'_+)\, t^{\alpha[u'_+]}\, x \;= \phi(\delta y),\end{aligned}$$

where we sum respectively over x, x' and x, y'. Hence ϕ is a chain map. □

A similar argument, by mimicking the proof of the untwisted case, shows that the twisted continuation maps compose well: given homotopies from H_- to K and from K to H_+ satisfying the conditions required in the untwisted case, the composite $CF^*(H_+;\underline{\Lambda}_\alpha) \to CF^*(K;\underline{\Lambda}_\alpha) \to CF^*(H_-;\underline{\Lambda}_\alpha)$ is chain homotopic to ϕ.

2.3.17 Novikov-symplectic cohomology

If we use the groups $HF^*(H;\underline{\Lambda}_\alpha)$ from (3.5.3) in place of $HF^*(H)$ in our discussion (2.3.10-2.3.13) of the symplectic cohomology groups of a Liouville domain, and we use the twisted continuation maps constructed in (2.3.16), then we obtain the $\Lambda-$modules

$$SH^*(M;H_\infty;\underline{\Lambda}_\alpha) \text{ and } SH^*(M;H_\infty;\underline{\Lambda}_\alpha;A,B),$$

which we call Novikov-symplectic cohomology groups.

So for H_∞ such that $H_\infty = h(\mathrm{x})$ for $\mathrm{x} \gg 0$ and $h'(\mathrm{x}) \to \infty$ as $\mathrm{x} \to \infty$, we define

$$SH^*(M;\underline{\Lambda}_\alpha) = HF^*(H_\infty;\underline{\Lambda}_\alpha).$$

Alternatively, we may use the Hamiltonians H which equal $h^m_{c,C}(\mathrm{x}) = m(\mathrm{x}-c)+C$ for $\mathrm{x} \gg 0$, and we take the direct limit over the twisted continuation maps between the corresponding twisted Floer cohomologies as the slopes $m > 0$ increase,

$$SH^*(M;\underline{\Lambda}_\alpha) = \varinjlim HF^*(H;\underline{\Lambda}_\alpha).$$

2.4 Abbondandolo-Schwarz isomorphism

For a closed (oriented) manifold N^n, the symplectic cohomology of the cotangent disc bundle $M^{2n} = DT^*N$ is isomorphic to the homology of the free loopspace,

$$SH^*(DT^*N) \cong H_{n-*}(\mathcal{L}N).$$

This was first proved by Viterbo [26], and there are now two alternative approaches by Abbondandolo-Schwarz [1] and Salamon-Weber [20]. We will use the Abbondandolo-Schwarz isomorphism and show that it carries over to twisted coefficients, but similar arguments could be carried out using either of the other approaches. We will recall the construction [1] of the chain isomorphism

$$(CM_*(\mathcal{E}), \partial^{\mathcal{E}}) \to (CF^{n-*}(H), \partial^H),$$

between the Morse complex of the Hilbert manifold $\mathcal{L}^1 N = W^{1,2}(S^1, N)$ with respect to a certain Lagrangian action functional $\mathcal{E}$ and the Floer complex of T^*N with respect to an appropriate Hamiltonian $H \in C^\infty(S^1 \times T^*N, \mathbb{R})$.

Let $\pi\colon T^*N \to N$ denote the projection. We use the standard symplectic structure $\omega = d\theta$ and Liouville field Z on T^*N, which in local coordinates (q, p) are

$$\theta = p\,dq \qquad \omega = dp \wedge dq \qquad Z = p\,\partial_p.$$

A metric on N induces metrics and Levi-Civita connections on TN and T^*N, and it defines a splitting $T_{(q,p)}T^*N \cong T_qN \oplus T_q^*N \cong T_qN \oplus T_qN$ into horizontal and vertical vectors and a connection $\nabla = \nabla_q \oplus \nabla_p$, and similarly for $T_{(q,v)}TN$. For this splitting our preferred $\omega-$compatible almost complex structure is $J\partial_q = -\partial_p$.

Remark. *Our action A_H is opposite to the action $\mathcal{A}$ used in [1], so our Floer trajectory $u(s,t)$ corresponds to $u(-s,t)$ in [1]. Our grading is $\mu(x) = n - \mu_{CZ}(x)$ (see [21], where the sign of H is opposite to ours), the one used in [1] is $\mu_{CZ}(x)$ and that in [22] is $-\mu_{CZ}(x)$. In our convention the index $\mu(x)$ agrees with the Morse index*

$\mathrm{ind}_H(x)$ *for* $x \in \mathrm{Crit}(H)$ *when* H *is a* C^2*-small Morse Hamiltonian.*

2.4.1 The Lagrangian Morse functional

The Morse function one considers on $\mathcal{L}^1 N = W^{1,2}(S^1, N)$ is the Lagrangian action functional

$$\mathcal{E}(q) = \int_0^1 L(t, q(t), \dot{q}(t))\, dt,$$

where the Lagrangian $L \in C^\infty(S^1 \times TN, \mathbb{R})$ is generic and satisfies certain growth conditions and a strong convexity assumption that ensure that: $\mathcal{E}$ is bounded below; the critical points of $\mathcal{E}$ are non-degenerate with finite Morse index; and $\mathcal{E}$ satisfies the Palais-Smale condition (any sequence of $q_n \in \mathcal{L}^1 N$ with bounded actions $\mathcal{E}(q_n)$ and with energies $\|\nabla \mathcal{E}(q_n)\|_{W^{1,2}} \to 0$ has a convergent subsequence). By an appropriate generic perturbation it is possible to obtain a metric G which is uniformly equivalent to the $W^{1,2}$ metric on $\mathcal{L}^1 N$ and for which $(\mathcal{E}, G)$ is a Morse-Smale pair. Denote by $\mathcal{M}^{\mathcal{E}}(q_-, q_+) = \mathcal{M}'_{\mathcal{E}}(q_-, q_+)/\mathbb{R}$ the unparametrized trajectories, where

$$\mathcal{M}'_{\mathcal{E}}(q_-, q_+) = \{v\colon \mathbb{R} \to \mathcal{L}^1 N : \partial_s v(s) = -\nabla \mathcal{E}(v(s)),\ \lim_{s \to \pm\infty} v(s) = q_\pm\}.$$

Under these assumptions, infinite dimensional Morse theory can be applied to $(\mathcal{L}^1 N, \mathcal{E}, G)$ and the Morse homology is isomorphic to the singular homology of $\mathcal{L}^1 N$ (which is isomorphic to the singular homology of $\mathcal{L}N$, since $\mathcal{L}^1 N$ and $\mathcal{L}N$ are homotopy equivalent). This isomorphism respects the filtration by action: the homology of the Morse complex generated by the $x \in \mathrm{Crit}(\mathcal{E})$ with $\mathcal{E}(x) < a$ is isomorphic to $H_*(\{q \in \mathcal{L}^1 N : \mathcal{E}(q) < a\})$. The isomorphism also respects the splitting of the Morse complex and the singular complex into subcomplexes corresponding to the components of $\mathcal{L}^1 N$ (which are indexed by the conjugacy classes of $\pi_1(N)$).

2.4.2 Legendre transform

L defines a Hamiltonian $H \in C^\infty(S^1 \times T^*N, \mathbb{R})$ by

$$H(t,q,p) = \max_{v \in T_q N} (p \cdot v - L(t,q,v)).$$

The strong convexity assumption on L ensures that there is a unique maximum precisely where $p = d_v L(t,q,v)$ is the differential of L restricted to the vertical subspace $T^{\text{vert}}_{(q,v)} TN \cong T_q N$, and it ensures that the Legendre transform

$$\mathfrak{L}\colon\ S^1 \times TN \to S^1 \times T^*N,\ (t,q,v) \mapsto (t,q,d_v L(t,q,v))$$

is a fiber-preserving diffeomorphism.

Pull back (ω, H, X_H) via $\mathfrak{L}$ to obtain $(\mathfrak{L}^*\omega, H \circ \mathfrak{L}, Y_L)$, so $\mathfrak{L}^*\omega(Y_L, \cdot) = -d(H \circ \mathfrak{L})$. The critical points of $\mathcal{E}$ are precisely the 1-periodic orbits $(q, \dot{q})$ of Y_L in TN, and these bijectively correspond to 1-periodic orbits x of X_H in T^*N via

$$(t,x) = \mathfrak{L}(t,q,\dot{q}).$$

Under this correspondence the Morse index of q is $m(q) = n - \mu(x)$ (in the conventions of [1], $m(q) = \mu_{CZ}(x)$). Moreover, for any $W^{1,2}-$path $x\colon\ [0,1] \to T^*N$,

$$\mathcal{E}(\pi x) \geq -A_H(x),$$

which becomes an equality iff $(t,x) = \mathfrak{L}(t, \pi x, \partial_t(\pi x))$ for all t.

2.4.3 The moduli spaces $\mathcal{M}^+(\mathbf{q},\mathbf{x})$

For 1-periodic orbits q of Y_L and x of X_H, define $\mathcal{M}^+(q,x)$ to be the collection of all maps $u \in C^\infty((-\infty,0) \times S^1, T^*N)$ which are of class $W^{1,3}$ on $(-1,0) \times S^1$ and which solve Floer's equation

$$\partial_s u + J(t,u)(\partial_t u - X_H(t,u)) = 0,$$

with the following boundary conditions:

i) as $s \to -\infty$, $u(s,\cdot) \to x$ uniformly in t;

ii) as $s \to 0$, u will converge to some loop $u(0,\cdot)$ of class $W^{2/3,3}$, and we require that the projection $\overline{q}(t) = \pi \circ u(0,t)$ in N flows backward to $q \in \text{Crit}(\mathcal{E})$ along the negative gradient flow $\phi^s_{-\nabla\mathcal{E}}$ of $\mathcal{E}$: $\phi^s_{-\nabla\mathcal{E}}(\overline{q}) \to q$ as $s \to -\infty$.

Loosely speaking, $\mathcal{M}^+(q,x)$ consists of pairs of trajectories (w, u_+) where w is a $-\nabla\mathcal{E}$ trajectory in N flowing out of q, and u_+ is a Floer solution in T^*N flowing out of x, such that w and πu_+ intersect in a loop $\overline{q}(t) = \pi u_+(0,t)$ in N.

2.4.4 Transversality and compactness

The assumption on H and L is that there are constants $c_i > 0$ such that for all $(t,q,p) \in S^1 \times T^*N$, $(t,q,v) \in S^1 \times TN$,

$$dH(p\partial_p) - H \geq c_0|p|^2 - c_1, \quad |\nabla_p H| \leq c_2(1+|p|), \quad |\nabla_q H| \leq c_2(1+|p|^2);$$
$$\nabla_{vv} L \geq c_3 \,\text{Id}, \ \ |\nabla_{vv} L| \leq c_4, \quad |\nabla_{qv} L| \leq c_4(1+|v|), \quad |\nabla_{qq} L| \leq c_4(1+|v|^2).$$

We also assume that a small generic perturbation of L (and hence H) are made so that the nondegeneracy condition (see 2.3.7) holds for 1-periodic orbits of Y_L and X_H. We call such H, L regular. For regular H, there are only finitely many 1-periodic orbits x of X_H with action $A_H(x) \geq a$, for $a \in \mathbb{R}$. After a small generic perturbation of J, the compactness and transversality results of (2.3.7) hold for the spaces $\mathcal{M}^H(x,y) = \mathcal{M}'(x,y)/\mathbb{R}$ of unparametrized Floer solutions in T^*N converging to $x, y \in \text{Crit}(A_H)$ at the ends, and similar results hold for $\mathcal{M}^+(q,x)$ by using the $W^{1,3}$ condition in the definition to generalize the proofs used for $\mathcal{M}'(x,y)$.

When all of the above assumptions are satisfied, we call (L, G, H, J) regular. In this case, $\mathcal{M}^{\mathcal{E}}(p,q)$, $\mathcal{M}^H(x,y)$ and $\mathcal{M}^+(q,x)$ are smooth manifolds with compactifications by broken trajectories, and their dimensions are:

$$\dim \mathcal{M}^{\mathcal{E}}(p,q) = m(p) - m(q) - 1$$
$$\dim \mathcal{M}^H(x,y) = \mu(x) - \mu(y) - 1$$
$$\dim \mathcal{M}^+(q,x) = m(q) + \mu(x) - n$$

and we denote by $\mathcal{M}_k^{\mathcal{E}}(p,q)$, $\mathcal{M}_k^H(x,y)$ and $\mathcal{M}_k^+(q,x)$ the k-dimensional ones.

Theorem (Abbondandolo-Schwarz [1]). *If (L,G,H,J) is regular then there is a chain-complex isomorphism $\varphi\colon (CM_*(\mathcal{E}),\partial^{\mathcal{E}}) \to (CF^{n-*}(H),\partial^H)$, which on a generator $q \in \mathrm{Crit}(\mathcal{E})$ is defined as*

$$\varphi(q) = \sum_{u_+ \in \mathcal{M}_0^+(q,x)} \epsilon(u_+)\, x,$$

where $\epsilon(u_+) \in \{\pm 1\}$ are orientation signs. The isomorphism is compatible with the splitting into subcomplexes corresponding to different conjugacy classes of $\pi_1(N)$, and it is compatible with the action filtrations: for any $a \in \mathbb{R}$ it induces an isomorphism on the subcomplexes generated by the q, x with $\mathcal{E}(q) < a$ and $-A_H(x) < a$.

2.4.5 Proof that φ is an isomorphism.

Since actions decrease along orbits and $\mathcal{E}(\pi x) \geq -A_H(x)$ with equality iff $(t,x) = \mathfrak{L}(t, \pi \circ x, \partial_t \pi x)$, we deduce that

$$\mathcal{E}(q) \geq \mathcal{E}(\overline{q}) \geq -A_H(u_+(0,\cdot)) \geq -A_H(x),$$

so $\mathcal{E}(q) \geq -A_H(x)$ with equality iff $q \equiv \overline{q}$, $u_+ \equiv x$, $q = \pi x$ and $(t,x) = \mathfrak{L}(t,q,\dot{q})$. Therefore if $\mathcal{E}(q) < -A_H(x)$ then $\mathcal{M}^+(q,x) = \emptyset$, and if $\mathcal{E}(q) = -A_H(x)$ then $\mathcal{M}^+(q,x)$ is either empty or, when $(t,x) = \mathfrak{L}(t,q,\dot{q})$, it consists of $u_+ \equiv x$. Now order the generators of $CM_*(\mathcal{E})$ according to increasing action and those of $CF^*(H)$ according to decreasing action, and so that the order is compatible with the correspondence $(t,x) = \mathfrak{L}(t,q,\dot{q})$. Then φ is a (possibly infinite) upper triangular matrix with ± 1 along the diagonal, so φ is an isomorphism.

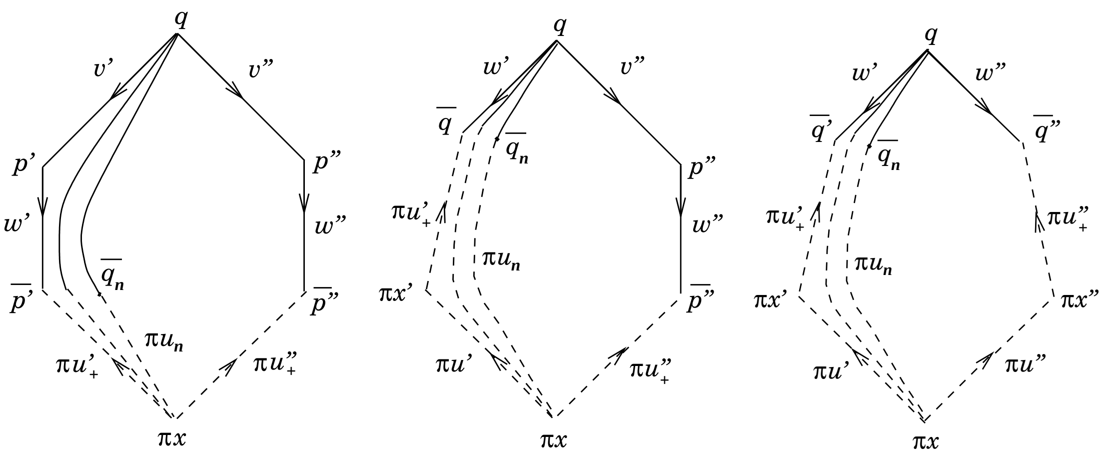


Figure 2-3: Solid lines are $-\nabla\mathcal{E}$ trajectories in N, dotted lines are the projections under $\pi\colon\, T^*N \to N$ of Floer solutions.

2.4.6 Proof that φ is a chain map

The differentials for the complexes $(CM_*(\mathcal{E}), \partial^{\mathcal{E}})$ and $(CF^*(H), \partial^H)$ are defined on generators $q \in \mathrm{Crit}(\mathcal{E})$, $y \in \mathrm{Crit}(A_H)$ by

$$\partial^{\mathcal{E}}(q) = \sum_{v \in \mathcal{M}_0^{\mathcal{E}}(q,p)} \epsilon(v)\, p \qquad\qquad \partial^H(y) = \sum_{u \in \mathcal{M}_0^H(x,y)} \epsilon(u)\, x$$

where $\epsilon(v), \epsilon(u) \in \{\pm 1\}$ depend on orientations. Observe Figure 2-3. A compactness argument shows that the broken trajectories that compactify $\mathcal{M}_1^+(q,x)$ are of two types: either (i) the $-\nabla\mathcal{E}$ trajectory breaks, or (ii) the Floer trajectory breaks. More precisely, if $u_n \in \mathcal{M}_1^+(q,x)$ and $\overline{q}_n(t) = \pi(u_n(0,t))$, then either

(i) there are $[v] \in \mathcal{M}_0^{\mathcal{E}}(q,p)$; $u'_+ \in \mathcal{M}_0^+(p,x)$; and reals $t_n \to -\infty$ with

$$\phi^{t_n}_{-\nabla\mathcal{E}}(\overline{q}_n) \to v(0) \text{ in } W^{1,2}, \text{ and } u_n \to u'_+ \text{ in } C^\infty_{\mathrm{loc}};$$

(ii) or there are $[u'] \in \mathcal{M}_0^H(x,x')$; $u'_+ \in \mathcal{M}_0^+(q,x')$; and reals $s_n \to -\infty$ with

$$u_n(s_n + \cdot, \cdot) \to u' \text{ and } u_n \to u'_+ \text{ both in } C^\infty_{\mathrm{loc}}.$$

Conversely, given (v, u'_+) or (u', u'_+) as above, there is a smooth curve $u\colon [0,1) \to \mathcal{M}_1^+(q,x)$, unique up to reparametrization and up to the choice of $u(0)$, which approaches the given broken trajectory as $r \to 1$, and the curve is orientation preserving iff respectively $\epsilon(v)\epsilon(u'_+) = 1$ and $\epsilon(u')\epsilon(u'_+) = -1$.

Thus the boundary of $\mathcal{M}_1^+(q,x)$ is parametrized by $\mathcal{M}_0^{\mathcal{E}}(q,p) \times \mathcal{M}_0^+(p,x)$ and by $-\mathcal{M}_0^H(x,x') \times \mathcal{M}_0^+(q,x')$. Figure 2-3 shows the possible components of $\mathcal{M}_1^+(q,x)$: in the first and third figures, the broken trajectories contribute zero respectively to $\varphi(\partial^{\mathcal{E}}(q))$ and $\partial^H(\varphi(q))$; in the second figure we see that $\epsilon(u')\epsilon(u'_+) = \epsilon(v'')\epsilon(u''_+)$, so the broken trajectories contribute $\pm x$ to both $\partial^H(\varphi(q))$ and $\varphi(\partial^{\mathcal{E}}(q))$. Therefore $\partial^H(\varphi(q)) = \varphi(\partial^{\mathcal{E}}(q))$, so φ is a chain map.

2.4.7 The twisted version of the Abbondandolo-Schwarz isomorphism

Let α be a singular cocyle representing a class in $H^1(\mathcal{L}^1 N) \cong H^1(\mathcal{L}N)$. We will use the bundles $\underline{\Lambda}_\alpha$ on $\mathcal{L}^1 N$ and $\underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}$ on $\mathcal{L}^1 T^*N$ (see 3.5.2), where $\mathcal{L}\pi\colon \mathcal{L}^1 T^*N \to \mathcal{L}^1 N$ is induced by $\pi\colon T^*N \to N$. The twisted complexes $(CM_*(\mathcal{E};\underline{\Lambda}_\alpha), \delta^{\mathcal{E}})$ and $(CF^*(H;\underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}), \delta^H)$ are freely generated over Λ respectively by the $q \in \mathrm{Crit}(\mathcal{E})$ and the $y \in \mathrm{Crit}(A_H)$, and the twisted differentials are defined by

$$\delta^{\mathcal{E}}(q) = \sum_{v \in \mathcal{M}_0^{\mathcal{E}}(q,p)} \epsilon(v)\, t^{-\alpha[v]}\, p \qquad\qquad \delta^H(y) = \sum_{u \in \mathcal{M}_0^H(x,y)} \epsilon(u)\, t^{\alpha[\mathcal{L}\pi(u)]}\, x$$

since $\alpha[\mathcal{L}\pi(u)] = (\mathcal{L}\pi)^*\alpha[u]$. The sign difference in the powers of t arises because $\delta^{\mathcal{E}}$ is a differential and δ^H is a codifferential. For simplicity, we write $\pi u = \mathcal{L}\pi(u)$.

Theorem 9. *If (L,G,H,J) is regular then for all $\alpha \in H^1(\mathcal{L}N)$ there is a chain-complex isomorphism $\varphi\colon (CM_*(\mathcal{E};\underline{\Lambda}_\alpha), \delta^{\mathcal{E}}) \to (CF^{n-*}(H;\underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}), \delta^H)$, which on a generator q is defined as*

$$\varphi(q) = \sum_{u_+ \in \mathcal{M}_0^+(q,x)} \epsilon(u_+)\, t^{-\alpha[w] + \alpha[\pi u_+]}\, x,$$

where $w\colon (-\infty, 0] \to \mathcal{L}^1 N$ *is the negative gradient trajectory* $w(s) = \phi^s_{-\nabla\mathcal{E}}(\overline{q})$ *connecting* q *to* $\overline{q}(\cdot) = \pi u_+(0, \cdot)$*. The isomorphism is compatible with the splitting into subcomplexes corresponding to different conjugacy classes of* $\pi_1(N)$*, and it is compatible with the action filtrations: for any* $a \in \mathbb{R}$ *it induces an isomorphism on the subcomplexes generated by the* q, x *with* $\mathcal{E}(q) < a$ *and* $-A_H(x) < a$*.*

After identifying Morse cohomology with singular cohomology, the map φ *induces an isomorphism*

$$SH^*(DT^*N; \underline{\Lambda}_\alpha) \cong H_{n-*}(\mathcal{L}N; \underline{\Lambda}_\alpha).$$

Proof. Figure 2-3 shows the possible connected components of $\mathcal{M}_1^+(q, x)$. Evaluating $d\alpha = 0$ on a component equals the sum of the values of α on the broken trajectories. For instance, in the second figure

$$-\alpha[w'] + \alpha[\pi u'_+] + \alpha[\pi u'] = -\alpha[v''] - \alpha[w''] + \alpha[\pi u''_+],$$

and therefore, since $\epsilon(u')\epsilon(u'_+) = \epsilon(v'')\epsilon(u''_+)$,

$$\epsilon(u')\epsilon(u'_+)\, t^{-\alpha[w'] + \alpha[\pi u'_+]}\, t^{\alpha[\pi u']} = \epsilon(v'')\epsilon(u''_+)\, t^{-\alpha[v'']}\, t^{-\alpha[w''] + \alpha[\pi u''_+]}.$$

Thus the broken trajectories contribute equally to $\delta^H(\varphi(q))$ and $\varphi(\delta^{\mathcal{E}}(q))$. A similar computation shows that in the first and third figures the broken trajectories contribute zero respectively to $\varphi(\delta^{\mathcal{E}}(q))$ and $\delta^H(\varphi(q))$. Hence

$$\begin{aligned} \delta^H(\varphi(q)) = & \sum_{(u', u'_+) \in \mathcal{M}_0^H(x, x') \times \mathcal{M}_0^+(q, x')} \epsilon(u')\, t^{\alpha[\pi u']} \cdot \epsilon(u'_+)\, t^{-\alpha[w'] + \alpha[\pi u'_+]}\, x \ = \\ = & \sum_{(v'', u''_+) \in \mathcal{M}_0^{\mathcal{E}}(q, p) \times \mathcal{M}_0^+(p, x)} \epsilon(v'')\, t^{-\alpha[v'']} \cdot \epsilon(u''_+)\, t^{-\alpha[w''] + \alpha[\pi u''_+]}\, x = \varphi(\delta^{\mathcal{E}}(q)), \end{aligned}$$

where we sum respectively over x, x' and over x, p, and where w', w'' are the $-\nabla\mathcal{E}$ trajectories ending in $\pi u'_+(0, \cdot)$, $\pi u''_+(0, \cdot)$. Hence φ is a chain map.

That φ is an isomorphism follows just as in the untwisted case, because for $\mathcal{E}(q) \leq -A_H(x)$ the only nonempty $\mathcal{M}_0^+(q, x)$ occurs when $(t, x) = \mathfrak{L}(t, q, \dot{q})$, and in this case $\mathcal{M}_0^+(q, x) = \{u_+\}$ where $u_+ \equiv x$ and $w \equiv q$ are independent of $s \in \mathbb{R}$ and so the coefficient of x in $\varphi(q)$ is

$$\epsilon(u_+)\, t^{-\alpha[w]+\alpha[\pi u_+]} = \epsilon(u_+) = \pm 1.$$

The last statement in the claim is a consequence of the identification of the Morse cohomology of $(\mathcal{L}^1 N, \mathcal{E}, G)$ with the singular cohomology of $\mathcal{L}^1 N$ just as in [1], after introducing the system $\underline{\Lambda}_\alpha$ of local coefficients. $\square$

2.5 Viterbo Functoriality

Let (M^{2n}, θ) be a Liouville domain (2.3.1), and suppose

$$i\colon\ (W^{2n}, \theta') \hookrightarrow (M^{2n}, \theta)$$

is a Liouville embedded subdomain, that is we require that $i^*\theta - e^{\rho}\theta'$ is exact for some $\rho \in \mathbb{R}$. For example the embedding $DT^*L \hookrightarrow DT^*N$, obtained by extending an exact Lagrangian embedding $L \hookrightarrow DT^*N$ to a neighbourhood of L, is of this type. We fix $\delta > 0$ with

$$0 < \delta < \min\{\text{periods of the nonconstant Reeb orbits on } \partial M \text{ and } \partial W\}.$$

We will now recall the construction of Viterbo's commutative diagram ([25]):

$$\begin{array}{ccc} SH^*(W) & \xleftarrow{SH^*(i)} & SH^*(M) \\ \uparrow{\scriptstyle c_*} & & \uparrow{\scriptstyle c_*} \\ H^*(W) & \xleftarrow{i^*} & H^*(M) \end{array}$$

2.5.1 Hamiltonians with small slopes

We now consider Hamiltonians H^0 as in (2.3.12), which are C^2-close to a constant on $\hat{M} \setminus (0,\infty) \times \partial M$; $H^0 = h(\mathrm{x})$ with slopes $h'(\mathrm{x}) \leq \delta$ for $\mathrm{x} \geq 0$; and which have constant slope $h'(\mathrm{x}) = m > 0$ for $\mathrm{x} \geq \mathrm{x}_0$.

A standard result in Floer homology is that (after a generic C^2-small time-independent perturbation of (H^0, J)) the 1-periodic orbits of X_{H^0} and the Floer trajectories connecting them inside $\hat{M} \setminus \{\mathrm{x} \geq \mathrm{x}_0\}$ are both independent of $t \in S^1$, and so these orbits correspond to critical points of H^0 and these Floer trajectories correspond to negative gradient trajectories of H^0. By the maximum principle, the Floer trajectories connecting these orbits do not enter the region $\{\mathrm{x} \geq \mathrm{x}_0\}$, and by the choice of δ there are no 1-periodic orbits in $\{\mathrm{x} \geq \mathrm{x}_0\}$ since there $0 < h'(\mathrm{x}) \leq \delta$.

The Floer complex $CF^*(H^0)$ is therefore canonically identified with the Morse complex $CM^*(H^0)$, which is generated by $\mathrm{Crit}(H^0)$ and whose differential counts the $-\nabla H^0$ trajectories. The Morse cohomology $HM^*(H^0)$ is isomorphic to the singular cohomology of $\hat{M}$ (which is homotopy equivalent to M), so

$$HF^*(H^0) \cong HM^*(H^0) \cong H^*(M).$$

Moreover, by Morse cohomology, a different choice $H^{0\prime}$ of H^0 yields an isomorphism $HM^*(H^{0\prime}) \cong H^*(M)$ which commutes with $HM^*(H^0) \cong H^*(M)$ via the continuation isomorphism $HM^*(H^0) \to HM^*(H^{0\prime})$.

2.5.2 Construction of $\mathbf{c}_*$

Recall from (2.3.13) that

$$SH^*(M) = \varinjlim HF^*(H),$$

where the direct limit is over the continuation maps for Hamiltonians H which equal $h^m_{c,C}(\mathrm{x}) = m(\mathrm{x} - c) + C$ for $\mathrm{x} \gg 0$, ordered by increasing slopes $m > 0$.

Since H^0 is such a Hamiltonian, there is a natural map $HF^*(H^0) \to \lim HF^*(H)$ arising as a direct limit of continuation maps. By 2.5.1, this defines a map

$$c_* \colon\ H^*(M) \to SH^*(M).$$

A different choice $H^{0\prime}$ yields a map $HF^*(H^{0\prime}) \to SH^*(M)$ which commutes with the map $HF^*(H^0) \to SH^*(M)$ via the continuation isomorphism $HF^*(H^0) \to$

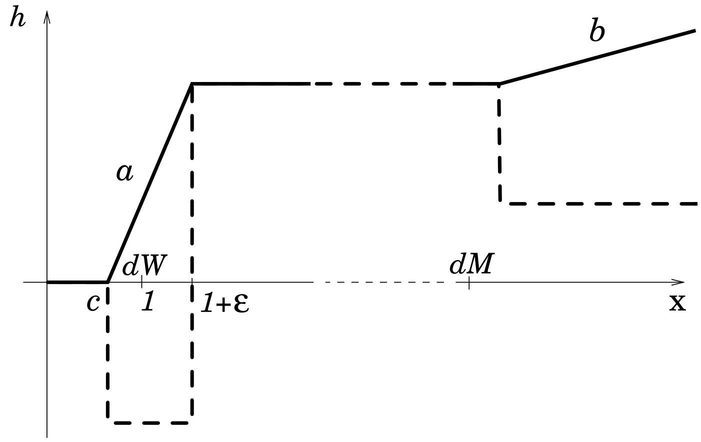


Figure 2-4: The solid line is a diagonal-step shaped Hamiltonian $h = h_c^{a,b}$ with slopes $a \gg b$. The dashed line is the action function $A_h(\mathrm{x}) = -\mathrm{x}h'(\mathrm{x}) + h(\mathrm{x})$.

$HF^*(H^{0\prime})$. Together with 2.5.1, this shows that c_* is independent of the choice of H^0.

2.5.3 Diagonal-step shaped Hamiltonians

We now consider the Liouville subdomain $i\colon W \hookrightarrow M$. The ∂_r−Liouville flow for θ' defines a tubular neighbourhood $(0, 1+\epsilon) \times \partial W$ of ∂W inside $\hat{M}$, where ∂W corresponds to $\mathrm{x} = e^r = 1$. This coordinate x may not extend to $\hat{M} \setminus W$, and it should not be confused with the x we previously used to parametrize $(0, \infty) \times \partial M \subset \hat{M}$.

We consider diagonal-step shaped Hamiltonians H as in Figure 3-2, which are zero on $W \setminus \{\mathrm{x} \geq c\}$ and which equal $h_c^{a,b}(\mathrm{x})$ on $\{\mathrm{x} \geq c\}$, where $h_c^{a,b}$ is piecewise linear with slope b at infinity; with slope $a \gg b$ on $(c, 1+\epsilon)$; and which is constant elsewhere. We assume that $0 \leq c \leq 1$ and that a, b are chosen generically so that they are not periods of Reeb orbits (see 2.3.2).

As usual, before we take Floer complexes we replace H by a generic C^2-small time-dependent perturbation of it, and the orbits and action values that we will mention take this into account. Let $M' \subset \hat{M}$ be the compact subset where h does not have slope b. Observe Figure 3-2: the 1-periodic orbits of X_H that can arise are:

1. critical points of H inside $W \setminus \{\mathrm{x} \geq c\}$ of action very close to 0;

2. nonconstant orbits near $\mathrm{x} = c$ of action in $(-ac, -\delta c)$;

3. nonconstant orbits near $\mathrm{x} = 1 + \epsilon$ of action in $(-ac, a(1 + \epsilon - c))$;

4. critical points of H in $M' \setminus (W \cup \{\mathrm{x} \leq 1 + \epsilon\})$ of action close to $a(1 + \epsilon - c)$;

5. nonconstant orbits near $\partial M'$ of action $\gg 0$ provided $a \gg b$.

Since the complement of the Reeb periods is open, there are no Reeb periods in $(a - \nu_a, a + \nu_a)$ for some small $\nu_a > 0$. Thus the actions in case (3) will be at least

$$-(a - \nu_a)(1 + \epsilon) + a(1 + \epsilon - c) = \nu_a(1 + \epsilon) - ac,$$

and for sufficiently small c, depending on a, we can ensure that this is at least ν_a. Hence (after a suitable perturbation of H) we can ensure that if $a \gg b$ and $c \ll a^{-1}$ then the actions of (1), (2) are negative and those of (3), (4), (5) are positive.

2.5.4 Construction of $\mathbf{SH^*(i)}$

Suppose H is a (perturbed) diagonal-step shaped Hamiltonian, with $a \gg b$ and $c \ll a^{-1}$ so that the orbits in W have negative actions and those outside W have positive actions. We write $CF^*(M, H)$ to emphasize that the Floer complex is computed for M. Consider the action-restriction map (2.3.8)

$$CF^*(M, H; -\infty, 0) \leftarrow CF^*(M, H).$$

Given two diagonal-step shaped Hamiltonians H, H' with $H \leq H'$ everywhere, pick a homotopy H_s from H' to H which is monotone ($\partial_s H_s \leq 0$). The induced continuation map $\phi\colon CF^*(M, H) \to CF^*(M, H')$ restricts to a map on the quotient complexes $\phi\colon CF^*(M, H; -\infty, 0) \to CF^*(M, H'; -\infty, 0)$ because the action decreases along Floer trajectories when H_s is monotone (see 2.3.9).

Consider the Hamiltonian H_W on the completion $\hat{W} = W \cup_{\partial W} [0, \infty) \times \partial W$ which equals H inside W and which is linear with slope a outside W. Then the quotient complex $CF^*(M, H; -\infty, 0)$ can be identified with $CF^*(W, H_W)$ by showing that there are no Floer trajectories connecting 1-periodic orbits of X_H in $\hat{M}$ which exit $W \cup \{\mathrm{x} \leq 1+\epsilon\}$. Therefore we obtain the commutative diagram

$$\begin{array}{ccc} CF^*(W, H'_W) & \longleftarrow & CF^*(M, H') \\ \uparrow & & \uparrow \\ CF^*(W, H_W) & \longleftarrow & CF^*(M, H) \end{array}$$

where the vertical maps are continuation maps and where the horizontal maps arose from action-restriction maps. Taking cohomology, and then taking the direct limit as $a \gg b \to \infty$ (so $c \ll a^{-1} \to 0$) defines the map $SH^*(i)$,

$$SH^*(i)\colon\ SH^*(W) \leftarrow SH^*(M).$$

2.5.5 Viterbo functoriality

Consider a (perturbed) diagonal-step shaped Hamiltonian $H = H^0$ with $c = 1$ and slopes $0 < b \ll a < \delta$ so that the orbits inside W have negative actions and those outside W have positive actions. Then H^0 and the corresponding H^0_W are of the type described in (2.5.1) for M and W respectively. The action-restriction map $CF^*(W, H^0_W) \leftarrow CF^*(M, H^0)$ is then identified with the map on Morse complexes $CM^*(W, H^0|_W) \leftarrow CM^*(M, H^0)$ which restricts to the generators $x \in \mathrm{Crit}(H^0)$ with $H^0(x) < 0$. In cohomology this map corresponds to the pullback on singular cohomology $i^*\colon\ H^*(W) \leftarrow H^*(M)$.

This identifies $CM^*(W, H^0|_W) \leftarrow CM^*(M, H^0)$ with the bottom map of the diagram in (2.5.4) when we take $H = H^0$, and so taking the direct limit over the H' we obtain Viterbo's commutative diagram in cohomology:

$$\begin{array}{ccc} SH^*(W) & \xleftarrow{SH^*(i)} & SH^*(M) \\ \uparrow c_* & & \uparrow c_* \\ H^*(W) & \xleftarrow{i^*} & H^*(M) \end{array}$$

2.5.6 Twisted Viterbo functoriality

We now introduce the twisted coefficients $\underline{\Lambda}_\alpha$ for some $\alpha \in H^1(\mathcal{L}\hat{M}) \cong H^1(\mathcal{L}M)$, as explained in (3.5.3) and (3.5.4). Recall that we have constructed twisted continuation maps (2.3.16) which compose well, so the discussion of (2.5.2) and (2.5.4) will hold in the twisted case provided that we understand how the local systems restrict.

Suppose H^0 is a Hamiltonian with small slope as in (2.5.1). In the twisted case the canonical identification of $CF^*(H^0)$ with the Morse complex $CM^*(H^0)$ becomes

$$CF^*(H^0; \underline{\Lambda}_\alpha) = CM^*(H^0; c^*\underline{\Lambda}_\alpha),$$

where $c^*\underline{\Lambda}_\alpha$ is the restriction of $\underline{\Lambda}_\alpha$ to the local system on $\hat{M} \subset \mathcal{L}_0\hat{M}$ which consists of a copy Λ_m of Λ over each $m \in \hat{M}$ and of the multiplication isomorphism $t^{\alpha[c\circ v]} = t^{c^*\alpha[v]} \colon \Lambda_m \to \Lambda_{m'}$ for every path $v(s)$ in $\hat{M}$ joining m to m', and where the twisted Morse differential is defined on $q_+ \in \mathrm{Crit}(H^0)$ analogously to the Floer case:

$$\delta q_+ = \sum \{\epsilon(v)\, t^{c^*\alpha[v]}\, q_- : q_- \in \mathrm{Crit}(H^0), \partial_s v = -\nabla H^0(v), \lim_{s\to\pm\infty} v(s) = q_\pm\}.$$

By mimicking the proof that $HM^*(H^0) \cong H^*(M)$, for twisted coefficients we have $HM^*(H^0; c^*\underline{\Lambda}_\alpha) \cong H^*(M; c^*\underline{\Lambda}_\alpha)$ (singular cohomology with coefficients in the local system $c^*\underline{\Lambda}_\alpha$, as defined in [28]).

As in (2.5.2), we get twisted continuation maps $CF^*(H^0; \underline{\Lambda}_\alpha) \to CF^*(H; \underline{\Lambda}_\alpha)$ for Hamiltonians H linear at infinity. In cohomology these maps yield a morphism $HF^*(H^0; \underline{\Lambda}_\alpha) \to \lim HF^*(H; \underline{\Lambda}_\alpha)$, where the direct limit is taken over twisted continuation maps as the slopes at infinity of the H increase. This defines

$$c_* \colon H^*(M; c^*\underline{\Lambda}_\alpha) \to SH^*(M; \underline{\Lambda}_\alpha).$$

In (2.5.4) we get action-restriction maps $CF^*(M, H; \underline{\Lambda}_\alpha; -\infty, 0) \leftarrow CF^*(M, H; \underline{\Lambda}_\alpha)$, and two choices of diagonal-step shaped Hamiltonians H, H' with $H \le H'$ induce a continuation map $\phi\colon CF^*(M, H; \underline{\Lambda}_\alpha) \to CF^*(M, H'; \underline{\Lambda}_\alpha)$ which restricts to the quotient complexes $\phi\colon CF^*(M, H; \underline{\Lambda}_\alpha; -\infty, 0) \to CF^*(M, H'; \underline{\Lambda}_\alpha; -\infty, 0)$.

Let $\mathcal{L}i\colon \mathcal{L}W \to \mathcal{L}M$ be the map induced by i. As in (2.5.4), the quotient complex $CF^*(M,H;\underline{\Lambda}_\alpha;-\infty,0)$ can be identified with $CF^*(W,H_W;\underline{\Lambda}_{(\mathcal{L}i)^*\alpha})$ because there are no Floer trajectories connecting 1-periodic orbits of X_H which exit $W\cup\{\mathrm{x}\le 1+\epsilon\}$ in $\hat{M}$ and so the twisted differentials of the two complexes agree since $(\mathcal{L}i)^*\alpha$ and α agree on the common Floer trajectories inside $W\cup\{\mathrm{x}\le 1+\epsilon\}$.

As in (2.5.4), the direct limit over the twisted continuation maps for diagonal-step shaped H of the action-restriction maps

$$CF^*(W,H_W;\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) \leftarrow CF^*(M,H;\underline{\Lambda}_\alpha)$$

as $a \gg b \to \infty$ will define a twisted map $SH^*(i)$ in cohomology,

$$SH^*(i)\colon\ SH^*(W;\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) \leftarrow SH^*(M;\underline{\Lambda}_\alpha).$$

As in (2.5.5), the action-restriction maps fit into a commutative diagram

$$\begin{array}{ccc}
CF^*(W,H'_W;\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) & \longleftarrow & CF^*(M,H';\underline{\Lambda}_\alpha) \\
\uparrow & & \uparrow \\
CM^*(W,H^0_W;c^*\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) & \longleftarrow & CM^*(M,H^0;c^*\underline{\Lambda}_\alpha)
\end{array}$$

and taking the direct limit over the H' yields the following result in cohomology.

Theorem 10. *Let (M^{2n},θ) be a Liouville domain. Then for all $\alpha\in H^1(\mathcal{L}M)$ there exists a map $c_*\colon H^*(M;c^*\underline{\Lambda}_\alpha)\to SH^*(M;\underline{\Lambda}_\alpha)$, where $c\colon M\to\mathcal{L}M$ is the inclusion of constant loops. Moreover, for any Liouville embedding $i\colon (W^{2n},\theta')\to(M^{2n},\theta)$ there exists a map $SH^*(i)\colon SH^*(W;\underline{\Lambda}_{(\mathcal{L}i)^*\alpha})\leftarrow SH^*(M;\underline{\Lambda}_\alpha)$ which fits into the commutative diagram*

$$\begin{array}{ccc}
SH^*(W;\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) & \xleftarrow{SH^*(i)} & SH^*(M;\underline{\Lambda}_\alpha) \\
\uparrow{\scriptstyle c_*} & & \uparrow{\scriptstyle c_*} \\
H^*(W;c^*\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) & \xleftarrow{i^*} & H^*(M;c^*\underline{\Lambda}_\alpha)
\end{array}$$

2.6 Proof of the Main Theorem

Lemma 11. *Let N^n be a closed manifold and let $L \to DT^*N$ be an exact Lagrangian embedding. Then for all $\alpha \in H^1(\mathcal{L}N)$, the composite*

$$H_*(N; c^*\underline{\Lambda}_\alpha) \xrightarrow{\sim} H^{n-*}(N; c^*\underline{\Lambda}_\alpha) \xrightarrow{c_*} SH^{n-*}(DT^*N; \underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}) \xrightarrow{\varphi^{-1}} H_*(\mathcal{L}N; \underline{\Lambda}_\alpha)$$

of Poincaré duality, the map c_* from (2.5.6) and the inverse of φ (Theorem 9), is equal to the ordinary map $c_*\colon H_*(N; c^*\underline{\Lambda}_\alpha) \to H_*(\mathcal{L}N; \underline{\Lambda}_\alpha)$ induced by the inclusion of constants $c\colon N \to \mathcal{L}N$.

In the untwisted case, the lemma was proved by Viterbo [27] using his construction of the isomorphism φ, and it can be proved in the Abbondandolo-Schwarz setup by using small perturbations of $L(q,v) = \frac{1}{2}|v|^2$ and $H(q,p) = \frac{1}{2}|p|^2$ and by considering the restriction of the isomorphism φ to the orbits of action close to zero. The twisted version is proved analogously.

Theorem 12. *Let N^n be a closed manifold and let $L \to DT^*N$ be an exact Lagrangian embedding. Then for all $\alpha \in H^1(\mathcal{L}N)$ there exists a commutative diagram*

$$\begin{array}{ccc} H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & \xleftarrow{\mathcal{L}p_!} & H_*(\mathcal{L}N; \underline{\Lambda}_\alpha) \\ \uparrow c_* & & \uparrow c_* \\ H_*(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & \xleftarrow{p_!} & H_*(N; c^*\underline{\Lambda}_\alpha) \end{array}$$

where $c\colon N \to \mathcal{L}N$ is the inclusion of constant loops, $p\colon L \to T^*N \to N$ is the projection and $p_!$ is the ordinary transfer map. Moreover, the diagram can be restricted to the components $\mathcal{L}_0L$ and $\mathcal{L}_0N$ of contractible loops.

If $c^*\alpha = 0$ then the bottom map becomes $p_! \otimes 1\colon H_*(L) \otimes \Lambda \leftarrow H_*(N) \otimes \Lambda$.

Proof. Let θ_N be the canonical 1-form which makes $(DT^*N, d\theta_N)$ symplectic. By Weinstein's theorem a neighbourhood of L is symplectomorphic to a small disc cotangent bundle DT^*L. Therefore the exact Lagrangian embedding $j\colon L^n \hookrightarrow DT^*N$ yields a Liouville embedding $i\colon (DT^*L, \theta_L) \hookrightarrow (DT^*N, \theta_N)$.

By Theorem 9 there are twisted isomorphisms

$$\begin{aligned} \varphi_N \colon\ & H_*(\mathcal{L}N; \underline{\Lambda}_\alpha) \to SH^{n-*}(DT^*N; \underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}) \\ \varphi_L \colon\ & H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) \to SH^{n-*}(DT^*L; \underline{\Lambda}_{(\mathcal{L}i)^*(\mathcal{L}\pi)^*\alpha}) \end{aligned}$$

We define $\mathcal{L}p_! = \varphi_L^{-1} \circ SH^*(i) \circ \varphi_N$ so that the following diagram commutes

$$\begin{CD} H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) @<{\mathcal{L}p_!}<< H_*(\mathcal{L}N; \underline{\Lambda}_\alpha) \\ @A{\varphi_L^{-1}}A{\wr}A @V{\varphi_N}V{\wr}V \\ SH^{n-*}(DT^*L; \underline{\Lambda}_{(\mathcal{L}i)^*(\mathcal{L}\pi)^*\alpha}) @<{SH^*(i)}<< SH^{n-*}(DT^*N; \underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}) \end{CD}$$

Recall that the ordinary transfer map $p_!$ is defined using Poincaré duality and the pullback p^* so that the following diagram commutes,

$$\begin{CD} H^{n-*}(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) @<{p^*}<< H^{n-*}(N; c^*\underline{\Lambda}_\alpha) \\ @A{}A{\wr}A @V{}V{\wr}V \\ H_*(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) @<{p_!}<< H_*(N; c^*\underline{\Lambda}_\alpha) \end{CD}$$

Finally, Theorem 35 for the map i yields another commutative diagram whose horizontal maps are the bottom and top rows respectively of the above two diagrams (in the second diagram we use that L, N are homotopy equivalent to DT^*L, DT^*N). By combining these diagrams we obtain a commutative diagram

$$\begin{CD} H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) @<{\mathcal{L}p_!}<< H_*(\mathcal{L}N; \underline{\Lambda}_\alpha) \\ @AAA @AAA \\ H_*(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) @<{p_!}<< H_*(N; c^*\underline{\Lambda}_\alpha) \end{CD}$$

Lemma 11 shows that the vertical maps are indeed the maps c_* in ordinary homology. Since $c\colon N \to \mathcal{L}N$ maps into the component of contractible loops $\mathcal{L}_0N$, the diagram restricts to $\mathcal{L}_0L$ and $\mathcal{L}_0N$ by restricting $\mathcal{L}p_!$ and projecting to $H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha})$ (not all loops in T^*L that are contractible in T^*N need be contractible in T^*L). $\square$

2.7 Proof of the Corollary

2.7.1 Transgressions

Given $\beta \in H^2(N)$, let $f\colon N \to \mathbb{CP}^\infty$ be a classifying map for β. Let $ev\colon \mathcal{L}_0N \times S^1 \to N$ be the evaluation map. Define

$$\tau = \pi \circ ev^*\colon H^2(N) \xrightarrow{ev^*} H^2(\mathcal{L}_0N \times S^1) \xrightarrow{\pi} H^1(\mathcal{L}_0N)\ ,$$

where π is the projection to the Künneth summand. If N is simply connected, then τ is an isomorphism. Let u be a generator of $H^2(\mathbb{CP}^\infty)$, then $v = \tau(u)$ generates $H^1(\mathcal{L}\mathbb{CP}^\infty) \cong H^1(\Omega\mathbb{CP}^\infty)$ and $\tau(\beta) = (\mathcal{L}f)^*v$. Identify $H^1(\mathcal{L}_0N) \cong \mathrm{Hom}(\pi_1(\mathcal{L}_0N), \mathbb{Z})$ and $\pi_1(\mathcal{L}_0N) \cong \pi_2(N) \rtimes \pi_1(N)$, then the class $\tau(\beta)$ vanishes on $\pi_1(N)$ and corresponds to

$$f_*\colon \pi_2(N) \to \pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}.$$

Similarly, define $\tau_b\colon H^2(N) \to H^1(\Omega_0N)$ for the space Ω_0N of contractible based loops. Then $\Omega f\colon \Omega_0N \to \Omega\mathbb{CP}^\infty$ is a classifying map for $\tau_b(\beta)$. The inclusion $\Omega_0N \to \mathcal{L}_0N$ induces a bijection $\tau(\beta) \mapsto \tau_b(\beta)$ between transgressed forms.

We will assume throughout that the transgression $\alpha = \tau(\beta) \in H^1(\mathcal{L}_0N)$ is nonzero, or equivalently that $f_*\colon \pi_2(N) \to \mathbb{Z}$ is not the zero map.

2.7.2 Novikov homology of the free loopspace

Denote by $\overline{\mathcal{L}_0N}$ the infinite cyclic cover of $\mathcal{L}_0N$ corresponding to $\alpha\colon \pi_1(\mathcal{L}_0N) \to \mathbb{Z}$, and let t denote a generator of the group of deck transformations of $\overline{\mathcal{L}_0N}$. The group ring of the cover is $R = \mathbb{Z}[t, t^{-1}]$, and $\Lambda = \mathbb{Z}((t)) = \mathbb{Z}[[t]][t^{-1}]$ is the Novikov ring of α (see 3.5.2).

The Novikov homology of $\mathcal{L}_0N$ with respect to α is defined as the homology of

$\mathcal{L}_0N$ with local coefficients in the bundle $\underline{\Lambda}_\alpha$, which by [28] can be calculated as

$$H_*(\mathcal{L}_0N;\underline{\Lambda}_\alpha) \cong H_*(C_*(\overline{\mathcal{L}_0N}) \otimes_R \Lambda).$$

Say that a space X is of *finite type* if $H_k(X)$ is finitely generated for each k.

Theorem 13. *For a compact manifold N, if $\tau(\beta) \neq 0$ and $\pi_m(N)$ is finitely generated for each $m \geq 2$ then $\overline{\mathcal{L}_0N}$ is of finite type.*

Proof. **Claim 1.** *If $\overline{\Omega_0N}$ is of finite type then so is $\overline{\mathcal{L}_0N}$.*
Proof. Consider the fibration $\Omega_0N \to \mathcal{L}_0N \to N$, and take cyclic covers corresponding to $\tau_b(\beta)$ and $\tau(\beta)$ to obtain the fibration $\overline{\Omega_0N} \to \overline{\mathcal{L}_0N} \to N$. By compactness, N is homotopy equivalent to a finite CW complex and Claim 1 follows by a Leray-Serre spectral sequence argument.

After replacing N by a homotopy equivalent space, we may assume that we have a fibration $f\colon\ N \to \mathbb{CP}^\infty$ with fibre $F = f^{-1}(*)$, and taking the spaces of contractible based loops gives a fibration $\Omega f\colon\ \Omega_0N \to \Omega\mathbb{CP}^\infty$.

Claim 2. *The fibre of Ωf is a union $(\Omega F)_K$ of finitely many components of ΩF, indexed by the finite set $K = \mathrm{Coker}(f_*\colon\ \pi_2N \to \pi_2\mathbb{CP}^\infty)$.*
Proof. Consider the homotopy LES for the fibration f,

$$\pi_2N \xrightarrow{f_*} \pi_2\mathbb{CP}^\infty \longrightarrow \pi_1F \longrightarrow \pi_1N$$

then $(\Omega f)^{-1}(*) = \Omega F \cap \Omega_0N$ consists of loops $\gamma \in \Omega F$ whose path component lies in the kernel of $\pi_1F \to \pi_1N$, which is isomorphic to the cokernel of f_*. Since $\tau(\beta) \neq 0$, also f_* is nonzero and so K is finite.

Claim 3. *$\overline{\Omega j}\colon\ (\Omega F)_K \to \overline{\Omega_0N}$ is a homotopy equivalence.*
Proof. Observe that $\overline{\Omega_0N}$ is the pull-back under Ωf of the cyclic cover of $\Omega\mathbb{CP}^\infty$ corresponding to the transgression $v = \tau_b(u) \in H^1(\Omega\mathbb{CP}^\infty)$ of a generator $u \in H^2(\mathbb{CP}^\infty)$

(see 3.5.1). We obtain the commutative diagram

$$\begin{array}{ccccccc}
(\Omega F)_K & \xrightarrow{\overline{\Omega j}} & \overline{\Omega_0 N} & \xrightarrow{\overline{\Omega f}} & \overline{\Omega\mathbb{CP}^\infty} & \xrightarrow[\simeq]{\overline{\varphi}} & \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\Omega F)_K & \xrightarrow{\Omega j} & \Omega_0 N & \xrightarrow{\Omega f} & \Omega\mathbb{CP}^\infty & \xrightarrow[\simeq]{\varphi} & S^1
\end{array}$$

where the homotopy equivalence φ corresponds to $\tau_b(u) \in H^1(\Omega\mathbb{CP}^\infty) \cong [\Omega\mathbb{CP}^\infty, S^1]$. The claim follows since $\mathbb{R}$ is contractible.

Claim 4. *$\overline{\Omega_0 N}$ is of finite type iff $\Omega_0 F = \Omega\widetilde{F}$ is of finite type.*

Proof. Each component of ΩF is homotopy equivalent to $\Omega_0 F$ via composition with an appropriate fixed loop. The claim follows from Claims 3 and 2 since K is finite. Note that we may identify $\Omega_0 F = \Omega\widetilde{F}$ since the loops of F that lift to closed loops of the universal cover $\widetilde{F}$ are precisely the contractible ones.

Claim 5. *$\Omega\widetilde{F}$ is of finite type iff $\pi_m N$ is finitely generated for each $m \geq 2$.*

Proof. Since $\widetilde{F}$ is simply connected, $\Omega\widetilde{F}$ is of finite type iff $\widetilde{F}$ is of finite type, by a Leray-Serre spectral sequence argument applied to the path-space fibration $\Omega\widetilde{F} \to P\widetilde{F} \to \widetilde{F}$ (see [24, 9.6.13]). Moreover $\widetilde{F}$ is of finite type iff $\pi_m(\widetilde{F}) = \pi_m(F)$ is finitely generated for all $m \geq 2$ (see [24, 9.6.16]). The claim follows from the homotopy LES for $F \to N \to \mathbb{CP}^\infty$. □

Corollary 14. *For a compact manifold N, if $\tau(\beta) \neq 0$ and $\pi_m(N)$ is finitely generated for each $m \geq 2$, then $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) = 0$.*

Proof. We need to show that each $HN_k = H_k(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)})$ vanishes. Since $\mathbb{Z}[t]$ is Noetherian, its $(t)-$adic completion $\mathbb{Z}[[t]]$ is flat over $\mathbb{Z}[t]$ (see [14, Theorem 8.8]). Therefore, localizing at the multiplicative set S generated by t, $\Lambda = S^{-1}\mathbb{Z}[[t]]$ is flat over $R = S^{-1}\mathbb{Z}[t]$. Thus $HN_k \cong H_k(\overline{\mathcal{L}_0 N}) \otimes_R \Lambda$, which is the localization of $H_k = H_k(\overline{\mathcal{L}_0 N}) \otimes_{\mathbb{Z}[t]} \mathbb{Z}[[t]]$. Observe that $t \cdot H_k = H_k$ since t acts invertibly on $H_k(\overline{\mathcal{L}_0 N})$. So if H_k were finitely generated over $\mathbb{Z}[t]$, then $H_k = 0$ by Nakayama's lemma [14, Theorem 2.2] since t lies in the radical of $\mathbb{Z}[[t]]$. By Theorem 13, H_k is in fact finitely generated over $\mathbb{Z}$, so this concludes the proof. $\square$

Remark 15. *The idea behind the proof of Corollary 14 is not original. I later realized that it is a classical result that if $H_*(X;\mathbb{Z})$ is finitely generated in each degree then the Novikov homology $H_*(C_*(\overline{X}) \otimes_R \Lambda_\alpha)$ vanishes for $0 \neq \alpha \in H^1(X)$. The basic idea dates back to [16] and a very general version of this result is proved in [5, Prop. 1.35].*

Corollary 16. *If N is a compact simply connected manifold, then $H_*(\mathcal{L}_0 N; \underline{\Lambda}_\alpha) = 0$ for any nonzero $\alpha \in H^1(\mathcal{L}_0 N)$.*

Proof. N is simply connected so its homotopy groups are finitely generated in each dimension because its homology groups are finitely generated by compactness (see [24, 9.6.16]). Since N is simply connected, any α in $H^1(\mathcal{L}_0 N)$ is the transgression of some $\beta \in H^2(N)$. The result now follows from Corollary 14. $\square$

2.7.3 Proof of the Main Corollary

Corollary 17. *Let N^n be a closed simply connected manifold. Let $L \to DT^*N$ be an exact Lagrangian embedding. Then the image of $p_*\colon \pi_2(L) \to \pi_2(N)$ has finite index and $p^*\colon H^2(N) \to H^2(L)$ is injective.*

Proof. A non-zero class $\beta \in H^2(N)$ yields a non-zero transgression $\tau(\beta) \in H^1(\mathcal{L}_0 N)$ (see 3.5.1). Suppose by contradiction that $\tau(p^*\beta) = 0$. Then the local system $(\mathcal{L}p)^*\underline{\Lambda}_{\tau(\beta)}$ is trivial (see 3.5.2). Moreover $c^*\tau(\beta) = 0$ since $\tau(\beta)$ vanishes on $\pi_1(N)$. Therefore the diagram of Theorem 12, restricted to contractible loops, becomes

$$\begin{array}{ccc} H_*(\mathcal{L}_0 L) \otimes \Lambda & \xleftarrow{\mathcal{L}p_!} & H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) \\ {\scriptstyle c_*}\uparrow \;\downarrow{\scriptstyle q_*} & & \uparrow{\scriptstyle c_*} \\ H_*(L) \otimes \Lambda & \xleftarrow{p_!} & H_*(N) \otimes \Lambda \end{array}$$

where $q\colon \mathcal{L}_0 L \to L$ is the evaluation at 0. By Corollary 16, $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) = 0$, so the fundamental class $[N] \in H_n(N)$ maps to $c_*[N] = 0$. But $\mathcal{L}p_!(c_*[N]) = c_* p_![N] = c_*[L] \neq 0$ since c_* is injective on $H_*(L)$.

Therefore $\tau(p^*\beta)$ cannot vanish, and so $\tau_b \circ p^*\colon H^2N \to H^1(\Omega L)$ is injective. Consider the commutative diagram

$$\begin{array}{ccc} H^2(N) & \xrightarrow[\sim]{\tau_b} & \mathrm{Hom}(\pi_2(N),\mathbb{Z}) \cong H^1(\Omega N) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle (\Omega p)^*} \\ H^2(L) & \xrightarrow{\tau_b} & \mathrm{Hom}(\pi_2(L),\mathbb{Z}) \cong H^1(\Omega L) \end{array}$$

where the top map τ_b is an isomorphism since N is simply connected. We deduce from the injectivity of $\tau_b \circ p^* = (\Omega p)^* \circ \tau_b$ that $p^*\colon H^2(N) \to H^2(L)$ and $\mathrm{Hom}(\pi_2(N),\mathbb{Z}) \to \mathrm{Hom}(\pi_2(L),\mathbb{Z})$ are both injective, so in particular the image of $p_*\colon \pi_2(L) \to \pi_2(N)$ has finite index. □

2.8 Non-simply connected cotangent bundles

We will prove that for non-simply connected N the map $\pi_2(L) \to \pi_2(N)$ still has finite index provided that the homotopy groups $\pi_m(N)$ are finitely generated for each $m \geq 2$.

This time we consider transgressions induced from the universal cover $\widetilde{N}$ of N,

$$\tau\colon H^2(\widetilde{N}) \to H^1(\mathcal{L}\widetilde{N}) = H^1(\mathcal{L}_0 N) \cong \mathrm{Hom}(\pi_2 N, \mathbb{Z}).$$

The homomorphism $\widetilde{f}_*\colon \pi_2(\widetilde{N}) = \pi_2(N) \to \mathbb{Z}$ corresponding to such a transgression $\tau(\widetilde{\beta})$ is induced by a classifying map $\widetilde{f}\colon \widetilde{N} \to \mathbb{CP}^\infty$ for $\widetilde{\beta} \in H^2(\widetilde{N})$. Since $\Omega\widetilde{N} = \Omega_0 N$ and $\mathcal{L}\widetilde{N} = \mathcal{L}_0 N$, the transgressions $\tau_b(\widetilde{\beta})$ and $\tau(\widetilde{\beta})$ define cyclic covers $\overline{\Omega_0 N}$ and $\overline{\mathcal{L}_0 N}$. We will use these in the construction of the Novikov homology.

Theorem 18. *Let N be a compact manifold with finitely generated $\pi_m(N)$ for each $m \geq 2$. If $\tau(\widetilde{\beta}) \neq 0$ then $\overline{\mathcal{L}_0 N}$ is of finite type and $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\widetilde{\beta})}) = 0$.*

Proof. Revisit the proof of Theorem 13. It suffices to prove that $\overline{\Omega_0 N}$ has finite type.

This time we have the commutative diagram

$$\begin{array}{ccccc}
\Omega F & \xrightarrow{\overline{\Omega j}} & \overline{\Omega_0 N} & \xrightarrow{\overline{\Omega \widetilde{f}}} & \overline{\Omega\mathbb{CP}^\infty} \simeq \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega F & \xrightarrow{\Omega j} & \Omega\widetilde{N} = \Omega_0 N & \xrightarrow{\Omega\widetilde{f}} & \Omega\mathbb{CP}^\infty \simeq S^1 \\
\downarrow & & \downarrow & & \downarrow \\
F & \xrightarrow{j} & \widetilde{N} & \xrightarrow{\widetilde{f}} & \mathbb{CP}^\infty
\end{array}$$

Since $\Omega F \simeq \overline{\Omega_0 N}$, it suffices to show that ΩF has finite type. Observe that

$$\Omega F \cong \oplus_K \Omega_0 F$$

where $K = \mathrm{Coker}(\widetilde{f}_*\colon \pi_2 N \to \pi_2\mathbb{CP}^\infty)$ is a finite set since $\widetilde{f}_* \neq 0$. So we just need to show that $\Omega_0 F = \Omega\widetilde{F}$ is of finite type. The same argument as in Theorem 13 proves that $\Omega\widetilde{F}$ is of finite type iff $\pi_m N = \pi_m \widetilde{N}$ is finitely generated for each $m \geq 2$. The same proof as for Corollary 14 yields the vanishing of the Novikov homology. □

Corollary 19. *Let N be a closed manifold with finitely generated $\pi_m(N)$ for each $m \geq 2$. Let $L \to DT^*N$ be an exact Lagrangian embedding. Then the image of $p_*\colon \pi_2(L) \to \pi_2(N)$ has finite index and $\widetilde{p}^*\colon H^2(\widetilde{N}) \to H^2(\widetilde{L})$ is injective.*

Proof. The proof is analogous to that of Corollary 17: $(\mathcal{L}p)^*$ in the diagram

$$\begin{array}{ccc}
H^2(\widetilde{N}) & \xrightarrow[\sim]{\tau} & \mathrm{Hom}(\pi_2(N), \mathbb{Z}) \cong H^1(\mathcal{L}_0 N) \\
\downarrow \scriptstyle{\widetilde{p}^*} & & \downarrow \scriptstyle{(\mathcal{L}p)^*} \\
H^2(\widetilde{L}) & \xrightarrow[\sim]{\tau} & \mathrm{Hom}(\pi_2(L), \mathbb{Z}) \cong H^1(\mathcal{L}_0 L)
\end{array}$$

is injective because if, by contradiction, $\tau(\widetilde{p}^*\widetilde{\beta}) \in H^1(\mathcal{L}_0 L)$ vanished then the functoriality diagram of Theorem 12 would not commute. □

2.9 Unoriented theory

So far we assumed that all manifolds were oriented. By using $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients instead of $\mathbb{Z}$ coefficients one no longer needs the Floer and Morse moduli spaces to be oriented in order to define the differentials and continuation maps. For the twisted setup, we change the Novikov ring to

$$\Lambda = \mathbb{Z}_2((t)) = \mathbb{Z}_2[[t]][t^{-1}],$$

the ring of formal Laurent series with $\mathbb{Z}_2$ coefficients. The bundle $\underline{\Lambda}_\alpha$ is now a bundle of $\mathbb{Z}_2((t))$ rings, however the singular cocycle α is still integral: $[\alpha] \in H^1(\mathcal{L}_0 N; \mathbb{Z})$.

Using these coefficients, all our theorems hold true without the orientability assumption on N and L. The following is an interesting application of Corollary 17 in this setup.

Corollary 20. *There are no unorientable exact Lagrangians in T^*S^2.*

Proof. For unorientable L, $H^2(L;\mathbb{Z}) = \mathbb{Z}_2$. Therefore the transgression τ vanishes on $H^2(L;\mathbb{Z})$ since its range $\mathrm{Hom}(\pi_2(L), \mathbb{Z})$ is torsion-free. But for S^2 there is a non-zero transgression. This contradicts the proof of Corollary 17. □

Chapter 3

Deformations of symplectic cohomology and ADE singularities

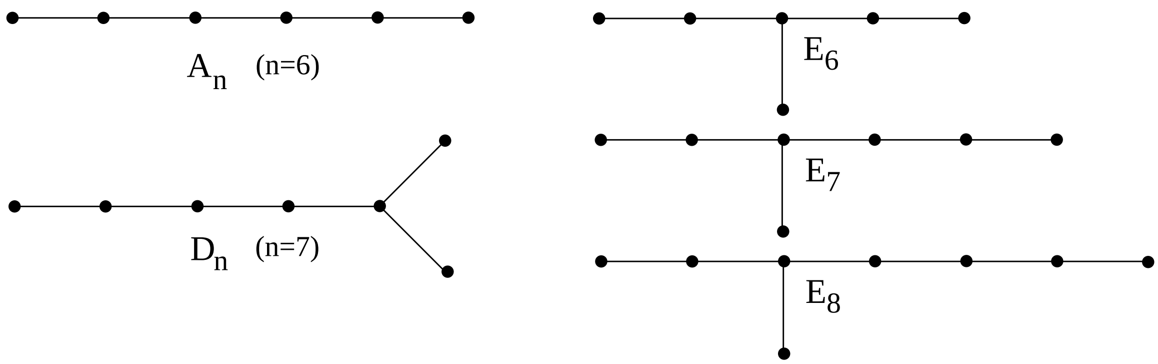
3.1 Summary

Let X be the plumbing of copies of the cotangent bundle of a $2-$sphere as prescribed by an ADE Dynkin diagram. We prove that the only exact Lagrangian submanifolds in X are spheres. Our approach involves studying X as an ALE hyperkähler manifold and observing that the symplectic cohomology of X will vanish if we deform the exact symplectic form to a generic non-exact one. We will construct the symplectic cohomology for non-exact symplectic manifolds with contact type boundary, and we will prove a general deformation theorem: if the non-exact symplectic form is sufficiently close to an exact one then the non-exact symplectic cohomology coincides with the natural Novikov symplectic cohomology for the exact form.

3.2 Introduction

An *ALE hyperkähler* manifold M is a non-compact simply-connected hyperkähler $4-$manifold which asymptotically looks like the standard Euclidean quotient $\mathbb{C}^2/\Gamma$ by a finite subgroup $\Gamma \subset SU(2)$. These spaces can be explicitly described and classified by a hyperkähler quotient construction due to Kronheimer [11].

ALE spaces have been studied in a variety of contexts. In theoretical physics they arise as gravitational instantons in the work of Gibbons and Hawking. In singularity theory they arise as the minimal resolution of the simple singularity $\mathbb{C}^2/\Gamma$. In symplectic geometry they arise as plumbings of cotangent bundles $T^*\mathbb{C}P^1$ according to ADE Dynkin diagrams:



Recall that the finite subgroups $\Gamma \subset SU(2)$ are the preimages under the double cover $SU(2) \to SO(3)$ of the cyclic group $\mathbb{Z}_n$, the dihedral group $\mathbb{D}_{2n}$, or one of the groups $\mathbb{T}_{12}$, $\mathbb{O}_{24}, \mathbb{I}_{60}$ of rigid motions of the Platonic solids. These choices of Γ will make $\mathbb{C}^2/\Gamma$ respectively a singularity of type A_{n-1}, D_{n+2}, E_6, E_7, E_8. The singularity is described as follows. The $\Gamma-$invariant complex polynomials in two variables are generated by three polynomials x, y, z which satisfy precisely one polynomial relation $f(x, y, z) = 0$. The hypersurface $\{f = 0\} \subset \mathbb{C}^3$ has precisely one singularity at the origin. The minimal resolution of this singularity over the singular point 0 is a connected union of copies of $\mathbb{C}P^1$ with self-intersection -2, which intersect each other transversely according to the corresponding ADE Dynkin diagram. Each vertex of the diagram corresponds to a $\mathbb{C}P^1$ and an edge between C_i and C_j means that $C_i \cdot C_j = 1$. We suggest Slodowy [23] or Arnol'd [2] for a survey of this construction.

In the symplectic world these spaces can be described as the plumbing of copies of $T^*\mathbb{C}P^1$ according to ADE Dynkin diagrams. Each vertex of the Dynkin diagram corresponds to a disc cotangent bundle $DT^*\mathbb{C}P^1$ and each edge of the Dynkin diagram corresponds to identifying the fibre directions of one bundle with the base directions of the other bundle over a small patch, and vice-versa. The boundary can be arranged to be a standard contact S^3/Γ, and along this boundary we attach an infinite symplectic cone $S^3/\Gamma \times [1, \infty)$ to form M as an exact symplectic manifold.

We are interested in the question: *what are the exact Lagrangian submanifolds inside the manifold M obtained by an ADE plumbing of copies of $T^*\mathbb{C}P^1$?*

Recall that a submanifold $j : L^n \hookrightarrow M^{2n}$ inside an exact symplectic manifold $(M, d\theta)$ is called *exact Lagrangian* if $j^*\theta$ is exact. For example, the A_1-plumbing is just $M = T^*S^2$ and the graph of any exact $1-$form on S^2 is an exact Lagrangian sphere inside T^*S^2. Viterbo [27] proved that there are no exact tori in T^*S^2 and so, for homological reasons, the only orientable exact Lagrangians in T^*S^2 are spheres. For exact spheres $L \subset T^*S^2$, it is known that L is isotopic to the zero section (Eliashberg-Polterovich [3]), indeed it is Hamiltonian isotopic (Hind [7]).

Theorem. *Let M be an ADE plumbing of copies of $T^*\mathbb{C}P^1$. Then the only exact Lagrangians inside M are spheres. In particular, there are no unorientable exact Lagrangians in M.*

We approach this problem via *symplectic cohomology*, which is an invariant of symplectic manifolds with contact type boundary. It is constructed as a direct limit of Floer cohomology groups for Hamiltonians which become steep near the boundary. Symplectic cohomology can be thought of as an obstruction to the existence of exact Lagrangians in the following sense.

Viterbo [25] proved that an exact $j : L \hookrightarrow (M, d\theta)$ yields a commutative diagram

$$\begin{array}{ccc} H_{n-*}(\mathcal{L}L) \cong SH^*(T^*L, d\theta) & \xleftarrow{SH^*(j)} & SH^*(M, d\theta) \\ \uparrow c_* & & \uparrow c_* \\ H_{n-*}(L) \cong H^*(L) & \xleftarrow{\quad j^* \quad} & H^*(M) \end{array}$$

where $\mathcal{L}L = C^\infty(S^1, L)$ is the space of free loops in L and the left vertical map is induced by the inclusion of constants $c : L \to \mathcal{L}L$. The element $c_*(j^*1)$ cannot vanish, and thus the vanishing of $SH^*(M, d\theta)$ would contradict the existence of L.

For an ADE plumbing M the symplectic cohomology $SH^*(M, d\theta)$ is never zero, indeed it contains a copy of the ordinary cohomology $H^*(M) \hookrightarrow SH^*(M, d\theta)$. However, we will show that if we make a generic infinitesimal perturbation of the closed form $d\theta$, then the symplectic cohomology will vanish. From this it will be easy to deduce that the only exact Lagrangians $L \subset M$ must be spheres.

We constructed the infinitesimally perturbed symplectic cohomology in Chapter 2 as follows. For any $\alpha \in H^1(\mathcal{L}_0 N)$, we constructed the associated Novikov homology theory for $SH^*(M, d\theta)$, which involves introducing twisted coefficients in the bundle of Novikov rings of formal Laurent series $\Lambda = \mathbb{Z}((t))$ associated to a singular cocycle representing α. Let's denote the bundle by $\underline{\Lambda}_\alpha$ and the twisted symplectic cohomology by $SH^*(M, d\theta; \underline{\Lambda}_\alpha)$.

We proved that the above functoriality diagram holds in this context – with the understanding that for unorientable L we use $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients and the Novikov ring $\mathbb{Z}_2((t))$ instead.

Consider a transgressed form $\alpha = \tau\beta$, where $\tau : H^2(M) \to H^1(\mathcal{L}M)$ is the transgression. The functoriality diagram simplifies to

$$\begin{array}{ccc} H_{n-*}(\mathcal{L}L; \underline{\Lambda}_{\tau j^*\beta}) & \xleftarrow{SH^*(j)} & SH^*(M, d\theta; \underline{\Lambda}_{\tau\beta}) \\ \uparrow c_* & & \uparrow c_* \\ H_{n-*}(L) \otimes \Lambda \cong H^*(L) \otimes \Lambda & \xleftarrow{j^*} & H^*(M) \otimes \Lambda \end{array}$$

For surfaces L which aren't spheres the transgression vanishes, so $H_*(\mathcal{L}L; \underline{\Lambda}_{\tau j^*\beta})$ simplifies to $H_*(\mathcal{L}L) \otimes \Lambda$ and the left vertical arrow c_* becomes injective. Thus $c_*(j^*1)$ cannot vanish, which contradicts the commutativity of the diagram if we can show that $SH^*(M, d\theta; \underline{\Lambda}_{\tau\beta}) = 0$.

Theorem. *Let M be an ADE plumbing of copies of $T^*\mathbb{C}P^1$. Then for generic β,*

$$SH^*(M, d\theta; \underline{\Lambda}_{\tau\beta}) = 0.$$

It turns out that it is quite easy to prove that the non-exact symplectic cohomology $SH^*(M, \omega)$ vanishes for a generic form ω. So to prove the above vanishing result, we will need to relate the twisted symplectic cohomology to the non-exact symplectic cohomology. We will prove the following general result.

Theorem. *Let $(M, d\theta)$ be an exact symplectic manifold with contact type boundary and let β be a closed two-form compactly supported in the interior of M. Then, at least for $\|\beta\| < 1$, there is an isomorphism*

$$SH^*(M, d\theta + \beta) \to SH^*(M, d\theta; \underline{\Lambda}_{\tau\beta}).$$

For our ADE plumbing M, we actually show that this result applies to a large non-compact deformation from $d\theta$ to a non-exact symplectic form ω which has a lot of symmetry. This symmetry will be the key to proving the vanishing of $SH^*(M, \omega)$ and therefore the vanishing of $SH^*(M, d\theta; \underline{\Lambda}_{\tau\beta}) = 0$, which concludes the proof of the non-existence of exact Lagrangians which aren't spheres.

The deformation from $d\theta$ to ω is best described by viewing M as a hyperkähler manifold via Kronheimer's construction. This has the advantage that on M we have an explicit S^2-worth of symplectic forms. We will start with an exact symplectic form lying on the equator of this S^2, and we deform it into the non-exact form ω lying at the North Pole. This deformation is not trivial, for instance the exact Lagrangian $\mathbb{C}P^1$ zero sections will turn into symplectic submanifolds. We will show that (M, ω) has a global Hamiltonian S^1-action, which at infinity looks like the action $(a, b) \mapsto (e^{2\pi i t}a, e^{2\pi i t}b)$ on $\mathbb{C}^2/\Gamma$.

We will show that the grading of the $1-$periodic orbits grows to negative infinity when we accelerate this Hamiltonian S^1-action, and this will imply that $SH^*(M, \omega) = 0$ because a generator would have to have arbitrarily negative grading. This concludes the argument.

The hyperkähler construction of M depends on certain parameters, and the cohomology class of ω varies linearly with these parameters. Thus, the form ω can be chosen to represent a generic class in $H^2(M; \mathbb{R})$ and so we conclude the following.

Theorem. *Let M be an ALE hyperkähler manifold. Then for generic ω,*

$$SH^*(M, \omega) = 0.$$

The outline of the Chapter is as follows. In section 3.3 we recall the basic terminology of symplectic manifolds with contact type boundary and we define the moduli spaces used to define symplectic cohomology. In section 3.4 we construct the symplectic cohomology $SH^*(M, \omega)$ for a (possibly non-exact) symplectic form ω, in particular in 3.4.1 we define the underlying Novikov ring Λ that we use throughout. In section 3.5 we define the twisted symplectic cohomology $SH^*(M, d\theta; \underline{\Lambda}_\alpha)$, in particular the Novikov bundle $\underline{\Lambda}_\alpha$ is defined in 3.5.2 and the functoriality property is described in 3.5.7. In section 3.6 we define the grading on symplectic cohomology, which is a $\mathbb{Z}-$grading if $c_1(M) = 0$. In section 3.7 we prove the deformation theorem which relates the twisted symplectic cohomology to the non-exact symplectic cohomology. In section 3.8 we recall Kronheimer's hyperkähler quotient construction of ALE spaces, and we describe the details of the proof outlined above.

3.3 Symplectic manifolds with contact boundary

3.3.1 Symplectic manifolds with contact type boundary

Let (M^{2n}, ω) be a compact symplectic manifold with boundary. The contact type boundary condition requires that there is a Liouville vector field Z defined near the boundary ∂M which is strictly outwards pointing along ∂M. The Liouville condition is that near the boundary $\omega = d\theta$, where $\theta = i_Z\omega$. This definition is equivalent to requiring that $\alpha = \theta|_{\partial M}$ is a contact form on ∂M, that is $d\alpha = \omega|_{\partial M}$ and $\alpha \wedge (d\alpha)^{n-1} > 0$ with respect to the boundary orientation on ∂M.

The Liouville flow of Z is defined for small negative times r, and it parametrizes a collar $(-\epsilon, 0] \times \partial M$ of ∂M inside M. So we may glue an infinite symplectic cone $([0, \infty) \times \partial M, d(e^r\alpha))$ onto M along ∂M, so that Z extends to $Z = \partial_r$ on the cone. This defines the completion $\widehat{M}$ of M,

$$\widehat{M} = M \cup_{\partial M} [0, \infty) \times \partial M.$$

We call $(-\epsilon, \infty) \times \partial M$ the collar of $\widehat{M}$. We extend θ and ω to the entire collar by

$\theta = e^r \alpha$ and $\omega = d\theta$.

Let J be an ω−compatible almost complex structure on $\widehat{M}$ and denote by $g = \omega(\cdot, J\cdot)$ the J−invariant metric. We always assume that J is of *contact type* on the collar, that is $J^*\theta = e^r dr$ or equivalently $J\partial_r = \mathcal{R}$ where $\mathcal{R}$ is the Reeb vector field. This implies that J restricts to an almost complex structure on the contact distribution $\ker \alpha$. We will only need the contact type condition for J to hold for $e^r \gg 0$ so that a certain maximum principle applies there.

From now on, we make the change of coordinates $R = e^r$ on the collar so that, redefining ϵ, the collar will be parametrized as the tubular neighbourhood $(\epsilon, \infty) \times \partial M$ of ∂M in $\widehat{M}$, so that the contact hypersurface ∂M corresponds to $\{R = 1\}$.

In the exact setup, that is when $\omega = d\theta$ on all of M, we call $(M, d\theta)$ a *Liouville domain*. In this case Z is defined on all of $\widehat{M}$ by $i_Z\omega = \theta$, and $\widehat{M}$ is the union of the infinite symplectic collar $((-\infty, \infty) \times \partial M, d(R\alpha))$ and the zero set of Z.

3.3.2 Reeb and Hamiltonian dynamics

The Reeb vector field $\mathcal{R} \in C^\infty(T\partial M)$ on ∂M is defined by $i_{\mathcal{R}} d\alpha = 0$ and $\alpha(\mathcal{R}) = 1$. The periods of the Reeb vector field form a countable closed subset of $[0, \infty)$.

For $H \in C^\infty(\widehat{M}, \mathbb{R})$ we define the Hamiltonian vector field X_H by

$$\omega(\cdot, X_H) = dH.$$

If inside M the Hamiltonian H is a C^2-small generic perturbation of a constant, then the 1-periodic orbits of X_H inside M are constants corresponding precisely to the critical points of H.

Suppose $H = h(R)$ depends only on $R = e^r$ on the collar. Then $X_H = h'(R)\mathcal{R}$. It follows that every non-constant 1-periodic orbit $x(t)$ of X_H which intersects the collar must lie in $\{R\} \times \partial M$ for some R and must correspond to a Reeb orbit $z(t) = x(t/T) : [0, T] \to \partial M$ with period $T = h'(R)$. Since the Reeb periods are countable, if we choose h to have a generic constant slope $h'(R)$ for $R \gg 0$ then there will be no 1-periodic orbits of X_H outside of a compact set of $\widehat{M}$.

3.3.3 Action 1-form

Let $\mathcal{L}\widehat{M} = C^\infty(S^1, \widehat{M})$ be the space of free loops in $\widehat{M}$. Suppose for a moment that $\omega = d\theta$ were exact on all of $\widehat{M}$, then one could define the $H-$perturbed action functional for $x \in \mathcal{L}\widehat{M}$ by

$$A_H(x) = -\int x^*\theta + \int_0^1 H(x(t))\,dt.$$

If $H = h(R)$ on the collar then this reduces to $A_H(x) = -Rh'(R) + h(R)$ where x is a 1-periodic orbit of X_H in $\{R\} \times \partial M$. The differential of A_H at $x \in \mathcal{L}\widehat{M}$ in the direction $\xi \in T_x\mathcal{L}\widehat{M} = C^\infty(S^1, x^*T\widehat{M})$ is

$$dA_H \cdot \xi = -\int_0^1 \omega(\xi, \dot{x} - X_H)\,dt.$$

In the case when ω is not exact on all of $\widehat{M}$, A_H is no longer well-defined, however the formula for dA_H still gives a well-defined $1-$form on $\mathcal{L}\widehat{M}$. The zeros x of dA_H are precisely the 1-periodic Hamiltonian orbits $\dot{x}(t) = X_H(x(t))$.

It also meaningful to say how A_H varies along a smooth path u in $\mathcal{L}\widehat{M}$ by defining

$$\partial_s A_H(u) = dA_H \cdot \partial_s u,$$

but the total variation $\int \partial_s A_H(u)\,ds$ will depend on u, not just on the ends of u.

3.3.4 Floer's equation

With respect to the L^2-metric $\int_0^1 g(\cdot,\cdot)\,dt$ the gradient corresponding to dA_H is $\nabla A_H = J(\dot{x} - X_H)$. For $u : \mathbb{R} \times S^1 \to M$, the negative L^2-gradient flow equation $\partial_s u = -\nabla A_H(u)$ in the coordinates $(s,t) \in \mathbb{R} \times S^1$ is

$$\partial_s u + J(\partial_t u - X_H) = 0 \quad \text{(Floer's equation)}.$$

Let $\mathcal{M}'(x_-, x_+)$ denote the moduli space of solutions u to Floer's equation, which at the ends converge uniformly in t to the 1-periodic orbits $x_\pm$:

$$\lim_{s \to \pm\infty} u(s,t) = x_\pm(t).$$

These solutions u occur in $\mathbb{R}-$families because we may reparametrize the $\mathbb{R}$ coordinate by adding a constant. Denote the quotient by $\mathcal{M}(x_-, x_+) = \mathcal{M}'(x_-, x_+)/\mathbb{R}$. To emphasize the context, we may also write $\mathcal{M}^H(x_-, x_+)$ or $\mathcal{M}(x_-, x_+; \omega)$.

The action A_H decreases along u since

$$\partial_s(A_H(u)) = dA_H \cdot \partial_s u = -\int_0^1 \omega(\partial_s u, \partial_t u - X_H)\, dt = -\int_0^1 |\partial_s u|_g^2\, dt \leq 0.$$

If ω is exact on M, the action decreases by $A_H(x_-) - A_H(x_+)$ independently of the choice of $u \in \mathcal{M}(x_-, x_+)$.

3.3.5 Energy

For a Floer solution u the energy is defined as

$$\begin{aligned} E(u) &= \int |\partial_s u|^2\, ds \wedge dt = \int \omega(\partial_s u, \partial_t u - X_H)\, ds \wedge dt \\ &= \int u^*\omega + \int H(x_-)\, dt - \int H(x_+)\, dt. \end{aligned}$$

If ω is exact on M then for $u \in \mathcal{M}(x_-, x_+)$ there is an a priori energy estimate, $E(u) = A_H(x_-) - A_H(x_+)$.

3.3.6 Transversality and compactness

Standard Floer theory methods can be applied to show that for a generic time-dependent perturbation (H_t, J_t) of (H, J) there are only finitely many $1-$periodic Hamiltonian orbits and the moduli spaces $\mathcal{M}(x_-, x_+)$ are smooth manifolds. We write $\mathcal{M}_k(x_-, x_+) = \mathcal{M}'_{k+1}(x_-, x_+)/\mathbb{R}$ for the k-dimensional part of $\mathcal{M}(x_-, x_+)$.

As explained in detail in Viterbo [25] and Seidel [22], there is a maximum principle which prevents Floer trajectories $u \in \mathcal{M}(x_-, x_+)$ from escaping to infinity.

Lemma 21 (Maximum principle). *If on the collar $H = h(R)$ and J is of contact type, then for any local Floer solution $u : \Omega \to [1, \infty) \times \partial M$ defined on a compact $\Omega \subset \mathbb{R} \times S^1$, the maxima of $R \circ u$ are attained on $\partial\Omega$. If $H_s = h_s(R)$ and $J = J_s$ depend on s, the result continues to hold provided that $\partial_s h_s' \leq 0$. In particular, Floer solutions of $\partial_s u + J(\partial_t u - X_H) = 0$ or $\partial_s u + J_s(\partial_t u - X_{H_s}) = 0$ converging to $x_\pm$ at the ends are entirely contained in the region $R \leq \max R(x_\pm)$.*

Proof. On the collar $u(s,t) = (R(s,t), m(s,t)) \in [1,\infty) \times \partial M$ and we can orthogonally decompose

$$T([1,\infty) \times \partial M) = \mathbb{R}\partial_r \oplus \mathbb{R}\mathcal{R} \oplus \xi$$

where $\xi = \ker \alpha$ is the contact distribution. By the contact type condition, $J\partial_r = \mathcal{R}$, $J\mathcal{R} = -\partial_r$ and J restricts to an endomorphism of ξ. Since $X_H = h'(R)\mathcal{R}$, Floer's equation in the first two summands $\mathbb{R}\partial_r \oplus \mathbb{R}\mathcal{R}$ after rescaling by R is

$$\partial_s R - \theta(\partial_t u) + Rh' = 0 \qquad \partial_t R + \theta(\partial_s u) = 0.$$

Adding ∂_s of the first and ∂_t of the second equation, yields $\partial_s^2 R + \partial_t^2 R + R\partial_s h' = |\partial_s u|^2$. So $LR \geq 0$ for the elliptic operator $L = \partial_s^2 + \partial_t^2 + Rh''(R)\partial_s$, thus a standard result in PDE theory [4, Theorem 6.4.4] ensures the maximum principle for $R \circ u$.

If h_s depends on s and $\partial_s h_s' \leq 0$, then we get $LR = |\partial_s u|^2 - R(\partial_s h_s')(R) \geq 0$ which guarantees the maximum principle for R. $\square$

If ω were exact on M, then the a priori energy estimate for $\mathcal{M}(x_-, x_+)$ described in 3.3.5 together with the maximum principle would ensure that the moduli spaces $\mathcal{M}(x_-, x_+)$ have compactifications $\overline{\mathcal{M}}(x_-, x_+)$ whose boundaries are defined in terms of broken Floer trajectories (Figure 3-1). In the proof of compactness, the exactness of ω excludes the possibility of bubbling-off of $J-$holomorphic spheres.

In the non-exact case if we assume that no bubbling-off of J-holomorphic spheres occurs, then the same techniques guarantee that the moduli space

$$\mathcal{M}(x_-, x_+; K) = \{u \in \mathcal{M}(x_-, x_+) : E(u) \leq K\}$$

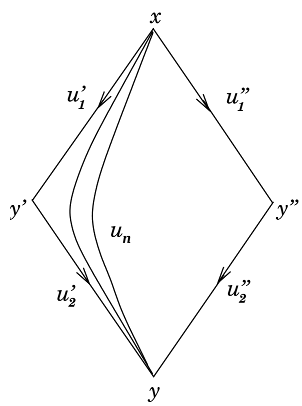


Figure 3-1: The x, y', y'', y are 1-periodic orbits of X_H, the lines are Floer solutions in M. The $u_n \in \mathcal{M}_1(x, y)$ are converging to the broken trajectory $(u_1', u_2') \in \mathcal{M}_0(x, y') \times \mathcal{M}_0(y', y)$.

of bounded energy solutions has a compactification by broken trajectories, for any given constant $K \in \mathbb{R}$.

Assumptions. *We assume henceforth that no bubbling occurs. This holds if* $c_1(M) = 0$ *by Hofer-Salamon [10], as in all our applications. To keep the notation under control, we will continue to write* (H, J) *even though we should always use a perturbed* (H_t, J_t).

3.4 Symplectic cohomology

3.4.1 Novikov symplectic chain complex

Let Λ denote the Novikov ring,

$$\Lambda = \left\{ \sum_{j=0}^{\infty} n_j t^{a_j} : n_j \in \mathbb{Z}, a_j \in \mathbb{R}, \lim_{j \to \infty} a_j = \infty \right\}.$$

In Chapter 2 we allowed only integer values of a_j because we were always using integral forms. In that setup Λ was just the ring of formal integral Laurent series. In the present Chapter the a_j will arise from integrating real forms so we use real a_j.

For an abelian group G the Novikov completion $G(\!(t)\!)$ is the $\Lambda-$module of formal sums $\sum_{j=0}^{\infty} g_j t^{a_j}$ where $g_j \in G$ and the real numbers $a_j \to \infty$.

Let $H \in C^\infty(\widehat{M}, \mathbb{R})$ be a Hamiltonian which on the collar is of the form $H = h(R)$, where h is linear at infinity. Define CF^* to be the abelian group freely generated by 1-periodic orbits of X_H,

$$CF^*(H) = \bigoplus \left\{ \mathbb{Z}x : x \in \mathcal{L}\widehat{M},\ \dot{x}(t) = X_H(x(t)) \right\}.$$

It is always understood that we first make a generic C^2-small time-perturbation H_t of H, so that there are only finitely many $1-$periodic orbits of X_{H_t} and therefore $CF^*(H)$ is finitely generated.

The symplectic chain complex $SC^*(H)$ is the Novikov completion of $CF^*(H)$:

$$\begin{aligned} SC^* = CF^*(\!(t)\!) &= \left\{ \textstyle\sum_{j=0}^{\infty} c_j t^{a_j} : c_j \in CF^*, \lim a_j = \infty \right\} \\ &= \left\{ \textstyle\sum_0^N \lambda_i y_i : \lambda_i \in \Lambda, N \in \mathbb{N}, y_i \text{ is a 1–periodic orbit of } X_H \right\}. \end{aligned}$$

The differential δ is defined by

$$\delta\left(\sum_{i=0}^{N} \lambda_i y_i\right) = \sum_{i=0}^{N} \sum_{u \in \mathcal{M}_0(x, y_i)} \epsilon(u)\, t^{E(u)} \lambda_i x$$

where $\mathcal{M}_0(x, y_i)$ is the $0-$dimensional component of the Floer trajectories connecting x to y_i, and $\epsilon(u)$ are signs depending on orientations. The sum is well-defined because there are only finitely many generators x, and below any energy bound $E(u) \leq K$ the moduli space $\mathcal{M}_0(x, y_i)$ is compact and therefore finite.

Lemma 22. *$SC^*(H)$ is a chain complex, i.e. $\delta \circ \delta = 0$. We denote the cohomology of $(SC^*(H), \delta)$ by $SH^*(H)$.*

Proof. This involves a standard argument (see Salamon [21]). Observe Figure 3-1. The $1-$dimensional moduli space $\mathcal{M}_1(x, y)$ has a compactification, such that the boundary consists of pairs of Floer trajectories joined at one end. Observe that $E(\cdot)$ is additive with respect to concatenation and $E(u)$ is invariant under homotoping u relative ends. Therefore, in Figure 3-1, $E(u_1') + E(u_2') = E(u_1'') + E(u_2'')$. Since

$\epsilon(u_1')\epsilon(u_2') = -\epsilon(u_1'')\epsilon(u_2'')$, we deduce

$$\epsilon(u_1')\, t^{E(u_1')}\, \epsilon(u_2')\, t^{E(u_2')} = -\epsilon(u_1'')\, t^{E(u_1'')}\, \epsilon(u_2'')\, t^{E(u_2'')}.$$

Thus the broken trajectories contribute opposite $\Lambda-$multiples of x to $\delta(\delta y)$ for each connected component of $\mathcal{M}_1(x, y)$. Hence, summing over x, y',

$$\delta(\delta y) = \sum_{(u_1', u_2') \in \mathcal{M}_0(x,y') \times \mathcal{M}_0(y',y)} \epsilon(u_1')\, t^{E(u_1')}\, \epsilon(u_2')\, t^{E(u_2')}\, x = 0. \quad \square$$

3.4.2 Continuation Maps

Under suitable conditions on two Hamiltonians $H_\pm$, it is possible to define a *continuation homomorphism*

$$\varphi : SC^*(H_+) \to SC^*(H_-).$$

This involves counting *parametrized Floer trajectories*, the solutions of

$$\partial_s v + J_s(\partial_t v - X_{H_s}) = 0.$$

Here J_s are $\omega-$compatible almost complex structures of contact type and H_s is a homotopy from H_- to H_+, such that $(H_s, J_s) = (H_-, J_-)$ for $s \ll 0$ and $(H_s, J_s) = (H_+, J_+)$ for $s \gg 0$. The conditions on H_s will be described in Theorem 23.

If x and y are respectively 1-periodic orbits of X_{H_-} and X_{H_+}, then let $\mathcal{M}(x, y)$ be the moduli space of such solutions v which converge to x and y at the ends. This time there is no freedom to reparametrize v in the $s-$variable.

The continuation map φ on a generator $y \in \mathrm{Zeros}(dA_{H_+})$ is defined by

$$\varphi(y) = \sum_{v \in \mathcal{M}_0(x,y)} \epsilon(v)\, t^{E_0(v)}\, x$$

where $\mathcal{M}_0(x, y)$ is the $0-$dimensional part of $\mathcal{M}(x, y)$, $\epsilon(v) \in \{\pm 1\}$ are orientation signs and the power of t in the above sum is

$$\begin{aligned} E_0(v) &= -\textstyle\int_{-\infty}^{\infty} \partial_s A_{H_s}(v)\, ds \\ &= \textstyle\int |\partial_s v|_{g_s}^2\, ds \wedge dt - \int (\partial_s H_s)(v)\, ds \wedge dt \\ &= \textstyle\int v^*(\omega - dK \wedge dt), \end{aligned}$$

where $K(s,m) = H_s(m)$. The last expression shows that $E_0(v)$ is invariant under homotoping v relative ends.

3.4.3 Energy of parametrized Floer trajectories

Let H_s be a homotopy of Hamiltonians. For an H_s-Floer trajectory the above weight $E_0(v)$ will be positive if H_s is monotone decreasing, $\partial_s H_s \leq 0$. The energy is

$$E(v) = E_0(v) + \int (\partial_s H_s)(v)\, ds \wedge dt.$$

If $\partial_s H_s \leq 0$ outside of a compact subset of $\widehat{M}$, then a bound on $E_0(v)$ imposes a bound on $E(v)$. Note that $E(v)$ is not invariant under homotoping v relative ends.

3.4.4 Properties of continuation maps

Theorem 23 (Monotone homotopies)**.** *Let H_s be a homotopy between $H_\pm$ such that*

1. *on the collar $H_s = h_s(R)$ for large R;*

2. *$\partial_s h_s' \leq 0$ for $R \geq R_\infty$, some R_∞;*

3. *h_s is linear for $R \geq R_\infty$ (the slope may be a Reeb period, but not for $h_\pm$).*

Then, after a generic C^2-small time-dependent perturbation of (H_s, J_s),

1. *all parametrized Floer trajectories lie in the compact subset*

$$C = M \cup \{R \leq R_\infty\} \subset \widehat{M};$$

2. *$\mathcal{M}(x;y)$ is a smooth manifold;*

3. *$\mathcal{M}(x,y;K) = \{v \in \mathcal{M}(x,y) : E_0(v) \leq K\}$ has a smooth compactification by broken trajectories, for any constant $K \in \mathbb{R}$;*

4. *the continuation map $\varphi : SC^*(H_+) \to SC^*(H_-)$ is well-defined;*

5. *φ is a chain map.*

Proof. (1) is a consequence of the maximum principle, Lemma 21, and (2) is a standard transversality result. Let

$$B_C = \max_{x \in C}\{\partial_s H_s(x), 0\}.$$

Suppose H_s varies in s precisely for $s \in [s_0, s_1]$. Since all $v \in \mathcal{M}(x,y;K)$ lie in C, $\int \partial_s H_s(v)\, ds \wedge dt \leq (s_1 - s_0)B_C$, so there is an a priori energy bound

$$E(v) \leq K + (s_1 - s_0)B_C.$$

From this the compactness of $\mathcal{M}(x,y;K)$ follows by standard methods.

The continuation map φ involves a factor of $t^{E_0(v)}$. The lower bound $E_0(v) \geq E(v) - (s_1 - s_0)B_C$ guarantees that as the energy $E(v)$ increases also the powers $t^{E_0(v)}$ increase, which proves (4).

Showing that φ is a chain map is a standard argument which involves investigating the boundaries of broken trajectories of the $1-$dimensional moduli spaces $\mathcal{M}_1(x,y;K)$. A sequence v_n in some $1-$dimensional component of $\mathcal{M}_1(x,y;K)$ will converge (after reparametrization) to a concatenation of two trajectories $u^+\#v$ or $v\#u^-$, where $u^+ \in \mathcal{M}_0^{H_+}(x,x')$, $v \in \mathcal{M}_0(x',y)$, or respectively $v \in \mathcal{M}_0(x,y')$, $u^- \in \mathcal{M}_0^{H_-}(y',y)$. Such solutions get counted with the same weight

$$E_0(v_n) = E_0(u^+\#v) = E_0(v\#u^-)$$

because E_0 is a homotopy invariant relative ends and $v_n, u^+\#v, v\#u^-$ are homotopic since they belong to the compactification of the same $1-$dimensional component of $\mathcal{M}_1(x,y)$. Therefore, $\partial_{H_-} \circ \varphi = \varphi \circ \partial_{H_+}$ as required. □

3.4.5 Chain homotopies

Theorem 24.

1. *Given monotone homotopies H_s, K_s from H_- to H_+, there is a chain homotopy $Y : SC^*(H_+) \to SC^*(H_-)$ between the respective continuation maps: $\varphi - \psi = \partial_{H_-} Y + Y \partial_{H_+}$;*

2. *the chain map φ defines a map on cohomology,*

$$[\varphi] : SH^*(H_+) \to SH^*(H_-),$$

which is independent of the choice of the homotopy H_s;

3. *the composite of the maps induced by homotoping H_- to K and K to H_+,*

$$SC^*(H_+) \to SC^*(K) \to SC^*(H_-),$$

is chain homotopic to φ and equals $[\varphi]$ on $SH^*(H_+)$;

4. *the constant homotopy $H_s = H$ induces the identity on $SC^*(H)$;*

5. *if $H_\pm$ have the same slope at infinity, then $[\varphi]$ is an isomorphism.*

Proof. Let $(H_{s,\lambda})_{0\leq\lambda\leq 1}$ be a linear interpolation of H_s and K_s, so that $H_{s,\lambda}$ is a monotone Hamiltonian for each λ. Consider the moduli spaces $\mathcal{M}(x, y, \lambda)$ of parametrized Floer solutions for $H_{s,\lambda}$. Let Y be the oriented count of the pairs (λ, v), counted with weight $t^{E_0(v)}$, where $0 < \lambda < 1$ and v is in a component of $\mathcal{M}(x, y, \lambda)$ of virtual dimension -1 (generically $\mathcal{M}_{-1}(x, y, \lambda)$ is empty, but in the family $\cup_\lambda \mathcal{M}_{-1}(x, y, \lambda)$ such isolated solutions (λ, v) can arise).

Consider a sequence (λ_n, v_n) inside some $1-$dimensional component of $\cup_\lambda \mathcal{M}(x, y, \lambda)$, such that $\lambda_n \to \lambda$. If $\lambda = 0$ or 1, then the limit of the v_n can break by giving rise to an H_s or K_s Floer trajectory, and this breaking is counted by $\varphi - \psi$. If $0 < \lambda < 1$, then the v_n can break by giving rise to $u^- \# v$ or $v \# u^+$, where $u^\pm$ are $H_\pm-$Floer trajectories and the v are as in the definition of Y. This type of breaking is therefore counted by $\partial_{H_-} Y + Y \partial_{H^+}$.

Both sides of the relation $\varphi - \psi = \partial_{H_-} Y + Y \partial_{H_+}$ will count a (broken) trajectory with the same weight because $E_0(\cdot)$ is a homotopy invariant relative ends and the broken trajectories are all homotopic, since they arise as the boundary of the same $1-$dimensional component of $\cup_\lambda \mathcal{M}(x, y, \lambda)$.

Claims (2) and (3) are standard consequences of (1) (see Salamon [21]). Claim (4) is a consequence of the fact that any non-constant Floer trajectory for $H_s = H$ would come in a $1-$dimensional family of solutions, due to the translational freedom in s. Claim (5) follows from (3) and (4): we can choose H_s to have constant slope for large R, therefore H_{-s} is also a monotone homotopy, and the composite of the chain maps induced by H_s and H_{-s} is chain homotopic to the identity. □

3.4.6 Hamiltonians linear at infinity

Consider Hamiltonians H^m which equal

$$h^m(R) = mR + C$$

for $R \gg 0$, where the slope $m > 0$ is not the period of any Reeb orbit. Up to isomorphism, $SH^*(H)$ is independent of the choice of C by Theorem 24.

For $m_+ < m_-$, a monotone homotopy H_s defines a continuation map

$$\phi^{m_+ m_-} : SC^*(H^{m_+}) \to SC^*(H^{m_-}),$$

for example the homotopy $h_s(R) = m_s R + C_s$ for $R \gg 0$, with $\partial_s m_s \leq 0$.

By Theorem 24 the continuation map $[\phi^{m_+ m_-}] : SH^*(H^{m_+}) \to SH^*(H^{m_-})$ on cohomology does not depend on the choice of homotopy h_s. Moreover, such continuation maps compose well: $\phi^{m_2 m_3} \circ \phi^{m_1 m_2}$ is chain homotopic to $\phi^{m_1 m_3}$ where $m_1 < m_2 < m_3$, and so $[\phi^{m_2 m_3}] \circ [\phi^{m_1 m_2}] = [\phi^{m_1 m_3}]$.

3.4.7 Symplectic cohomology

Definition 25. *The symplectic cohomology is defined to be the direct limit*

$$SH^*(M,\omega) = \varinjlim SH^*(H)$$

taken over the continuation maps between Hamiltonians linear at infinity.

Observe that $SH^*(M,\omega)$ can be calculated as the direct limit

$$\lim_{k\to\infty} SH^*(H_k)$$

over the continuation maps $SH^*(H_k) \to SH^*(H_{k+1})$, where the slopes at infinity of the Hamiltonians H_k increase to infinity as $k \to \infty$.

3.4.8 The maps $\mathbf{c}_*$ from ordinary cohomology

The symplectic cohomology comes with a map from the ordinary cohomology of M with coefficients in Λ,

$$c_* : H^*(M;\Lambda) \to SH^*(M,\omega).$$

We sketch the construction here, and refer to Chapter 2 for a detailed construction. Fix a $\delta > 0$ which is smaller than all periods of the nonconstant Reeb orbits on ∂M. Consider Hamiltonians H^δ which are C^2-close to a constant on M and such that on the collar $H^\delta = h(R)$ with constant slope $h'(R) = \delta$.

A standard result in Floer cohomology is that, after a generic C^2-small time-independent perturbation of (H^δ, J), the 1-periodic orbits of X_{H^δ} and the connecting Floer trajectories are both independent of $t \in S^1$. By the choice of δ there are no 1-periodic orbits on the collar, and by the maximum principle no Floer trajectory leaves M. The Floer complex $CF^*(H^\delta)$ is therefore canonically identified with the Morse complex $CM^*(H^\delta)$, which is generated by $\mathrm{Crit}(H^\delta)$ and whose differential counts the negative gradient trajectories of H^δ with weights $t^{H^\delta(x_-)-H^\delta(x_+)}$. After the change of basis $x \mapsto t^{H^\delta(x)}x$, the differential reduces to the ordinary Morse complex defined over the ring Λ which is isomorphic to the singular cochain complex of M with coefficients in Λ. Thus

$$SH^*(H^\delta) \cong HM^*(H^\delta; \Lambda) \cong H^*(M; \Lambda).$$

Since $SH^*(H^\delta)$ is part of the direct limit construction of $SH^*(M, \omega)$, this defines a map $c_* : H^*(M; \Lambda) \to SH^*(M, \omega)$ independently of the choice of H^δ.

3.4.9 Invariance under contactomorphisms

Definition 26. *Let M, N be symplectic manifolds with contact type boundary. A symplectomorphism $\varphi : \widehat{M} \to \widehat{N}$ is of* contact type at infinity *if on the collar*

$$\varphi^*\theta_N = \theta_M + d(\textit{compactly supported function})$$

and at infinity

$$\varphi(e^r, y) = (e^{r-f(y)}, \psi(y)),$$

for some smooth function $f : \partial M \to \mathbb{R}$ where $\psi : \partial M \to \partial N$ is a contactomorphism (that is a diffeomorphism with $\psi^*\alpha_N = e^f \alpha_M$).

Under such a map $\varphi : \widehat{M} \to \widehat{N}$, the Floer equations on $\widehat{M}$ for (φ^*H, ω_M) correspond precisely to the Floer solutions on $\widehat{N}$ for (H, ω_N). However, for a Hamiltonian H on $\widehat{N}$ which is linear at infinity, the Hamiltonian $\varphi^*H(e^r, y) = h(e^{r-f(y)})$ is not linear at infinity. Thus we want to show that for this new class of Hamiltonians we obtain the same symplectic cohomology.

In order to relate the two symplectic cohomologies, we need a maximum principle for homotopies of Hamiltonians which equal $H_s = h_s(R_s)$ on the collar, where

$$R_s(e^r, y) = e^{r-f_s(y)},$$

and $f_s = f_-$, $h_s = h_-$ for $s \ll 0$ and $f_s = f_+, h_s = h_+$ for $s \gg 0$. We prove that if $h'_- \gg h'_+$ then one can choose h_s so that the maximum principle applies. We denote by X_s the Hamiltonian vector field for h_s and we assume that the almost complex structures J_s satisfy the contact type condition $J_s^* \theta = dR_s$ for $e^r \gg 0$.

Lemma 27 (Maximum principle)**.** *There is a constant $K > 0$ depending only on f_s such that if $h'_- \geq K h'_+$ then it is possible to choose a homotopy h_s from h_- to h_+ in such a way that the maximum principle applies to the function*

$$\rho(s,t) = R_s(u(s,t)) = e^{r(u) - f_s(y(u))}$$

where u is any local solution of Floer's equation $\partial_s u + J_s(\partial_t u - X_s) = 0$ which lands in the collar $e^r \gg 0$. In particular, a continuation map $SH^*(h_+) \to SH^*(h_-)$ can then be defined.

Proof. We will seek an equation satisfied by $\Delta \rho$. Using $J_s^* dR_s = -\theta$ we obtain

$$\begin{aligned} \partial_s \rho &= \partial_s R_s(u) + dR_s \cdot \partial_s u = -\rho \partial_s f_s + dR_s \cdot \partial_s u \\ &= -\rho \partial_s f_s + dR_s \cdot (J_s(X_s - \partial_t u)) = -\rho \partial_s f_s - \theta(X_s) + \theta(\partial_t u), \\ \partial_t \rho &= dR_s \cdot \partial_t u = dR_s \cdot (X_s + J_s \partial_s u) = J_s^* dR_s \cdot \partial_s u = -\theta(\partial_s u). \end{aligned}$$

Since $X_s = h'_s(\rho)\mathcal{R}$ and $\theta(\mathcal{R}(u)) = \rho$, we deduce $\theta(X_s) = \rho h'_s(\rho)$ so

$$d^c \rho = d\rho \circ i = -\partial_s \rho \, dt + \partial_t \rho \, ds = -u^* \theta + \rho h'_s(\rho) \, dt + \rho \partial_s f_s \, dt.$$

Therefore $dd^c \rho = -\Delta \rho \, ds \wedge dt = -u^* \omega + F \, ds \wedge dt$ where

$$F = h'_s \partial_s \rho + \rho \partial_s h'_s + \rho h''_s \partial_s \rho + \partial_s \rho \partial_s f_s + \rho \partial_s^2 f_s + \rho d(\partial_s f_s) \cdot \partial_s u.$$

We now try to relate $u^* \omega$ with $|\partial_s u|^2$:

$$\begin{aligned} |\partial_s u|^2 &= \omega(\partial_s u, \partial_t u - X_s) = u^* \omega - dH_s \cdot \partial_s u = u^* \omega - h'_s dR_s \cdot \partial_s u \\ &= u^* \omega - h'_s \partial_s \rho - h'_s \rho \partial_s f_s \end{aligned}$$

where we used that $\partial_s \rho = -\rho \partial_s f_s + dR_s \cdot \partial_s u$.

Thus, $\Delta\rho = u^*\omega - F$ equals

$$|\partial_s u|^2 + h_s'\rho\partial_s f_s - \rho\partial_s h_s' - \rho h_s''\partial_s\rho - \partial_s\rho\partial_s f_s - \rho\partial_s^2 f_s - \rho d(\partial_s f_s)\cdot\partial_s u.$$

We may assume that f is C^2-bounded by a constant $C>0$. Then in particular

$$|d(\partial_s f_s)\cdot\partial_s u| \le \|d(\partial_s f_s)\|_{\rho\times\partial M}\cdot|\partial_s u| \le \rho^{-1}\|d(\partial_s f_s)\|_{1\times\partial M}\cdot|\partial_s u| \le \rho^{-1}C|\partial_s u|.$$

We deduce an inequality for $\Delta\rho$,

$$\begin{aligned}\Delta\rho + \text{ first order terms } &\ge |\partial_s u|^2 - \rho\partial_s h_s' - \rho(h_s'C + C) - C|\partial_s u| \\ &\ge (|\partial_s u| - \tfrac{1}{2}C)^2 - \rho(\partial_s h_s' + h_s'C + C) - \tfrac{1}{4}C^2.\end{aligned}$$

Therefore a maximum principle will apply for ρ if we can ensure that $\partial_s h_s' \le 0$ everywhere and that $\partial_s h_s' + h_s'C + C + C^2 \le 0$ on the (finite) interval I of all s such that $\partial_s f_s \ne 0$. Multiplying the latter condition by e^{Cs} and integrating in s, we deduce that these conditions can be satisfied provided that

$$h_+' e^{C\text{length}(I)} - h_-' \le C'$$

where C' is a constant depending only on C and the length of I. □

Theorem 28. *If $\varphi : \widehat{M} \to \widehat{N}$ is a symplectomorphism of contact type at infinity, then $SH^*(M) \cong SH^*(N)$.*

Proof. By identifying the Floer solutions via φ, the claim reduces to showing that the symplectic cohomology $SH^*(M) = \lim SH^*(h)$ is isomorphic to the symplectic cohomology $SH_f^*(M) = \lim SH_f^*(h)$ which is calculated for Hamiltonians of the form $H(e^r, y) = h(e^{r-f(y)})$, where the h are linear at infinity and $f : \partial M \to \mathbb{R}$ is a fixed smooth function.

Pick an interpolation f_s from f to 0, constant in s for large $|s|$. We can inductively construct Hamiltonians h_n and k_n on $\widehat{M}$ with $h_n' \gg k_n'$ and $k_{n+1}' \gg h_n'$, which by

Lemma 27 yield continuation maps

$$\phi_n : SH^*_f(k_n) \to SH^*(h_n),\ \psi_n : SH^*(h_n) \to SH^*_f(k_{n+1}).$$

We can arrange that the slope at infinity of the h_n, k_n grow to infinity as $n \to \infty$, so that $SH^*_f(M) = \lim SH^*_f(k_n)$ and $SH^*(M) = \lim SH^*(h_n)$.

The composites $\psi_n \circ \phi_n$ and $\phi_{n+1} \circ \psi_n$ are equal to the ordinary continuation maps $SH^*_f(k_n) \to SH^*_f(k_{n+1})$ and $SH^*(h_n) \to SH^*(h_{n+1})$.

Therefore the maps ϕ_n and ψ_n form a compatible family of maps and so define

$$\phi : SH^*_f(M) \to SH^*(M),\ \psi : SH^*(M) \to SH^*_f(M).$$

The composites $\psi \circ \phi$ and $\phi \circ \psi$ are induced by the families $\psi_n \circ \phi_n$ and $\phi_{n+1} \circ \psi_n$, which are the ordinary continuation maps defining the direct limits $SH^*_f(M)$ and $SH^*(M)$. Hence $\phi \circ \psi$, $\psi \circ \phi$ are identity maps, and so ϕ, ψ are isomorphisms. □

3.4.10 Independence from choice of cohomology representative

Lemma 29. *Let η be a one-form supported in the interior of M. Suppose there is a homotopy ω_λ through symplectic forms from ω to $\omega + d\eta$. By Moser's lemma this yields an isomorphism $\varphi : (\widehat{M}, \omega + d\eta) \to (\widehat{M}, \omega)$, and therefore a chain isomorphism*

$$\varphi : SC^*(H, \omega + d\eta) \to SC^*(\varphi^* H, \omega),$$

which is the identity on orbits outside M and sends the orbits x in M to $\varphi^{-1}x$.

3.5 Twisted symplectic cohomology

3.5.1 Transgressions

Let $ev : \mathcal{L}M \times S^1 \to \mathcal{L}M$ be the evaluation map. Define

$$\tau = \pi \circ ev^* : H^2(M;\mathbb{R}) \xrightarrow{ev^*} H^2(\mathcal{L}M \times S^1;\mathbb{R}) \xrightarrow{\pi} H^1(\mathcal{L}M;\mathbb{R}) \,,$$

where π is the projection to the Künneth summand. Explicitly, $\tau\beta$ evaluated on a smooth path u in $\mathcal{L}M$ is given by

$$\tau\beta(u) = \int \beta(\partial_s u, \partial_t u)\, ds \wedge dt.$$

In particular, $\tau\beta$ vanishes on time-independent paths in $\mathcal{L}M$. If M is simply connected, then τ is an isomorphism. After identifying

$$H^1(\mathcal{L}M;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(H_1(\mathcal{L}M;\mathbb{R}),\mathbb{R}) \cong \mathrm{Hom}(\pi_1(\mathcal{L}M),\mathbb{R})$$

and $\pi_1(\mathcal{L}M) = \pi_2(M) \rtimes \pi_1(M)$, the $\tau\beta$ correspond precisely to homomorphisms $\pi_2(M) \to \mathbb{R}$. In particular, if β is an integral class then this homomorphism is $f_* : \pi_2(M) \to \mathbb{Z}$ where $f : M \to \mathbb{CP}^\infty$ is a classifying map for β.

3.5.2 Novikov bundles of coefficients

We suggest [28] as a reference on local systems. Let α be a singular smooth real cocycle representing $a \in H^1(\mathcal{L}M;\mathbb{R})$. The Novikov bundle $\underline{\Lambda}_\alpha$ is the local system of coefficients on $\mathcal{L}M$ defined by a copy Λ_γ of Λ over each loop $\gamma \in \mathcal{L}M$ and by the multiplication isomorphism

$$t^{-\alpha[u]} \colon \Lambda_\gamma \to \Lambda_{\gamma'}$$

for each path u in $\mathcal{L}M$ connecting γ to γ'. Here $\alpha[\cdot]\colon C_1(\mathcal{L}M;\mathbb{R}) \to \mathbb{R}$ denotes evaluation on smooth singular one-chains, which is given explicitly by

$$\alpha[u] = \int \alpha(\partial_s u)\, ds.$$

Changing α to $\alpha + df$ yields a change of basis isomorphism $x \mapsto t^{f(x)}x$ for the local systems, so by abuse of notation we write $\underline{\Lambda}_a$ and $a[u]$ instead of $\underline{\Lambda}_\alpha$ and $\alpha[u]$.

Remark 30. *In Chapter 2 we used the opposite sign convention $t^{\alpha[u]}$. In this Chapter we changed it for the following reason. For a Liouville domain $(M, d\theta)$, the local system for the action $1-$form $\alpha = dA_H$ acts on Floer solutions $u \in \mathcal{M}(x, y; d\theta)$ by*

$$t^{-dA_H[u]} = t^{A_H(x)-A_H(y)}.$$

Therefore large energy Floer solutions will occur with high powers of t.

We will be considering the (co)homology of M or $\mathcal{L}M$ with local coefficients in the Novikov bundles, and we now mention two recurrent examples. First consider a transgressed form $\alpha = \tau\beta$ (see 3.5.1). Since $\tau(\beta)$ vanishes on time-independent paths, $\underline{\Lambda}_{\tau\beta}$ pulls back to a trivial bundle via the inclusion of constant loops $c\colon M \to \mathcal{L}M$. So for the bundle $c^*\underline{\Lambda}_{\tau\beta}$ we just get ordinary cohomology with underlying ring Λ,

$$H^*(M; c^*\underline{\Lambda}_{\tau(\beta)}) \cong H^*(M; \Lambda).$$

Secondly, consider a map $j\colon L \to M$. This induces a map $\mathcal{L}j\colon \mathcal{L}L \to \mathcal{L}M$ which by the naturality of τ satisfies $(\mathcal{L}j)^*\underline{\Lambda}_{\tau(\beta)} \cong \underline{\Lambda}_{\tau(j^*\beta)}$. For example if $\tau(j^*\beta) = 0 \in H^1(\mathcal{L}L;\mathbb{R})$ then this is a trivial bundle, so the corresponding Novikov homology is

$$H_*(\mathcal{L}L; (\mathcal{L}j)^*\underline{\Lambda}_{\tau(\beta)}) \cong H_*(\mathcal{L}L) \otimes \Lambda.$$

3.5.3 Twisted Floer cohomology

Let (M^{2n}, θ) be a Liouville domain. Let α be a singular cocyle representing a class in $H^1(\mathcal{L}M; \mathbb{R}) \cong H^1(\mathcal{L}\widehat{M}; \mathbb{R})$. The Floer chain complex for $H \in C^\infty(\widehat{M}, \mathbb{R})$ with twisted coefficients in $\underline{\Lambda}_\alpha$ is the $\Lambda-$module freely generated by the 1-periodic orbits of X_H,

$$CF^*(H; \underline{\Lambda}_\alpha) = \bigoplus \left\{ \Lambda x : x \in \mathcal{L}\widehat{M},\ \dot{x}(t) = X_H(x(t)) \right\},$$

and the differential δ on a generator $y \in \mathrm{Crit}(A_H)$ is defined as

$$\delta y = \sum_{u \in \mathcal{M}_0(x,y)} \epsilon(u)\, t^{-\alpha[u]}\, x,$$

where $\epsilon(u) \in \{\pm 1\}$ are orientation signs and $\mathcal{M}_0(x, y)$ is the $0-$dimensional component of Floer trajectories connecting x to y. It is always understood that we perturb (H, J) as explained in 3.3.6.

The ordinary Floer complex (with underlying ring Λ) has no weights $t^{-\alpha[u]}$ in δ. These appear in the twisted case because they are the multiplication isomorphisms $\Lambda_x \to \Lambda_y$ of the local system $\underline{\Lambda}_\alpha$ which identify the $\Lambda-$fibres over x and y (see 3.5.2).

Proposition/Definition 31. *$CF^*(H; \underline{\Lambda}_\alpha)$ is a chain complex: $\delta \circ \delta = 0$, and its cohomology $HF^*(H; \underline{\Lambda}_\alpha)$ is a $\Lambda-$module called twisted Floer cohomology.*

3.5.4 Twisted symplectic cohomology

Proposition 32 (Twisted continuation maps)**.** *For the twisted Floer cohomology of $(M, d\theta)$, Theorem 23 continues to hold for the continuation maps $\phi : CF^*(H_+; \underline{\Lambda}_\alpha) \to CF^*(H_-; \underline{\Lambda}_\alpha)$ defined on generators $y \in \mathrm{Crit}(A_{H_+})$ by*

$$\phi(y) = \sum_{v \in \mathcal{M}_0(x,y)} \epsilon(v)\, t^{-\alpha[v]}\, x.$$

Definition 33. *The twisted symplectic cohomology of $(M, d\theta; \alpha)$ is*

$$SH^*(M, d\theta; \underline{\Lambda}_\alpha) = \varinjlim HF^*(H, d\theta; \underline{\Lambda}_\alpha),$$

where the direct limit is over the twisted continuation maps between Hamiltonians H which are linear at infinity.

3.5.5 Independence from choice of cohomology representative

Lemma 34. *Let $f_H \in C^\infty(\mathcal{L}\widehat{M}, \mathbb{R})$ be an $H-$dependent function. Then the change of basis isomorphisms $x \mapsto t^{f_H(x)}x$ of the local systems induce chain isomorphisms*

$$SC^*(H, d\theta; \underline{\Lambda}_\alpha) \cong SC^*(H, d\theta; \underline{\Lambda}_{\alpha+df_H})$$

which commute with the twisted continuation maps.

3.5.6 Twisted maps $\mathbf{c}_*$ from ordinary cohomology

The twisted symplectic cohomology comes with a map from the induced Novikov cohomology of M,

$$c_* : H^*(M; c^*\underline{\Lambda}_\alpha) \to SH^*(M; d\theta, \underline{\Lambda}_\alpha).$$

The construction is analogous to 3.4.8, and was carried out in detail in Chapter 2. The map c_* comes automatically with the direct limit construction of $SH^*(M; d\theta, \underline{\Lambda}_\alpha)$, since for the Hamiltonian H^δ described in 3.4.8 we have

$$HF^*(H^\delta; \underline{\Lambda}_\alpha) \cong HM^*(H^\delta; c^*\underline{\Lambda}_\alpha) \cong H^*(M; c^*\underline{\Lambda}_\alpha).$$

3.5.7 Twisted Functoriality

In Chapter 2 we proved the following variant of Viterbo functoriality [25], which holds for *Liouville subdomains* $(W^{2n}, \theta') \subset (M^{2n}, \theta)$. These are Liouville domains for which $\theta - e^\rho\theta'$ is exact for some $\rho \in \mathbb{R}$. The standard example is the Weinstein embedding $DT^*L \hookrightarrow DT^*N$ of a small disc cotangent bundle of an exact Lagrangian $L \hookrightarrow DT^*N$ (see Chapter 2).

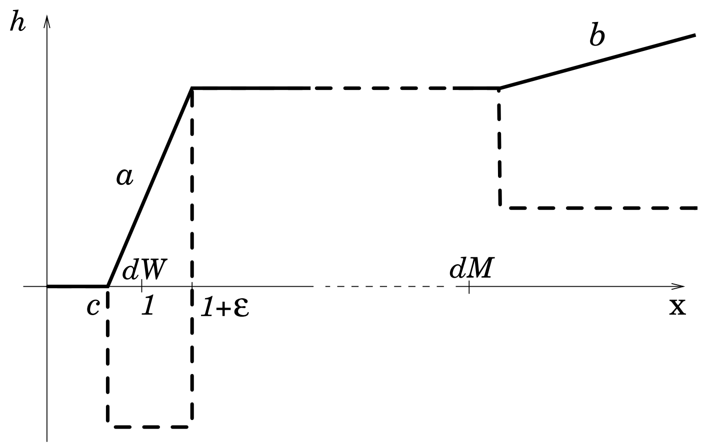


Figure 3-2: The solid line is a diagonal-step shaped Hamiltonian with $a \gg b$. The dashed line is the action $A(\mathrm{x}) = -\mathrm{x}h'(\mathrm{x}) + h(\mathrm{x})$.

Theorem 35. *Let $i : (W^{2n}, \theta') \hookrightarrow (M^{2n}, \theta)$ be a Liouville embedded subdomain. Then there exists a map*

$$SH^*(i) : SH^*(W, d\theta'; \underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) \leftarrow SH^*(M, d\theta; \underline{\Lambda}_\alpha)$$

which fits into the commutative diagram

$$\begin{array}{ccc} SH^*(W, d\theta'; \underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) & \xleftarrow{SH^*(i)} & SH^*(M, d\theta; \underline{\Lambda}_\alpha) \\ \uparrow c_* & & \uparrow c_* \\ H^*(W; c^*\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) & \xleftarrow{i^*} & H^*(M; c^*\underline{\Lambda}_\alpha) \end{array}$$

The map $SH^*(i)$ is constructed using a "step-shaped" Hamiltonian, as in Figure 3-2, which grows near ∂W and reaches a slope a, then becomes constant up to ∂M where it grows again up to slope b. By a careful construction, with $a \gg b$, one can arrange that all orbits in W have negative action with respect to $(d\theta, H)$, and for orbits outside of W they have positive actions. The map $SH^*(i)$ is then the limit, as $a \gg b \to \infty$, of the action restriction maps which quotient out by the generators of positive action.

Theorem 36. *Let $(M, d\theta)$ be a Liouville domain and let $L \subset M$ be an exact orientable Lagrangian submanifold. By Weinstein's Theorem, this defines a Liouville embedding*

$j : (DT^*L, d\theta) \to (M, d\theta)$ of a small disc cotangent bundle of L. Then for all $\alpha \in H^1(\mathcal{L}M;\mathbb{R})$ there exists a commutative diagram

$$\begin{array}{ccc} H_{n-*}(\mathcal{L}L;\underline{\Lambda}_{(\mathcal{L}j)^*\alpha}) \cong SH^*(T^*L, d\theta;\underline{\Lambda}_{(\mathcal{L}j)^*\alpha}) & \longleftarrow & SH^*(M, d\theta;\underline{\Lambda}_{\alpha}) \\ \uparrow c_* & & \uparrow c_* \\ H_{n-*}(L; c^*\underline{\Lambda}_{(\mathcal{L}j)^*\alpha}) \cong H^*(L; c^*\underline{\Lambda}_{(\mathcal{L}j)^*\alpha}) & \longleftarrow & H^*(M; c^*\underline{\Lambda}_{\alpha}) \end{array}$$

where the left vertical map is induced by the inclusion of constant loops $c : L \to \mathcal{L}L$. If $c^*\alpha = 0$ then the bottom map is the pullback $H^*(L) \otimes \Lambda \leftarrow H^*(M) \otimes \Lambda$.

Corollary 37. *Let $(M, d\theta)$ be a Liouville domain and let $L \subset M$ be an exact orientable Lagrangian. Suppose $\beta \in H^2(\widehat{M};\mathbb{R})$ is such that $\tau(j^*\beta) = 0 \in H^1(\mathcal{L}L;\mathbb{R})$. Then there is a commutative diagram*

$$\begin{array}{ccc} H_{n-*}(\mathcal{L}L) \otimes \Lambda & \longleftarrow & SH^*(M, d\theta;\underline{\Lambda}_{\tau\beta}) \\ c_* \uparrow & & \uparrow c_* \\ H_{n-*}(L) \otimes \Lambda \cong H^*(L) \otimes \Lambda & \xleftarrow{j^*} & H^*(M) \otimes \Lambda \end{array}$$

Therefore $SH^*(M, d\theta;\underline{\Lambda}_{\tau\beta})$ cannot vanish since $c_* j^* 1 = c_* 1 \neq 0$.

Remark 38. ***Unorientable exact Lagrangians.*** *In Theorem 36 we assumed that the Lagrangian is orientable. However, the result easily extends to the unorientable case: instead of using $\mathbb{Z}$ coefficients we use $\mathbb{Z}_2$ coefficients. This means that the moduli spaces do not need to be oriented and we can drop all orientation signs in the definitions of the differentials for the Floer complexes and the Morse complexes. The Novikov ring is now defined by*

$$\Lambda = \{\textstyle\sum_{n=0}^{\infty} a_n t^{r_n} : a_n \in \mathbb{Z}_2, r_n \in \mathbb{R}, r_n \to \infty\}.$$

Note that the Novikov one-form α is still chosen in $H^1(\mathcal{L}M;\mathbb{R})$.

This is particularly interesting in dimension four since $H^2(L;\mathbb{R}) = 0$ for unorientable $L^2 \subset M^4$, therefore the transgression vanishes. In particular the pullback of any transgression from M will vanish on L. This immediately contradicts

Corollary 37 if $SH^*(M, d\theta; \underline{\Lambda}_{\tau\beta}) = 0$. *For example in Chapter 2 we proved that* $SH^*(T^*S^2, d\theta; \underline{\Lambda}_{\tau\beta}) = 0$ *for any non-zero* $\beta \in H^2(S^2; \mathbb{R})$. *Therefore there can be no unorientable exact Lagrangians in* T^*S^2.

3.6 Grading of symplectic cohomology

3.6.1 Maslov index and Conley-Zehnder grading

We assume that $c_1(M) = 0$: this condition will ensure that the symplectic cohomology has a $\mathbb{Z}-$grading defined by the Conley-Zehnder index.

Since $c_1(M) = 0$, we can choose a trivialization of the canonical bundle $\mathcal{K} = \Lambda^{n,0}T^*M$. Then over any $1-$periodic Hamiltonian orbit γ, trivialize γ^*TM so that it induces an isomorphic trivialization of $\mathcal{K}$. Let ϕ_t denote the linearization $D\varphi^t(\gamma(0))$ of the time t Hamiltonian flow written in a trivializing frame for γ^*TM.

Let $\mathrm{sign}(t)$ denote the signature of the quadratic form

$$\omega(\cdot, \partial_t\phi_t\cdot) : \ker(\phi_t - \mathrm{id}) \to \mathbb{R},$$

assuming we perturbed ϕ_t relative endpoints to make the quadratic form non-degenerate and to make $\ker(\phi_t - \mathrm{id}) = 0$ except at finitely many t.

The Maslov index $\mu(\gamma)$ of γ is

$$\mu(\gamma) = \frac{1}{2}\,\mathrm{sign}(0) + \sum_{0<t<1} \mathrm{sign}(t) + \frac{1}{2}\,\mathrm{sign}(1).$$

The Maslov index is invariant under homotopy relative endpoints, and it is additive with respect to concatenations. If ϕ_t is a loop of unitary transformations, then its Maslov index is the winding number of the determinant, $\det \phi_t : \mathcal{K} \to \mathcal{K}$. For example $\phi_t = e^{2\pi i t} \in U(1)$ for $t \in [0, 1]$ has Maslov index 1.

In our applications, γ will often not be an isolated orbit. It will typically lie in an S^1-worth or an S^3-worth of orbits. In this case it is possible to make a small time-dependent perturbation of H so that γ breaks up into two isolated orbits whose

Maslov indices get shifted by $\pm\dim(S^1)/2$ or $\pm\dim(S^3)/2$ respectively.

The grading we use on SH^* is the Conley-Zehnder index, defined by

$$|\gamma| = \frac{\dim(M)}{2} - \mu(\gamma).$$

This grading agrees with the Morse index when H is a generic C^2-small Hamiltonian and γ is a critical point of H.

3.7 Deformation of the Symplectic cohomology

Let β be a compactly supported two-form representing a class in $H^2(M;\mathbb{R})$ such that $d\theta + \beta$ is symplectic. We want to construct an isomorphism between the non-exact symplectic cohomology and the twisted symplectic cohomology

$$SH^*(H, d\theta + \beta) \cong SH^*(H, d\theta; \underline{\Lambda}_{\tau\beta}).$$

We will show that this holds if $d\theta + s\beta$ is symplectic for $0 \leq s \leq 1$. For example, it will always hold if $\|\beta\| < 1$.

3.7.1 Outline of the argument

Let H^m denote a Hamiltonian which only depends on R on the collar and which has slope m at infinity. Choosing H^m generic and C^2-small inside M ensures that the only $1-$periodic Hamiltonian orbits inside M are the critical points of H^m. We will prove that we may assume that the critical points lie outside the support of β. Therefore $SC^*(H^m, d\theta + \beta)$ and $SC^*(H^m, d\theta; \underline{\Lambda}_{\tau\beta})$ have the same generators: the critical points of H^m and the $1-$periodic Hamiltonian orbits lying in the collar (we used that $\operatorname{supp}\beta \subset M$).

We will build chain isomorphisms

$$\psi_\mu^m : SC^*(H^m, d\theta + \beta) \to SC^*(H^m, d\theta; \underline{\Lambda}_{\tau\beta})$$

which are defined for a sufficiently large parameter μ; which are independent of μ on homology, say $\psi^m = \psi^m_\mu$; and which commute with the continuation maps

$$\begin{array}{ccc} SH^*(H^m, d\theta+\beta) & \xrightarrow{\psi^m} & SH^*(H^m, d\theta; \underline{\Lambda}_{\tau\beta}) \\ \downarrow & & \downarrow \\ SH^*(H^{m'}, d\theta+\beta) & \xrightarrow{\psi^{m'}} & SH^*(H^{m'}, d\theta; \underline{\Lambda}_{\tau\beta}) \end{array}$$

Therefore, by exactness of direct limits, $\psi = \lim \psi^m$ is the desired isomorphism

$$\psi : SH^*(M, d\theta+\beta) \to SH^*(M, d\theta, \underline{\Lambda}_{\tau\beta}).$$

The parameter μ arises in the construction of the maps ψ^m_μ because for large μ the identity map provides a natural chain isomorphism

$$\mathrm{id} : SC^*(H^m, d\theta+\mu^{-1}\beta) \cong SC^*(H^m, d\theta; \underline{\Lambda}_{\mu^{-1}\tau\beta}).$$

This is proved by showing that the moduli spaces $\mathcal{M}(x, y; d\theta+\lambda\beta)$ form a $1-$parameter family joining $\mathcal{M}(x, y; d\theta+\mu^{-1}\beta)$ to $\mathcal{M}(x, y; d\theta)$.

To define the maps ψ^m_μ we therefore just need to deform $d\theta+\beta$ to $d\theta+\mu^{-1}\beta$. On the twisted side, there are no difficulties:

$$SH^*(H^m, d\theta; \underline{\Lambda}_{\tau\beta}) \cong SH^*(H^m, d\theta; \underline{\Lambda}_{\mu^{-1}\tau\beta})$$

is just a rescaling $t \mapsto t^{(\mu^{-1})}$.

For the non-exact symplectic cohomology we first combine the Liouville flow φ_μ for time $\log\mu$ and a rescaling of the metric by μ^{-1}. This will change $d\theta+\beta$ to $d\theta+\mu^{-1}\varphi_\mu^*\beta$. Then we want to make a Moser deformation from $d\theta+\mu^{-1}\varphi_\mu^*\beta$ to $d\theta+\mu^{-1}\beta$, so we need a deformation through symplectic forms without changing the cohomology class. This is possible if $d\theta+s\beta$ is symplectic for $0 \leq s \leq 1$.

Lemma 39. *If $d\theta+s\beta$ is symplectic for $0 \leq s \leq 1$, then it is possible to deform $d\theta+\mu^{-1}\beta$ to $d\theta+\mu^{-1}\varphi_\mu^*\beta$ through symplectic forms within its cohomology class.*

Proof. Since $d\theta + s\beta$ are symplectic for $0 \leq s \leq 1$, so are

$$\omega_s = (s\mu)^{-1}\varphi_{s\mu}^*(d\theta + s\beta) = d\theta + \mu^{-1}\varphi_{s\mu}^*\beta$$

for $\frac{1}{\mu} \leq s \leq 1$. It remains to show that $\partial_s\omega_s$ is exact. By Cartan's formula,

$$\partial_s\omega_s = \varphi_{s\mu}^*\mathcal{L}_{Z/s\mu}\beta = \varphi_{s\mu}^*(i_{Z/s\mu}d\beta + di_{Z/s\mu}\beta) = d\varphi_{s\mu}^*(i_{Z/s\mu}\beta). \quad \square$$

The argument hides a small technical challenge. The changes in symplectic forms will change the Hamiltonian H^m (without affecting the slope at infinity). Since the 1−parameter family argument heavily depends on H^m, it is not clear that the same large μ works for all Hamiltonians of a given slope. Therefore we first apply a continuation isomorphism to change the Hamiltonian back to the original H^m. Now it is no longer clear that ψ_μ^m is independent of μ on homology, and when we take the direct limit of continuation maps as $m \to \infty$ it is not clear that the same choice of μ will work for different Hamiltonians. Thus it is necessary to prove that the construction is independent of μ.

The 1−parameter family of moduli spaces argument is presented in 3.7.7. We will need several preliminary results: the Palais-Smale Lemma (3.7.3); the Lyapunov property for the action functional (3.7.4); an a priori energy estimate (3.7.5) and a transversality result (3.7.6). In section 3.7.9 we will construct the maps ψ_μ^m.

3.7.2 Metric rescaling

Lemma 40. *Let $\mu > 0$. There is a natural identification*

$$SC^*(H,\omega) \to SC^*(\mu H, \mu\,\omega),$$

induced by the change of ring isomorphism $\Lambda \to \Lambda, t \mapsto t^\mu$.

Proof. Under the rescaling, X_H does not change, so the Floer equations don't change. The energy functional gets rescaled by μ, so a Floer trajectory contributes a factor

$t^{\mu E(u)} = (t^{\mu})^{E(u)}$ to the differential instead of $t^{E(u)}$. $\square$

3.7.3 Palais-Smale Lemma

Let X_t be a time-dependent vector field. Define

$$F : \mathcal{L}M \to \bigcup_{x \in \mathcal{L}M} x^*TM, \; F(x)(t) = \dot{x}(t) - X_t(x(t)).$$

The solutions of $F(x) = 0$ are precisely the $1-$periodic orbits of X_t. The following standard result (see Salamon [21]) ensures that F is small only near such solutions.

Lemma 41. *Let M be a compact Riemannian manifold, and X_t a time-dependent vector field on M whose $1-$periodic orbits form a discrete set. Then*

1. *A sequence $x_n \in \mathcal{L}M$ with $\|F(x_n)\|_{L^2} \to 0$ has a subsequence converging in C^0 to a solution of $F(x) = 0$.*

2. *For any $\epsilon > 0$ there is a $\delta > 0$ such that $\|F(y)\|_{L^2} < \delta$ implies that there is some solution of $F(x) = 0$ close to y, $\sup_{t \in S^1} dist(x(t), y(t)) < \epsilon$.*

Corollary 42. *Let $(M, d\theta)$ be a Liouville domain. Fix J on $\widehat{M}$ as defined in 3.3.1. Let H_t be a time-dependent Hamiltonian on $\widehat{M}$ such that $H_t = h(R)$ is linear with generic slope for $R \gg 0$. Then for any $\delta > 0$ there is an $\epsilon > 0$ such that any smooth loop $x : S^1 \to \widehat{M}$ with $\|F(x)\| < \delta$ will be within distance ϵ of some $1-$periodic orbit of H_t.*

3.7.4 Lyapunov property of the action functional

Let $(M, d\theta, J)$ be a Liouville domain. The metric we use will be $d\theta(\cdot, J\cdot)$, and denote by $|\cdot|$ the norm and by $\|\cdot\|$ the L^2-norm integrating over time. Let X be the Hamiltonian vector field for $(H, d\theta)$, where H is linear at infinity, and recall $F(x) = \partial_t x - X(x)$.

Let β be a closed two-form compactly supported in M such that $d\theta + \beta$ is symplectic. Denote X_β the Hamiltonian vector field for $(H, d\theta + \beta)$, and let

$$F_\beta(x) = \partial_t x - X_\beta(x).$$

Let $\|\beta\| = \sup|\beta(Y, Z)|$ taken over all vectors Y, Z of norm 1. We will also use the notation $Y_{\operatorname{supp}\beta}$ for a vector field Y, where

$$Y_{\operatorname{supp}\beta}(m) = Y(m) \text{ if } m \in \operatorname{supp}\beta, \text{ and } Y_{\operatorname{supp}\beta}(m) = 0 \text{ otherwise.}$$

Lemma 43. *Let V be a neighbourhood containing the $1-$periodic orbits of X in M, and let β be a closed 2-form compactly supported in M and vanishing on V.*

1. *If $\|\beta\| < 1$ then $\|F_\beta(x) - F(x)\| \leq \dfrac{\|\beta\|}{1 - \|\beta\|}\, \|X_{supp\,\beta}\|$.*

2. *There is a $\delta > 0$ depending on $(M, H, d\theta, J, V)$, but not on β, such that*

$$\|F(x)\| < \delta \Longrightarrow x \textit{ lies in } V \textit{ or outside } M, \textit{ so } F_\beta(x) = F(x).$$

3. *If $\|\beta\|$ is sufficiently small, then $\|F_\beta - F\| \leq \frac{1}{3}\|F\|$ and $\|F_\beta\| \leq 2\|F\|$.*

Proof. Observe that $F_\beta - F = X - X_\beta$ and that $d\theta(X - X_\beta, .) = \beta(X_\beta, .)$, so

$$|X - X_\beta|^2 = \beta(X_\beta, J(X - X_\beta)) \leq \|\beta\| \cdot |X_\beta| \cdot |X - X_\beta|.$$

Dividing out by $|X - X_\beta|$ gives $|X - X_\beta| \leq \|\beta\| \cdot |X_\beta|$. From $|X_\beta| \leq |X| + |X - X_\beta|$ we deduce that $|X_\beta| \leq \frac{1}{1-\|\beta\|}\,|X|$. Therefore

$$|F_\beta(x) - F(x)| \leq \frac{\|\beta\|}{1 - \|\beta\|}\, |X_{\operatorname{supp}\beta}(x)|,$$

since $F_\beta - F = X - X_\beta$ vanishes at (x, t) if the loop x lies outside the support of β at time t. The first claim follows, and the second follows by Corollary 42.

Let $C = \sup |X_{\mathrm{supp}\,\beta}|$. Then, whenever $\|F\| \geq \delta$,

$$\|F_\beta - F\| \leq \frac{\|\beta\|}{1-\|\beta\|} C \leq \frac{\|\beta\|}{1-\|\beta\|}\frac{C}{\delta}\|F\| \leq \frac{1}{3}\|F\|$$

for small enough $\|\beta\|$. The last claim then follows from (2). □

For $\|\beta\| < 1$, $d\theta + \beta$ is symplectic and $(d\theta+\beta)(\cdot, J\cdot)$ is positive definite but may not be symmetric. By symmetrizing, we obtain a metric

$$\widetilde{g}_\beta(V,W) = \frac{1}{2}[(d\theta+\beta)(V,JW) + (d\theta+\beta)(W,JV)].$$

There is a unique endomorphism B such that $\widetilde{g}_\beta(BV,W) = (d\theta+\beta)(V,W)$, and this yields an almost complex structure $J_\beta = (-B^2)^{-1/2}B$ compatible with $d\theta+\beta$, inducing the metric

$$g_\beta(V,W) = (d\theta+\beta)(V,J_\beta W) = \widetilde{g}_\beta((-B^2)^{1/2}V,W).$$

For sufficiently small $\|\beta\|$, J_β is C^2-close to J and is equal to J outside the support of β, so in particular g_β induces a norm $|\cdot|_\beta$ which is equivalent to the norm $|\cdot|$. Moreover, on the support of β we may perturb J_β among $(d\theta+\beta)-$compatible almost complex structures so that transversality holds for $(d\theta+\beta)-$Floer trajectories. For convenience, we use the abbreviations

$$\delta J = J_\beta - J \qquad \delta F = F_\beta - F.$$

Theorem 44. *Let V be a neighbourhood containing the $1-$periodic orbits of X in M, and let β be a closed 2-form compactly supported in M and vanishing on V. Then for sufficiently small $\|\beta\|$,*

$$\partial_s A(u) \leq -\frac{1}{2}\|F(u)\|^2$$

for all $u \in \mathcal{M}(x,y;d\theta+\beta,H)$, where $A(x) = -\int x^*\theta + \int H(x)\,dt$ is the action functional for $(H,d\theta)$. In particular, A is a Lyapunov function for the action $1-$form *for* $(H, d\theta + \beta)$.

Proof. The action A for $(d\theta, H)$ varies as follows on $u \in \mathcal{M}(x, y; d\theta + \beta, H)$,

$$\begin{aligned}
-\partial_s A(u) &= \int_0^1 d\theta(\partial_s u, F(u))\, dt \\
&= \int_0^1 d\theta(F(u), J_\beta F_\beta(u))\, dt \\
&= \int_0^1 d\theta(F(u), (J + \delta J)(F + \delta F)(u))\, dt \\
&\geq \|F(u)\|^2 - \|\delta J\|\, \|F(u)\|^2 - \|\delta F(u)\|\, \|F(u)\| - \|\delta F(u)\|\, \|\delta J\|\, \|F(u)\| \\
&\geq \left(1 - \|\delta J\| - \tfrac{1}{3} - \tfrac{1}{3}\|\delta J\|\right) \|F(u)\|^2,
\end{aligned}$$

using Lemma 43 in the last line. □

3.7.5 A priori energy estimate

We now want an a priori energy estimate for all $u \in \mathcal{M}(x, y; d\theta + \beta, H)$ when $\|\beta\|$ is small. The key idea is to reparametrize the action A by energy and then use the Lyapunov inequality $\partial_s A(u) \leq -\frac{1}{2}\|F(u)\|^2$ of Theorem 44. Let $e(s)$ denote the energy up to s calculated with respect to $(d\theta + \beta, J_\beta)$,

$$e(s) = \int_{-\infty}^{s} \int_0^1 |\partial_s u|_\beta^2\, dt\, ds = \int_{-\infty}^{s} \|\partial_s u\|_\beta^2\, ds$$

where $|\cdot|_\beta$ is the norm corresponding to the metric $(d\theta + \beta)(\cdot, J_\beta \cdot)$, and $\|\cdot\|_\beta$ is the L^2 norm integrated over time.

Theorem 45. *Let β be as in Theorem 44. Then there is a constant $k > 1$ such that for all $u \in \mathcal{M}(x, y; d\theta + \beta, H)$,*

$$E(u) \leq k(A(x) - A(y)).$$

Proof. $\partial_s e = \|\partial_s u\|_\beta^2$ vanishes at s precisely if $F_\beta(u) = 0$. By ignoring those s for which $\partial_s e = 0$, we can assume that $\partial_s e > 0$. Let $s(e)$ be the inverse of the function $e(s)$. Then reparametrize the trajectory u by

$$\tilde{u}(e, t) = u(s(e), t).$$

Since $\partial_e s = \dfrac{1}{\|\partial_s u\|_\beta^2}$, we deduce $\partial_e(A \circ \tilde{u}) = \dfrac{\partial_s A(u)}{\|\partial_s u\|_\beta^2} = \dfrac{\partial_s A(u)}{\|F_\beta(u)\|_\beta^2}$.
Now apply respectively Theorem 44, Lemma 43 and the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_\beta$,

$$\partial_e(A \circ \tilde{u}) \leq \frac{-\|F(u)\|^2}{2\|F_\beta(u)\|_\beta^2} \leq \frac{-\|F_\beta(u)\|^2}{\text{constant} \cdot \|F_\beta(u)\|^2} =: -\frac{1}{k}.$$

Integrate in e over $(e(-\infty), e(\infty)) = (0, E(u))$ to get $A(y) - A(x) \leq (-1/k)E(u)$. By making $\|\beta\|$ sufficiently small, one can actually make k arbitrarily close to 1. □

3.7.6 Transversality for deformations

We now prove a general result which guarantees transversality for a 1−parameter deformation $\mathcal{G}$ of a map $\mathcal{F}$ for which transversality holds. We need a preliminary lemma.

Lemma 46. *Let $L : B_1 \to B_2$ be a surjective bounded operator of Banach spaces, and consider a perturbation $L + P_\varepsilon : B_1 \to B_2$ where P_ε is a bounded operator which depends on a topological parameter ε, with $P_0 = 0$ and $\|P_\varepsilon\| \to 0$ as $\varepsilon \to 0$.*

1. *If L is Fredholm then so is $L + P_\varepsilon$ for small ε.*

2. *If L is Fredholm and surjective, then so is $L + P_\varepsilon$ for small ε.*

Proof. The Fredholm property is a norm-open condition, hence (1). Recall some general results relating an operator $L : B_1 \to B_2$ to its Banach dual $L^* : B_2^* \to B_1^*$:

i) L is surjective if and only if L^* is injective and $\operatorname{im} L$ is closed;

ii) if L is Fredholm then L^* is Fredholm;

iii) a Fredholm operator is injective if and only if it is bounded below.

In (2), L^* is bounded below, say $\|L^* v\| \geq \delta_L \|v\|$ for all $v \in B_2^*$, so

$$\|(L + P_\varepsilon)^* v\| \geq \|L^* v\| - \|P_\varepsilon^* v\| \geq (\delta_L - \|P_\varepsilon^*\|)\, \|v\|.$$

If ε is so small that $\delta_L > \|P_\varepsilon^*\| = \|P_\varepsilon\|$, then $(L+P_\varepsilon)^*$ is bounded below and so $L+P_\varepsilon$ is surjective. □

Theorem 47. *Let $Y \to X$ be a Banach vector bundle. Suppose that a differentiable section $\mathcal{F} : X \to Y$ is transverse to the zero section with Fredholm differential $D_u\mathcal{F}$ at all $u \in \mathcal{F}^{-1}(0)$. Let $\mathcal{S} : \mathbb{R} \times X \to Y$ be a differentiable parameter-valued section with $\mathcal{S}(0,\cdot) = 0$. Then for the deformation $\mathcal{G} = \mathcal{F} + \mathcal{S} : \mathbb{R} \times X \to Y$,*

1. *$\mathcal{G}^{-1}(0)$ is a smooth submanifold near $0 \times \mathcal{F}^{-1}(0)$;*

2. *$\mathcal{G}^{-1}(0)$ is transverse to $\{\lambda = 0\}$.*

3. *If 0 is an index zero regular value of $\mathcal{F}$ and $\mathcal{G}^{-1}(0)$ is compact near $\lambda = 0$, then the deformation $\mathcal{G}^{-1}(0)$ of $\mathcal{F}^{-1}(0)$ is trivial near $\lambda = 0$,*

$$\mathcal{G}^{-1}(0) \cap \{\lambda \in [-\lambda_0, \lambda_0]\} \cong [-\lambda_0, \lambda_0] \times \mathcal{F}^{-1}(0).$$

Proof. The first claim essentially follows from the implicit function theorem and Lemma 46 applied to the operators $L = D_u\mathcal{F}$ and $P_\varepsilon = D_{\lambda,u}\mathcal{S}$ with parameter $\varepsilon = (\lambda, u)$. More precisely, we reduce to the local setup by choosing an open neighbourhood U of u so that $T_U X \cong U \times B_1$, $T_U Y \cong U \times B_2$,

$$T_{[-\lambda_0,\lambda_0]\times U}(\mathbb{R} \times X) \cong ([-\lambda_0, \lambda_0] \times U) \times \mathbb{R} \times B_1,$$

so locally $D_u\mathcal{F} : B_1 \to B_2$ and $D_{\lambda,u}\mathcal{S} : \mathbb{R} \times B_1 \to B_2$. Suppose $\mathcal{F}(u_0) = 0$, then apply Lemma 46 to $L = D_{u_0}\mathcal{F}$ and $P_{(\lambda,u)} = D_u\mathcal{F} - D_{u_0}\mathcal{F} + D_{(\lambda,u)}\mathcal{S}$. Therefore $D_{\lambda,u}\mathcal{G} = L + P_{(\lambda,u)}$ is Fredholm and surjective, so by the implicit function theorem $\mathcal{G}^{-1}(0)$ is a smooth submanifold for u close to u_0. Thus claim (1) follows.

Observe that at $(\eta, \xi) \in T\mathbb{R} \oplus TX$,

$$\begin{aligned} D_{0,u}\mathcal{G} \cdot (\eta, \xi) &= D_u\mathcal{F} \cdot \xi + D_{0,u}\mathcal{S} \cdot \xi + \partial_\lambda|_{\lambda=0}\, \mathcal{S} \cdot \eta \\ &= D_u\mathcal{F} \cdot \xi + \partial_\lambda|_{\lambda=0}\, \mathcal{S} \cdot \eta. \end{aligned}$$

Therefore, $D_{0,u}\mathcal{G}\cdot(0,\xi) = D_u\mathcal{F}\cdot\xi$. We deduce that $\mathrm{im} D_u\mathcal{F} \subset \mathrm{im} D_{0,u}\mathcal{G}$ and $\ker D_u\mathcal{F} \subset \ker D_{0,u}\mathcal{G}$. Since $D_u\mathcal{F}$ is surjective whenever $\mathcal{F}(u) = 0$ $(= \mathcal{G}(0,u))$, also $D_{0,u}\mathcal{G}$ is surjective and therefore $T_{0,u}\mathcal{G}^{-1}(0) \cong \ker D_{0,u}\mathcal{G}$ must be 1 dimension larger than $\ker D_u\mathcal{F}$, so it contains some vector $(1,\xi)$, which implies claim (2).

This also relates the indices at solutions of $\mathcal{F}(u) = 0$:

$$\mathrm{ind}\, D_{0,u}\mathcal{G} = \dim\ker\, D_{0,u}\mathcal{G} = \dim\ker\, D_u\mathcal{F} + 1 = \mathrm{ind}\, D_u\mathcal{F} + 1.$$

If 0 is an index zero regular value of $\mathcal{F}$, then $\mathcal{F}^{-1}(0)$ is $0-$dimensional and $\mathcal{G}^{-1}(0)$ is a $1-$dimensional submanifold near $0\times\mathcal{F}^{-1}(0)$ diffeomorphic to a product $[-\lambda_0,\lambda_0]\times\mathcal{F}^{-1}(0)$, for some small λ_0. If $\mathcal{G}^{-1}(0)$ is compact near $\lambda = 0$ then for sufficiently small λ_0 all solutions of $\mathcal{G}(\lambda,u) = 0$ with $|\lambda| \le \lambda_0$ will be close to $0\times\mathcal{F}^{-1}(0)$, proving claim (3). □

3.7.7 The $1-$parameter family of moduli spaces

Let H be a Hamiltonian which is linear at infinity. In this section we will prove

Theorem 48. *For β as in Theorem 44 the family of moduli spaces*

$$\mathcal{M}_\lambda(x,y) = \mathcal{M}(x,y;d\theta+\lambda\beta,H)$$

is smoothly trivial near $\lambda = 0$,

$$\bigsqcup_{-\lambda_0<\lambda<\lambda_0} \mathcal{M}_\lambda(x,y) \cong \mathcal{M}(x,y;d\theta,H)\times(-\lambda_0,\lambda_0).$$

In particular, the identity map

$$id : SC^*(H,d\theta+\lambda\beta) \to SC^*(H,d\theta;\underline{\Lambda}_{dA+\lambda\tau\beta})$$

is a chain isomorphism for all small λ, where $A(x) = -\int x^*\theta + \int H(x)\,dt$ is the action functional for $(H,d\theta)$.

Proof. Let $X_{\lambda\beta}$ be the Hamiltonian vector field determined by $(H, d\theta + \lambda\beta)$. We want to compare the following two maps,

$$\mathcal{F}(u) = \partial_s u + J(\partial_t u - X) \quad \text{and} \quad \mathcal{G}(u) = \partial_s u + J_{\lambda\beta}(\partial_t u - X_{\lambda\beta}),$$

since $\mathcal{F}^{-1}(0) = \mathcal{M}(x, y)$ and $\mathcal{G}^{-1}(0) = \cup_\lambda \mathcal{M}_\lambda(x, y)$.

These maps can be extended to sections $X \to Y$ of an appropriate Banach vector bundle and generically $\mathcal{F}$ is a Fredholm map (its linearizations are Fredholm operators). Indeed for $k \geq 1$ and $p > 2$, we can take Y to be the $W^{k-1,p}$ completion of the space of smooth sections of u^*TM with suitable exponential decay at the ends. The base X is the space of $W^{k,p}$ maps $u : \mathbb{R} \times S^1 \to M$ connecting two fixed $1-$periodic Hamiltonian orbits. We refer to Salamon [21] and McDuff-Salamon [15] for a precise description.

For convenience, denote $\delta J = J_{\lambda\beta} - J$ and $\delta X = X - X_{\lambda\beta}$. We may assume that δJ is C^2-small, and we showed in Lemma 43 that

$$|\delta X| \leq \frac{|\lambda| \, \|\beta\|}{1 - |\lambda| \, \|\beta\|} \, |X_{\operatorname{supp}\beta}|.$$

So δJ, δX are small for small λ. We can rewrite $\mathcal{G}(\lambda, u) = \mathcal{F}(u) + \mathcal{S}(\lambda, u)$, where

$$\mathcal{S}(\lambda, u) = \delta J \cdot (F(u) + \delta X) + J\delta X,$$

where $F(u) = \partial_t u - X(u)$. $\mathcal{S}$ is supported at those (u, s, t) with $u(s, t) \in \operatorname{supp}\beta$, and $\mathcal{S} : X \to Y$ is a differentiable parameter-valued section vanishing at $\lambda = 0$.

By the a priori energy estimate of Theorem 45, $\mathcal{G}^{-1}(0)$ is compact near $\lambda = 0$. Theorem 47 implies that if 0 is an index zero regular value of $\mathcal{F}$ then $\mathcal{G}^{-1}(0)$ is a trivial $1-$dimensional family in the parameter λ, for small λ.

Thus, for sufficiently small λ_0, there is a natural bijection between the moduli spaces which define the differentials of $SC^*(H, d\theta + \lambda_0\beta)$ and $SC^*(H, d\theta; \underline{\Lambda}_{dA+\lambda_0\tau\beta})$.

Indeed, if $u_{\lambda_0} \in \mathcal{M}_0(x, y; H, d\theta + \lambda_0\beta)$ then there is a natural $1-$parameter family

$$u_\lambda \in \mathcal{M}_0(x, y; H, d\theta + \lambda\beta)$$

connecting u_{λ_0} to some $u_0 \in \mathcal{M}_0(x, y; H, d\theta)$. Since u_{λ_0} is homotopic to u_0 relative endpoints via u_λ, the local system $\underline{\Lambda}_{dA+\lambda_0\tau\beta}$ yields the same isomorphism for u_{λ_0} as for u_0, which is multiplication by

$$t^{-\int u^* d\theta + \int_0^1 (H(x)-H(y))\, dt - \int u^*(\lambda_0\beta)} = t^{-\int u^*(d\theta+\lambda_0\beta) + \int_0^1 (H(x)-H(y))\, dt}$$

and which is the same weight used in the definition of ∂y for $SC^*(H, d\theta+\lambda_0\beta)$. Therefore the two complexes have exactly the same generators and the same differential, and in particular the identity map between them is a chain isomorphism. $\square$

3.7.8 Continuation of the $1-$parameter family

Theorem 49. *Let β be as in Theorem 44. Let H_s be a monotone homotopy. Then the family of moduli spaces of parametrized Floer trajectories*

$$\mathcal{M}_\lambda(x, y; H_s) = \mathcal{M}(x, y; d\theta + \lambda\beta, H_s)$$

is smoothly trivial near $\lambda = 0$. In particular, the following diagram commutes for all small enough λ,

$$\begin{array}{ccc} SC^*(H_+, d\theta + \lambda\beta) & \xrightarrow{id} & SC^*(H_+, d\theta; \underline{\Lambda}_{dA+\tau\lambda\beta}) \\ \downarrow{\scriptstyle continuation} & & \downarrow{\scriptstyle continuation} \\ SC^*(H_-, d\theta + \lambda\beta) & \xrightarrow{id} & SC^*(H_-, d\theta; \underline{\Lambda}_{dA+\tau\lambda\beta}) \end{array}$$

Proof. Let $X_{s,\lambda\beta}$ be the Hamiltonian vector field determined by $(H_s, d\theta + \lambda\beta)$, and let $X_s = X_{s,0}$. The claim follows by mimicking the proof of Theorem 48 for

$$\mathcal{F}(u) = \partial_s u + J_s(\partial_t u - X_s) \quad \text{and} \quad \mathcal{G}(u) = \partial_s u + J_{s,\lambda\beta}(\partial_t u - X_{s,\lambda\beta}). \quad \square$$

Theorem 50. *Let β be as in Theorem 44. Let λ be so small that Theorem 48 holds for H. Let φ_ε be a smooth parameter-valued isotopy of $\widehat{M}$, with $\varphi_0 = id$, such that $\varphi_\varepsilon^* H$ is a monotone homotopy in ε. Let $H_{s,\varepsilon} = \varphi_{s\varepsilon}^* H$ for $s \in [0,1]$ be the homotopy from H to $\varphi_\varepsilon^* H$. Then the family of moduli spaces of parametrized Floer trajectories $\mathcal{M}_\varepsilon(x,y;H_{s,\varepsilon}) = \mathcal{M}(x,y;d\theta+\lambda\beta,H_{s,\varepsilon})$ is smoothly trivial near $\varepsilon = 0$. So there is a commutative diagram of chain isomorphisms for all small ε,*

$$\begin{array}{ccc} SC^*(H, d\theta+\lambda\beta) & \xrightarrow{id} & SC^*(H, d\theta; \underline{\Lambda}_{dA+\tau\lambda\beta}) \\ \downarrow{\scriptstyle continuation} & & \downarrow{\scriptstyle continuation} \\ SC^*(\varphi_\varepsilon^* H, d\theta+\lambda\beta) & \xrightarrow{id} & SC^*(\varphi_\varepsilon^* H, d\theta; \underline{\Lambda}_{dA+\tau\lambda\beta}) \end{array}$$

where the vertical maps send the generators $x \mapsto \varphi_\varepsilon^{-1}(x)$.

Proof. Let $X_{s,\varepsilon}$ be the Hamiltonian vector field determined by $(H_{s,\varepsilon}, d\theta+\lambda\beta)$, and let $X = X_s = X_{s,0}$. The claim follows by mimicking the proof of Theorem 48 for

$$\mathcal{F}(u) = \partial_s u + J_s(\partial_t u - X) \quad \text{and} \quad \mathcal{G}(u) = \partial_s u + J_{s,\varepsilon}(\partial_t u - X_{s,\varepsilon}). \quad \square$$

3.7.9 Construction of the isomorphism

We now give the proof outlined in 3.7.1.

Let β be a closed two-form compactly supported in the interior of M, and suppose that $d\theta + s\beta$ is symplectic for all $0 \leq s \leq 1$ (so that Lemma 39 applies).

Let H^m be a Hamiltonian linear at infinity with slope m. Up to a continuation isomorphism on symplectic cohomologies, we may assume that all critical points of H^m in the interior of M lie in a neighbourhood V contained in $M \setminus \operatorname{supp}\beta$. This technical remark is explained in section 3.7.10.

Define ψ_μ^m by the diagram of isomorphisms

$$\begin{array}{ccc}
SC^*(H^m, d\theta + \beta) & \overset{\psi_\mu^m}{\dashrightarrow} & SC^*(H^m, d\theta; \underline{\Lambda}_{\tau\beta}) \\
\Big\downarrow {\scriptstyle (1)\ \text{Liouville } \varphi_\mu} & & \Big\downarrow {\scriptstyle (6)\ \text{rescale}} \\
SC^*(\varphi_\mu^* H^m, \mu d\theta + \varphi_\mu^* \beta) & & \\
\Big\downarrow {\scriptstyle (2)\ \text{Moser } \sigma_\mu} & & \\
SC^*(\phi_\mu^* H^m, \mu d\theta + \beta) & & \\
\Big\downarrow {\scriptstyle (3)\ \text{continuation}} & & \\
SC^*(\mu H^m, \mu d\theta + \beta) & & SC^*(H^m, d\theta; \underline{\Lambda}_{\mu^{-1}\tau\beta}) \\
\Big\downarrow {\scriptstyle (4)\ \text{rescale}} & & \Big\downarrow {\scriptstyle (7)\ \text{change of basis}} \\
SC^*(H^m, d\theta + \mu^{-1}\beta) & \xrightarrow[(5)]{\text{id}} & SC^*(H^m, d\theta; \underline{\Lambda}_{dA + \mu^{-1}\tau\beta})
\end{array}$$

where the maps are defined as follows:

1. apply φ_μ, the Liouville flow for time $\log \mu$ (see 3.3.1 for the definition of the Liouville vector field);

2. apply the Moser symplectomorphism $\sigma_\mu : (\widehat{M}, \mu\, d\theta + \beta) \to (\widehat{M}, \mu\, d\theta + \varphi_\mu^* \beta)$ obtained by Lemmas 39 and 29, and denote $\phi_\mu = \sigma_\mu \circ \varphi_\mu$;

3. observe that $\phi_\mu^* H^m$ has slope μm at infinity, so the linear interpolation from μH^m to $\phi_\mu^* H^m$ is a compactly supported homotopy and therefore induces a continuation isomorphism;

4. metric rescaling by μ^{-1} (Lemma 40), which changes t to $T = t^{(\mu^{-1})}$;

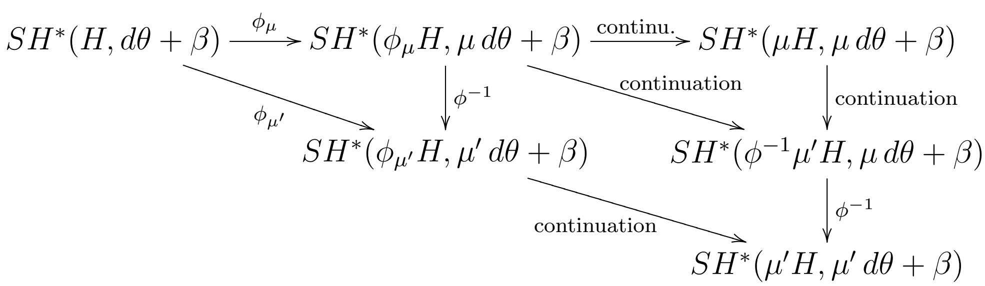
5. the identity map is a chain isomorphism by Theorem 48 provided μ is sufficiently large (depending on m);

6. rescale $\tau\beta$ to $\mu^{-1}\tau\beta$, so change t to $T = t^{(\mu^{-1})}$;

7. adding an exact form dA to $\mu^{-1}\tau\beta$, where A is the action $1-$form for $(H^m, d\theta)$, corresponds to a change of basis $x \mapsto T^{A(x)} x$ by Lemma 34.

Lemma 51. *The map* $\psi^m_\mu : SH^*(H^m, d\theta + \beta) \to SH^*(H^m, d\theta; \underline{\Lambda}_{\tau\beta})$ *on homology does not depend on the choice of large* μ*.*

Proof. In this proof we abbreviate H^m by H and pullbacks ϕ^* by ϕ. Consider μ' close to μ, and write $\phi = \phi_{\mu'}\phi_\mu^{-1}$ and $\varphi = \varphi_{\mu'}\varphi_\mu^{-1}$. Observe the following commutative diagram, in which the top row and bottom diagonal are part of the construction of the maps ψ^m_μ and $\psi^m_{\mu'}$, for $\mu' > \mu$.



The last vertical composite, after a metric rescaling, is the map

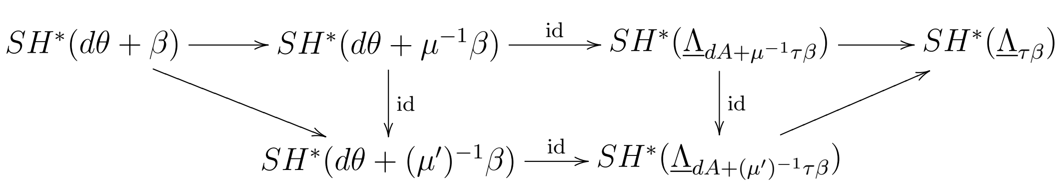
$$\phi^{-1} \circ C : SH^*(H, d\theta + \mu^{-1}\beta) \to SH^*(H, d\theta + {\mu'}^{-1}\beta)$$

where C is the continuation map

$$C : SH^*(H, d\theta + \mu^{-1}\beta) \to SH^*(\mu^{-1}\phi^{-1}\mu' H, d\theta + \mu^{-1}\beta).$$

For μ' sufficiently close to μ, ϕ^{-1} is an isotopy of $\widehat{M}$ close to the identity, therefore by Theorem 50, C maps the generators by ϕ. Thus $\phi^{-1} \circ C = \mathrm{id}$ for μ' close to μ.

For the twisted symplectic cohomology we just apply changes of basis so we deduce the following commutative diagram (using abbreviated notation),



We showed that this diagram holds for all μ' close to μ. Suppose it holds for all $\mu, \mu' \in [\mu_0, \mu_1)$, for some maximal such $\mu_1 < \infty$. Apply the above result to $\mu = \mu_1$, then the diagram holds for all $\mu, \mu' \in (\mu_1 - \epsilon, \mu_1 + \epsilon)$, for some $\epsilon > 0$. Thus it holds for all $\mu, \mu' \in [\mu_0, \mu_1 + \epsilon)$. So there is no maximal such μ_1 and the diagram must hold for all large enough μ, μ', and thus the map $SH^*(H, d\theta + \beta) \to SH^*(H, d\theta; \underline{\Lambda}_{\tau\beta})$ does not depend on the choice of (large) μ. □

Lemma 52. *The maps $\psi^m : SH^*(H^m, d\theta + \beta) \to SH^*(H^m, d\theta; \underline{\Lambda}_{\tau\beta})$ commute with the continuation maps induced by monotone homotopies.*

Proof. Let H_s be a monotone homotopy from $H^{m'}$ to H^m. By theorem 49, for sufficiently large μ the following diagram commutes

$$\begin{array}{ccc} SC^*(H_+, d\theta + \mu^{-1}\beta) & \xrightarrow{\text{id}} & SC^*(H_+, d\theta; \underline{\Lambda}_{dA+\mu^{-1}\tau\beta}) \\ \text{continuation}\downarrow & & \downarrow\text{continuation} \\ SC^*(H_-, d\theta + \mu^{-1}\beta) & \xrightarrow{\text{id}} & SC^*(H_-, d\theta; \underline{\Lambda}_{dA+\mu^{-1}\tau\beta}) \end{array}$$

and by Lemma 34 we deduce the required commutative diagram

$$\begin{array}{ccc} SC^*(H_+, d\theta + \beta) & \xrightarrow{\psi^m} & SC^*(H_+, d\theta; \underline{\Lambda}_{\tau\beta}) \\ \text{continuation}\downarrow & & \downarrow\text{continuation} \\ SC^*(H_-, d\theta + \beta) & \xrightarrow{\psi^{m'}} & SC^*(H_-, d\theta; \underline{\Lambda}_{\tau\beta}) \end{array}$$

Theorem 53. *Let β be a closed two-form compactly supported in the interior of M, and suppose that $d\theta + s\beta$ is symplectic for $0 \le s \le 1$. Then there is an isomorphism*

$$\psi : SH^*(M, d\theta + \beta) \to SH^*(M, d\theta; \underline{\Lambda}_{\tau\beta}).$$

Proof. By Lemma 51 the map $\psi^m = \psi^m_\mu$ on homology is independent of μ for large μ, and by Lemma 52 the maps ψ^m commute with continuation maps. The direct limit is an exact functor, so $\psi = \lim \psi^m$ is an isomorphism. □

Remark 54. *The theorem can sometimes be applied to deformations ω_s which are not compactly supported by using Gray's stability theorem e.g. see Lemma 68.*

Remark 55. *Let $\beta \in H^2(M; \mathbb{R})$ come from $H^2(\partial M; \mathbb{R})$ by the Thom construction. Then $SH^*(M, d\theta + \beta) \cong SH^*(M, d\theta; \Lambda)$, the ordinary symplectic cohomology with* *underlying ring* Λ*. Indeed, suppose* β *vanishes on*

$$Fix(\varphi_\mu) = \lim_{\mu\to-\infty} \varphi_\mu(M).$$

Let $H = h(R)$ *be a convex Hamiltonian defined in a neighbourhood* $R < R_0$ *of* $Fix(\varphi_\mu)$ *where* β *vanishes, such that* $h'(R) \to \infty$ *as* $R \to R_0$*. Let* $H^m = h$ *if* $h' \leq m$ *and let* H^m *be linear with slope* m *elsewhere. Then the Floer solutions concerned in the symplectic chain groups will all lie in the subset of* M *where* $\beta = 0$*.*

3.7.10 Technical remark

We assumed in 3.7.9 that all critical points of H in the interior of M lie in a neighbourhood $V \subset M \setminus \operatorname{supp}\beta$. We can do this as follows.

Pick a small neighbourhood V around $\mathrm{Crit}(H)$ so that $\beta|_V = d\alpha$ is exact. We may assume that α is supported in V. To construct the isomorphism of 3.7.9 we need to homotope $d\theta + \mu^{-1}\beta$ to $d\theta + \mu^{-1}(\beta - d\alpha)$ for all large μ. This can be done by a Moser isotopy compactly supported in V via the exact deformation $\omega_s = d\theta + \mu^{-1}(\beta - sd\alpha)$. Since $\partial_s\omega_s = -\mu^{-1}d\alpha$, for large μ the Moser isotopy ϕ_s is close to the identity. Therefore during the isotopy the critical points of ϕ_s^*H stay within V. This guarantees that the Palais-Smale Lemma 42 can be applied for V independently of large μ, and the construction 3.7.9 can be carried out with minor modifications.

3.8 ADE spaces

3.8.1 Hyperkähler manifolds

We suggest [8] for a detailed account of Hyperkähler manifolds and ADE spaces.

Recall that a symplectic manifold (M, ω) is *Kähler* if there is an integrable $\omega-$compatible almost complex structure I. Equivalently, a Riemannian manifold (M, g) is Kähler if there is an orthogonal almost complex structure I which is covariant constant with respect to the Levi-Civita connection. (M, g) is called *hyperkähler* if there are three orthogonal covariant constant almost complex structures I, J, K satisfying the quaternion relation $IJK = -1$.

The hyperkähler manifold (M, g) is therefore Kähler with respect to each of the (integrable) complex structures I, J, K, with corresponding Kähler forms

$$\omega_I = g(I\cdot, \cdot), \quad \omega_J = g(J\cdot, \cdot), \quad \omega_K = g(K\cdot, \cdot).$$

Indeed, there is an S^2 worth of Kähler forms: any vector $(u_I, u_J, u_K) \in S^2 \subset \mathbb{R}^3$ gives rise to a complex structure $I_u = u_I I + u_J J + u_K K$ and a Kähler form

$$\omega_u = u_I \omega_I + u_J \omega_J + u_K \omega_K.$$

We will always think of M as a complex manifold with respect to I, and we recall from [9] that $\omega_J + i\omega_K$ is a holomorphic symplectic structure on M (a non-degenerate closed holomorphic $(2,0)$ form). The form $\omega_J + i\omega_K$ determines a trivialization of the canonical bundle $\Lambda^{2,0}T^*M$, so $c_1(M) = 0$ and the Conley-Zehnder indices give a $\mathbb{Z}-$grading on symplectic cohomology (see 3.6.1).

Lemma 56. *Let $L^2 \subset \mathbb{H}$ be an $I-$complex vector subspace of the space of quaternions. Then L is a Lagrangian subspace with respect to ω_J and ω_K, and a symplectic subspace with respect to ω_I. After an automorphism of $\mathbb{H}$, L is identified with $\mathbb{C} \oplus 0 \subset \mathbb{H}$.*

Proof. L is a complex 1-dimensional vector subspace with respect to the $I-$holomorphic symplectic form $\omega_c = \omega_J + i\omega_K$, and therefore it is complex Lagrangian. Thus L is a real Lagrangian vector subspace with respect to ω_J and ω_K.

Moreover, given any vector $e_1 \in L$, let $e_2 = Ie_1$, $e_3 = Je_1$ and $e_4 = Ke_1$. Then $L = \text{span}\{e_1, e_2\}$ and $\omega_I(e_1, e_2) = g(e_2, e_2) > 0$, so L is symplectic with respect to ω_I and corresponds to $\mathbb{C} \oplus 0$ in the hyperkähler basis $e_1, \ldots, e_4$. □

3.8.2 Hyperkähler quotients

Let M be a simply connected hyperkähler manifold. Let G be a compact Lie group G acting on M and preserving g, I, J, K. Then corresponding to the forms ω_I, ω_J,

ω_K there exist moment maps μ_I, μ_J, μ_K. Recall that if ζ is in the Lie algebra $\mathfrak{g}$ of G, then it generates a vector field X_ζ on M. A moment map $\mu : M \to \mathfrak{g}^\vee$ is a $G-$equivariant map such that

$$d\mu_m(\zeta) = \omega(X_\zeta(m), \cdot) \;\text{ at }\; m \in M.$$

For simply connected M such a μ exists and is determined up to the addition of an element in $Z = (\mathfrak{g}^\vee)^G$, the invariant elements of the dual Lie algebra $\mathfrak{g}^\vee$.

Putting these moment maps together yields $\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathbb{R}^3 \otimes \mathfrak{g}^\vee$, and for $\zeta \in \mathbb{R}^3 \otimes Z$ we may define the hyperkähler quotient space

$$X_\zeta = \mu^{-1}(\zeta)/F.$$

If F acts freely on $\mu^{-1}(\zeta)$ then X_ζ is a smooth manifold of dimension $\dim M - 4\dim F$ and the structures g, I, J, K descend to X_ζ making it hyperkähler (see [9]).

3.8.3 ALE and ADE spaces

Definition 57. *An* ADE space *is a minimal resolution of the quotient singularity* $\mathbb{C}^2/\Gamma$ *for a finite subgroup* Γ *of* $SU(2)$*. An* ALE space *(asymptotically locally Euclidean) is a Riemannian* $4-$*manifold with precisely one end which at infinity is isometric to a quotient* $\mathbb{R}^4/\Gamma$ *by a finite group* Γ*, where* $\mathbb{R}^4/\Gamma$ *is endowed with a metric that differs from the Euclidean metric by order* $\mathcal{O}(r^{-4})$ *terms and which has the appropriate decay in the derivatives.*

Theorem 58 (Kronheimer, [12])**.** *Every ALE hyperkähler manifold is diffeomorphic to an ADE space.*

We now recall Kronheimer's construction [11] of ADE spaces as hyperkähler quotients. Let R be the left regular representation of $\Gamma \subset SU(2)$ endowed with the natural Euclidean metric,

$$R = \bigoplus_{\gamma\in\Gamma} \mathbb{C}e_\gamma \cong \mathbb{C}^{|\Gamma|}.$$

Denoting by $\mathbb{C}^2$ the natural left $SU(2)-$module, let

$$M = (\mathbb{C}^2 \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(R, R))^{\Gamma}$$

be the pairs of endomorphisms (α, β) of R, which are invariant under the induced left action of Γ. We make M into a hyperkähler vector space by letting I act by i and J by $J(\alpha, \beta) = (-\beta^*, \alpha^*)$.

The Lie group

$$F = \mathrm{Aut}_{\mathbb{C}}(R, R)^{\Gamma}/\{\text{scalar maps}\}$$

of unitary automorphisms of R which are $\Gamma-$invariant act by conjugation on M, $f \cdot (\alpha, \beta) = (f\alpha f^{-1}, f\beta f^{-1})$, where we quotiented by the scalar matrices since they act trivially. The corresponding Lie algebra $\mathfrak{f}$ corresponds to the traceless elements of $\mathrm{Hom}_{\mathbb{C}}(R, R)$, and the moment maps are:

$$\mu_I(\alpha, \beta) = \frac{1}{2} i([\alpha, \alpha^*] + [\beta, \beta^*]), \quad (\mu_J + i\mu_K)(\alpha, \beta) = [\alpha, \beta].$$

By McKay's correspondence, this description can be made explicit. Recall that $R = \oplus n_i R_i$, where the R_i are the complex irreducible representations of Γ of complex dimension n_i. Then $\mathbb{C}^2 \otimes R_i \cong \oplus_j A_{ij} R_j$ where A is the adjacency matrix describing an extended Dynkin diagram of ADE type (the correspondence between Γ and the type of diagram is described in the Introduction). It follows that

$$M = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$$

where each edge $i \to j$ of the extended Dynkin diagram appears twice, once for each choice of orientation. Moreover,

$$F = (\oplus_i U(n_i))/\{\text{scalar maps}\}$$

where the unitary group $U(n_i)$ acts naturally on $\mathbb{C}^{n_i}$.

The hyperkähler quotient for $\zeta \in Z = \text{centre}(\mathfrak{f}^\vee)$ is therefore

$$X_\zeta = \mu^{-1}(\zeta)/F.$$

Definition 59. *Let $\mathfrak{h}_\mathbb{R}$ denote the real Cartan algebra associated to the Dynkin diagram for Γ. Let the hyperplanes $D_\theta = \ker\theta$ denote the walls of the Weyl chambers, where the θ are the roots. We identify the centre Z with $\mathfrak{h}_\mathbb{R}$ by dualizing the map*

$$\textit{centre}(\mathfrak{f}) \to \mathfrak{h}_\mathbb{R}^\vee,\ i\,\pi_k \mapsto n_k\theta_k,$$

where $\pi_k : R \to \mathbb{C}^{n_k} \otimes R_k$ are the projections to the summands.

We call $\zeta \in \mathbb{R}^3 \otimes Z$ generic *if it does not lie in $\mathbb{R}^3 \otimes D_\theta$ for any root θ, i.e. $\theta(\zeta_1)$, $\theta(\zeta_2)$, $\theta(\zeta_3)$ are not all zero for any root θ.*

Theorem 60 (Kronheimer, [11])**.** *Let $\zeta \in \mathbb{R}^3 \otimes Z$ be generic. Then X_ζ is a smooth hyperkähler four-manifold with the following properties.*

1. *X_ζ is a continuous family of hyperkähler manifolds in the parameter ζ;*

2. *X_0 is isometric to $\mathbb{C}^2/\Gamma$;*

3. *there is a map $\pi : X_\zeta \to X_0$ which is an $I-$holomorphic minimal resolution of $\mathbb{C}^2/\Gamma$, and π varies continuously with ζ;*

4. *in particular, π is a biholomorphism away from $\pi^{-1}(0)$ and $\pi^{-1}(0)$ consists of a collection of $I-$holomorphic spheres with self-intersection -2 which intersect transversely according to the Dynkin diagram from Γ;*

5. *$H^2(X_\zeta;\mathbb{R}) \cong Z$ such that $[\omega_I], [\omega_J], [\omega_K]$ map to $\zeta_1, \zeta_2, \zeta_3$.*

6. *$H_2(X_\zeta;\mathbb{Z}) \cong \{$root lattice for $\Gamma\}$, such that the classes Σ with self-intersection -2 correspond to the roots;*

7. *X_ζ and $X_{\zeta'}$ are isometric hyperkähler manifolds if ζ, ζ' lie in the same orbit of the Weyl group;*

8. *Every hyperkähler ALE space asymptotic to $\mathbb{C}^2/\Gamma$ is isomorphic to X_ζ for some generic ζ.*

3.8.4 Plumbing construction of ADE spaces

Our goal is to prove that for any ADE space X, $SH^*(X;\omega) = 0$ for a generic choice of (non-exact) symplectic form ω. By Theorem 60.(5) the cohomology class $[\omega_I]$ ranges linearly in ζ_1 over all of $H^2(X;\mathbb{R})$. Therefore it suffices to consider the hyperkähler quotient $X = X_\zeta$ for all generic $\zeta = (\zeta_1, 0, 0) \in Z \otimes \mathbb{R}^3$.

Lemma 61. *The exceptional divisors in X are exact Lagrangian spheres with respect to ω_J and ω_K and they are symplectic spheres with respect to ω_I. Moreover, the areas $< \omega_I, \Sigma_m >$ of the exceptional spheres Σ_m range linearly in ζ_1 over all possible positive values.*

Proof. The first statement is an immediate consequence of Lemma 56, using the fact that the exceptional divisors in X are holomorphic spheres by Theorem 60.(4). Note that if a sphere is Lagrangian then it is exact since $H^1(S^2;\mathbb{R}) = 0$. The second statement is immediate since $[\omega_I]$ ranges linearly in ζ_1 over $H^2(X;\mathbb{R})$ and the Σ_m generate $H_2(X;\mathbb{Z})$, by Theorem 60.(6). □

The space X is the plumbing of copies of $T^*\mathbb{C}P^1$, plumbed according to the Dynkin diagram for Γ. Indeed, by mimicking the proof of Weinstein's Lagrangian neighbourhood theorem, one observes that a neighbourhood of the collection of exceptional Lagrangian spheres is symplectomorphic to a plumbing of copies of small disc cotangent bundles $DT^*\mathbb{C}P^1$. That neighbourhood can be chosen so that its complement is a symplectic collar diffeomorphic to $(S^3/\Gamma) \times [1, \infty)$, since X is biholomorphic to $\mathbb{C}^2/\Gamma$ away from 0.

3.8.5 Contact hypersurfaces inside ADE spaces

Lemma 62. *Recall that to any $(u_I, u_J, u_K) \in S^2 \subset \mathbb{R}^3$ gives rise to a Kähler form*

$$\omega_u = u_I\omega_I + u_J\omega_J + u_K\omega_K.$$

Then (X, ω_u) *is a symplectic manifold such that* $\pi^{-1}(S_r^3/\Gamma)$ *is a contact hypersurface in* X *for all sufficiently large* r*, so that* X *can be thought of as a symplectic manifold with contact type boundary with an infinite collar attached. Moreover,* X *is exact symplectic precisely when* $u_I = 0$*.*

Proof. Recall $\pi : X \to \mathbb{C}^2/\Gamma$ denotes the resolution. Let ω'_u denote the corresponding combination of forms for $\mathbb{C}^2/\Gamma = \mathbb{H}/\Gamma$. On $\mathbb{C}^2/\Gamma$ the Liouville vector field for any ω'_u is $Z = \partial_r$, and $\omega'_u = d\theta'_u$ where $\theta'_u = i_Z\omega'_u$. Restricted to any sphere S_r^3/Γ of radius $r > 0$, θ'_u is the corresponding contact one-form.

By Theorem 60, X is asymptotic to $\mathbb{C}^2/\Gamma = \mathbb{H}/\Gamma$ at infinity, therefore on $\pi^{-1}(S_r^3/\Gamma)$, $\omega_u = d\theta_u$ where θ_u can be chosen to be asymptotic to θ'_u. In particular, since $\theta'_u \wedge d\theta'_u > 0$ also $\theta_u \wedge d\theta_u > 0$ on $\pi^{-1}(S_r^3/\Gamma)$ for large r. Thus X can be thought of as a contact type manifold with boundary $\pi^{-1}(S_r^3/\Gamma)$ with the infinite collar $\pi^{-1}(\cup_{\rho\geq r} S_\rho^3/\Gamma)$ attached. The last statement follows by Lemma 61. □

3.8.6 An S^1-action on ADE spaces

Let $X = X_{\zeta_1,0,0}$ for generic $(\zeta_1, 0, 0)$. The resolution $\pi : X \to \mathbb{C}^2/\Gamma$ can be described explicitly as follows (following [8]). The moment map equations are $[\alpha, \beta] = 0$ and $[\alpha, \alpha^*] + [\beta, \beta^*] = -2i\zeta_1$. Since α, β commute by the first equation, they have a common eigenvector e, say $(\alpha, \beta)e = (a, b)e$. By $\Gamma-$invariance, $e^\gamma = R(\gamma) \cdot e$ is also a common eigenvector such that

$$(\alpha, \beta)e^\gamma = (\gamma \cdot (a, b))e^\gamma.$$

The map $X \to \mathbb{C}^2/\Gamma$, $(\alpha, \beta) \mapsto \Gamma \cdot (a, b)$ is then an $I-$holomorphic minimal resolution. In fact π is also compatible with J and K if we identify $\mathbb{C}^2/\Gamma = \mathbb{H}/\Gamma$.

Theorem 63. *The* S^1-*action* $\lambda\cdot(a, b) = (\lambda a, \lambda b)$ *on* $\mathbb{C}^2/\Gamma$ *lifts to a unique* $I-$*holomorphic* S^1-*action on* (X, ω_I)*. Moreover the* S^1-*action preserves the contact hypersurface* $\pi^{-1}(S_r^3/\Gamma)$ *inside* (X, ω_I) *described in Lemma 62, and the contact form* θ_I *can be chosen to be* S^1-*equivariant.*

Proof. Since Γ is a complex group, it commutes with the diagonal S^1-action on $\mathbb{C}^2$, therefore the action is well-defined on $\mathbb{C}^2/\Gamma$. The lift of the action is

$$\varphi_\lambda(\alpha, \beta) = (\lambda\alpha, \lambda\beta).$$

In particular, the S^1-action preserves ω_I because it preserves the metric g and it commutes with the action of I.

Let θ_I denote the contact form constructed in Lemma 62 for the hypersurface $\pi^{-1}(S^3_r/\Gamma)$ and the symplectic form ω_I. To make θ_I an S^1-equivariant contact form, we simply replace it by the S^1-averaged form $\overline{\theta}_I = \int_0^1 \varphi^*_{e^{2\pi it}}\theta_I\,dt$. Since $\varphi^*_\lambda\omega_I = \omega_I$, it satisfies $d\overline{\theta}_I = \omega_I$ and the positivity condition

$$\overline{\theta}_I \wedge d\overline{\theta}_I = \left(\int_0^1 \varphi^*_{e^{2\pi it}}\theta_I\,dt\right) \wedge \omega_I = \int_0^1 \varphi^*_{e^{2\pi it}}(\theta_I \wedge \omega_I)\,dt > 0. \quad \square$$

Remark 64. *The S^1-action does not preserve ω_J and ω_K. That is why the symplectic cohomology for ω_I will be very different from the one for ω_J or ω_K.*

3.8.7 Changing the contact hypersurface to a standard S^3/Γ

Our aim is to change the contact hypersurface in (X, ω_I) so that it becomes a standard S^3_r/Γ. We want to do this compatibly with the S^1-actions on X and $\mathbb{C}^2/\Gamma$, so that the S^1-action on (X, ω_I) will coincide with the new Reeb flow. To do this, we need an S^1-equivariant version of Gray's stability theorem.

Lemma 65 (S^1-equivariant Gray stability)**.** *For $t \in [0,1]$, let $\xi_t = \ker\alpha_t$ be a smooth family of contact structures on some closed manifold N^{2n-1}. Then there is an isotopy ψ_t of N and a family of smooth functions f_t such that*

$$\psi_t^*\alpha_t = e^{f_t}\alpha_0.$$

If there is an S^1-action on N preserving each α_t, then f_t and ψ_t are S^1-equivariant.

Proof. Let X_t be a vector field inducing a flow ψ_t. By Cartan's formula,

$$\partial_t \psi_t^* \alpha_t = \psi_t^*(\dot{\alpha}_t + \mathcal{L}_{X_t}\alpha_t) = \psi_t^*(\dot{\alpha}_t + d i_{X_t}\alpha_t + i_{X_t} d\alpha_t).$$

Observe now that if ψ_t satisfied the claim, then $\partial_t \psi_t^* \alpha_t = \dot{f}_t e^{f_t}\alpha_0 = \psi_t^*(\dot{f}_t(\psi_t^{-1})\,\alpha_t)$.

We can reverse the argument to obtain the required ψ_t if we can find a vector field X_t in ξ_t (so $i_{X_t}\alpha_t = 0$) satisfying the equation

$$i_{X_t} d\alpha_t = \dot{f}_t(\psi_t^{-1})\,\alpha_t - \dot{\alpha}_t.$$

Inserting the Reeb vector field $\mathcal{R}_t$ we obtain $0 = \dot{f}_t(\psi_t^{-1}) - \dot{\alpha}_t(\mathcal{R}_t)$. Solving the latter equation determines f_t with $f_0 = 0$. Then the original equation determines $X_t \in \xi_t$ since $d\alpha_t$ is non-degenerate on ξ_t.

Suppose we had an S^1-action φ_λ preserving α, $\varphi_\lambda^* \alpha_t = \alpha_t$. Applying φ_λ^* to the equation which determines X_t at x we obtain the equation

$$i_{\varphi_\lambda^* X_t} d\alpha_t = \dot{f}_t(\psi_t^{-1})\,\alpha_t - \dot{\alpha}_t$$

at $y = \varphi_\lambda^{-1}(x)$. The solution f_t does not change and so by uniqueness and $\varphi_\lambda^* X_t = X_t$, which proves that f_t and ψ_t are S^1-equivariant. □

Lemma 66. *The contact hypersurface $\pi^{-1}(S_r^3/\Gamma)$ can be deformed inside (X, ω_I) into a copy of the standard S_r^3/Γ via an S^1-equivariant contactomorphism.*

Proof. Consider $X_t = X_{t\zeta_1,0,0}$ and denote by ω_t its form ω_I, $(0 \leq t \leq 1)$, and let $\pi_t : X_t \to X_0 = \mathbb{H}/\Gamma$ denote the minimal resolution. By Lemma 62, each X_t comes with an S^1-equivariant contact form θ_t with $d\theta_t = \omega_t$ and such that θ_0 is the standard contact form on $S_r^3/\Gamma \subset X_0$. This defines a family of S^1-equivariant contact forms $\alpha_t = (\pi_t)_*\theta_t$ on S_r^3/Γ. By Lemma 65 there is an S^1-equivariant isomorphism $(S_r^3/\Gamma, e^{f_1}\alpha_1) \to (S_r^3/\Gamma, \alpha_0)$. In particular, this proves that X arises by attaching an infinite collar to the manifold

$$\{(R, x) : R \leq e^{f_1(\pi x)}, x \in \pi^{-1}(S_r^3/\Gamma)\} \subset X$$

along the boundary $\{(e^{f_1(\pi x)}, x)\} \subset X$ which is a standard contact S^3_r/Γ. □

3.8.8 Non-vanishing of the exact symplectic cohomology

Theorem 67. *$SH^*(X, \omega_u) \neq 0$ for $u = (0, u_J, u_K) \in S^2$, indeed $c_* : H^*(X) \otimes \Lambda \to SH^*(X, \omega_u)$ is an injection.*

Proof. The exceptional spheres in X are exact by Lemma 61. For each such S^2 we have a commuting diagram by Theorem 36, using the bundles described in 3.5.2:

$$\begin{array}{ccc} H_{4-*}(\mathcal{L}S^2) \otimes \Lambda \cong SH^*(T^*S^2, d\theta) & \longleftarrow & SH^*(X, \omega_u) \\ \uparrow c_* & & \uparrow c_* \\ H_{4-*}(S^2) \otimes \Lambda \cong H^*(S^2) \otimes \Lambda & \xleftarrow{i^*} & H^*(X) \otimes \Lambda \end{array}$$

The left vertical map is induced by the inclusion of constant loops and it is injective on homology because it has a left inverse by evaluation at 0. Since $H^*(X)$ is generated by the exceptional spheres by Theorem 60, and i^* is the projection to the summands of $H^*(X)$, the claim follows. □

3.8.9 Vanishing of the non-exact symplectic cohomology

Lemma 68. *The non-compactly supported deformation from ω_J to ω_I can be made to satisfy Theorem 53:*

$$SH^*(X, \omega_J; \underline{\Lambda}_{\tau\omega_I}) \cong SH^*(X, \omega_I).$$

Proof. Let $\omega_\varepsilon = \omega_J + \varepsilon\omega_I$. By the proof of Lemma 62, we can find a family of contact forms $\theta_\varepsilon|_S$ on $S = \pi^{-1}(S^3_r/\Gamma)$ with $d\theta_\varepsilon = \omega_\varepsilon$. By Gray's stability theorem, there is a family of contactomorphisms $\psi_\varepsilon : S \to S$ such that $\psi^*_\varepsilon(\theta_0|_S) = e^{f_\varepsilon}\theta_\varepsilon|_S$. As we deform ω_0 to ω_ε we simultaneously change the hypersurface in X by

$$S \to X, (R, x) \mapsto (e^{-f_\varepsilon(R,x)}R, \psi_\varepsilon(R, x)),$$

so that on the collar determined by this hypersurface the one-form is θ_0 instead of θ_ε. This change of hypersurface will change the symplectic cohomology by an isomorphism (Theorem 28). The "interior part" of X has changed by a diffeomorphism, and we have reduced the setup to the case where we deform ω_0 to a form ω'_ε which is cohomologous to ω_ε but which equals $d\theta_0$ on the collar.

Now it is possible to make a compactly supported deformation from ω_J to ω'_ε and, for small ε, Theorem 53 implies that

$$SH^*(X, \omega_J; \underline{\Lambda}_{\varepsilon\tau\omega_I}) \cong SH^*(X, \omega'_\varepsilon).$$

Rescale by $1/\varepsilon$ via $t \mapsto t^{1/\varepsilon}$ to deduce that

$$SH^*(X, \omega_J; \underline{\Lambda}_{\tau\omega_I}) \cong SH^*(X, \omega'_\varepsilon/\varepsilon).$$

Now $\omega'_\varepsilon/\varepsilon$ is cohomologous to ω_I. By applying Gray's theorem as above, we can change $\omega'_\varepsilon/\varepsilon$ within its cohomology class so that on the collar it becomes equal to ω_I. Finally we apply a compactly-supported Moser symplectomorphism as in Lemma 29 to deform the form to ω_I on all of X. Hence

$$SH^*(X, \omega'_\varepsilon/\varepsilon) \cong SH^*(X, \omega_I). \quad \square$$

Theorem 69. *$SH^*(X, \omega_I) \cong SH^*(X, \omega_u; \underline{\Lambda}_{\tau\omega_I}) = 0$ for all $u = (0, u_J, u_K) \in S^2$.*

Proof. By Theorem 28 the symplectic cohomology changes by an isomorphism if we choose a different contact hypersurface in the collar. By Lemma 66, we changed the hypersurface by an S^1-equivariant contactomorphism so that the collar of X (after metric rescaling) can be assumed to be the standard $S^3/\Gamma \times [1, \infty)$ with S^1-action $(a, b) \mapsto (\lambda a, \lambda b)$. The symplectic S^1-action φ_λ on (X, ω_I) defines a vector field

$$X_\varphi(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi_{e^{2\pi i t}}(x).$$

By Cartan's formula $0 = \partial_\lambda \varphi_\lambda^* \omega = \varphi_\lambda^* \mathcal{L}_{X_\varphi} \omega = d i_{X_\varphi} \omega$. Thus, since $H^1(X; \mathbb{R}) = 0$, we obtain a Hamiltonian via $i_{X_\varphi}\omega = -dH_\varphi$. Moreover, accelerating the flow by a factor k, we obtain an S^1-action $\varphi_{k\lambda}$ with Hamiltonian $H_k = kH_\varphi$. On the collar, $H_k(a,b) = k\pi(|a|^2+|b|^2)$ and since $R = |a|^2+|b|^2$, the Hamiltonian is linear at infinity: $h_k(R) = k\pi R$.

The $1-$periodic orbits of H_k either lie in $\pi^{-1}(0)$ or come from lifts of nonconstant $1-$periodic orbits on $\mathbb{C}^2/\Gamma$ for the flow $(a,b) \mapsto (\lambda^k a, \lambda^k b)$. But for generic k, there are no $1-$periodic orbits of H_k on $\mathbb{C}^2/\Gamma$ except for 0. So we reduce to calculating the Maslov indices of $1-$periodic orbits in $\pi^{-1}(0)$.

Since the flow φ_λ is holomorphic, the linearization over a periodic orbit will be a loop of unitary transformations. Its Maslov index can therefore be calculated as the winding number of the determinant of the linearization in the trivialization $\mathbb{C}\cdot(\omega_J + i\omega_K)$ of the canonical bundle. Since

$$\varphi_\lambda^*\omega_J(V,W) = g(J\varphi_{\lambda*}V, \varphi_{\lambda*}W) = g(J\lambda V, \lambda W) = \lambda^2\omega_J(V,W),$$

and similarly for K, we deduce that φ_λ acts on the canonical bundle by rotation by λ^2. Therefore the Maslov index increases by 2 for each full rotation of λ.

We deduce that the Maslov indices for H_k grow to infinity as $k \to \infty$. Therefore the generators of $SH^*(H_{k+N}, \omega_I)$ have arbitrarily negative Conley-Zehnder indices as $N \to \infty$, and so the image of $SH^m(H_k, \omega_I)$ under the continuation map

$$SH^m(H_k,\omega_I) \to SH^m(H_{k+N},\omega_I)$$

vanishes for large N. Thus the direct limit $SH^m(X,\omega_I) = 0$ for all m. Since $SH^*(X,\omega_I) = 0$ also $SH^*(X,\omega_u;\underline{\Lambda}_{\tau\omega_I})$ vanishes by Lemma 68. □

Corollary 70. *Let $(X, d\theta)$ be the plumbing of copies of T^*S^2 according to an ADE Dynkin diagram. Then*

$$SH^*(X, d\theta; \underline{\Lambda}_{\tau\omega}) \cong SH^*(X,\omega) = 0$$

for any generic symplectic form ω, where genericity refers to choosing $[\omega]$ in the *complement of certain finitely many hyperplanes in $H^2(X;\mathbb{R})$.*

Proof. All the $X_{\zeta_1,0,0}$ for generic ζ_1 are diffeomorphic to the plumbing (see section 3.8.4). We fix one such choice $X = X_{a,0,0}$, and we consider the family of forms ω_I induced on X by pull-back from $X_{\zeta_1,0,0}$ via the diffeomorphism $X \cong X_{\zeta_1,0,0}$. By Lemma 61, $[\omega_I]$ will range over all generic choices in $H^2(X;\mathbb{R})$ (genericity of ω_I corresponds to the genericity of ζ_1). The result now follows from Theorem 69. □

3.8.10 Exact Lagrangians in ADE spaces

Theorem 71. *Let $(X, d\theta)$ be the plumbing of copies of T^*S^2 according to an ADE Dynkin diagram. Let $j : L \to X$ be an exact Lagrangian submanifold. Then L must be a sphere, in particular L cannot be unorientable.*

Proof. By Corollary 70, $SH^*(X, d\theta; \underline{\Lambda}_{\tau\omega}) = 0$ for a generic symplectic form ω. Therefore by Corollary 37 the transgression $\tau(j^*[\omega])$ cannot vanish. But for orientable L which are not spheres all transgressions must vanish since $\pi_2(L) = 0$. Therefore the only allowable orientable exact Lagrangians are spheres. The unorientable case follows by Remark 38. □

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