REPRESENTATIONS OF SEMISIMPLE LIE GROUPS
AND AN ENVELOPING ALGEBRA DECOMPOSITION

by

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ABSTRACT

The theory of minimal types for representations of complex semisimple Lie groups [K. R. Parthasarathy, R. Ranga Rao and V. S. Varadarajan, Ann. of Math. (2) 85 (1967), 383-429, Chapters 1, 2 and 3] is reformulated so that it can be generalized, at least partially, to real semisimple Lie groups. A rather complete extension of the complex theory is obtained for the semisimple Lie groups of real rank 1.

More specifically, let G=NAK be an Iwasawa decomposition of a connected real semisimple Lie group with finite center, and let M be the centralizer of A in K. Suppose that G has real rank 1. Let $\alpha \in \hat{G}$ ($\hat{}$ denotes the set of equivalence classes of continuous finite dimensional complex irreducible representations), and let $\gamma \in \hat{M}$ be the class under which the highest restricted weight space of any member of α transforms. It is proved by means of an unpublished general formula of B. Kostant that there exists $\beta \in \hat{K}$ such that $m(\alpha,\beta) = m(\beta,\gamma) = 1$ (m denotes multiplicity). Moreover, β can be chosen so that it depends only on γ, and not on α. The corresponding complex-valued homomorphism on the centralizer of K in the complex enveloping algebra of the Lie algebra of G is computed. A similar approach is used to study a certain series of infinite dimensional irreducible representations of G related to a series of representations studied by Harish-Chandra.

The computation of the above-mentioned homomorphism is embedded in a general theory (for all real groups G) based on a certain enveloping algebra decomposition which generalizes a decomposition used to study the classical class 1 infinitesimal spherical functions. The general theory deals with arbitrary elements of $\hat{M}$ and $\hat{K}$ in the same sense that the class 1 theory deals with the trivial elements of $\hat{M}$ and $\hat{K}$. Furthermore, the general theory handles arbitrary

multiplicities, not just multiplicity 1. Partial results are obtained concerning all possible equivalences among the infinite dimensional representations mentioned above and concerning the concrete realization of abstract irreducible representations of G.

By means of Kostant's formula alluded to above, an explicit formula is obtained for $m(\alpha,\beta)$ $(\alpha \in \hat{G},\ \beta \in \hat{K})$ in two new cases: (1) G is a symplectic group of real rank 1 and (2) G is the rank 1 real form of F_4 and α is of class 1. The result for the symplectic case is expressed rather interestingly in terms of a certain combinatorial function - the number of ways of putting s indistinguishable balls into k distinguishable boxes of specified finite capacities. The following theorem is verified case-by-case: If G is arbitrary of real rank 1 and $\alpha \in \hat{G}$ is of class 1, then $m(\alpha,\beta) \leq 1$ for all $\beta \in \hat{K}$ (cf. [Kostant, Bull. Amer. Math. Soc. 75 (1969), 627-642, Theorem 6]).

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Introduction

The main purpose of this paper is to extend to real semisimple Lie groups some of the results of K. R. Parthasarathy, R. Ranga Rao and V. S. Varadarajan [11, Chapters 1, 2 and 3] on representations of complex semisimple Lie groups.

Let G be a connected real semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. We denote by $\mathfrak{G}$ the complex enveloping algebra of the Lie algebra of G, and by $\mathfrak{G}^K$ the centralizer of K in $\mathfrak{G}$ under the natural action.

Suppose now that G is complex. Let π be an irreducible quasi-simple representation of G (in the sense of [4]). In [11], an equivalence class of irreducible representations of K, called the "minimal type" of π, is defined by the condition that its highest weight is a weight of all the irreducible representations of K occurring in the restriction of π to K. If the minimal type of π exists, it is uniquely determined. If in addition it occurs in π with multiplicity 1, it gives rise to a complex-valued homomorphism, which we call $\eta(\pi)$, of $\mathfrak{G}^K$. In this case, $\eta(\pi)$ and the minimal type of π together determine π up to infinitesimal equivalence, and if the minimal type of π is trivial, $\eta(\pi)$ is the classical (infinitesimal) spherical function.

It is shown in [11] that if π is finite dimensional, the minimal type of π exists and does in fact occur in π with multiplicity 1. It is interesting to note that this assertion can be regarded as an extension of Schur's lemma, which implies that the trivial representation of K occurs in π with multiplicity at most 1.

Let $\mathfrak{h}$ be a Cartan subalgebra of the complexified Lie algebra of G and let $\mathcal{H}$ be the algebra of polynomial functions on the dual of $\mathfrak{h}$. For each system Q of positive roots of the complexified Lie algebra of K with respect to a fixed Cartan subalgebra, a homomorphism $h^Q: \mathcal{G}^K \to \mathcal{H}$ is defined in [11]. It is shown that each finite dimensional π selects at least one system Q, and for each such Q, $\eta(\pi)$ is given by the evaluation of h^Q at a certain integral point in the dual of $\mathfrak{h}$. Moreover, when the homomorphisms h^Q are evaluated at suitable nonintegral points in the dual of $\mathfrak{h}$, the result is of the form $\eta(\hat{\pi})$, where $\hat{\pi}$ ranges over a series of irreducible quasi-simple representations of G which are related to certain representations studied by Harish-Chandra. The images of the homomorphisms h^Q are studied in [11] in order to provide information on all the possible equivalences among the representations $\hat{\pi}$.

In attempting to generalize these results of [11] to real groups G, we find that the notion of minimal type does not extend naturally to representations of real semisimple Lie groups. Instead, we exploit the fact that the highest weight of the minimal type of the finite dimensional representation π can be obtained by restricting the highest weight of π to a certain subalgebra of $\mathfrak{h}$ isomorphic to a Cartan subalgebra of the complexified Lie algebra of K. The mapping from π to the highest weight of its minimal type takes on a canonical meaning for real G when we introduce the subgroup M, which is the centralizer in K of a maximal abelian subspace of $\mathfrak{p}_0$ (here $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G, where $\mathfrak{k}_0$ is the Lie algebra of K). Specifically, the analogue of this highest weight in the general context is the equivalence class $\gamma(\pi)$ of irreducible representations of M under which the highest restricted weight space of π transforms.

There is no natural way, however, of passing from a class γ of irreducible representations of M to a unique class of irreducible representations of K which contains it "extremally" (cf. highest weights). Instead, we ask only for a class $\mathcal{C}(\gamma)$ of irreducible representations of K which contains γ with multiplicity 1 and which is contained with multiplicity 1 in every finite dimensional irreducible representation π of G such that $\gamma(\pi) = \gamma$.

In [11], the homomorphisms h^Q are constructed from homomorphisms β^Q associated with the positive systems Q (cf. Lemma 1.1 in [11]). When we impose an analogous condition on our correspondence $\mathcal{C}$, we are led to our notion of "system of minimal types" defined in Chapter III, § 1. (The set of positive systems Q is not appropriate as an index set in general, so we replace it by an arbitrary finite index set I.)

Assuming the existence of a system of minimal types for the real group G, we generalize in § 1 of Chapter III many of the above-mentioned results obtained in [11] for complex groups. Furthermore, although our methods of proof are somewhat similar to those used in [11], our axiomatization of the procedures used in [11] enables us to simplify some of those results. For example, by using the Iwasawa decomposition, we redefine the homomorphisms h^Q in a more natural way than is done in [11].

By adding extra conditions to our definition of system of minimal types - conditions which are also satisfied in the complex case - we obtain the notion of "strong system of minimal types" defined in §1 of Chapter III. Using this, we generalize more of the above-mentioned results of [11].

The relationship between our development and that in [11] is discussed in detail in the Appendix to §1 of Chapter III.

In §2 of Chapter III, we construct strong systems of minimal types for the groups G of real rank 1 (i.e., of split rank 1). Thus the results of §1 apply to such groups. Our construction depends on Lemmas 1 - 6, which give a generalization of Lemma 1.1 in [11]. We then make use of the classification of the rank 1 groups. In order to justify the key fact that the multiplicity of $\mathcal{C}(\gamma(\pi))$ in π is 1 (in the above notation), we invoke our multiplicity formulas of Chapter II, which we shall discuss below. It would be desirable to have a simple direct proof of this key fact, and a generalization of Theorem 2.1 in [11]. In a Remark at the end of Chapter III, we outline an alternate description of most of the systems of minimal types which we have constructed.

The following interesting fact is an immediate corollary of the key fact mentioned above: For every finite dimensional irreducible representation π of G (assumed to be of real rank 1), there exists an irreducible representation of K, depending only on $\gamma(\pi)$, which is contained in π with multiplicity 1 and which contains $\gamma(\pi)$ with multiplicity 1.

Strong systems of minimal types are not unique as we have defined them. We do not know whether all real groups G admit systems of minimal types.

Let G be real. Let $\mathfrak{k}$ be the complexified Lie algebra of K, and let $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ be a complexified Iwasawa decomposition of the Lie algebra of G. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Choose a Cartan subalgebra $\mathfrak{h}_{\mathfrak{m}}$ of $\mathfrak{m}$. Then $\mathfrak{h} = \mathfrak{a} + \mathfrak{h}_{\mathfrak{m}}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathcal{O}\mathcal{L}$ and $\mathcal{H}$ be the algebras of polynomial functions on the duals of $\mathfrak{a}$ and $\mathfrak{h}$, respectively.

If a homomorphism from $\mathfrak{G}^K$ into $\mathcal{H}$ is evaluated at any point in the dual of $\mathfrak{h}_{\mathfrak{m}}$, the result is a homomorphism from $\mathfrak{G}^K$ into $\mathcal{O}\mathcal{L}$. It turns out that the natural way of studying the homomorphisms h^Q mentioned above, and our generalizations of them for real groups, is by means of homomorphisms from $\mathfrak{G}^K$ into $\mathcal{O}\mathcal{L}$ obtained by such evaluation. The study of these homomorphisms can in turn be embedded in a general theory which forms the subject of Chapter I.

This theory is designed to handle arbitrary positive multiplicities, not just multiplicity 1. Correspondingly, it is necessary to deal with linear maps from $\mathfrak{G}^K$ into $\mathfrak{A}$ which are not homomorphisms. Chapter I contains the heart of the proofs of the theorems in § 1 of Chapter III.

In § 1 of Chapter 1 we first describe some basic notation. We then define the key linear mapping $p_{\mathfrak{g}}: \mathfrak{G}^K \to \mathfrak{A}$ in terms of a simple but crucial enveloping algebra decomposition (formula (3)). Elementary consequences of the definition are derived. § 2 is devoted to some important notation and to some technical lemmas based mostly on linear algebra. The non-trivial parts of these lemmas are needed only to handle the case of multiplicities greater than 1. In § 3 we obtain the main theorem on finite dimensional representations. The first formula in the statement of Theorem 1 may be thought of as an analogue of the Frobenius reciprocity theorem for finite dimensional representations, and is undoubtedly known. If we impose the crucial condition that this inequality be an equality (formula (5)), we obtain formula (6), which demonstrates the important relation between the mapping $p_{\mathfrak{g}}$ and finite dimensional representations.

§ 4 of Chapter I deals with certain infinite dimensional representations. Following [11], we use several results and methods of Harish-Chandra ([4] and [5]). First we prove Theorem 2, which relates the mapping $p_{\mathfrak{g}}$ to a certain series of representations very similar to a series defined by Harish-Chandra. We then construct a new series of irreducible representations by generalizing a procedure of [11], and we show in Theorem 3 that these representations naturally extend the finite dimensional representations satisfying condition (5) (Theorem 1). This condition is replaced in the present case by the important irreducibility condition stated in Theorems 2 and 3.

In § 5 of Chapter I we give a simple independent proof of Theorem 1 in the special case in which the relevant multiplicities are 1. The resulting statement, Theorem 1', is the only part of Theorem 1 which is needed in § 1 of Chapter III. Similarly, a short proof of Theorems 2 and 3 could also be given in the case of multiplicity 1. In addition, we explain how the mapping $p_{\mathfrak{g}}$ generalizes the infinitesimal versions of the class 1 spherical functions.

In § 6 of Chapter I we discuss the class 1 theory relating to Harish-Chandra's formula for the spherical function, and we indicate directions in which it might be generalized by means of the mapping $p_{\mathfrak{g}}$. We prove Theorem 4 concerning Weyl group transformation properties of the mapping $p_{\mathfrak{g}}$. The idea for this proof is due to S. Helgason. We finally state two partial generalizations of certain aspects of the class 1 theory.

Chapter II is devoted to multiplicity formulas for the reduction under K of finite dimensional irreducible representations of G, when G has real rank 1. B. Kostant has derived a multiplicity formula [unpublished] for the reduction of a finite dimensional irreducible representation of an arbitrary compact connected Lie group under an arbitrary compact connected subgroup. In § 1 this theorem is formulated and proved in the special case in which a certain simplifying assumption holds. The result is applied case-by-case in §§ 2a-2e to the rank 1 simple groups to derive explicit multiplicity formulas. The proofs are highly combinatorial. A general fact about class 1 representations is obtained as a corollary of the resulting formulas. This fact, stated as Theorem 2 in the initial paragraphs of § 2, is essentially the same as a theorem ([10], Theorem 6) recently obtained by Kostant in a different way.

The multiplicity formulas for the unitary and orthogonal groups (§§ 2a-2c) are well known and classical (cf. [1]). They are included here because it is interesting to see how easily they follow from Kostant's formula, because they are used in the applications, and because it is interesting to compare them with the results obtained for the other groups. Since our proofs for these classical cases are extremely similar, we give the proof for only the unitary case.

On the other hand, our formulas for the remaining cases - the symplectic and exceptional cases - seem to be new (although C. G. Hegerfeldt [7] has obtained related results on symplectic groups). Our result for the symplectic case (§ 2d) is expressed rather interestingly in terms of a certain combinatorial function - the number of ways of putting s indistinguishable balls into k dis-tinguishable boxes of specified finite capacities. This gives an intuitive interpretation to the multiplicity formula and its applications. For example, the existence of systems of minimal types for the symplectic groups (see above) is reduced by formula (10) (Theorem 6) to the fact that the number of ways of putting 0 balls into boxes of capacity 0 is 1. Another interesting example is given in the Remark following the Corollary to Theorem 6.

We have succeeded in obtaining only partial results for the exceptional case (§ 2e), but these are sufficient for our applications. Specifically, Theorem 7 is the formula for class 1 representations, and Theorem 8 is precisely the statement which we need to prove the existence of a system of minimal types for the exceptional group (see above).

The proofs for the orthogonal and unitary cases are quite easy. The main reason is that the partition function (which is defined in § 1 of Chapter II) takes only the values 0 and 1, and the same is true of the multiplicity function. Neither of these statements is true, however, for the symplectic and exceptional cases. Correspondingly, our proofs for these two cases are relatively complicated. The proof of Theorem 8 continues along the lines of that of Theorem 7, but we have omitted it because it is extremely long, and because it does not seem to lead easily to a completely general multiplicity formula for the exceptional group. It would be desirable to have such a formula, but it would be even more desirable to have a uniform statement and proof of the multiplicity results for all the rank 1 groups (or for a more general class of real semisimple Lie groups), as opposed to a case-by-case analysis. Our proof for the symplectic case has some promise of pointing the way toward a general proof, in view of the key role played by the reflections with respect to the simple roots (see Lemma 10), but we do not see how to extend this proof to a general one.

As we remarked above, it would be desirable to find a way to avoid having to use our formulas of Chapter II in proving the existence of systems of minimal types in Chapter III. In any case, however, these formulas certainly have independent interest.

Chapters I and II of this paper are logically and notationally independent, and hence can be read separately. On the other hand, Chapters I and III form a unit in which the notation and results are cumulative. In addition, results of Chapter II are quoted in § 2 of Chapter III. §§ 2a-2e of Chapter II are completely independent of one another, but they all depend on § 1 of Chapter II. Theorems, Propositions, Lemmas, Definitions and formulas are numbered independently in each chapter.

The reader is referred to [8] and [9] for background material on semisimple Lie groups and Lie algebras.

The author would like to thank Prof. Kostant for his valuable advice on several aspects of this work. In particular, Prof.

Kostant suggested the original thesis question of extending the theory of minimal types in [11] to real semisimple Lie groups. He also provided much specific insight in the development of the ideas in Chapter I.

Chapter I. An enveloping algebra decomposition and some of its consequences

§1 The decomposition

Let G be a connected real semisimple Lie group with finite center, and let $\mathfrak{g}_0$ be its Lie algebra. Fix a Cartan involution θ of $\mathfrak{g}_0$, and let $\mathfrak{k}_0$ and $\mathfrak{p}_0$ be its +1 and -1 eigensubspaces, respectively, so that $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$. Choose a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$, and a system Σ_+ of positive restricted roots of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$. Let $\mathfrak{n}_0 \subset \mathfrak{g}_0$ be the sum of the restricted root spaces corresponding to the roots in Σ_+. Then we have the Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{n}_0 + \mathfrak{a}_0 + \mathfrak{k}_0$ (direct sum of vector spaces) of the real semisimple Lie algebra $\mathfrak{g}_0$.

Let N, A and K be the connected Lie subgroups of G corresponding to the Lie subalgebras $\mathfrak{n}_0$, $\mathfrak{a}_0$ and $\mathfrak{k}_0$, respectively, of $\mathfrak{g}_0$. Let M be the centralizer of A in K. We have the Iwasawa decomposition $G = NAK$ of G, and K is a maximal compact subgroup of G. Moreover, NAM is a closed subgroup of G, and M is compact.

We shall find it convenient to pass to the complexifications of the Lie algebras defined above. Let $\mathbb{C}$ denote the field of complex numbers. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, so that $\mathfrak{g}$ is a complex semisimple Lie algebra. Let $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ be the complex subspaces of $\mathfrak{g}$ generated by $\mathfrak{n}_0$, $\mathfrak{a}_0$ and $\mathfrak{k}_0$, respectively. Then $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ are subalgebras of $\mathfrak{g}$, and we have the complexified Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$.

Let $\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{g}$, and let $1 \in \mathfrak{G}$ be the identity element. Let $\mathfrak{N}$, $\mathfrak{A}$ and $\mathfrak{K}$ denote the subalgebras of $\mathfrak{G}$ generated by $\mathfrak{n}$ and 1, by $\mathfrak{a}$ and 1, and by $\mathfrak{k}$ and 1, respectively. Then $\mathfrak{N}$, $\mathfrak{A}$ and $\mathfrak{K}$ are isomorphic to the universal enveloping algebras of $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$, respectively. Moreover, the map of $\mathfrak{N} \otimes \mathfrak{A} \otimes \mathfrak{K}$ into $\mathfrak{G}$ given by $n \otimes a \otimes k \to nak$ is a linear isomorphism. If $\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_j$ are arbitrary linear subspaces of $\mathfrak{G}$, we shall denote by $\mathfrak{T}_1 \mathfrak{T}_2 \ldots \mathfrak{T}_j$ the subspace consisting of the sums of the elements of the form $t_1 t_2 \ldots t_j$ where $t_i \in \mathfrak{T}_i$ $(1 \leq i \leq j)$.

G acts in a canonical way as a group of algebra automorphisms of $\mathfrak{G}$, by unique extension of the adjoint representation Ad of G on $\mathfrak{g}_0$. We shall also denote the extension by Ad. For every subset S of G and every subset $\mathfrak{T}$ of $\mathfrak{G}$, let $\mathfrak{T}^S$ denote the centralizer of S in $\mathfrak{T}$ under Ad. Then $\mathfrak{G}^K$ commutes with $\mathfrak{K}$ in $\mathfrak{G}$.

Now $\mathfrak{N} = \mathfrak{n}\mathfrak{N} + \mathbb{C}\cdot 1$ (direct sum), so that $\mathfrak{G} = (\mathfrak{n}\mathfrak{N} + \mathbb{C}\cdot 1)\mathfrak{A}\mathfrak{K} = \mathfrak{A}\mathfrak{K} + \mathfrak{n}\mathfrak{N}\mathfrak{A}\mathfrak{K}$ (direct sum). Hence

(1) $$\mathfrak{G} = \mathfrak{A}\mathfrak{K} + \mathfrak{n}\mathfrak{G} \quad \text{(direct sum).}$$

Since AdM centralizes $\mathfrak{A}$ and normalizes $\mathfrak{n}$, we have from (1) that

$$\mathfrak{G}^M \subset \mathfrak{A}\mathfrak{K}^M + \mathfrak{n}\mathfrak{G}.$$

In particular,

(2) $$\mathfrak{G}^K \subset \mathfrak{A}\mathfrak{K}^M + \mathfrak{n}\mathfrak{G}.$$

Let $\mathcal{J}$ be an arbitrary linear complement of the constants in $\mathcal{K}^M$. From (2), we have the key decomposition

$$\mathcal{G}^K \subset \mathfrak{a}\mathcal{J} + \mathfrak{a} + n\mathcal{G} \quad \text{(direct sum)}. \tag{3}$$

Definition 1. For every linear complement $\mathcal{J}$ of the constants in $\mathcal{K}^M$, let $p_{\mathcal{J}} : \mathcal{G}^K \to \mathfrak{a}$ be the linear map defined as follows: For all $u \in \mathcal{G}^K$, $p_{\mathcal{J}}^u$ is the $\mathfrak{a}$-component of u with respect to the decomposition (3).

The corollary to the following simple proposition points out some of the significance of the mapping $p_{\mathcal{J}}$ (more of the significance is indicated in §§ 3, 4, 5 and 6):

Proposition 1. (cf. Harish-Chandra [5, p. 48, Lemma 10]). Suppose we identify $\mathfrak{a}\mathcal{K}^M$ with the algebra $\mathfrak{a}\otimes\mathcal{K}^M$. For all $u \in \mathcal{G}^K$, let q^u denote the $\mathfrak{a}\otimes\mathcal{K}^M$-component of u with respect to the decomposition (2). Then the linear map $q: \mathcal{G}^K \to \mathfrak{a}\otimes\mathcal{K}^M$ is an algebra antihomomorphism.

Proof. Choose a basis $\{H_1,\ldots,H_\ell\}$ of $\mathfrak{a}$. For every ℓ-tuple $(S) = (S_1,\ldots S_\ell)$ of non-negative integers, let

$$H_{(S)} = H_1^{S_1} \ldots H_\ell^{S_\ell},$$

so that $\{H_{(S)}\}$ is a basis of $\mathfrak{a}$.

Let $u,v \in \mathcal{G}^K$, and let

$$u \equiv \sum_{(S)} H_{(S)}x_{(S)} \pmod{n\mathcal{G}},$$

$$v \equiv \sum_{(S)} H_{(S)} y_{(S)} \pmod{\mathfrak{n}\mathcal{B}},$$

where $x_{(S)}, y_{(S)} \in \mathcal{K}^M$. Then

$$uv \equiv \sum_{(S)} H_{(S)} v x_{(S)} \pmod{\mathfrak{n}\mathcal{B}}$$

$$(\text{since } v \in \mathcal{B}^K)$$

$$\equiv \sum_{(S),(t)} H_{(S)} H_{(t)} y_{(t)} x_{(S)} \pmod{\mathfrak{n}\mathcal{B}}$$

$$(\text{since } \mathfrak{a} \text{ normalizes } \mathfrak{n})$$

$$\equiv q^v q^u \pmod{\mathfrak{n}\mathcal{B}}.$$

Hence $q^{uv} = q^v q^u$ in $\mathfrak{A} \otimes \mathcal{K}^M$, and this proves the proposition.

Corollary. Let $\mathcal{J}$ be a linear complement of the constants in $\mathcal{K}^M$. If $\mathcal{J}$ is an ideal in $\mathcal{K}^M$, $p_{\mathcal{J}}$ is a homomorphism. If $\mathcal{J}$ contains all commutators xy-yx of elements $x,y \in \mathcal{K}^M$, then $p_{\mathcal{J}}$ vanishes on all commutators uv-vu of elements $u,v \in \mathcal{B}^K$.

Proof. Let $f_{\mathcal{J}} : \mathcal{K}^M \to \mathbb{C}$ be the linear map such that $f_{\mathcal{J}}(\mathcal{J}) = 0$ and $f_{\mathcal{J}}(1) = 1$. Then $p_{\mathcal{J}} = (1 \otimes f_{\mathcal{J}}) \circ q$, and the corollary now follows from the proposition.

§ 2 Some lemmas

In this section, we obtain some lemmas which prepare the way for §§ 3 and 4. The reason for the somewhat involved notation in this section is that Lemma 1 is designed to apply to both pairs of groups

(G,K) and (K,M), and Lemmas 1, 4 and 5 are designed to apply to both finite and infinite dimensional representations of G (see § 3 and § 4, respectively). However, when we treat the special case of multiplicity 1 in § 5, we will see that all of the complications of § 2 can be avoided.

We retain the notation of § 1. We shall also use the following notation: If H is a Lie group (resp., an associative algebra over $\mathbb{C}$), we let $\hat{H}$ denote the set of equivalence classes of continuous (resp., all) finite dimensional complex irreducible representations of H. We recall that if Π is a continuous finite dimensional complex representation of a connected Lie group H, then Π may be naturally regarded as a representation of the complexified Lie algebra $\mathfrak{h}$ of H, and hence as a representation of the universal enveloping algebra of $\mathfrak{h}$.

If Π is a continuous finite dimensional complex representation of a Lie group H, and Π acts on the space V, then for all $\alpha \in \hat{H}$ we let V_α denote the α-primary subspace of V, that is, the space of vectors in V which transform under Π according to α. We recall that a representation of a group (or an algebra) may be naturally regarded as a representation of a subgroup (or subalgebra) by restriction. If H_1 and H_2 are Lie groups such that $H_2 \subset H_1$, and if $\alpha \in \hat{H}_1$ is such that the restriction to H_2 of every member of α splits into a direct sum of irreducible representations of H_2, then for all $\beta \in \hat{H}_2$ we let $m(\alpha,\beta)$ denote the multiplicity with which members of β occur in the restriction to H_2 of any representation in α.

If $\mathcal{A}$ is an associative algebra over $\mathbb{C}$ and $\eta \in \hat{\mathcal{A}}$, we let χ_η denote the complex-valued linear function on $\mathcal{A}$ which is the character of any member of η.

We now come to the lemmas. Lemma 1 follows from standard linear algebra, and we omit the proof.

<u>Lemma 1</u>. Let H be a group and Π_1 an irreducible representation of H on the finite dimensional complex vector space X. Let σ_1 be a representation of H on the finite dimensional complex vector space Z, and assume that σ_1 is equivalent to a direct sum of r copies of Π_1, where $r > 0$. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$, and σ_2 a representation of $\mathcal{A}$ on Z. Assume that $\sigma_2(\mathcal{A})$ is precisely the commuting ring of $\sigma_1(H)$ in End Z. Then σ_2 is equivalent to a direct sum of finitely many copies of an irreducible representation Π_2 of $\mathcal{A}$. More specifically:

(i) There exists a unique element $\eta \in \hat{\mathcal{A}}$ such that if $\Pi_2 \in \eta$ acts on the space Y, there is a unique linear isomorphism L from Z to $X \otimes Y$ which intertwines the actions of H and $\mathcal{A}$ (H acts on $X \otimes Y$ via the action of Π_1 on the first factor, and $\mathcal{A}$ acts on $X \otimes Y$ via the action of Π_2 on the second factor).

(ii) If P is any subspace of Z invariant and irreducible under σ_2, then the corresponding representation of $\mathcal{A}$ on P is a member of η.

(iii) If $z \in Z$ is a cyclic vector for σ_1, then $r = \dim Y \leq \dim X$, and there exist bases $\{x_i\}_{1 \leq i \leq \dim X}$ of X and $\{y_j\}_{1 \leq j \leq r}$ of Y such that $L(z) = \sum_{i=1}^{r} x_i \otimes y_i$.

In order to apply Lemma 1, we prove the following lemma:

Lemma 2. Let $\xi \in \hat{K}$, and let $\Pi \in \xi$ act on the space V. Fix $\omega \in \hat{M}$ such that $m(\xi,\omega) > 0$ (that is, $V_\omega \neq \{0\}$), and let σ_1 be the corresponding representation of M on V_ω. Regard Π as a representation of $\mathcal{K}$ on V. Then V_ω is invariant under $\Pi(\mathcal{K}^M)$. Let σ_2 denote the corresponding representation of $\mathcal{K}^M$ on V_ω. Then $\sigma_2(\mathcal{K}^M)$ is precisely the commuting ring of $\sigma_1(M)$ in End V_ω.

Proof. Let $\mathcal{R}$ be the commuting ring of $\sigma_1(M)$ in End V_ω. For all $m \in M$ and $k \in K$, we have

$$\Pi((\operatorname{Ad} m)(k)) = \Pi(m)\Pi(k)\Pi(m)^{-1}. \tag{4}$$

Hence V_ω is invariant under $\Pi(\mathcal{K}^M)$, and $\sigma_2(\mathcal{K}^M) \subset \mathcal{R}$. Conversely, let $\tau \in \mathcal{R}$. Then there exists $k \in \mathcal{K}$ such that $\Pi(k)v = \tau v$ for all $v \in V_\omega$. Thus

$$\Pi((\operatorname{Ad} m)(k))v = \tau v \text{ for all } m \in M,\ v \in V_\omega$$

by (4) and the fact that $\tau \in \mathcal{R}$. Let dm denote normalized Haar measure on M, and let $k' = \int_M (\operatorname{Ad} m)(k)\,dm$. Then $k' \in \mathcal{K}^M$, and

$$\Pi(k')v = \int_M \Pi((\mathrm{Ad}\ m)(k))v\ dm$$
$$= \int_M \tau v\ dm$$
$$= \tau v \text{ for all } v \in V_\omega.$$

This proves the lemma.

By Lemmas 1 and 2, we have:

Lemma 3. In the notation of Lemma 2, let $\Pi_1 \in \omega$ act on the space X. Then:

(i) There exists a unique element $\zeta(\xi,\omega) \in \widehat{\mathcal{K}^M}$, which depends only on ξ and ω, such that if $\Pi_2 \in \zeta(\xi,\omega)$ acts on the space Y, there is a unique linear isomorphism from V_ω to $X \otimes Y$ which intertwines the actions of M and $\mathcal{K}^M$.

(ii) If P is any subspace of V_ω invariant and irreducible under σ_2, then the corresponding representation of $\mathcal{K}^M$ on P is a member of $\zeta(\xi,\omega)$.

As we shall see later, the next lemma is designed to handle infinite dimensional as well as finite dimensional representations of G.

Lemma 4. Assume the hypotheses of Lemma 1, with H=K and $\mathcal{Q} = \mathcal{D}^K$. Fix $\xi \in \hat{K}$ such that $\Pi_1 \in \xi$. Regard σ_1 and Π_1 as representations of $\mathcal{K}$ on Z and X, respectively. Assume that σ_1 and σ_2 agree on $\mathcal{K} \cap \mathcal{D}^K$. Identify Z with $X \otimes Y$ via the isomorphism L. Suppose $z \in Z$ is a cyclic vector for σ_1, so that $z = \sum_{i=1}^{r} x_i \otimes y_i$ where r, $\{x_i\}$ and $\{y_j\}$

are as in Lemma 1(iii). Assume that $\sigma_2(\mathcal{D}^K)z \subset \sigma_1(\mathcal{K}^M)z$. Also assume that $z \in Z_\omega$ where $\omega \in \hat{M}$ is such that $m(\xi,\omega) > 0$, and where M is regarded as acting on Z via σ_1. Then:

(i) $r \leq m(\xi,\omega)$.

(ii) If $r = m(\xi,\omega)$, then the subspace P of X spanned by $\{x_i\}_{1 \leq i \leq r}$ is invariant and irreducible under the restriction of Π_1 to $\mathcal{K}^M$, and the corresponding action of $\mathcal{K}^M$ on P is a member of $\zeta(\xi,\omega)$.

<u>Proof</u>. Let $S \subset Z$ be the linear span of $\{x_i \otimes y_j\}_{1 \leq i,j \leq r}$. Since $\Pi_2(\mathcal{D}^K)$ acts irreducibly on Y, we have that $\sigma_2(\mathcal{D}^K)z = S$. By hypothesis, $\sigma_1(\mathcal{K}^M)z \supset S$, and so P is irreducible under the subalgebra of $\mathcal{K}^M$ which preserves P under Π_1.

Now $Z = \sum_{i=1}^{r} X \otimes y_i$ (direct sum). The corresponding projections of Z onto the $\sigma_1(K)$-invariant subspaces $X \otimes y_i$ $(1 \leq i \leq r)$ are intertwining operators for $\sigma_1(K)$, and hence for $\sigma_1(M)$. Thus $x_i \otimes y_i \in Z_\omega$ (where M is regarded as acting on Z via σ_1), and so $x_i \in X_\omega$ $(1 \leq i \leq r)$. Hence $P \subset X_\omega$. Let σ^* denote the representation of M on X_ω induced by Π_1. Then since $P \subset X_\omega$ is irreducible under the subalgebra of $\mathcal{K}^M$ which preserves P under Π_1, we have that the commuting ring of σ^* in End X_ω contains a subspace of dimension r^2. This proves statement (i).

If $r = m(\xi,\omega)$, then P is invariant and irreducible under all of $\mathcal{K}^M$. Indeed, if this were not true, then the commuting ring of σ^* in End X_ω would contain a subspace of dimension greater than $(m(\xi,\omega))^2$, a contradiction. Statement (ii) now follows immediately from Lemma 3(ii), and Lemma 4 is proved.

Lemma 5. Assume the hypotheses of Lemma 4, and suppose that $r = m(\xi,\omega)$. Let D be the linear form on Z which takes $\sum_{i,j} a_{ij}\, x_i \otimes y_j$ ($a_{ij} \in \mathbb{C}$) to $\sum_{i=1}^{r} a_{ii}$. Then

$$\chi_\eta(u) = D(\sigma_2(u)z) \quad \text{for all } u \in \mathcal{J}^K$$

(where $\eta \in \widehat{\mathcal{J}^K}$ is defined as in Lemma 1), and

$$\chi_{\zeta(\xi,\omega)}(v) = D(\sigma_1(v)z) \quad \text{for all } v \in \mathcal{K}^M.$$

Proof. The first statement follows immediately from the fact that $z = \sum_{i=1}^{r} x_i \otimes y_i$, and the second statement follows immediately from this fact and Lemma 4(ii).

§ 3 Finite dimensional representations

The purpose of this section is to prove Theorem 1, which indicates the connection between the mapping $p_{\mathcal{J}}$ defined in § 1 and finite dimensional representations. We make use of some ideas from [11]. We retain the notation of the preceding sections.

Let $\alpha \in \hat{G}$, and let $\Pi \in \alpha$ act on the space V. Let

$$V^{\mathfrak{n}} = \{v \in V \mid \Pi(X)v = 0 \text{ for all } X \in \mathfrak{n}\}.$$

It is well known that $V^{\mathfrak{n}}$, the highest restricted weight space of Π, is invariant and irreducible under $\Pi(M)$, and that A acts on $V^{\mathfrak{n}}$ according to multiplication by a (one-dimensional) character on A. (These facts follow from the fact that MA normalizes $\mathfrak{n}$, and from the irreducibility of $V^{\mathfrak{n}}$ under $\Pi(\mathfrak{m} + \mathfrak{a})$, where $\mathfrak{m}$ is the complexified Lie algebra of M. This irreducibility is easily proved by means of the decomposition $\mathfrak{g} = \theta\mathfrak{n} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ and the standard enveloping-algebra proof of the one-dimensionality of the highest weight space of an irreducible representation of a complex semisimple Lie algebra.) Let $\gamma(\alpha) \in \hat{M}$ and $\lambda(\alpha) \in \hat{A}$ denote the classes obtained in this way. When $\lambda(\alpha)$ is regarded as a linear form on $\mathfrak{a}$, it becomes the highest restricted weight of Π.

Let W be the Weyl group of $\mathfrak{g}_0$. It is well known that W acts naturally on $\hat{M}$ and $\hat{A}$. Let $s_0 \in W$ be the Weyl group element which takes the system Σ_+ of positive restricted roots into $-\Sigma_+$.

The proofs of Lemmas 2 and 3 (§ 2), and hence Lemmas 2 and 3 themselves, hold without change if K, M, $\mathcal{K}$ and $\mathcal{K}^M$ are replaced by G, K, $\mathcal{G}$ and $\mathcal{G}^K$, respectively. If $\delta \in \hat{G}$ and $\xi \in \hat{K}$ are such that $m(\delta,\xi) > 0$, we thus have the existence of a unique class $\eta(\delta,\xi) \in \widehat{\mathcal{G}^K}$ such that the action of K and $\mathcal{G}^K$ on the ξ-primary subspace of any representation in δ factors into the tensor product of a representation in ξ with a representation in $\eta(\delta,\xi)$; $\eta(\delta,\xi)$ depends only on δ and ξ.

Let $\xi \in \hat{K}$ and $\omega \in \hat{M}$ be such that $m(\xi,\omega) > 0$. Then $\chi_{\zeta(\xi,\omega)}$ is a linear form on $\mathcal{K}^M$, and its kernel $\mathcal{J}_{\xi,\omega}$ is a linear complement of the constants in $\mathcal{K}^M$ which contains all commutators xy-yx of elements $x,y \in \mathcal{K}^M$.

<u>Definition 2</u>. For every $\xi \in \hat{K}$ and $\omega \in \hat{M}$ such that $m(\xi,\omega) > 0$, let $p_{\xi,\omega} = p_{\mathcal{J}_{\xi,\omega}}$ (see Definition 1 (§ 1) for the definition of $p_{\mathcal{J}_{\xi,\omega}}$).

Thus $p_{\xi,\omega}$ is a linear mapping from $\mathfrak{D}^K$ into $\mathfrak{A}$, and $p_{\xi,\omega}$ vanishes on all commutators uv-vu of elements $u,v \in \mathfrak{D}^K$ (Corollary to Proposition 1 (§ 1)).

We recall that $\mathfrak{A}$ may be regarded as the algebra of polynomial functions on the dual of $\mathfrak{a}$.

We can now state:

<u>Theorem 1</u>. For all $\alpha \in \hat{G}$ and $\beta \in \hat{K}$, we have

$$m(\alpha,\beta) \leq m(\beta,\gamma(\alpha)).$$

Suppose that α and β satisfy the condition

$$m(\alpha,\beta) = m(\beta,\gamma(\alpha)) > 0. \tag{5}$$

Then

$$\chi_{\eta(\alpha,\beta)}(u) = m(\beta,\gamma(\alpha))\ p^u_{\beta,s_o\gamma(\alpha)}(s_o\lambda(\alpha)) \tag{6}$$

for all $u \in \mathfrak{G}^K$. Moreover, the linear mapping $p_{\beta,s_o\gamma(\alpha)}: \mathfrak{G}^K \to \mathfrak{A}$ vanishes on all commutators uv-vu of elements $u,v \in \mathfrak{G}^K$.

Proof. We note that $p_{\beta,s_o\gamma(\alpha)}$ is defined because $m(\beta,s_o\gamma(\alpha)) = m(\beta,\gamma(\alpha)) > 0$, by (5). The last statement of the theorem has been proved. For every class $\delta \in \hat{G}$, $\hat{K}$ or $\hat{M}$, let δ' denote the contragredient class.

Let $\Pi \in \alpha$ act on the space V, and let Π' be the contragredient module to Π, so that $\Pi' \in \alpha'$, and Π' acts on the dual V' of V.

Let $v' \in V'$ be a non-zero highest restricted weight vector of Π'. Let E' be the projection of V' onto $V'_{\beta'}$ with respect to the direct sum decomposition $V' = \sum_{\xi \in \hat{K}} V'_{\xi}$. Then $V'_{\beta'}$ is invariant under $\Pi'(K)$ and $\Pi'(\mathfrak{G}^K)$, and E' commutes with $\Pi'(\mathcal{K})$ and $\Pi'(\mathfrak{G}^K)$.

We note that $E'v'$ is a cyclic vector for the action of $\Pi'(K)$ on $V'_{\beta'}$. Indeed, $\mathfrak{G} = \mathcal{K}\mathfrak{A} + \mathfrak{G}\mathfrak{n}$, so that

$$V' = \Pi'(\mathfrak{G})v' = \Pi'(\mathcal{K})v',$$

and so

$$V'_{\beta'} = E'V' = \Pi'(\mathcal{K})E'v'.$$

Also, since $\mathfrak{G}^K \subset \mathcal{K}^M\mathfrak{A} + \mathfrak{G}\mathfrak{n}$, we have that

$$\begin{aligned}\Pi'(\mathfrak{G}^K)E'v' &= E'\Pi'(\mathfrak{G}^K)v' \\ &\subset E'\Pi'(\mathcal{K}^M)v' \\ &= \Pi'(\mathcal{K}^M)E'v'.\end{aligned}$$

Moreover, $v' \in V'_{(s_o\gamma(\alpha))'}$, so that $E'v' \in (V'_{\beta'})_{(s_o\gamma(\alpha))'}$. Now in proving the theorem, we may assume that $m(\alpha',\beta') = m(\alpha,\beta) > 0$. Then since $E'v'$ is a non-zero vector in $(V'_{\beta'})_{(s_o\gamma(\alpha))'}$, we have that $m(\beta,\gamma(\alpha)) = m(\beta',(s_o\gamma(\alpha))') > 0$.

We have thus verified all the hypotheses of Lemma 4, which we apply to the case $Z=V'_{\beta'}$, $\xi=\beta'$, $\omega = (s_o\gamma(\alpha))'$, $z = E'v'$, and σ_1 and σ_2 are the representations of K (or $\mathcal{K}$) and $\mathfrak{G}^K$, respectively, on $V'_{\beta'}$ induced by Π'. Lemma 4(i) now implies that

$$m(\alpha,\beta) = m(\alpha',\beta') = r \leq m(\beta',(s_o\gamma(\alpha))') = m(\beta,\gamma(\alpha)),$$

and the first statement of Theorem 1 is proved.

Now suppose that condition (5) holds. Then Lemma 5 applies, and so

$$(7) \qquad \chi_{\eta(\alpha',\beta')}(u) = D(\Pi'(u)E'v') \quad \text{for all } u \in \mathfrak{G}^K$$

and

$$(8) \qquad \chi_{\zeta(\beta',(s_o\gamma(\alpha))')}(v) = D(\Pi'(v)E'v') \quad \text{for all } v \in \mathcal{K}^M,$$

where D is a linear form on $V'_{\beta'}$ such that

$$(9) \qquad D(E'v') = m(\beta',(s_o\gamma(\alpha))') = m(\beta,\gamma(\alpha)).$$

Let $u \to u^t$ denote the transpose map from $\mathfrak{G}$ into itself, that is, the unique (involutive) antiautomorphism of $\mathfrak{G}$ which is -1 on $\mathfrak{g}$.

If σ is a representation of $\mathfrak{U}^K$ on the finite dimensional complex vector space Q, and Q' is the dual of Q, then the representation of $\mathfrak{U}^K$ contragredient to σ is the representation σ' defined on Q' by the condition

$$\langle\sigma(u)q,q'\rangle = \langle q,\sigma'(u^t)q'\rangle \text{ for all } u \in \mathfrak{U}^K,\ q \in Q,\ q' \in Q'.$$

The same statement holds for $\mathcal{K}^M$ in place of $\mathfrak{U}^K$. For every class $\delta \in \widehat{\mathfrak{U}^K}$ or $\widehat{\mathcal{K}^M}$, let δ' denote the contragredient class.

Since $(\eta(\alpha,\beta))' = \eta(\alpha',\beta')$, we have that

$$\chi_{\eta(\alpha',\beta')}(u^t) = \chi_{\eta(\alpha,\beta)}(u) \text{ for all } u \in \mathfrak{U}^K. \tag{10}$$

Similarly, since $(\zeta(\beta,s_o\gamma(\alpha)))' = \zeta(\beta',(s_o\gamma(\alpha))')$, we have that

$$\chi_{\zeta(\beta',(s_o\gamma(\alpha))')}(v^t) = \chi_{\zeta(\beta,s_o\gamma(\alpha))}(v) \quad \text{for all } v \in \mathcal{K}^M.$$

Hence

$$(\mathcal{J}_{\beta,s_o\gamma(\alpha)})^t = \mathcal{J}_{\beta',(s_o\gamma(\alpha))'}. \tag{11}$$

Let $p\colon \mathfrak{U}^K \to \mathfrak{A}$ be the linear map defined as follows: For all $u \in \mathfrak{U}^K$, $p_u = (p^{u^t}_{\beta,s_o\gamma(\alpha)})^t$. Then for all $u \in \mathfrak{U}^K$, we have by (11) that p_u is the $\mathfrak{A}$-component of u with respect to the decomposition

$$\mathfrak{U}^K \subset \mathcal{J}_{\beta',(s_o\gamma(\alpha))'}\mathfrak{A} + \mathfrak{A} + \mathfrak{U}\mathfrak{n}.$$

Thus for all $u \in \mathfrak{U}^K$, we have that

$$\begin{aligned}
D(\Pi'(u^t)E'v') &= D(E'\Pi'(u^t)v') \\
&\equiv D(E'\Pi'(p_{u^t})v') \bmod D(E'\Pi'(\mathcal{J}_{\beta',(s_o\gamma(\alpha))'})v') \\
&\equiv D(E'(p_{u^t}(-s_o\lambda(\alpha))v') \bmod D(\Pi'(\mathcal{J}_{\beta',(s_o\gamma(\alpha))'})E'v') \\
&\equiv p_{u^t}(-s_o\lambda(\alpha))D(E'v') \\
&\equiv p^u_{\beta,s_o\gamma(\alpha)}(s_o\lambda(\alpha))D(E'v').
\end{aligned}$$

Hence

$$\chi_{\eta(\alpha',\beta')}(u^t) = m(\beta,\gamma(\alpha))p^u_{\beta,s_o\gamma(\alpha)}(s_o\lambda(\alpha))$$

by (7), (8), (9) and the definition of $\mathcal{J}_{\beta',(s_o\gamma(\alpha))'}$. Theorem 1 now follows from (10).

Remark. The reasoning in § 3 could have been carried out more simply if we had worked entirely in V, and not in the dual V'. We then would have obtained an analogue of Theorem 1 based on the decomposition $\mathfrak{g}^K \subset \mathfrak{K}^M\mathfrak{a} + \mathfrak{g}\mathfrak{n}$ instead of the decomposition $\mathfrak{g}^K \subset \mathfrak{a}\mathfrak{K}^M + \mathfrak{n}\mathfrak{g}$. However, the latter decomposition is necessary for the infinite dimensional case (Theorem 2, § 4), and so our use of the dual module shows that the finite and infinite dimensional cases fit into the same pattern (compare Theorem 1 with Theorems 2 and 3).

§4 Infinite dimensional representations

In this section, we shall prove analogues of Theorem 1 (§3) for families of infinite dimensional representations of G which are closely related to the representations defined by Harish-Chandra in [4, §12]. We shall use several results and methods from [4] and [5], and several ideas from [11]. We again retain the notation of the preceding sections.

Let $\gamma \in \hat{M}$, and let $\nu \in \hat{A}$, so that ν is a (not necessarily unitary) complex one-dimensional character on A. We also regard ν as a linear form on $\mathfrak{a}$. We shall define a continuous (not necessarily unitary) permissible (see [4]) representation $\Pi_{\gamma,\nu}$ of G on a Hilbert space.

First we define a representation Π_ν of G on the Hilbert space $\mathcal{L}^2(K)$. For all $g \in G$ and $k \in K$, let

$$gk = k_g \exp H(g,k)n,$$

where $k_g \in K$, $H(g,k) \in \mathfrak{a}_0$, $n \in N$, and exp denotes the exponential mapping. Let ρ be half the sum of the positive restricted roots of $\mathfrak{g}_0$, so that ρ may be regarded as a linear form on $\mathfrak{a}$. For all $g \in G$, $f \in \mathcal{L}^2(K)$ and almost all $k \in K$, define

$$(\Pi_\nu(g)f)(k) = e^{(\nu-\rho)(H(g^{-1},k))} f(k_g-1).$$

Then $\Pi_\nu(g)f \in \mathcal{L}^2(K)$, and Π_ν is a continuous representation of G on $\mathcal{L}^2(K)$ (cf. [4, §12]).

Let $\gamma' \in \hat{M}$ denote the class contragredient to γ. Let ϕ be a diagonal matrix element of a representation in γ', so that ϕ is a continuous function on M. Let dm denote normalized Haar measure on M, let τ be the left regular representation of K on $\mathcal{L}^2(K)$, and let τ' be the right regular representation of K on $\mathcal{L}^2(K)$. Let d be the dimension of the representation space of any member of γ. Now the operator $F = d\int_M \phi(m^{-1})\tau'(m)dm$ is a projection in $\mathcal{L}^2(K)$, and F commutes with Π_ν. $\Pi_{\gamma,\nu}$ is defined as the representation of G which is the restriction of Π_ν to the invariant subspace $\mathcal{H} = F\mathcal{L}^2(K)$.

Remark. γ and ν define an element $\sigma_{\gamma,\nu}$ of $\widehat{NAM}$ in an obvious way. If ν is a unitary character of A, then $\Pi_{\gamma,\nu}$ is unitary and is unitarily equivalent to the representation of G induced by any member of $\sigma_{\gamma,\nu}$.

For all $\xi \in \hat{K}$, let $\mathcal{H}_\xi$ denote the space of vectors in $\mathcal{H}$ which transform under $\Pi_{\gamma,\nu}(K)$ according to ξ. Then $\mathcal{H}_\xi$ is finite dimensional, it consists of C^∞ functions, and $\sum_{\xi \in \hat{K}} \mathcal{H}_\xi$ (direct sum) is dense in $\mathcal{H}$. $\sum_{\xi \in \hat{K}} \mathcal{H}_\xi$ is well-behaved under $\Pi_{\gamma,\nu}(G)$, and $\Pi_{\gamma,\nu}$ induces an "infinitesimal" representation of $\mathcal{D}$ on $\sum_{\xi \in \hat{K}} \mathcal{H}_\xi$ in a natural way compatible with the exponential mapping (see [4]). We shall also denote this representation by $\Pi_{\gamma,\nu}$. Let $\alpha_{\gamma,\nu}$ denote the infinitesimal equivalence class of permissible Banach space representations of G defined by $\Pi_{\gamma,\nu}$, so that $\alpha_{\gamma,\nu}$ depends only on γ and ν.

For all $\xi \in \hat{K}$, let $\mathcal{D}_\xi$ denote the subalgebra of $\mathcal{D}$ which leaves $\mathcal{H}_\xi$ invariant under $\Pi_{\gamma,\nu}$. Suppose $\beta \in \hat{K}$ is such that $\mathcal{H}_\beta \neq \{0\}$ and $\mathcal{H}_\beta$ is irreducible under $\mathcal{D}_\beta$. Then the proof of Lemma 2 (§ 2) holds in the present situation, and implies that $\mathcal{H}_\beta$ is invariant under $\Pi_{\gamma,\nu}(\mathcal{D}^K)$ and that $\Pi_{\gamma,\nu}(\mathcal{D}^K)$ (restricted to $\mathcal{H}_\beta$) is precisely the commuting ring of $\Pi_{\gamma,\nu}(K)$ (restricted to $\mathcal{H}_\beta$) in End $\mathcal{H}_\beta$. Just as in Lemma 3, Lemma 1 now implies the existence of a unique class $\eta(\alpha_{\gamma,\nu},\beta) \in \widehat{\mathcal{D}^K}$ such that the action of K and $\mathcal{D}^K$ on $\mathcal{H}_\beta$ factors into the tensor product of a representation in β with a representation in $\eta(\alpha_{\gamma,\nu},\beta)$; $\eta(\alpha_{\gamma,\nu},\beta)$ depends only on $\alpha_{\gamma,\nu}$ and β.

For all $\xi \in \hat{K}$, let $m(\alpha_{\gamma,\nu},\xi)$ denote the multiplicity with which members of ξ occur in the restriction to K of any member of $\alpha_{\gamma,\nu}$, or, equivalently, in the representation of K on $\mathcal{H}_\xi$ defined by $\Pi_{\gamma,\nu}$.

We can now state an analogue of Theorem 1:

<u>Theorem 2</u>. For all $\gamma \in \hat{M}$, $\nu \in \hat{A}$ and $\beta \in \hat{K}$, we have

$$m(\alpha_{\gamma,\nu},\beta) = m(\beta,\gamma).$$

Suppose that γ, ν and β satisfy the condition

$$m(\alpha_{\gamma,\nu},\beta) = m(\beta,\gamma) > 0 \tag{12}$$

and suppose that $\mathcal{H}_\beta$ is irreducible under $\Pi_{\gamma,\nu}(\mathcal{D}_\beta)$ (in the notation above). Then

(13) $$\chi_{\eta(\alpha_{\gamma,\nu},\beta)}(u) = m(\beta,\gamma)\ p^{u}_{\beta,\gamma}(-\nu+\rho)$$

for all $u \in \mathcal{D}^K$.

Proof. The first statement follows from the Frobenius reciprocity theorem.

Let $\psi \in \mathcal{L}^2(K)$ be the character of any member of β. Now $\Pi_\nu(k) = \tau(k)$ for all $k \in K$ (in the above notation). Hence $\theta = F\psi$ is a cyclic vector for the action of $\Pi_{\gamma,\nu}(K)$ on $\mathcal{H}_\beta$.

Let F' denote the projection of $\mathcal{L}^2(K)$ onto the subspace of all vectors which transform under $\tau'(M)$ according to γ', so that F' commutes with $\tau'(M)$, $\tau(K)$ and $\Pi_\nu(G)$, and $F = FF'$. Now $\psi(kk') =$ $= \psi(k'k)$ for all $k,k' \in K$. Hence for all $m \in M$, we have

$$\begin{aligned}\Pi_\nu(m)F'\psi = \tau(m)F'\psi = F'\tau(m)\psi &= \\ = F'\tau'(m^{-1})\psi = \tau'(m^{-1})F'\psi.&\end{aligned}$$

Let (,) denote scalar product in $\mathcal{L}^2(K)$. Then

$$\begin{aligned}(\Pi_\nu(m)F'\psi,F'\psi) = (\tau'(m^{-1})F'\psi,F'\psi) &= \\ = (F'\psi,\tau'(m)F'\psi) = \overline{(\tau'(m)F'\psi,F'\psi)}&\end{aligned}$$

(where bar denotes complex conjugate), a matrix element for γ. Thus $F'\psi$ transforms under $\Pi_\nu(M)$ according to γ. But since F commutes with $\Pi_\nu(M)$, we have that $\theta = F\psi = FF'\psi$ transforms under $\Pi_{\gamma,\nu}(M)$ according to γ.

We now use a technique of Harish-Chandra [5, Lemma 11 (p. 49)]. For all $g \in G$, $u \in \mathcal{G}$ and $k \in K$, let $g^k = kgk^{-1}$ and let $u^k = (\mathrm{Ad}\ k)u$. For all $f \in \mathcal{L}^2(K)$ and almost all $k \in K$, we have

$$(\Pi_\nu(n^k)f)(k) = f(k) \qquad \text{for all } n \in N,$$

$$(\Pi_\nu(a^k)f)(k) = e^{(-\nu+\rho)(\log a)}f(k) \qquad \text{for all } a \in A,$$

$$(\Pi_\nu(k_o^k)f)(k) = f(kk_o^{-1}) \qquad \text{for all } k_o \in K,$$

where log denotes the inverse of the exponential mapping from $\mathfrak{a}_o$ to A. Thus if f is a $\mathcal{C}^\infty$ function on K which is well-behaved under Π_ν, then

$$(14) \qquad (\Pi_\nu(X^k)f)(k) = 0 \qquad \text{for all } X \in \mathfrak{n},$$

$$(15) \qquad (\Pi_\nu(H^k)f)(k) = (-\nu+\rho)(H)f(k) \qquad \text{for all } H \in \mathfrak{a}.$$

In addition,

$$(\Pi_\nu(k_o^k)\psi)(k) = \psi(kk_o^{-1}) = \psi(k_o^{-1}k) = (\Pi_\nu(k_o)\psi)(k),$$

so that

$$(16) \qquad (\Pi_\nu(z^k)\psi)(k) = (\Pi_\nu(z)\psi)(k) \qquad \text{for all } z \in \mathcal{K}.$$

We add to Harish-Chandra's reasoning the observation that for all $z \in \mathcal{K}^M$ and all $k \in K$, we have

$$(17) \qquad (\Pi_{\gamma,\nu}(z^k)\theta)(k) = (\Pi_{\gamma,\nu}(z)\theta)(k).$$

Indeed,

$$
\begin{aligned}
(\Pi_{\gamma,\nu}(z)\theta)(k) &= (\Pi_\nu(z)F\psi)(k) \\
&= F(\Pi_\nu(z)\psi)(k) \\
&= F(\Pi_\nu(z^k)\psi)(k) \quad \text{(by (16))} \\
&= d\int_M \phi(m^{-1})\Pi_\nu(z^{km})\psi(km)\,dm \\
&= d\int_M \phi(m^{-1})\Pi_\nu(z^{k})\psi(km)\,dm \\
&\qquad\qquad (\text{since } z\in\mathcal{K}^M) \\
&= (\Pi_\nu(z^k)F\psi)(k) \\
&= (\Pi_{\gamma,\nu}(z^k)\theta)(k).
\end{aligned}
$$

Now $\mathfrak{J}^K \subset \mathfrak{A}\mathcal{K}^M + n\mathfrak{J}$. Choose a basis $\{H_1,\ldots,H_\ell\}$ of $\mathfrak{a}$. For every ℓ-tuple $(s) = (s_1,\ldots,s_\ell)$ of non-negative integers, let

$$H_{(s)} = H_1^{s_1} \ldots H_\ell^{s_\ell},$$

so that $\{H_{(s)}\}$ is a basis of $\mathfrak{A}$. Let $u\in\mathfrak{J}^K$, and write

$$u = \sum_{(s)} H_{(s)}z_{(s)} + \sum_i X_i y_i \ , \tag{18}$$

where $z_{(s)}\in\mathcal{K}^M$, $X_i\in n$ and $y_i\in\mathfrak{J}$. Now let $k\in K$. Since $u\in\mathfrak{J}^K$, we have

$$u = u^k = \sum_{(s)} H^k_{(s)}z^k_{(s)} + \sum_i X^k_i y^k_i \ .$$

Hence

$$(\Pi_{\gamma,\nu}(u)\theta)(k) = \sum_{(s)} (\Pi_\nu(H^k_{(s)})\Pi_\nu(z^k_{(s)})\theta)(k) + \sum_i (\Pi_\nu(X^k_i)\Pi_\nu(y^k_i)\theta)(k)$$

$$= \sum_{(s)} H_{(s)}(-\nu+\rho)(\Pi_{\gamma,\nu}(z^k_{(s)})\theta)(k)$$

(by (14) and (15))

Thus

$$\Pi_{\gamma,\nu}(u)\theta = \sum_{(s)} H_{(s)}(-\nu+\rho)\Pi_{\gamma,\nu}(z_{(s)})\theta \tag{19}$$

by (17). In particular,

$$\Pi_{\gamma,\nu}(\mathcal{G}^K)\theta \subset \Pi_{\gamma,\nu}(\mathcal{K}^M)\theta.$$

We now suppose that (12) holds (see the statement of the theorem) and that $\mathcal{H}_\beta$ is irreducible under $\Pi_{\gamma,\nu}(\mathcal{G}_\beta)$. We have thus verified all the hypotheses of Lemma 5 (§ 2), which we apply to the case $Z = \mathcal{H}_\beta$, $\xi = \beta$, $\omega = \gamma$, $z = \theta$, and σ_1 and σ_2 are the representations of K (or $\mathcal{K}$) and $\mathcal{G}^K$, respectively, on $\mathcal{H}_\beta$ induced by $\Pi_{\gamma,\nu}$. Hence Lemma 5 implies that

$$\chi_{\eta(\alpha_{\gamma,\nu},\beta)}(u) = D(\Pi_{\gamma,\nu}(u)\theta) \qquad \text{for all } u \in \mathcal{G}^K \tag{20}$$

and

$$\chi_{\zeta(\beta,\gamma)}(v) = D(\Pi_{\gamma,\nu}(v)\theta) \qquad \text{for all } v \in \mathcal{K}^M, \tag{21}$$

where D is a certain linear form on $\mathcal{H}_\beta$.

Combining (19), (20) and (21), we have

$$(22) \qquad \chi_{\eta(\alpha_{\gamma,\nu},\beta)}(u) = \sum_{(s)} H_{(s)}(-\nu+\rho)\chi_{\zeta(\beta,\gamma)}(z_{(s)}).$$

If we rewrite (18) in the form

$$u = \sum_{(s)} H_{(s)} z'_{(s)} + p^u_{\beta,\gamma} + \sum_i X_i y_i,$$

where $z'_{(s)} \in \mathcal{Y}_{\beta,\gamma}$, and if we recall that $\chi_{\zeta(\beta,\gamma)}(1) = m(\beta,\gamma)$, then we see that (22) implies the theorem.

We shall now give a generalization of the procedure used in [11, § 2.4] to construct a series of irreducible representations:

Assume the hypotheses of Theorem 2, so that $m(\alpha_{\gamma,\nu},\beta) > 0$ and $\mathcal{H}_\beta$ is irreducible under $\Pi_{\gamma,\nu}(\mathcal{G}_\beta)$. Let $\mathcal{H}'$ be the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{H}_\beta$ and invariant under $\Pi_{\gamma,\nu}(G)$. If $\mathcal{P}$ is any closed subspace of $\mathcal{H}'$ invariant under $\Pi_{\gamma,\nu}(G)$ and such that $\mathcal{P} \cap \mathcal{H}_\beta = \{0\}$, then $\mathcal{P} \subset \bigoplus_{\xi \neq \beta} \mathcal{H}_\xi$ (Hilbert space direct sum), and so $\mathcal{P} \perp \mathcal{H}_\beta$. Let $\mathcal{H}''$ be the closed linear span of all such $\mathcal{P}$, so that $\mathcal{H}'' \perp \mathcal{H}_\beta$. Since $\mathcal{H}_\beta$ is irreducible under $\Pi_{\gamma,\nu}(\mathcal{G}_\beta)$, we have that $\mathcal{H}''$ is a maximal closed $\Pi_{\gamma,\nu}(G)$-invariant subspace of $\mathcal{H}'$. Hence the quotient representation $\hat{\Pi}^\beta_{\gamma,\nu}$ of G on the Hilbert space $\hat{\mathcal{H}} = \mathcal{H}'/\mathcal{H}''$ is irreducible. Also, $\hat{\Pi}^\beta_{\gamma,\nu}$ is quasi-simple (see [4]).

We digress to describe a general picture developed by Harish-Chandra in [4] and [5]. Let Π be an irreducible quasi-simple representation of G on a Banach space $\mathcal{B}$. For all $\xi \in \hat{K}$, let $\mathcal{B}_\xi$

denote the space of vectors in $\mathcal{B}$ which transform under $\Pi(K)$ according to ξ. Then $\mathcal{B}_\xi$ is finite dimensional, and $\sum_{\xi \in \hat{K}} \mathcal{B}_\xi$ is dense in $\mathcal{B}$. $\sum_{\xi \in \hat{K}} \mathcal{B}_\xi$ is well-behaved under $\Pi(G)$, and Π induces a representation of $\mathcal{Y}$ on $\sum_{\xi \in \hat{K}} \mathcal{B}_\xi$ in a natural way compatible with the exponential mapping. This infinitesimal representation is also irreducible, and will also be denoted by Π. Let α denote the corresponding infinitesimal equivalence class of quasi-simple Banach space representations of G.

For all $\xi \in \hat{K}$, let $\mathcal{Y}_\xi$ denote the subalgebra of $\mathcal{Y}$ which leaves $\mathcal{B}_\xi$ invariant under Π. Then for every $\beta \in \hat{K}$ such that $\mathcal{B}_\beta \neq \{0\}$, we have that $\mathcal{B}_\beta$ is irreducible under $\Pi(\mathcal{Y}_\beta)$ (see [5, §2, Corollary 2]). Thus for any such β, the proofs of Lemmas 2 and 3 (§2) again apply, and we again obtain a unique class $\eta(\alpha,\beta) \in \widehat{\mathcal{Y}^K}$ such that the action of K and $\mathcal{Y}^K$ on $\mathcal{B}_\beta$ factors into the tensor product of a representation in β with a representation in $\eta(\alpha,\beta)$.

The significance of $\eta(\alpha,\beta)$ is that if α' is a second infinitesimal equivalence class of irreducible quasi-simple Banach space representations, if β also occurs with positive multiplicity in any member of α', and if $\eta(\alpha',\beta) = \eta(\alpha,\beta)$, then $\alpha' = \alpha$ (see [5, §2, Corollary 2]).

We now return to the special situation which we were discussing above. Let $\hat{\alpha}^\beta_{\gamma,\nu}$ denote the infinitesimal equivalence class of irreducible quasi-simple Banach space representations of G defined by $\hat{\Pi}^\beta_{\gamma,\nu}$. Now the action of $\Pi_{\gamma,\nu}(\mathcal{Y}^K)$ on $\mathcal{H}_\beta$ is equivalent to the action of $\hat{\Pi}^{\beta}_{\gamma,\nu}(\mathcal{Y}^K)$ on $\hat{\mathcal{H}}_{\beta}$. Hence formula (13) (Theorem 2) remains valid when $\alpha_{\gamma,\nu}$ is replaced by $\hat{\alpha}^{\beta}_{\gamma,\nu}$. Thus we have:

Theorem 3. Let $\gamma \in \hat{M}$, $\nu \in \hat{A}$ and $\beta \in \hat{K}$. Suppose that $m(\beta,\gamma) > 0$ and that $\mathcal{H}_{\beta}$ is irreducible under $\Pi_{\gamma,\nu}(\mathcal{Y}_{\beta})$, in the notation of Theorem 2. Then $\hat{\alpha}^{\beta}_{\gamma,\nu}$ is an infinitesimal equivalence class of irreducible quasi-simple Banach space representations of G such that

$$\chi_{\eta(\hat{\alpha}^{\beta}_{\gamma,\nu},\beta)}(u) = m(\beta,\gamma)p^{u}_{\beta,\gamma}(-\nu+\rho) \tag{23}$$

for all $u \in \mathcal{Y}^K$.

The above remarks indicate the significance of formula (23). Comparing Theorems 1 and 3, we see that the representations $\hat{\alpha}^{\beta}_{\gamma,\nu}$ "extend" the finite dimensional representations satisfying condition (5) (Theorem 1), in a certain natural way.

Even if $\mathcal{H}_{\beta}$ is not irreducible under $\mathcal{Y}_{\beta}$, it seems possible that the action of $\mathcal{Y}^K$ on $\mathcal{H}_{\beta}$ can be used to obtain information about the "composition series" of the (not necessarily irreducible) representations $\Pi_{\gamma,\nu}$.

§ 5 Special cases

The proofs of Theorems 1 and 2 (§ 3 and § 4, respectively) can be greatly simplified if the relevant multiplicities are assumed to be 1. In this section, we shall show how this can be done. We shall also show how the enveloping algebra formalism which is used to describe the infinitesimal class 1 spherical functions (see [8, Chapter X, § 6.3]) can be regarded as a special case of our formalism. We again retain the previous notation.

Suppose $\delta \in \hat{G}$ and $\xi \in \hat{K}$ are such that $m(\delta,\xi) = 1$. Then by Schur's Lemma, $\mathfrak{G}^K$ acts as scalars on the ξ-primary part of any representation in δ, and $\chi_{\eta(\delta,\xi)}$ is the corresponding homomorphism of $\mathfrak{G}^K$ into $\mathbb{C}$. Similarly, if $\xi \in \hat{K}$ and $\omega \in \hat{M}$ are such that $m(\xi,\omega) = 1$, then the same observation shows that $\chi_{\zeta(\xi,\omega)}$ is a homomorphism of $\mathcal{K}^M$ into $\mathbb{C}$.

We now give a simple proof of Theorem 1 in the case in which $m(\alpha,\beta) = m(\beta,\gamma(\alpha)) = 1$:

Theorem 1'. Let $\alpha \in \hat{G}$ and $\beta \in \hat{K}$ be such that

$$m(\alpha,\beta) = m(\beta,\gamma(\alpha)) = 1.$$

Then

$$\chi_{\eta(\alpha,\beta)}(u) = p^u_{\beta,s_o\gamma(\alpha)}(s_o\lambda(\alpha)) \quad \text{for all } u \in \mathfrak{G}^K.$$

Moreover, the linear mapping $p_{\beta,s_o\gamma(\alpha)}: \mathfrak{G}^K \to \mathfrak{A}$ is a homomorphism.

Proof. Since $m(\beta,s_o\gamma(\alpha)) = m(\beta,\gamma(\alpha)) = 1$, and since the kernel of $\chi_{\zeta(\beta,s_o\gamma(\alpha))}$ is thus an ideal in $\mathcal{K}^M$, the last statement of the theorem follows from the Corollary to Proposition 1 (§ 1). For every class $\delta \in \hat{G}$, $\hat{K}$ or $\hat{M}$, let δ' denote the contragredient class.

Let $\Pi \in \alpha$ act on the space V, and let Π' be the contragredient module to Π, so that $\Pi' \in \alpha'$, and Π' acts on the dual V' of V.

Let $v' \in V'$ be a non-zero highest restricted weight vector of Π'. Let E' be the projection of V' onto $V'_{\beta'}$ with respect to the direct sum decomposition $V' = \sum_{\xi \in \hat{K}} V'_{\xi}$.

As in the proof of Theorem 1, we see immediately that $E'v' \neq 0$ and that $E'v' \in (V'_{\beta'})_{(s_o\gamma(\alpha))'}$.

Choose $w \in V_\beta$ such that $\langle w, E'v' \rangle = 1$. Then for all $u \in \mathfrak{G}^K$, we have

$$\begin{aligned} \chi_{\eta(\alpha,\beta)}(u) &= \langle \Pi(u)w, E'v' \rangle \\ &= \langle w, \Pi'(u^t)E'v' \rangle \\ &= \langle w, E'\Pi'(u^t)v' \rangle \end{aligned}$$

where $u \to u^t$ denotes the unique antiautomorphism of $\mathfrak{G}$ which is -1 on $\mathfrak{g}$.

Now

$$(\mathfrak{f}_{\beta, s_o\gamma(\alpha)})^t = \mathfrak{f}_{\beta', (s_o\gamma(\alpha))'}.$$

Hence for all $u \in \mathfrak{G}^K$, $p_u = (p^{u^t}_{\beta, s_o\gamma(\alpha)})^t$ is the $\mathfrak{a}$-component of u with respect to the decomposition

$$\mathfrak{G}^K \subset \mathfrak{f}_{\beta', (s_o\gamma(\alpha))'}\mathfrak{a} + \mathfrak{a} + \mathfrak{G}\mathfrak{n}\ .$$

Thus for all $u \in \mathfrak{U}^K$, we have

$$\begin{aligned}
\chi_{\eta(\alpha,\beta)}(u) &\equiv \langle w, E'\Pi'(p_u^t)v'\rangle \bmod \langle w, E'\Pi'(\mathcal{J}_{\beta',(s_o\gamma(\alpha))'})v'\rangle \\
&\equiv p_u^t(-s_o\lambda(\alpha)) \bmod \langle w, \Pi'(\mathcal{J}_{\beta',(s_o\gamma(\alpha))'})E'v'\rangle \\
&\equiv (p^u_{\beta,s_o\gamma(\alpha)})^t(-s_o\lambda(\alpha)) \\
&\equiv p^u_{\beta,s_o\gamma(\alpha)}(s_o\lambda(\alpha)).
\end{aligned}$$

But

$$\begin{aligned}
\Pi'(\mathcal{J}_{\beta',(s_o\gamma(\alpha))'})E'v' &= \chi_{\zeta(\beta',(s_o\gamma(\alpha))')}(\mathcal{J}_{\beta',(s_o\gamma(\alpha))'})E'v' \\
&= 0
\end{aligned}$$

by the definition of $\mathcal{J}_{\beta',(s_o\gamma(\alpha))'}$. This proves Theorem 1'.

The proof of Theorem 2 can of course also be simplified if we assume that $m(\beta,\gamma) = 1$. In this case, we can omit the hypothesis that $\mathcal{H}_\beta$ is irreducible under $\Pi_{\gamma,\nu}(\mathfrak{U}_\beta)$ in Theorems 2 and 3, since this hypothesis is automatically satisfied.

Let β_o and γ_o be the classes of the trivial one-dimensional representations of K and M, respectively. Then

$$\mathcal{J}_{\beta_o,\gamma_o} = \mathcal{K}^M \cap \mathcal{K}\mathfrak{k},$$

and so for all $u \in \mathfrak{U}^K$, $p^u_{\beta_o,\gamma_o}$ is the $\mathfrak{a}$-component of u with respect to the decomposition

$$\mathcal{D}^K \subset \mathfrak{a}(\mathcal{K}^M \cap \mathcal{K}k) + \mathfrak{a} + n\mathcal{D}.$$

Thus p_{β_o,γ_o} is essentially the same as the mapping γ defined in [8, p. 430] for the study of the class 1 spherical functions. Hence our results can be regarded as generalizations of certain aspects of the class 1 theory. In § 6, we shall discuss the generalization of other aspects of the class 1 theory.

§ 6 Further questions

We again retain the previous notation. As we pointed out at the end of the last section, the mapping p_{β_o,γ_o} is essentially the same as the mapping γ used in [8, Chapter X, § 6.3] to study the class 1 spherical functions. One of the main results there is the theorem of Harish-Chandra which can be stated in our terminology as follows:

$$0 \to \mathcal{D}^K \cap \mathcal{D}k \xrightarrow{\iota} \mathcal{D}^K \xrightarrow{p_{\beta_o,\gamma_o}} \mathfrak{a}^{\tilde{W}} \to 0 \tag{25}$$

is an exact sequence. Here ι is the injection, and $\mathfrak{a}^{\tilde{W}}$ is the set of elements in $\mathfrak{a}$ which are fixed by the translated Weyl group $\tilde{W}$, which in turn is defined as follows: For all $s \in W$, let s^A denote the affine transformation of the dual of $\mathfrak{a}$ given by $\lambda \to s\lambda - s\rho + \rho$; then s^A can be regarded as an algebra automorphism of $\mathfrak{a}$, and $\tilde{W} = \{s^A | s \in W\}$.

One significance of (25) is that it yields Harish-Chandra's formula for the class 1 spherical function (see [8, loc. cit.]), and it also leads to a concrete realization of an abstract quasi-simple irreducible representation of class 1. Specifically, if Π is a quasi-simple irreducible representation of G with a K-fixed vector v, then the corresponding action of $\mathfrak{G}^K$ on v annihilates $\mathfrak{G}^K \cap \mathfrak{G}\mathfrak{k}$, and so induces a homomorphism of $\mathfrak{a}^{\tilde{W}}$ into $\mathbb{C}$ by (25). But since $\mathfrak{a}$ is integral over $\mathfrak{a}^{\tilde{W}}$, this homomorphism extends to $\mathfrak{a}$, and the action of $\mathfrak{G}^K$ on v is thus given by evaluation of p_{β_o,γ_o} at some point in the dual of $\mathfrak{a}$. By Theorem 3 in the class 1 case, the action of $\mathfrak{G}^K$ on v is the same as the action of $\mathfrak{G}^K$ on a K-fixed vector in a representation of the form $\hat{\Pi}^{\beta_o}_{\gamma_o,\nu}$. By Harish-Chandra's theorem stated just before Theorem 3, Π is infinitesimally equivalent to a representation of the form $\hat{\Pi}^{\beta_o}_{\gamma_o,\nu}$. This yields the desired results.

Another significance of (25) is that it determines all possible equivalences between representations of the form $\hat{\Pi}^{\beta_o}_{\gamma_o,\nu}$.

Hopefully, a similar study of the mapping $p_{\beta,\gamma}$ for arbitrary β and γ will yield a formula for general spherical functions and a theorem on the concrete realization of irreducible representations outside the class 1 case, as well as information on the possible equivalences between representations of the form $\hat{\Pi}^{\beta}_{\gamma,\nu}$. Harish-Chandra has obtained such theorems (see [5, Theorem 4 (p. 63)] and

[6]; cf. also R. Godement, A theory of spherical functions, I, Trans. Amer. Math. Soc. 73 (1952), 496-556), but it is hoped that a study of the mapping $p_{\beta,\gamma}$ and a generalization of (25) will lead to a sharpening of his results.

We now briefly indicate the results which we have obtained relating to the generalization of (25). We have:

Theorem 4. Let $\beta \in \hat{K}$ and $\gamma \in \hat{M}$ be such that $m(\beta,\gamma) > 0$. Then for all $s \in W$ and $u \in \mathfrak{D}^K$, we have

$$s^A p^u_{\beta,\gamma} = p^u_{\beta,s\gamma}. \tag{26}$$

Moreover, if $\nu \in \hat{A}$ and if β satisfies the usual irreducibility condition (see Theorem 2) with respect to $\Pi_{\gamma,\nu}$ and $\Pi_{s\gamma,s\nu}$, then $\hat{\alpha}^\beta_{\gamma,\nu}$ and $\hat{\alpha}^\beta_{s\gamma,s\nu}$ are defined and

$$\hat{\alpha}^\beta_{\gamma,\nu} = \hat{\alpha}^\beta_{s\gamma,s\nu}.$$

Proof. The second statement follows from the first and from Theorem 3. Concerning the first statement, it follows from a theorem of F. Bruhat [2, Theorem 7.2 (p. 193)] that if ν assumes pure imaginary values on $\mathfrak{a}_0$ and is regular, then $\Pi_{\gamma,\nu}$ and $\Pi_{s\gamma,s\nu}$ are irreducible and equivalent. This gives a Zariski dense set on which (26) (regarded as an equality of polynomials on the dual of $\mathfrak{a}$) holds, in view of Theorem 2. Thus (26) itself holds, and Theorem 4 is proved.

Remark. The idea for the proof of Theorem 4 is due to S. Helgason.

By slightly modifying the proofs of Lemmas 6.12 and 6.14 in [8, Chapter X, § 6.3], and by using Theorem 4 above, we can prove the following generalization of Lemma 6.14 [loc. cit.]:

Proposition 2. Let $\beta \in \hat{K}$ and $\gamma \in \hat{M}$ be such that $m(\beta,\gamma) > 0$, and assume that $s\gamma = \gamma$ for all $s \in W$. Then the image of $p_{\beta,\gamma}$ in $\mathfrak{A}$ is precisely $\mathfrak{A}^{\tilde{W}}$.

Remark. Proposition 2 yields information on the non-equivalence of certain pairs of representations of the form $\hat{\Pi}^{\beta}_{\gamma,\nu}$. Further information of this type would follow from the following conjecture, a generalization of the statement of Proposition 2: Let $\beta \in \hat{K}$ and $\gamma \in \hat{M}$ be such that $m(\beta,\gamma) > 0$. Let $W_{\gamma} = \{s \in W | s\gamma = \gamma\}$. Then the image of $p_{\beta,\gamma}$ in $\mathfrak{A}$ is $\{x \in \mathfrak{A} | s^{A}x = x$ for all $s \in W_{\gamma}\}$ (cf. [11, § 3.6]).

By applying [5, Lemma 1 (p. 28)] and Proposition 2, we can prove:

Proposition 3. Let G be $SU(1,n)$ $(n \geq 1)$ or the universal covering group of $SO_{o}(1,2n)(n \geq 2)$ (see [8, Chapter IX] for the notation). Let Π be an arbitrary quasi-simple irreducible representation of G, and let $\beta \in \hat{K}$ occur in Π. Then β occurs with multiplicity exactly 1 in Π, and Π is infinitesimally equivalent to $\hat{\Pi}^{\beta}_{\gamma,\nu}$ for some $\gamma \in \hat{M}$ and $\nu \in \hat{A}$. Moreover, for all $\beta \in \hat{K}$, $\gamma \in \hat{M}$ and $\nu,\nu' \in \hat{A}$ such that $m(\beta,\gamma) > 0$, $\hat{\alpha}^{\beta}_{\gamma,\nu}$ and $\hat{\alpha}^{\beta}_{\gamma,\nu'}$ are defined, and they are equal if and only if $\nu' = s^{A}\nu$ for some $s \in W$.

The properties of G that we have used to prove Proposition 3 are:

1) $m(\beta,\gamma) \leq 1$ for all $\beta \in \hat{K}$ and $\gamma \in \hat{M}$.

2) $s\gamma = \gamma$ for all $\gamma \in \hat{M}$ and $s \in W$ (cf. Proposition 2).

3) For every $\beta \in \hat{K}$, there exists $\alpha \in \hat{G}$ such that $m(\alpha,\beta) > 0$.

(This hypothesis is required for the application of Harish-Chandra's result [5, loc. cit.].)

Chapter II. Multiplicity formulas

§1 Kostant's multiplicity formula

Let U be a compact connected Lie group and K a compact connected Lie subgroup of U. Let T and S be maximal tori of U and K, respectively, such that $S \subset T$. Let $\mathfrak{u}$ be the Lie algebra of U, and let $\mathfrak{k}_o$, $\mathfrak{t}$ and $\mathfrak{s}$ be the Lie subalgebras of $\mathfrak{u}$ corresponding to K, T and S, respectively. Then $\mathfrak{t}$ and $\mathfrak{s}$ are maximal abelian subalgebras of $\mathfrak{u}$ and $\mathfrak{k}_o$, respectively, and $\mathfrak{s} \subset \mathfrak{t}$. Let $\underline{i}$ be a fixed complex number whose square is -1. Then the complexifications of $\mathfrak{u}$, $\mathfrak{k}_o$, $\mathfrak{t}$ and $\mathfrak{s}$ can be denoted $\mathfrak{g} = \mathfrak{u} + \underline{i}\,\mathfrak{u}$, $\mathfrak{k} = \mathfrak{k}_o + \underline{i}\,\mathfrak{k}_o$, $\mathfrak{h} = \mathfrak{t} + \underline{i}\,\mathfrak{t}$ and $\mathfrak{h}^* = \mathfrak{s} + \underline{i}\,\mathfrak{s}$, respectively. $\mathfrak{g}$ and $\mathfrak{k}$ are complex reductive Lie algebras, and $\mathfrak{h}$ and $\mathfrak{h}^*$ are Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{k}$, respectively. For every linear form ν on $\mathfrak{h}$, let ν^* denote its restriction to $\mathfrak{h}^*$.

Assumption. We assume that $\underline{i}\,\mathfrak{s}$ contains a regular element of $\underline{i}\,\mathfrak{t}$.

We fix the unique Weyl chambers in $\underline{i}\,\mathfrak{t}$ and $\underline{i}\,\mathfrak{s}$ (for $\mathfrak{g}$ and $\mathfrak{k}$, respectively) which contain such an element, which is regular for both $\mathfrak{g}$ and $\mathfrak{k}$. All notions of positivity and dominance of roots and weights will be taken with respect to these chambers.

Let $\omega_1, \dots \omega_r$ be the positive weights of the canonical representation of $\mathfrak{k}$ on $\mathfrak{g}/\mathfrak{k}$, repeated according to multiplicity if necessary. For every integral linear form μ on $\mathfrak{h}^*$, let $P(\mu)$ be the number of non-negative integral r-tuples $n_1, \dots n_r$ such that $\mu = \sum_{i=1}^{r} n_i \omega_i$. P is called the partition function.

Let ρ be half the sum of the positive roots of $\mathfrak{g}$. Let W be the Weyl group of $\mathfrak{g}$, regarded as a group of linear transformations of the dual of $\mathfrak{h}$. For all $\sigma \in W$, let det σ denote the determinant of σ. Let D_U and D_K denote the sets of dominant integral linear forms for U and K, respectively.

For all $\lambda \in D_U$ and $\mu \in D_K$, we define $m_\lambda(\mu)$ to be the multiplicity with which finite dimensional (continuous complex) irreducible representations of K with highest weight μ occur in the restriction to K of any finite dimensional (continuous complex) irreducible representation of U with highest weight λ.

We can now state the multiplicity formula:

Theorem 1 (Kostant). In the above notation,

$$(1) \qquad m_\lambda(\mu) = \sum_{\sigma \in W} (\det \sigma)\, P((\sigma(\lambda+\rho))^* - (\mu+\rho^*))$$

for all $\lambda \in D_U$ and $\mu \in D_K$.

Proof (cf. P. Cartier [3] and N. Jacobson [9, Chapter VIII, §5] for special cases). U can be embedded as a Lie subgroup in a (complex) Lie group U_1 whose Lie algebra is $\mathfrak{g}$. Let K_1 be the connected Lie subgroup of U_1 with Lie algebra $\mathfrak{k}$. Let Π be a finite dimensional irreducible representation of U with highest weight λ. Then Π can be extended uniquely to a holomorphic representation Π_1 of U_1. For all $H \in \mathfrak{h}$, let

$$\chi_\lambda(H) = \text{Trace } (\Pi_1(\exp H))$$

where exp denotes the exponential mapping from $\mathfrak{g}$ into U_1. For all $\nu \in D_K$ and $H' \in \mathfrak{h}^*$, we define $\chi_\nu(H')$ analogously, using K_1 in place of U_1.

By Weyl's character formula (cf. [9], p. 255),

$$(2) \qquad \chi_\lambda(H) = \frac{\sum_{\sigma \in W} (\det \sigma) e^{\sigma(\lambda+\rho)(H)}}{\sum_{\sigma \in W} (\det \sigma) e^{(\sigma\rho)(H)}} \quad \text{for all } H \in \mathfrak{h}.$$

Let $\Delta_+^{\mathfrak{g}}$ and $\Delta_+^{\mathfrak{k}}$ denote the sets of positive roots for $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The denominator in (2) can be rewritten:

$$(3) \qquad \sum_{\sigma \in W} (\det \sigma) e^{(\sigma\rho)(H)} = e^{\rho(H)} \prod_{\alpha \in \Delta_+^{\mathfrak{g}}} (1-e^{-\alpha(H)})$$

(cf. [9], p. 252, Lemma 4). There is a bijection f from $\Delta_+^{\mathfrak{g}}$ to $\Delta_+^{\mathfrak{k}} \cup \{1,\ldots r\}$ such that for all $\alpha \in \Delta_+^{\mathfrak{g}}$, $\alpha^* = f(\alpha)$ if $f(\alpha) \in \Delta_+^{\mathfrak{k}}$, and $\alpha^* = \omega_i$ if $f(\alpha) = i$ $(1 \leq i \leq r)$. Thus

$$(4) \qquad e^{\rho(H')} \prod_{\alpha \in \Delta_+^{\mathfrak{g}}} (1-e^{-\alpha(H')}) =$$

$$= e^{\rho(H')} \prod_{\beta \in \Delta_+^{\mathfrak{k}}} (1-e^{-\beta(H')}) \prod_{i=1}^{r} (1-e^{-\omega_i(H')})$$

for all $H' \in \mathfrak{h}^*$.

Now the left-hand side of (4) represents a <u>non-zero</u> trigonometric polynomial on $\mathfrak{h}^*$. Hence (2) can be regarded as an equality of trigonometric polynomials on $\mathfrak{h}^*$. Let $\mathcal{L}$ denote the set of integral linear forms on $\mathfrak{h}^*$. Then for all $\nu \in \mathcal{L}$, $P(\nu)$ is the coefficient of $e^{-\nu}$ in the Fourier series for

$$\prod_{i=1}^{r} \frac{1}{1-e^{-\omega_i}}.$$

Thus by (2), (3) and (4), we have

$$\chi_\lambda(H') = \frac{\sum_{\sigma \in W} \sum_{\nu \in \mathcal{L}} (\det \sigma) P(\nu) e^{((\sigma(\lambda+\rho))^* - (\nu+\rho^*))(H')}}{\prod_{\beta \in \Delta_+^{\mathfrak{k}}} (1-e^{-\beta(H')})}$$

for all $H' \in \mathfrak{h}^*$, so that

$$\text{(5)} \qquad \chi_\lambda(H') \prod_{\beta \in \Delta_+^{\mathfrak{k}}} (1-e^{-\beta(H')}) =$$

$$= \sum_{\nu \in \mathcal{L}} \Big(\sum_{\sigma \in W} (\det \sigma)\ P((\sigma(\lambda+\rho))^* - (\nu+\rho^*))\Big) e^{\nu(H')}$$

for all $H' \in \mathfrak{h}^*$.

On the other hand,

$$\text{(6)} \qquad \chi_\lambda(H') = \sum_{\nu \in D_K} m_\lambda(\nu)\ \chi_\nu(H') \quad \text{for all } H' \in \mathfrak{h}^*.$$

Let W' denote the Weyl group of $\mathfrak{k}$, and let ρ' be half the sum of the positive roots of $\mathfrak{k}$. Exactly as above, we get

$$(7) \qquad \chi_\nu(H') = \frac{\sum_{\sigma' \in W'} (\det \sigma')\ e^{\sigma'(\nu+\rho')(H')}}{e^{\rho'(H')} \prod_{\beta \in \Delta_+^{\mathfrak{k}}} (1-e^{-\beta(H')})}$$

for all $\nu \in D_K$ and $H' \in \mathfrak{h}^*$. By (6) and (7), we have

$$\chi_\lambda(H') \prod_{\beta \in \Delta_+^{\mathfrak{k}}} (1-e^{-\beta(H')}) =$$

$$= \sum_{\nu \in D_K} \sum_{\sigma' \in W'} m_\lambda(\nu)(\det \sigma')\ e^{(\sigma'(\nu+\rho')-\rho')(H')}$$

for all $H' \in \mathfrak{h}^*$. The sum on the right-hand side may be taken over all $\nu \in \mathcal{L}$, if we define $m_\lambda(\nu)$ to be 0 for all $\nu \in \mathcal{L} - D_K$. We get

$$\chi_\lambda(H') \prod_{\beta \in \Delta_+^{\mathfrak{k}}} (1-e^{-\beta(H')}) =$$

$$= \sum_{\sigma' \in W'} \sum_{\nu \in \mathcal{L}} m_\lambda((\sigma')^{-1}(\nu+\rho')-\rho')(\det \sigma')e^{\nu(H')}$$

$$= \sum_{\nu \in \mathcal{L}} \sum_{\sigma' \in W'} m_\lambda(\sigma'(\nu+\rho')-\rho')(\det \sigma')e^{\nu(H')}$$

$$= \sum_{\nu \in \mathcal{L}} m_\lambda(\nu)e^{\nu(H')}$$

for all $H' \in \mathfrak{h}^*$. In view of (5), we see that (1) follows by comparison of Fourier coefficients, and Theorem 1 is proved.

§2 Application to the rank 1 groups

Let G be a connected real semisimple Lie group with finite center, and let H be a maximal compact subgroup of G. It is well known that the solution of the problem of computing the multiplicities with which finite dimensional irreducible representations of H occur in the restriction to H of finite dimensional irreducible representations of G is contained in the solution of the "dualized" problem in which G is replaced by an appropriate simply connected compact group U and H by an appropriate compact subgroup K of U.

In this section, we show how Kostant's multiplicity formula (§ 1) can be used to obtain explicit multiplicity formulas for the pairs (G,H) where G is simple and of real rank 1. We do this by means of case-by-case analysis (see [8, Chapter IX] for the notation and classification). The dualized problems correspond to the pairs $(U,K) = (SU(n+1), S(U_1 \times U_n))$, $(U,K) = (SO(2n+1), SO(2n))$ (or, rather, their respective covering groups), $(U,K) = (SO(2n), SO(2n-1))$ (or, rather, their respective covering groups), $(U,K) = (Sp(n), Sp(1) \times Sp(n-1))$, and $(U,K) = (F_4, \text{Spin}(9))$. These cases are treated respectively in the subsections §§2a-2e below. As explained in the Introduction, we obtain complete results for all but the pair $(F_4, \text{Spin}(9))$, for which we obtain partial results, and we omit certain proofs.

A finite dimensional irreducible representation of G or U is said to be of <u>class 1</u> if it contains the trivial one-dimensional representation of H or K, respectively. Our formulas in the subsections below immediately yield the following theorem for rank 1 simple groups, and hence for rank 1 semisimple groups G:

Theorem 2 (cf. Kostant [10], Theorem 6). Let G be a connected rank 1 real semisimple Lie group with finite center, and H a maximal compact subgroup of G. Then the multiplicity with which an arbitrary finite dimensional irreducible representation of H occurs in the restriction to H of an arbitrary class 1 finite dimensional irreducible representation of G is either 0 or 1.

In §§2a-2e, we use the notation $\mathbb{Z}$ for the integers, $\mathbb{Z}_+$ for the non-negative integers, $\mathbb{R}$ for the real numbers, and $\mathbb{C}$ for the complex numbers. §§2a-2e all depend on §1, but they are independent of one another.

§2a The formula for $(SU(n+1), S(U_1 \times U_n))$

Let $n = 1,2,\ldots$ Let $U = SU(n+1)$, $K = S(U_1 \times U_n)$, the set of matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where $A \in U(1)$, $B \in U(n)$ and $\det A \det B = 1$. U is simply connected.

Let T be the set of diagonal matrices in U. T is a maximal torus of U and K. Let $\mathfrak{u}$ be the Lie algebra of U; we identify $\mathfrak{u}$ with the Lie algebra of traceless skew hermitian $(n+1) \times (n+1)$ matrices. Let $\mathfrak{k}_0$ and $\mathfrak{t}$ be the subalgebras of $\mathfrak{u}$ corresponding to K and T, respectively. Then $\mathfrak{h} = \mathfrak{t} + \underline{i}\mathfrak{t}$ is a Cartan subalgebra of the complexification $\mathfrak{g} = \mathfrak{u} + \underline{i}\mathfrak{u}$, and also of $\mathfrak{k} = \mathfrak{k}_o + \underline{i}\mathfrak{k}_o$.

Let X_i $(0 \leq i \leq n)$ be the $(n+1) \times (n+1)$ diagonal matrix which is 1 in the i^{th} diagonal entry and 0 everywhere else. Then $\{X_i\}_{0 \leq i \leq n}$ is a basis of the complex vector space $\hat{\mathfrak{h}}$ of complex diagonal matrices. Let $\{\lambda_i\}_{0 \leq i \leq n}$ be the dual basis. If ν is a linear functional on $\hat{\mathfrak{h}}$, let $\bar{\nu}$ denote its restriction to $\mathfrak{h}$. ($\mathfrak{h}$ is the space of traceless complex diagonal matrices.)

The roots of the complex simple Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$ are $\pm(\bar{\lambda}_i - \bar{\lambda}_j)$ $(0 \leq i < j \leq n)$. The roots of the complex reductive Lie algebra $\mathfrak{k}$ with respect to $\mathfrak{h}$ are $\pm(\bar{\lambda}_i - \bar{\lambda}_j)$ $(1 \leq i < j \leq n)$.

$\{\bar{\lambda}_i - \bar{\lambda}_{i+1}\}_{0 \leq i \leq n-1}$ is a system of simple roots for $\mathfrak{g}$. The corresponding system of positive roots is $\{\bar{\lambda}_i - \bar{\lambda}_j\}_{0 \leq i < j \leq n}$ and the corresponding Weyl chamber in $\underline{i}\mathfrak{t}$ is $\{\sum_{i=0}^{n} a_i X_i \mid a_i \in \mathbb{R},$

$a_o \geq a_1 \geq \cdots \geq a_n, \sum_{i=0}^{n} a_i = 0\}$. The Weyl chamber for $\mathfrak{k}$ in $\underline{i}\mathfrak{t}$ which contains this $\mathfrak{g}$-chamber is $\{\sum_{i=0}^{n} a_i X_i \mid a_i \in \mathbb{R},$

$a_1 \geq a_2 \geq \cdots \geq a_n, \sum_{i=0}^{n} a_i = 0\}$. We take these two chambers to be the dominant chambers for $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The positive system for $\mathfrak{k}$ is $\{\bar{\lambda}_i - \bar{\lambda}_j\}_{1 \leq i < j \leq n}$.

Let α be a root of U or K. The root normal $H_\alpha \in \mathfrak{h}$ is defined as usual as the unique element of $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ (or $[\mathfrak{k}^\alpha, \mathfrak{k}^{-\alpha}]$) such that $\alpha(H_\alpha) = 2$ (where $\mathfrak{g}^{\pm\alpha}$ and $\mathfrak{k}^{\pm\alpha}$ denote the root spaces of $\mathfrak{g}$ and $\mathfrak{k}$ for the roots $\pm\alpha$). The root normals for $\mathfrak{g}$ are: $H_{\pm(\bar\lambda_i - \bar\lambda_j)} = \pm(X_i - X_j)$ $(0 \le i < j \le n)$. The root normals for $\mathfrak{k}$ are: $H_{\pm(\bar\lambda_i - \bar\lambda_j)} = \pm(X_i - X_j)$ $(1 \le i < j \le n)$. Hence the dominant linear forms for $\mathfrak{g}$ are the forms $\sum_{i=0}^{n} a_i\bar\lambda_i$ where $a_i \in \mathbb{R}$ $(0 \le i \le n)$ and $a_0 \ge a_1 \ge \ldots \ge a_n$. The dominant linear forms for $\mathfrak{k}$ are the forms $\sum_{i=0}^{n} a_i\bar\lambda_i$ where $a_i \in \mathbb{R}$ $(0 \le i \le n)$ and $a_1 \ge a_2 \ge \ldots \ge a_n$. The integral linear forms for the simply connected group U, and hence also for K, are the forms $\sum_{i=0}^{n} a_i\bar\lambda_i$ where $a_i \in \mathbb{Z}$ $(0 \le i \le n)$. Let D_U and D_K denote the sets of dominant integral linear forms for U and K, respectively.

(We note that if $a_i, b_i \in \mathbb{C}$ $(0 \le i \le n)$, then $\sum_{i=0}^{n} a_i\bar\lambda_i = \sum_{i=0}^{n} b_i\bar\lambda_i \iff$ there exists a complex constant c such that $a_i = b_i + c$ $(0 \le i \le n)$.)

We can now state:

<u>Theorem 3</u>. Let $\lambda = \sum_{i=0}^{n} a_i\bar\lambda_i \in D_U$ and $\mu = \sum_{i=0}^{n} b_i\bar\lambda_i \in D_K$. We

assume $a_i, b_i \in \mathbb{Z}$ $(0 \leq i \leq n)$. If $\sum_{i=0}^{n} a_i \not\equiv \sum_{i=0}^{n} b_i \pmod{(n+1)}$ then $m_\lambda(\mu) = 0$. If $\sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} b_i \pmod{(n+1)}$, we may add an integral constant to the b_i and assume $\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} b_i$. Then $m_\lambda(\mu) = 1 \Longleftrightarrow$ $a_o \geq b_1 \geq a_1 \geq \ldots \geq b_n \geq a_n$; otherwise, $m_\lambda(\mu) = 0$.

Proof. Let ρ be half the sum of the positive roots of $\mathfrak{g}$. Then $\rho = n\bar{\lambda}_o + (n-1)\bar{\lambda}_1 + \ldots + 2\bar{\lambda}_{n-2} + \bar{\lambda}_{n-1}$.

The Weyl group W of $\mathfrak{g}$ is the group of transformations of the dual of $\mathfrak{h}$ of the form σ_Π: $\sum_{i=0}^{n} c_i\bar{\lambda}_i \to \sum_{i=0}^{n} c_{\Pi(i)}\bar{\lambda}_i$, where Π is an arbitrary permutation of the set $\{0,1,\ldots n\}$; distinct permutations give distinct elements of W. We have that $\det \sigma_\Pi = \operatorname{sign} \Pi$.

The positive weights of the canonical representation of $\mathfrak{k}$ on $\mathfrak{g}/\mathfrak{k}$ are $\bar{\lambda}_o - \bar{\lambda}_i$ $(1 \leq i \leq n)$. Let P be the partition function, and let ν be an integral linear form. Then $P(\nu) = 1 \Longleftrightarrow \nu$ can be expressed in the form $\sum_{i=0}^{n} c_i\bar{\lambda}_i$, where $c_i \in \mathbb{Z}$ $(0 \leq i \leq n)$, $\sum_{i=0}^{n} c_i = 0$ and $c_i \leq 0$ $(1 \leq i \leq n)$; $P(\nu) = 0$ otherwise. According to Kostant's formula, $m_\lambda(\mu) = \sum_{\sigma \in W} (\det \sigma)\ P(\sigma(\lambda+\rho)-(\mu+\rho))$.

Suppose $\sum_{i=0}^{n} a_i \not\equiv \sum_{i=0}^{n} b_i \pmod{n+1}$. Then for every $\sigma \in W$, $\sigma(\lambda+\rho)-(\mu+\rho)$ may be expressed as $\sum_{i=0}^{n} c_i\bar{\lambda}_i$, where $c_i \in \mathbb{Z}$ and $\sum_{i=0}^{n} c_i = \sum_{i=0}^{n} a_i - \sum_{i=0}^{n} b_i$, which is not a multiple of $n+1$. Hence $P(\sigma(\lambda+\rho)-(\mu+\rho)) = 0$ for all $\sigma \in W$, so that $m_\lambda(\mu) = 0$.

Suppose $\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} b_i = d(n+1)$ $(d \in \mathbb{Z})$. Replacing b_i $(0 \leq i \leq n)$ by b_i+d, we may assume $\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} b_i$. Let $a_i' = a_i + (n-i)$, $b_i' = b_i+(n-i)$ $(0 \leq i \leq n)$. Let Π be an arbitrary permutation of $\{0,1,\ldots n\}$. Then $P(\sigma_\Pi(\lambda+\rho)-(\mu+\rho)) = 1$ $\Longleftrightarrow$ $a'_{\Pi(i)} \leq b_i'$ $(1 \leq i \leq n)$.

Assume $a_o \geq b_1 \geq a_1 \geq b_2 \geq \cdots \geq b_n \geq a_n$. Then $a_o' > b_1' \geq a_1' > b_2' \geq \cdots > b_n' \geq a_n'$. $P(\sigma(\lambda+\rho)-(\mu+\rho)) = 0$ unless σ is the identity, in which case the value is 1. Thus $m_\lambda(\mu) = 1$.

Assume $b_j < a_j$ for some $j=1,2,\ldots n$. Then $a_o', a_1', \ldots a_j' > b_j'$. Hence $a_i' \leq b_j'$ for at most $n-j$ values of i $(0 \leq i \leq n)$. Suppose Π is a permutation of $\{0,1,\ldots n\}$ such that $a'_{\Pi(i)} \leq b_i'$ $(1 \leq i \leq n)$. Then $a'_{\Pi(i)} \leq b_j'$ for $i=j,j+1,\ldots n$, so that $a_i' \leq b_j'$ for $n-j+1$ values of i, a contradiction. Hence $P(\sigma(\lambda+\rho)-(\mu+\rho)) = 0$ for all $\sigma \in W$, and so $m_\lambda(\mu) = 0$.

The only remaining case is that in which $b_j > a_{j-1}$ for some $j=1,2,\ldots n$. We assume this. Then $b'_j \geq a'_{j-1}, a'_j, \ldots a'_n$. Let $\mathcal{S}$ be the set of permutations Π of $\{0,1,\ldots n\}$ such that $a'_{\Pi(i)} \leq b'_i$ $(1 \leq i \leq n)$. We define a map $\mathcal{f}$ from $\mathcal{S}$ into the set of permutations of $\{0,1,\ldots n\}$ as follows: Let $\Pi \in \mathcal{S}$. For some i such that $1 \leq i \leq j$, we have $j-1 \leq \Pi(i-1) \leq n$. Then $a'_{\Pi(i-1)} \leq b'_j \leq b'_i$. Let k be the smallest value of i $(1 \leq i \leq n)$ such that $a'_{\Pi(i-1)} \leq b'_i$. Let $\mathcal{f}(\Pi) = \Pi\tau_{k-1,k}$, where $\tau_{k-1,k}$ is the transposition of k-1 and k.

We claim that $\mathcal{f}$ is an involutive bijection of $\mathcal{S}$ onto itself. In fact, to see that $\mathcal{f}$ takes $\mathcal{S}$ into itself, let $\Pi \in \mathcal{S}$ and choose k as above. We must show that $a'_{\Pi\tau_{k-1,k}(i)} \leq b'_i$ $(1 \leq i \leq n)$. This is clear if $i \neq k, k-1$, since $\Pi \in \mathcal{S}$. The remaining possibilities are taken care of by the fact that $a'_{\Pi(k-1)} \leq b'_k$ and that $a'_{\Pi(k)} \leq b'_k \leq b'_{k-1}$ if $k > 1$. It remains to prove that $\mathcal{f}(\mathcal{f}(\Pi)) = \Pi$. But $a'_{\Pi\tau_{k-1,k}(k-1)} = a'_{\Pi(k)} \leq b'_k$, and $a'_{\Pi\tau_{k-1,k}(\ell-1)} = a'_{\Pi(\ell-1)} > b'_\ell$ if $1 \leq \ell < k$. Thus k is the smallest value with the required property, and so $\mathcal{f}(\mathcal{f}(\Pi)) = \Pi\tau_{k-1,k}\tau_{k-1,k} = \Pi$. This proves the claim.

For all $\Pi \in \mathcal{S}$, sign $\mathcal{f}(\Pi) = -$sign Π. Thus

$$m_\lambda(\mu) = \sum_{\Pi \in \mathcal{S}} (\det \sigma_\Pi)\, P(\sigma_\Pi(\lambda+\rho)-(\mu+\rho)) = \sum_{\Pi \in \mathcal{S}} (\text{sign }\Pi) =$$

$= \frac{1}{2} \sum_{\Pi \in \mathcal{S}} (\text{sign }\Pi + \text{sign }\mathcal{f}(\Pi)) = 0$, and the theorem is proved.

§ 2b The formula for (SO(2n+1), SO(2n))

In order to handle the case of the simply connected covering group Spin(2n+1) of SO(2n+1), we deal directly with the Lie algebras.

Let n=1,2,... Let $\mathfrak{u} = \mathfrak{so}(2n+1)$, and let $\mathfrak{k}_0$ be the subalgebra of $\mathfrak{u}$ obtained by taking the first row and column to be zero, so that $\mathfrak{k}_0 = \mathfrak{so}(2n)$. We take U to be the simply connected group corresponding to $\mathfrak{u}$, and K its connected subgroup corresponding to $\mathfrak{k}_0$. Let $\mathfrak{g} = \mathfrak{u} + \underline{i}\,\mathfrak{u} = \mathfrak{so}(2n+1, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{k}_0 + \underline{i}\,\mathfrak{k}_0 =$ $= \mathfrak{so}(2n, \mathbb{C})$.

We shall describe the multiplicity formula for the pair (U,K). Let E_{ij} $(1 \leq i,j \leq 2n+1)$ denote the $(2n+1) \times (2n+1)$ matrix which is 1 in the i,j entry and 0 everywhere else. Let X_j $(1 \leq j \leq n)$ denote the matrix $\underline{i}(E_{2j,2j+1} - E_{2j+1,2j})$. Let $\mathfrak{t}$ be the real span of $\underline{i}X_1, \underline{i}X_2, \ldots \underline{i}X_n$. Then $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{u}$ and $\mathfrak{k}_0$, and $\mathfrak{h} = \mathfrak{t} + \underline{i}\,\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$. Let $\{\lambda_i\}_{1 \leq i \leq n}$ be the basis of the dual of $\mathfrak{h}$ dual to $\{X_i\}_{1 \leq i \leq n}$.

The roots of the complex simple Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$ are $\underline{+}\lambda_i \underline{+}\lambda_j$ $(1 \leq i < j \leq n)$ and $\underline{+}\lambda_i$ $(1 \leq i \leq n)$. The roots of the complex reductive Lie algebra $\mathfrak{k}$ with respect to $\mathfrak{h}$ are $\underline{+}\lambda_i \underline{+}\lambda_j$ $(1 \leq i < j \leq n)$.

$\{\lambda_1-\lambda_2, \lambda_2-\lambda_3, \ldots \lambda_{n-1}-\lambda_n, \lambda_n\}$ is a system of simple roots for $\mathfrak{g}$. The corresponding positive system is $\{\lambda_i \pm \lambda_j$ $(1 \leq i < j \leq n)$, $\lambda_k (1 \leq k \leq n)\}$. The positive system for $\mathfrak{k}$ contained in this positive system for $\mathfrak{g}$ is $\{\lambda_i \pm \lambda_j\ (1 \leq i \leq n)\}$. We take these to be the positive systems for $\mathfrak{g}$ and $\mathfrak{k}$.

The root normals for $\mathfrak{g}$ are: $H_{\pm\lambda_i\pm\lambda_j} = \pm X_i \pm X_j\ (1 \leq i < j \leq n)$, $H_{\pm\lambda_i} = \pm 2X_i\ (1 \leq i \leq n)$; the root normals for $\mathfrak{k}$ are: $H_{\pm\lambda_i\pm\lambda_j} = \pm X_i \pm X_j\ (1 \leq i < j \leq n)$. Hence the dominant linear forms for $\mathfrak{g}$ are the forms $\sum_{i=1}^{n} a_i\lambda_i$ where $a_i \in \mathbb{R}\ (1 \leq i \leq n)$ and $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$; the dominant linear forms for $\mathfrak{k}$ are the forms $\sum_{i=1}^{n} a_i\lambda_i$ where $a_i \in \mathbb{R}\ (1 \leq i \leq n)$ and $a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq |a_n|$. The integral linear forms for U and K are the forms $\sum_{i=1}^{n} a_i\lambda_i$ where either $a_i \in \mathbb{Z}\ (1 \leq i \leq n)$ or $a_i \in \mathbb{Z} + \frac{1}{2}\ (1 \leq i \leq n)$.

Let D_U and D_K be the sets of dominant integral linear forms for U and K, respectively. The multiplicity formula states:

<u>Theorem 4</u>. Let $\lambda = \sum_{i=1}^{n} a_i\lambda_i \in D_U$ and $\mu = \sum_{i=1}^{n} b_i\lambda_i \in D_K$. If $a_i - b_j \notin \mathbb{Z}$ for some $i,j=1,\ldots n$, or, equivalently, for all i,j, then $m_\lambda(\mu) = 0$. If $a_i - b_j \in \mathbb{Z}$, then $m_\lambda(\mu) = 1 \Longleftrightarrow a_1 \geq b_1 \geq a_2 \geq b_2 \geq \ldots \geq a_n \geq |b_n|$; otherwise, $m_\lambda(\mu) = 0$.

The proof is extremely similar to that for the pair $(SU(n+1), S(U_1 \times U_n))$ (see § 2a), except for the slight complication that the Weyl group involves sign changes as well as permutations of coefficients. We omit the proof.

§2c The formula for (SO(2n), SO(2n-1))

As in § 2b, we deal directly with the Lie algebras, in order to handle the simply connected covering group Spin(2n) of SO(2n).

Let n=2,3,... Let $\mathfrak{u} = \mathfrak{so}(2n)$, and let $\mathfrak{k}_o$ be the subalgebra of $\mathfrak{u}$ obtained by taking the first row and column to be zero, so that $\mathfrak{k} = \mathfrak{so}(2n-1)$. Let U be the simply connected group corresponding to $\mathfrak{u}$, and K its connected subgroup corresponding to $\mathfrak{k}_o$. Let $\mathfrak{g} = \mathfrak{u} + \underline{i}\mathfrak{u} = \mathfrak{so}(2n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{k}_o + \underline{i}\mathfrak{k}_o = \mathfrak{so}(2n-1, \mathbb{C})$.

We shall describe the multiplicity formula for the pair (U,K). Let E_{ij} $(1 \leq i,j \leq 2n)$ denote the $2n \times 2n$ matrix which is 1 in the i,j entry and 0 everywhere else. Let X_j $(1 \leq j \leq n-1)$ be the matrix $\underline{i}(E_{2j+1,2j+2} - E_{2j+2,2j+1})$, and let X_n denote the matrix $\underline{i}(E_{12}-E_{21})$. Let $\mathfrak{t}$ be the real span of $\underline{i}X_1, \underline{i}X_2, \ldots \underline{i}X_n$, and let $\mathfrak{s}$ be the real span of $\underline{i}X_1, \underline{i}X_2, \ldots \underline{i}X_{n-1}$. Then $\mathfrak{t}$ and $\mathfrak{s}$ are maximal abelian subalgebras of $\mathfrak{u}$ and $\mathfrak{k}_o$, respectively. Also, $\mathfrak{h} = \mathfrak{t} + \underline{i}\mathfrak{t}$ and $\mathfrak{h}^* = \mathfrak{s} + \underline{i}\mathfrak{s}$ are Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Let $\{\lambda_i\}_{1 \leq i \leq n}$ be the basis of the dual of $\mathfrak{h}$ dual to $\{X_i\}_{1 \leq i \leq n}$. If ν is a linear form on $\mathfrak{h}$, let ν^* denote its restriction to $\mathfrak{h}^*$.

The roots of the complex semisimple Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$ are $\pm\lambda_i \pm \lambda_j$ $(1 \leq i < j \leq n)$. The roots of the complex simple

Lie algebra $\mathfrak{k}$ with respect to $\mathfrak{h}^*$ are $\pm\lambda_i^*\pm\lambda_j^*$ $(1 \leq i < j \leq n-1)$, $\pm\lambda_i^*$ $(1 \leq i \leq n-1)$.

$\{\lambda_1-\lambda_2,\ldots\lambda_{n-1}-\lambda_n,\lambda_{n-1}+\lambda_n\}$ is a simple system for $\mathfrak{g}$. The corresponding positive system is $\{\lambda_i \pm \lambda_j\}_{1 \leq i < j \leq n}$. The corresponding Weyl chamber in $\underline{i}\mathfrak{t}$ is $\{\sum_{i=1}^{n} a_i X_i \mid a_i \in \mathbb{R}, a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq |a_n|\}$. The interior of this chamber intersects $\underline{i}\mathfrak{s}$ in the set $\{\sum_{i=2}^{n} a_i X_i \mid a_i \in \mathbb{R}, a_1 > a_2 > \ldots > a_{n-1} > 0\}$, the interior of a Weyl chamber in $\underline{i}\mathfrak{s}$. The corresponding positive system for $\mathfrak{k}$ is $\{\lambda_i^* \pm\lambda_j^*$ $(1 \leq i < j \leq n-1)$, λ_k^* $(1 \leq k \leq n-1)\}$. We take the above systems to be the positive systems for $\mathfrak{g}$ and $\mathfrak{k}$.

The root normals for $\mathfrak{g}$ are: $H_{\pm\lambda_i\pm\lambda_j} = \pm X_i \pm X_j$ $(1 \leq i < j \leq n)$; the root normals for $\mathfrak{k}$ are $H_{\pm\lambda_i^* \pm\lambda_j^*} = \pm X_i \pm X_j$ $(1 \leq i < j \leq n-1)$, $H_{\pm\lambda_i^*} = \pm 2X_i$ $(1 \leq i \leq n-1)$. Hence the dominant linear forms for $\mathfrak{g}$ are: $\sum_{i=1}^{n} a_i\lambda_i$ where $a_i \in \mathbb{R}$ $(1 \leq i \leq n)$ and $a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq |a_n|$; the dominant linear forms for $\mathfrak{k}$ are: $\sum_{i=1}^{n-1} a_i\lambda_i^*$ where $a_i \in \mathbb{R}$ $(1 \leq i \leq n-1)$ and $a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq 0$. The integral linear forms for U are: $\sum_{i=1}^{n} a_i\lambda_i$ where either $a_i \in \mathbb{Z}$ $(1 \leq i \leq n)$ or

$a_i \in \mathbb{Z} + \frac{1}{2}$ $(1 \leq i \leq n)$; the integral linear forms for K are:

$\sum_{i=1}^{n-1} a_i\lambda_i^*$ where either $a_i \in \mathbb{Z}$ $(1 \leq i \leq n-1)$ or $a_i \in \mathbb{Z} + \frac{1}{2}$ $(1 \leq i \leq n-1)$.

Let D_U and D_K denote the sets of dominant integral linear forms for U and K, respectively. We have:

Theorem 5. Let $\lambda = \sum_{i=1}^{n} a_i\lambda_i \in D_U$, $\mu = \sum_{i=1}^{n-1} b_i\lambda_i^* \in D_K$. If $a_i - b_j \notin \mathbb{Z}$, then $m_\lambda(\mu) = 0$. If $a_i - b_j \in \mathbb{Z}$, then $m_\lambda(\mu) = 1 \Longleftrightarrow$ $a_1 \geq b_1 \geq a_2 \geq b_2 \geq \ldots \geq a_{n-1} \geq b_{n-1} \geq |a_n|$; otherwise, $m_\lambda(\mu) = 0$.

As in § 2b, the proof of this theorem is extremely similar to that for the pair $(SU(n+1), S(U_1 \times U_n))$, except that in this case, the Weyl group involves even numbers of sign changes as well as permutations of coefficients. We omit the proof.

§ 2d The formula for $(Sp(n), Sp(1) \times Sp(n-1))$

Let n=2,3,... Let U be Sp(n), the group of unitary matrices in $GL(2n,\mathbb{C})$ which leave invariant the exterior form $z_1 \wedge z_{n+1} + z_2 \wedge z_{n+2} + \ldots + z_n \wedge z_{2n}$ ($(z_1,\ldots z_{2n})$ is a variable point in $\mathbb{C}^{2n}$). U is simply connected. Let K be the subgroup of U consisting of those matrices $(a_{ij}) \in Sp(n)$ which are 0 in the 1^{st} and $(n+1)^{st}$ row and column, except perhaps for the entries a_{11}, $a_{1,n+1}$, $a_{n+1,1}$, $a_{n+1,n+1}$. Then $K = Sp(1) \times Sp(n-1)$.

Let T be the subgroup of U consisting of the diagonal matrices in U. T is a maximal torus for both U and K. We identify the Lie algebra $\mathfrak{u}$ of U with $\mathfrak{sp}(n)$, the Lie algebra of complex $2n \times 2n$ matrices of the form $\begin{pmatrix} C & D \\ -\bar{D} & \bar{C} \end{pmatrix}$ where $C = (c_{ij})$ is an $n \times n$ skew-hermitian matrix and $D = (d_{ij})$ is an $n \times n$ complex symmetric matrix. The Lie subalgebra $\mathfrak{k}_o$ of $\mathfrak{u}$ corresponding to K is the set of matrices in $\mathfrak{u}$ such that $c_{12} = c_{13} = \dots = c_{1n} = d_{12} = d_{13} = \dots = d_{1n} = 0$. $\mathfrak{t}$, the Lie subalgebra of $\mathfrak{u}$ corresponding to T, is the set of diagonal matrices in $\mathfrak{u}$. We have that $\mathfrak{h} = \mathfrak{t} + \underline{i}\,\mathfrak{t}$ is a Cartan subalgebra of the complexification $\mathfrak{g} = \mathfrak{u} + \underline{i}\,\mathfrak{u}$ of $\mathfrak{u}$, and also of $\mathfrak{k} = \mathfrak{k}_o + \underline{i}\,\mathfrak{k}_o$.

Let X_i $(1 \leq i \leq n)$ be the $2n \times 2n$ diagonal matrix $(a_{k\ell})$ such that $a_{kk} = 1$ for $k=i$, $a_{kk} = -1$ for $k = n+i$, and $a_{kk} = 0$ for all other k. $\{X_i\}_{1 \leq i \leq n}$ is a basis of the real space $\underline{i}\,\mathfrak{t}$, and hence of the complex space $\mathfrak{h}$. Let $\{\lambda_i\}_{1 \leq i \leq n}$ be the basis of the dual of $\mathfrak{h}$ dual to $\{X_i\}_{1 \leq i \leq n}$.

The roots of the complex simple Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$ are $\pm\lambda_i \pm\lambda_j$ $(1 \leq i < j \leq n)$ and $\pm 2\lambda_i$ $(1 \leq i \leq n)$. The roots of the complex semisimple Lie algebra $\mathfrak{k}$ with respect to $\mathfrak{h}$ are $\pm\lambda_i \pm\lambda_j$ $(2 \leq i < j \leq n)$ and $\pm 2\lambda_i$ $(1 \leq i \leq n)$.

$\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots \lambda_{n-1} - \lambda_n, 2\lambda_n\}$ is a simple system for $\mathfrak{g}$. The corresponding positive system is $\{\lambda_i \pm\lambda_j$ $(1 \leq i < j \leq n)$,

$2\lambda_k$ $(1 \leq k \leq n)\}$, and the corresponding Weyl chamber in $\underline{i}\mathfrak{t}$ is $\{\sum_{i=1}^{n} a_i X_i \mid a_i \in \mathbb{R},\ a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}$. The $\mathfrak{k}$-chamber in $\underline{i}\mathfrak{t}$ which contains this $\mathfrak{g}$-chamber is $\{\sum_{i=1}^{n} a_i X_i \mid a_i \in \mathbb{R},\ a_1 \geq 0,\ a_2 \geq a_3 \geq \cdots \geq a_n \geq 0\}$ We take these chambers to be the dominant chambers for $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The positive system for $\mathfrak{k}$ is $\{\lambda_i \pm \lambda_j\ (2 \leq i < j \leq n),\ 2\lambda_k\ (1 \leq k \leq n)\}$.

The root normals for $\mathfrak{g}$ are: $H_{\pm\lambda_i\pm\lambda_j} = \pm X_i \pm X_j$ $(1 \leq i < j \leq n)$, $H_{\pm 2\lambda_i} = \pm X_i$ $(1 \leq i \leq n)$. The root normals for $\mathfrak{k}$ are: $H_{\pm\lambda_i\pm\lambda_j} = \pm X_i \pm X_j$ $(2 \leq i < j \leq n)$, $H_{\pm 2\lambda_i} = \pm X_i$ $(1 \leq i \leq n)$. Thus the dominant linear forms for $\mathfrak{g}$ are: $\sum_{i=1}^{n} a_i \lambda_i$ where $a_i \in \mathbb{R}$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$; the dominant linear forms for $\mathfrak{k}$ are: $\sum_{i=1}^{n} a_i \lambda_i$ where $a_i \in \mathbb{R}$ and $a_1 \geq 0$, $a_2 \geq a_3 \geq \cdots \geq a_n \geq 0$. The integral linear forms for U, and also for K, are: $\sum_{i=1}^{n} a_i \lambda_i$ where $a_i \in \mathbb{Z}$ $(1 \leq i \leq n)$.

Let D_U and D_K denote the sets of dominant integral linear forms for U and K, respectively.

Before stating the multiplicity formula, we discuss the combinatorial function F_k defined as follows:

<u>Definition 1</u>. Let $k,s \in \mathbb{Z}$, $k \geq 1$, and let $t_1,t_2,\ldots t_k \in \mathbb{Z}_+$. Define $F_k(s;t_1,t_2,\ldots t_k)$ to be the number of ways of putting s indistinguishable balls into k distinguishable boxes with capacities $t_1,t_2,\ldots t_k$, respectively. (We note that s can be less than 0.)

We prove two simple lemmas about the function F_k:

<u>Lemma 1</u>. $F_k(s;t_1,t_2,\ldots t_k) =$

$$= \sum_{L \subset \{1,2,\ldots k\}} (-1)^{|L|} \binom{k-1-|L|+s - \sum_{i \in L} t_i}{k-1}$$, where $|L|$ denotes

the number of elements in $|L|$, and $\binom{x}{y}$ denotes the binomial coefficient, which is defined to be 0 if $x-y \notin \mathbb{Z}_+$.

<u>Proof</u>. The number of ways of putting s indistinguishable balls into k distinguishable boxes of infinite capacity is $\binom{k-1+s}{k-1}$. Let $L \subset \{1,2,\ldots k\}$. The number of ways of putting s indistinguishable balls into k distinguishable boxes of infinite capacity in such a way that for all $i \in L$, the i^{th} box has at least t_i+1 balls, is $\binom{k-1-|L|+s - \sum_{i \in L} t_i}{k-1}$. The lemma follows easily.

<u>Lemma 2</u>. $F_k(s;t_1,t_2,\ldots t_k) = F_k((\sum_{i=1}^{k} t_i)-s;\ t_1,t_2,\ldots t_k)$.

<u>Proof</u>. $F_k(s;t_1,t_2,\ldots t_k)$ is clearly equal to the number of ways of putting s indistinguishable white balls <u>and</u> $(\sum_{i=1}^{k} t_i)-s$ indistinguishable black balls into k distinguishable boxes with capacities $t_1,t_2,\ldots t_k$, respectively. The lemma follows easily.

Kostant's formula is expressed as a sum over the Weyl group of $\mathfrak{g}$, which has order $2^n n!$ in the present case, as we shall see below. By means of a series of combinatorial arguments, we shall reduce this sum to a sum over a 2^n-element set, the set of subsets of $\{1,2,\ldots n\}$ (see formula (8) below). We split this sum into the sum over those subsets which do not contain 1 and the sum over those subsets which do contain 1. Applying Lemmas 1 and 2 to the result, we immediately obtain the interesting expressions (9) and (10) below for the multiplicities. The multiplicity theorem states:

<u>Theorem 6</u>. Let $\lambda = \sum_{i=1}^{n} a_i\lambda_i \in D_U$ and let $\mu = \sum_{i=1}^{n} b_i\lambda_i \in D_K$. Define

$$
\begin{aligned}
A_1 &= a_1 - \max(a_2,b_2)\\
A_2 &= \min(a_2,b_2) - \max(a_3,b_3)\\
A_3 &= \min(a_3,b_3) - \max(a_4,b_4)\\
&\vdots\\
A_{n-1} &= \min(a_{n-1},b_{n-1}) - \max(a_n,b_n)\\
A_n &= \min(a_n,b_n).
\end{aligned}
$$

Then $m_\lambda(\mu) = 0$ unless $b_1 + \sum_{i=1}^{n} A_i \in 2\mathbb{Z}$ (that is, $\sum_{i=1}^{n} (a_i+b_i) \in 2\mathbb{Z}$) and $A_1,A_2,\ldots A_{n-1} \geq 0$ ($A_n \geq 0$ automatically). If these conditions hold, then $m_\lambda(\mu) =$

$$(8)\quad = \sum_{L \subset \{1,2,\ldots n\}} (-1)^{|L|} \binom{n-2-|L| + \frac{1}{2}(-b_1 + \sum_{i=1}^{n} A_i) - \sum_{i \in L} A_i}{n-2} =$$

$$(9)\quad = F_{n-1}(\tfrac{1}{2}(-b_1 + \sum_{i=1}^{n} A_i);\ A_2, A_3, \ldots A_n) -$$

$$- F_{n-1}(\tfrac{1}{2}(-b_1 + \sum_{i=1}^{n} A_i) - (A_1+1);\ A_2, A_3, \ldots A_n) =$$

$$(10)\quad = F_{n-1}(\tfrac{1}{2}(b_1 - A_1 + \sum_{i=2}^{n} A_i);\ A_2, A_3, \ldots A_n) -$$

$$- F_{n-1}(\tfrac{1}{2}(-b_1 - A_1 + \sum_{i=2}^{n} A_i) - 1;\ A_2, A_3, \ldots A_n).$$

The following corollary follows easily from (10):

Corollary. Let $\lambda = \sum_{i=1}^{n} a_i \lambda_i \in D_U$, $\mu = \sum_{i=1}^{n} b_i \lambda_i \in D_K$. Then λ is the highest weight of a class 1 finite dimensional irreducible representation of $U \Longleftrightarrow a_1 - a_2 = a_3 = a_4 = \ldots = a_n = 0$. In this case, $m_\lambda(\mu) = 1 \Longleftrightarrow b_1 - (b_2 - b_3) = b_4 = b_5 = \ldots = b_n = 0$ and $b_2 \leq a_2$ (if $n=2$, the condition is $b_1 = b_2 \leq a_2$); otherwise, $m_\lambda(\mu) = 0$.

Remark. Suppose $\lambda \in D_U$ is the highest weight of a class 1 finite dimensional irreducible representation of U, so that λ has the form indicated in the corollary. In view of (10), the fact that $m_\lambda(\mu) \leq 1$ for all $\mu \in D_K$ is a consequence of the intuitive fact that the number of ways of putting finitely many balls into one box of finite capacity is ≤ 1.

Proof of Theorem 6. By the above remarks, it suffices to prove formula (8). We shall apply Kostant's formula to the pair (U,K).

Let ρ be half the sum of the positive roots of $\mathfrak{g}$. Then $\rho = n\lambda_1 + (n-1)\lambda_2 + \ldots + 2\lambda_{n-1} + \lambda_n$.

We shall now describe the partition function P by means of three lemmas. The positive weights of the canonical representation of $\mathfrak{k}$ on $\mathfrak{g}/\mathfrak{k}$ are $\lambda_1 \pm \lambda_i$ $(2 \leq i \leq n)$. We have:

Lemma 3. Let $\xi = \sum_{i=1}^{n} x_i\lambda_i$ be an integral linear form. Then $P(\xi) = P((x_1 - \sum_{i=2}^{n} |x_i|)\lambda_1)$.

Proof. For each i $(2 \leq i \leq n)$, let $x_i^+ = \max(x_i,0)$ and let $x_i^- = \max(-x_i,0)$. Then

$$\xi = \sum_{i=2}^{n} x_i^+(\lambda_1+\lambda_i) + \sum_{i=2}^{n} x_i^-(\lambda_1-\lambda_i) + (x_1 - \sum_{i=2}^{n} |x_i|)\lambda_1 .$$

For every integral linear form ν, let $S(\nu)$ be the set of non-negative integral 2(n-1)-tuples $(y_2,y_3,\ldots y_n,\ z_2,z_3,\ldots z_n)$ such that $\nu = \sum_{i=2}^{n} y_i(\lambda_1+\lambda_i) + \sum_{i=2}^{n} z_i(\lambda_1-\lambda_i)$. We define a map from $S((x_1 - \sum_{i=2}^{n} |x_i|)\lambda_1)$ to $S(\xi)$ as follows: $(y_2,\ldots y_n,z_2,\ldots z_n) \to (y_2+x_2^+,\ldots y_n+x_n^+,\ z_2+x_2^-,\ldots z_n+x_n^-)$. This map is a bijection. Indeed, if $(d_2,\dots d_n, e_2,\dots e_n) \in S(\xi)$, then $d_2 \geq x_2^+,\dots d_n \geq x_n^+$, $e_2 \geq x_2^-,\dots e_n \geq x_n^-$, so that $(d_2-x_2^+,\dots d_n-x_n^+, e_2-x_2^-,\dots e_n-x_n^-) \in S((x_1 - \sum_{i=2}^{n} |x_i|)\lambda_1)$. This proves the lemma.

Lemma 4. Let $a \in \mathbb{Z}$. Then $P(a\lambda_1) = \binom{n-2+\frac{a}{2}}{n-2}$. (We recall that this binomial coefficient is 0 if $\frac{a}{2} \notin \mathbb{Z}_+$.)

Proof. Clearly $P(a\lambda_1) = 0$ unless $a \in 2\mathbb{Z}_+$, so we assume $a \in 2\mathbb{Z}_+$. The elements of $S(a\lambda_1)$ (S is defined as in the proof of the last lemma) are exactly the $2(n-1)$-tuples $(y_2,\dots y_n, z_2,\dots z_n)$ such that $y_i = z_i \in \mathbb{Z}_+$ $(1 \leq i \leq n)$ and $\sum_{i=2}^{n} y_i = \frac{a}{2}$. Thus $P(a\lambda_1)$ is the number of ways of inserting $\frac{a}{2}$ indistinguishable balls into n-1 distinguishable boxes of infinite capacity. Hence $P(a\lambda_1) =$

$$= \binom{n-2+\frac{a}{2}}{n-2}.$$

Thus we have:

Lemma 5. Let $\xi = \sum_{i=1}^{n} x_i\lambda_i$ be an integral linear form. Define

$$\ell(\xi) = \frac{x_1 - \sum_{i=2}^{n} |x_i|}{2}.$$ Then $P(\xi) = \binom{n-2+\ell(\xi)}{n-2}$. (In particular, $P(\xi) = 0$ unless $\ell(\xi) \in \mathbb{Z}_+$.)

The Weyl group W of $\mathfrak{g}$, regarded as a group of linear transformations of the dual of $\mathfrak{h}$, is the semidirect product of a normal subgroup $\mathcal{U}$ and a subgroup $\mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are described as follows: $\mathcal{U}$ is the 2^n-element group of transformations u:

$$\sum_{i=1}^{n} x_i\lambda_i \to \sum_{i=1}^{n} (-1)^{\varepsilon_i} x_i\lambda_i, \text{ where } \varepsilon_i = \varepsilon_i(u) = 0 \text{ or } 1 \ (1 \leq i \leq n);$$

the $\varepsilon_i(u)$ are uniquely determined by u. $\mathcal{V}$ is the n!-element group of transformations v: $\sum_{i=1}^{n} x_i\lambda_i \to \sum_{i=1}^{n} x_{\Pi(i)}\lambda_i$, where $\Pi = \Pi(v)$ is a permutation of the set $\{1,2,\ldots n\}$; $\Pi(v)$ is uniquely determined by v.

Every $\sigma \in W$ can be uniquely written $\sigma = u_\sigma v_\sigma = v_\sigma u'_\sigma$ where $u_\sigma, u'_\sigma \in \mathcal{U}$ and $v_\sigma \in \mathcal{V}$. We define $\varepsilon_i(\sigma) = \varepsilon_i(u_\sigma)$ $(1 \leq i \leq n)$, $\varepsilon'_i(\sigma) = \varepsilon_i(u'_\sigma)$ $(1 \leq i \leq n)$ and $\Pi(\sigma) = \Pi(v_\sigma)$. Then det σ =

$$= (-1)^{\sum_{i=1}^{n} \varepsilon_i(\sigma)} \text{ sign } \Pi(\sigma) = (-1)^{\sum_{i=1}^{n} \varepsilon'_i(\sigma)} \text{ sign } \Pi(\sigma).$$

We note that if $\sigma, \sigma' \in W$, then $\Pi(\sigma\sigma') = \Pi(\sigma')\Pi(\sigma)$. For all i,j such that $1 \leq i,j \leq n$, we define $v_{i,j} \in \mathcal{V}$ by the condition that $\Pi(v_{i,j})$ is the transposition $\Pi_{i,j}$ of i and j, which is defined to be the identity if i=j. For all i=1,2,...n, we define $u_i \in \mathcal{U}$ by the condition that $\varepsilon_j(u_i) = \delta_{ij}$ $(1 \leq j \leq n)$. Hence $v_{i,j}$ is the Weyl group element which transposes the i^{th} and j^{th} coordinates, and u_i is the Weyl group element which changes the sign of the i^{th} coordinate.

Now let $a'_i = a_i + (n-i+1)$, and $b'_i = b_i + (n-i+1)$ $(1 \leq i \leq n)$, so that a'_i and b'_i are the i^{th} components of $\lambda+\rho$ and $\mu+\rho$, respectively. For all $\sigma \in W$, let $\nu_\sigma = \sigma(\lambda+\rho)-(\mu+\rho)$. We define $\ell_\sigma = \ell(\nu_\sigma)$ and

$P_\sigma = P(\nu_\sigma)$. Now

$$(11) \qquad \ell_\sigma = \frac{1}{2}((\sigma(\lambda+\rho)_1 - b_1') - \sum_{i=2}^{n} |\sigma(\lambda+\rho)_i - b_i'|) ,$$

where $\sigma(\lambda+\rho)_i$ is the i^{th} component of $\sigma(\lambda+\rho)$ $(1 \leq i \leq n)$. Since $\sigma = u_\sigma v_\sigma = v_\sigma u_\sigma'$, we have

$$(12) \qquad \sigma(\lambda+\rho)_i = (-1)^{\varepsilon_i(\sigma)} a'_{\Pi(\sigma)(i)} \quad (1 \leq i \leq n)$$

and

$$(12') \qquad \sigma(\lambda+\rho)_i = (-1)^{\varepsilon'_{\Pi(\sigma)(i)}(\sigma)} a'_{\Pi(\sigma)(i)} \quad (1 \leq i \leq n).$$

Also, $P_\sigma = \binom{n-2+\ell_\sigma}{n-2}$ by Lemma 5, and $m_\lambda(\mu) = \sum_{\sigma \in W} (\det \sigma) P_\sigma$ by Kostant's formula. We shall use the following fact several times: If $\sigma, \sigma' \in W$, $\ell_\sigma = \ell_{\sigma'}$, and $\det \sigma = -\det \sigma'$, then $(\det \sigma)P_\sigma +$ $+ (\det \sigma')P_{\sigma'} = 0$, so that σ and σ' can be canceled out of the formula for $m_\lambda(\mu)$. In order to check that $\ell_\sigma = \ell_{\sigma'}$, we shall use formula (11) and either formula (12) or formula (12') above.

We now prove two lemmas which state that $m_\lambda(\mu) = 0$ under the conditions listed in the statement of the theorem.

<u>Lemma 6</u>. $m_\lambda(\mu) = 0$ unless $b_1 + \sum_{i=1}^{n} A_i \in 2\mathbb{Z}$ (that is, unless $\sum_{i=1}^{n} (a_i+b_i) \in 2\mathbb{Z}$).

<u>Proof</u>. If $\sum(a_i+b_i) \notin 2\mathbb{Z}$, then by (11) and (12) we have that $\ell_\sigma \in \mathbb{Z} + \frac{1}{2}$ for all $\sigma \in W$, and so $P_\sigma = 0$. This shows that $m_\lambda(\mu) = 0$.

<u>Lemma 7</u>. $m_\lambda(\mu) = 0$ unless $A_1, A_2, \ldots A_{n-1} \geq 0$ (that is, unless $a_i \geq b_{i+1}$ $(1 \leq i \leq n-1)$ and $b_i \geq a_{i+1}$ $(2 \leq i \leq n-1)$).

<u>Proof</u>. Suppose $a_j < b_{j+1}$ for some j $(1 \leq j \leq n-1)$. Then

$$b'_2, b'_3, \ldots b'_{j+1} \geq a'_j, a'_{j+1}, \ldots a'_n. \tag{13}$$

For all $\sigma \in W$, at least two elements of the set $\{\Pi(\sigma)(1), \Pi(\sigma)(2), \ldots \Pi(\sigma)(j+1)\}$ lie in the set $\{j, j+1, \ldots n\}$. Let $k(\sigma)$ and $\ell(\sigma)$ denote the smallest two distinct numbers such that $1 \leq k(\sigma), \ell(\sigma) \leq j+1$ and $j \leq \Pi(\sigma)(k(\sigma)), \Pi(\sigma)(\ell(\sigma)) \leq n$. We define a function $\mathfrak{f}: W \to W$ as follows: $\mathfrak{f}(\sigma) = v_{k(\sigma),\ell(\sigma)}\sigma$. $\mathfrak{f}$ is an involution since $\Pi(\mathfrak{f}(\sigma)) = \Pi(\sigma)\Pi_{k(\sigma),\ell(\sigma)}$.

Also, $\ell_{\mathfrak{f}(\sigma)} = \ell_\sigma$. To see this, we show that expression (11) has the same value for ℓ_σ and $\ell_{\mathfrak{f}(\sigma)}$. Now $\varepsilon'_i(\mathfrak{f}(\sigma)) = \varepsilon'_i(\sigma)$ $(1 \leq i \leq n)$. Hence by (12') we have

$$f(\sigma)(\lambda+\rho)_i = (-1)^{\varepsilon'_{\Pi(\mathfrak{f}(\sigma))(i)}(\mathfrak{f}(\sigma))} a'_{\Pi(\mathfrak{f}(\sigma))(i)} =$$

$$= (-1)^{\varepsilon'_{\Pi(\sigma)\Pi_{k(\sigma),\ell(\sigma)}(i)}(\sigma)} a'_{\Pi(\sigma)\Pi_{k(\sigma),\ell(\sigma)}(i)} \tag{14}$$

for all i $(1 \leq i \leq n)$. If $i \neq k(\sigma), \ell(\sigma)$, (14) becomes

$(-1)^{\varepsilon'_{\Pi(\sigma)(i)}(\sigma)}$ $a'_{\Pi(\sigma)(i)}$, which is $\sigma(\lambda+\rho)_i$, by (12'). Similarly, for $i=k(\sigma)$ and $\ell(\sigma)$, we get

$$\mathcal{f}(\sigma)(\lambda+\rho)_{k(\sigma)} = \sigma(\lambda+\rho)_{\ell(\sigma)} \quad \text{and}$$

$$\mathcal{f}(\sigma)(\lambda+\rho)_{\ell(\sigma)} = \sigma(\lambda+\rho)_{k(\sigma)}.$$

Hence the expressions (11) for ℓ_σ and $\ell_{\mathcal{f}(\sigma)}$ differ only by the transposition of the terms $\sigma(\lambda+\rho)_{k(\sigma)}$ and $\sigma(\lambda+\rho)_{\ell(\sigma)}$. Thus in order to show that $\ell_{\mathcal{f}(\sigma)} = \ell_\sigma$, it suffices to show that (11) has the same value when $\sigma(\lambda+\rho)_{k(\sigma)}$ and $\sigma(\lambda+\rho)_{\ell(\sigma)}$ are permuted. What has to be checked is that the absolute value in (11) behaves correctly. Since $k(\sigma)$ or $\ell(\sigma)$ can equal 1, it suffices to show that $\sigma(\lambda+\rho)_{k(\sigma)}$ and $\sigma(\lambda+\rho)_{\ell(\sigma)}$ are both equal to or less than each of the quantities $b'_{k(\sigma)}$ and $b'_{\ell(\sigma)}$ whose subscript is greater than 1. By the definition of $k(\sigma)$ and $\ell(\sigma)$, it suffices to show that $\sigma(\lambda+\rho)_{k(\sigma)}$, $\sigma(\lambda+\rho)_{\ell(\sigma)} \leq b'_2, b'_3, \ldots b'_{j+1}$. But we have

$$\sigma(\lambda+\rho)_{k(\sigma)} \leq a'_{\Pi(\sigma)(k(\sigma))} \quad \text{(by (12) or (12'))}$$

$$\leq b'_2, b'_3, \ldots b'_{j+1}$$

by the definition of $k(\sigma)$ and by (13). The corresponding inequality holds for $\ell(\sigma)$. This shows that $\ell_{\mathcal{f}(\sigma)} = \ell_\sigma$.

Since det $\mathfrak{f}(\sigma) = -\det \sigma$ for all $\sigma \in W$, we have that $m_\lambda(\mu) = 0$ if $a_j < b_{j+1}$ for some j $(1 \leq j \leq n-1)$. This completes the proof of the first half of the lemma.

The situation is similar if $b_j < a_{j+1}$ for some j $(2 \leq j \leq n-1)$. In this case,

$$a'_1, a'_2, \ldots a'_{j+1} \geq b'_j, b'_{j+1}, \ldots b'_n. \tag{15}$$

For all $\sigma \in W$, at least two elements of the set $\{\Pi(\sigma)(j), \Pi(\sigma)(j+1), \ldots \ldots \Pi(\sigma)n\}$ lie in the set $\{1,2,\ldots j+1\}$. Let $k'(\sigma)$ and $\ell'(\sigma)$ denote the largest two distinct numbers such that $j \leq k'(\sigma), \ell'(\sigma) \leq n$ and $1 \leq \Pi(\sigma)(k'(\sigma)),\ \Pi(\sigma)(\ell'(\sigma)) \leq j+1$. We define a function $\mathfrak{f}'$: $W \to W$ as follows: $\mathfrak{f}'(\sigma) = u_\sigma v_{k'(\sigma),\ell'(\sigma)} v_\sigma$. $\mathfrak{f}'$ is an involution since $\Pi(\mathfrak{f}'(\sigma)) = \Pi(\sigma)\Pi_{k'(\sigma),\ell'(\sigma)}$.

Now $\ell_{\mathfrak{f}'(\sigma)} = \ell_\sigma$. To see this, we show that expression (11) has the same value for ℓ_σ and $\ell_{\mathfrak{f}'(\sigma)}$. Now $\varepsilon_i(\mathfrak{f}'(\sigma)) = \varepsilon_i(\sigma)$ $(1 \leq i \leq n)$. Exactly as above, we have by (12) that the expressions (11) for ℓ_σ and $\ell_{\mathfrak{f}'(\sigma)}$ differ only by the substitution of $\mathfrak{f}'(\sigma)(\lambda+\rho)_{k'(\sigma)}$ for $\sigma(\lambda+\rho)_{k'(\sigma)}$, and of $\mathfrak{f}'(\sigma)(\lambda+\rho)_{\ell'(\sigma)}$ for $\sigma(\lambda+\rho)_{\ell'(\sigma)}$. Hence it suffices to show that

$$|\sigma(\lambda+\rho)_{k'(\sigma)} - b'_{k'(\sigma)}| + |\sigma(\lambda+\rho)_{\ell'(\sigma)} - b'_{\ell'(\sigma)}| =$$

$$= |\mathfrak{f}'(\sigma)(\lambda+\rho)_{k'(\sigma)} - b'_{k'(\sigma)}| + |\mathfrak{f}'(\sigma)(\lambda+\rho)_{\ell'(\sigma)} - b'_{\ell'(\sigma)}|.$$

By (12), this is equivalent to:

$$|(-1)^{\varepsilon_{k'(\sigma)}(\sigma)} a'_{\Pi(\sigma)(k'(\sigma))} - b'_{k'(\sigma)}| + |(-1)^{\varepsilon_{\ell'(\sigma)}(\sigma)} a'_{\Pi(\sigma)(\ell'(\sigma))} - b'_{\ell'(\sigma)}| =$$

$$= |(-1)^{\varepsilon_{k'(\sigma)}(\sigma)} a'_{\Pi(\sigma)(\ell'(\sigma))} - b'_{k'(\sigma)}| + |(-1)^{\varepsilon_{\ell'(\sigma)}(\sigma)} a'_{\Pi(\sigma)(k'(\sigma))} - b'_{\ell'(\sigma)}|.$$

But this follows immediately from (15), the definition of $k'(\sigma)$ and $\ell'(\sigma)$, and a trivial check of the four possibilities for the pair $(\varepsilon_{k'(\sigma)}(\sigma), \varepsilon_{\ell'(\sigma)}(\sigma))$. Thus $\ell_{\mathcal{f}'(\sigma)} = \ell_\sigma$.

Since $\det \mathcal{f}'(\sigma) = -\det \sigma$ for all $\sigma \in W$, we have that $m_\lambda(\mu) = 0$ if $b_j < a_{j+1}$ for some j $(2 \leq j \leq n-1)$. This proves the lemma.

By Lemmas 6 and 7, we may, and do, assume that $b_1 + \sum_{i=1}^{n} A_i \in 2\mathbb{Z}$ and that $A_1, A_2, \ldots A_{n-1} \geq 0$. The first of these assumptions is not important and not necessary, since formula (8) in the statement of the theorem holds without it. The second assumption, however, <u>is</u> essential, and will be used after the statement of Lemma 10 below.

Now $m_\lambda(\mu) = \sum_{\sigma \in W} (\det \sigma) P_\sigma$. For an arbitrary subset W' of W, let $m_\lambda^{W'}(\mu) = \sum_{\sigma \in W'} (\det \sigma) P_\sigma$. By means of the next three lemmas, we shall construct inductively a 2^n-element subset $\tilde{W}$ of W such that $m_\lambda(\mu) = m_\lambda^{\tilde{W}}(\mu)$. $\tilde{W}$ will be defined in terms of the set of simple roots of $\mathfrak{g}$.

We recall that for all $\sigma \in W$, we have $\sigma = u_1^{\varepsilon_1(\sigma)} u_2^{\varepsilon_2(\sigma)} \ldots$

$\ldots u_n^{\varepsilon_n(\sigma)} v_\sigma = v_\sigma u_1^{\varepsilon'_1(\sigma)} u_2^{\varepsilon'_2(\sigma)} \ldots u_n^{\varepsilon'_n(\sigma)}$, by the above definitions.

If $S_1, S_2, \ldots S_p$ are subsets of W, let $S_1 S_2 \ldots S_p$ denote the set of products $\{s_1 s_2 \ldots s_p | s_i \in S_i \ (1 \leq i \leq p)\}$. Let $e \in W$ be the identity element.

<u>Lemma 8</u>. Let $A \subset W$ be the two-element subset $\{e, u_n\}$. If $a_n \geq b_n$, then $m_\lambda(\mu) = m_\lambda^{A\mathcal{V}}(\mu)$. If $a_n \leq b_n$, then $m_\lambda(\mu) = m_\lambda^{\mathcal{V}A}(\mu)$.

<u>Proof</u>. Suppose $a_n \geq b_n$. Now $A\mathcal{V} = \{\sigma \in W | \varepsilon_1(\sigma) = \varepsilon_2(\sigma) = \ldots$ $\ldots = \varepsilon_{n-1}(\sigma) = 0\}$. Let $\mathcal{S} = W - A\mathcal{V}$. We define a map $f : \mathcal{S} \to W$ as follows: For all $\sigma \in \mathcal{S}$, let $j(\sigma)$ denote the lowest index i such that $1 \leq i \leq n-1$ and $\varepsilon_i(\sigma) = 1$. Let $f(\sigma) = u_\sigma v_{j(\sigma),n} v_\sigma$. f is clearly an involutive bijection of $\mathcal{S}$ onto itself. We show that $\ell_{f(\sigma)} = \ell_\sigma$: As above, we have by (12) that the expressions (11) for ℓ_σ and $\ell_{f(\sigma)}$ differ only by the substitution of $f(\sigma)(\lambda+\rho)_{j(\sigma)} =$ $= - a'_{\Pi(\sigma)(n)}$ for $\sigma(\lambda+\rho)_{j(\sigma)} = - a'_{\Pi(\sigma)(j(\sigma))}$, and of $f(\sigma)(\lambda+\rho)_n =$ $= (-1)^{\varepsilon_n(\sigma)} a'_{\Pi(\sigma)(j(\sigma))}$ for $\sigma(\lambda+\rho)_n = (-1)^{\varepsilon_n(\sigma)} a'_{\Pi(\sigma)(n)}$. Hence it suffices to show that

$$-a'_{\Pi(\sigma)(j(\sigma))} - b'_{j(\sigma)} - |(-1)^{\varepsilon_n(\sigma)} a'_{\Pi(\sigma)(n)} - b'_n| =$$

$$= -a'_{\Pi(\sigma)(n)} - b'_{j(\sigma)} - |(-1)^{\varepsilon_n(\sigma)} a'_{\Pi(\sigma)(j(\sigma))} - b'_n|.$$

But this is trivial if $\varepsilon_n(\sigma) = 1$. If $\varepsilon_n(\sigma) = 0$, it is true because $a'_i \geq a'_n \geq b'_n$ for all i $(1 \leq i \leq n)$, by our assumption that $a_n \geq b_n$. Hence $\ell_{\mathfrak{f}(\sigma)} = \ell_\sigma$. Since $\det \mathfrak{f}(\sigma) = -\det \sigma$, we have proved the first half of the lemma.

Suppose $a_n \leq b_n$. $\mathcal{V}A = \{\sigma \in W | \varepsilon'_1(\sigma) = \varepsilon'_2(\sigma) = \ldots = \varepsilon'_{n-1}(\sigma) = 0\}$. Let $\mathcal{S}' = W - \mathcal{V}A$. We define a map $\mathfrak{f}': \mathcal{S}' \to W$ as follows: For all $\sigma \in \mathcal{S}'$, let $j'(\sigma)$ denote the lowest index i such that $1 \leq i \leq n-1$ and $\varepsilon'_i(\sigma) = 1$. Let $\mathfrak{f}'(\sigma) = v_\sigma v_{j'(\sigma),n} u'_\sigma$. $\mathfrak{f}'$ is clearly an involutive bijection of $\mathcal{S}'$ onto itself. Also, $\ell_{\mathfrak{f}'(\sigma)} = \ell_\sigma$. Indeed, as above we have by (12') that the expressions (11) for ℓ_σ and $\ell_{\mathfrak{f}'(\sigma)}$ differ only by the substitution of $\mathfrak{f}(\sigma)(\lambda+\rho)_{\Pi(\sigma)^{-1}(j'(\sigma))} =$ $= (-1)^{\varepsilon'_n(\sigma)} a'_n$ for $\sigma(\lambda+\rho)_{\Pi(\sigma)^{-1}(j'(\sigma))} = -a'_{j'(\sigma)}$, and of $f(\sigma)(\lambda+\rho)_{\Pi(\sigma)^{-1}(n)} = -a'_{j'(\sigma)}$ for $\sigma(\lambda+\rho)_{\Pi(\sigma)^{-1}(n)} = (-1)^{\varepsilon'_n(\sigma)} a'_n$. The result that $\ell_{\mathfrak{F}(\sigma)} = \ell_\sigma$ now follows as above by a trivial check of the two possibilities for $\varepsilon'_n(\sigma)$, and by the assumption that $a_n \leq b_n$. Also, $\det \mathfrak{f}'(\sigma) = -\det \sigma$. This proves the lemma.

We make two more definitions. For all k $(1 \leq k \leq n)$, let $\mathcal{V}_k = \{v \in \mathcal{V} | \ \Pi(v)$ is a permutation of $\{1,2,\ldots k\}\}$, and let $W_k = \{\sigma \in W | \ \varepsilon_i(\sigma) = 0$ for all $i=1,2,\ldots k-1$, $\Pi(\sigma)$ is a permutation of $\{k,k+1,\ldots n\}\}$.

The next lemma is designed to handle the inductive step.

Lemma 9. Let $2 \leq i \leq n-1$. Let A and B be subsets of W_{i+1}, and suppose that every element $\sigma \in A\mathcal{V}_{i+1}B$ can be expressed uniquely as $\sigma = avb$ where $a \in A$, $v \in \mathcal{V}_{i+1}$, $b \in B$. Suppose $m_\lambda(\mu) = m_\lambda^{A\mathcal{V}_{i+1}B}(\mu)$. Let $C \subset W$ be the two-element subset $\{e, v_{i,i+1}\}$. Then we have the following: If $a_i \geq b_i$, then $m_\lambda(\mu) = m_\lambda^{AC\mathcal{V}_i B}$, and if $a_i \leq b_i$, then $m_\lambda(\mu) = m_\lambda^{A\mathcal{V}_i C B}$.

Proof. Suppose $a_i \geq b_i$. Every element $\sigma \in \mathcal{V}_{i+1}$ may be uniquely written $\sigma = v_{k,i+1}v$ where $v \in \mathcal{V}_i$ and $1 \leq k \leq i+1$ (we recall that $v_{j,j} = e$ $(1 \leq j \leq n)$). Let $\mathcal{S} = \{\sigma \in A\mathcal{V}_{i+1}B \mid \sigma = av_{k,i+1}vb$ where $a \in A$, $b \in B$, $v \in \mathcal{V}_i$ and $1 \leq k \leq i-1\}$. We define a map $f: \mathcal{S} \to W$ as follows: Let $\sigma = av_{k,i+1}vb \in \mathcal{S}$. Define $f(\sigma) = av_{i,i+1}v_{k,i+1}vb$. f is well defined by our uniqueness assumption. Also, $f(\sigma) \in \mathcal{S}$ since $v_{i,i+1}v_{k,i+1} = v_{k,i+1}v_{k,i}$. Hence f is an involutive bijection of $\mathcal{S}$ onto itself.

We show that $\ell_{f(\sigma)} = \ell_\sigma$. Let $\sigma = av_{k,i+1}vb$ as above, write $b = u_{i+1}^{\varepsilon_{i+1}(b)} u_{i+2}^{\varepsilon_{i+2}(b)} \cdots u_n^{\varepsilon_n(b)} v_b$, let $w_b = u_{i+2}^{\varepsilon_{i+2}(b)} \cdots u_n^{\varepsilon_n(b)}$ and let $v_a w_b = w_b' v_a$. Then

$$\sigma = (u_a w_b' u_k^{\varepsilon_{i+1}(b)})(v_a v_{k,i+1} v v_b).$$

Similarly,

$$\mathcal{f}(\sigma) = (u_a w_b' u_k^{\varepsilon_{i+1}(b)})(v_a v_{i,i+1} v_{k,i+1} v v_b).$$

Applying (12), we see that the expressions (11) for ℓ_σ and $\ell_{\mathcal{f}(\sigma)}$ differ only by the substitution of $\mathcal{f}(\sigma)(\lambda+\rho)_i = a'_{\Pi(v)(k)}$ for $\sigma(\lambda+\rho)_i = a'_{\Pi(v)(i)}$, and of $\mathcal{f}(\sigma)(\lambda+\rho)_{\Pi(v_a)^{-1}(i+1)} = (-1)^{\varepsilon} a'_{\Pi(v)(i)}$ for $\sigma(\lambda+\rho)_{\Pi(v_a)^{-1}(i+1)} = (-1)^{\varepsilon} a'_{\Pi(v)(k)}$ (where ε=0 or 1). The fact that $\ell_{\mathcal{f}(\sigma)} = \ell_\sigma$ now follows immediately from (11) and the fact that $a'_{\Pi(v)(k)}$ and $a'_{\Pi(v)(i)} \geq a'_i \geq b'_i \geq b'_{\Pi(v_a)^{-1}(i+1)}$.

Since det $\mathcal{f}(\sigma)$ = -det σ, we have the first half of the lemma.

Suppose $a_i \leq b_i$. Every element $\sigma \in \mathcal{V}_{i+1}$ may be uniquely written $\sigma = vv_{k,i+1}$ where $v \in \mathcal{V}_i$ and $1 \leq k \leq i+1$. Let $\mathcal{S}' = \{\sigma \in A\mathcal{V}_{i+1}B \mid \sigma = av\, v_{k,i+1}b$ where $a \in A$, $b \in B$, $v \in \mathcal{V}_i$ and $1 \leq k \leq i-1\}$. We define a map $\mathcal{f}': \mathcal{S}' \to W$ as follows: Let $\sigma = av\, v_{k,i+1}b \in \mathcal{S}'$. Define $\mathcal{f}'(\sigma) = av\, v_{k,i+1}v_{i,i+1}b$. As above, $\mathcal{f}'$ is a well defined involutive bijection of $\mathcal{S}$ onto itself since $v_{k,i+1}v_{i,i+1} = v_{k,i}v_{k,i+1}$.

Also, $\ell_{\mathcal{f}'(\sigma)} = \ell_\sigma$. Indeed, let $\sigma = av\, v_{k,i+1}b$ as above and let w_b and w_b' be as above. Then

$$\sigma = (u_a w_b' u_{\Pi(v)^{-1}(k)}^{\varepsilon_{i+1}(b)})(v_a v v_{k,i+1} v_b)$$

and

$$\mathcal{f}'(\sigma) = (u_a w_b' u_{\Pi(v)^{-1}(i)}^{\varepsilon_{i+1}(b)})(v_a v v_{k,i+1} v_{i,i+1} v_b).$$

As above, we see by (12) that the expressions (11) for ℓ_σ and $\ell_{\mathcal{f}'(\sigma)}$ differ only by the substitution of $\mathcal{f}'(\sigma)(\lambda+\rho)_{\Pi(v)^{-1}(k)} = a_i'$ for $\sigma(\lambda+\rho)_{\Pi(v)^{-1}(k)} = (-1)^{\varepsilon_{i+1}(b)} a'_{\Pi(v_b)(i+1)}$ and of

$\mathcal{f}'(\sigma)(\lambda+\rho)_{\Pi(v)^{-1}(i)} = (-1)^{\varepsilon_{i+1}(b)} a'_{\Pi(v_b)(i+1)}$ for

$\sigma(\lambda+\rho)_{\Pi(v)^{-1}(i)} = a_i'$. We now have by (11) that $\ell_{\mathcal{f}'(\sigma)} = \ell_\sigma$ since $a'_{\Pi(v_b)(i+1)} \leq a_i' \leq b_i' \leq b_s'$ for each of the indices $s = \Pi(v)^{-1}(k)$ and $\Pi(v)^{-1}(i)$ which is greater than 1.

Also, det $\mathcal{f}'(\sigma) = -\det \sigma$. This completes the proof of the lemma.

We define $\tau_1 = v_{1,2}$, $\tau_2 = v_{2,3}, \ldots \tau_{n-1} = v_{n-1,n}$ and $\tau_n = u_n$, so that $\tau_1, \tau_2, \ldots$ and τ_n are the Weyl reflections with respect to the simple roots $\lambda_1-\lambda_2$, $\lambda_2-\lambda_3, \ldots$ $\lambda_{n-1}-\lambda_n$ and $2\lambda_n$, respectively. We define the subset $\tilde{W} \subset W$ as follows: $\tilde{W}$ is the set of products of $\tau_1, \ldots \tau_n$, each τ_i taken at most once, in the order determined by the following conditions: For each i such that $2 \leq i \leq n$, if $a_i \geq b_i$, then τ_i must occur to the left of τ_{i-1}, and if $a_i < b_i$, then τ_i

must occur to the right of τ_{i-1} (we note that τ_k and τ_ℓ commute if $|k-\ell| > 1$). By Lemmas 8 and 9 and induction, we have:

<u>Lemma 10</u>. $m_\lambda(\mu) = m_\lambda^{\tilde{W}}(\mu)$. $\tilde{W}$ has exactly 2^n elements, indexed naturally by the 2^n subsets of the set $\{\tau_1, \ldots \tau_n\}$. ($\tau_i$ $(1 \leq i \leq n)$ are the Weyl reflections with respect to the simple roots of $\mathfrak{g}$.)

Now by (11) and (12), we have:

$$\ell_\sigma = \frac{1}{2}\left(-b_1' + \sum_{i=1}^{n} \pm a_i' + \sum_{i=2}^{n} \pm b_i'\right) \tag{16}$$

for all $\sigma \in W$.

Suppose $a_n \geq b_n$. Then all the elements σ of $\tilde{W}$ are of the form $\tau_n^\varepsilon v$ where $\varepsilon=0$ or 1 and $v \in \mathcal{V}$. If $\varepsilon=0$, then b_n' occurs with a plus sign in (16), since $\sigma(\lambda+\rho)_n \geq a_n' \geq b_n'$ by (12). If $\varepsilon=1$, then b_n' occurs with a minus sign in (16), since $\sigma(\lambda+\rho)_n \leq 0$ by (12). Suppose $a_n < b_n$. All the elements σ of $\tilde{W}$ are of the form $v\,\tau_n^\varepsilon$ where ε and v are as above. If $\varepsilon=0$, then a_n' occurs with a plus sign in (16), since a_n' occurs as $\sigma(\lambda+\rho)_j$ for some j, by (12'), and $a_n' \leq b_n' \leq b_j'$ if $j \geq 2$. If $\varepsilon=1$, then a_n' occurs with a minus sign in (16), since a_n' occurs as $-\sigma(\lambda+\rho)_j$ for some j, by (12'). Thus in any case, if τ_n does not occur in $\sigma \in \tilde{W}$, then $A_n+1 = \min(a_n', b_n')$ occurs with a plus sign in (16) (see the statement of the theorem for the definition of A_i), and if τ_n does occur in σ, then A_n+1 occurs with a minus sign in (16).

Let $2 \leq i \leq n-1$. Suppose $a_i \geq b_i$ and $a_{i+1} \geq b_{i+1}$. Then all the elements $\sigma \in \tilde{W}$ are of the form $a\tau_{i+1}^{\varepsilon_{i+1}} \tau_i^{\varepsilon_i} vb$, where the ε's are 0 or 1, $v \in \mathcal{V}_i$, and $a, b \in W_{i+2}$. If $\varepsilon_i = 0$, then b_i' occurs with a plus sign in (16), since $\sigma(\lambda+\rho)_i \geq a_i' \geq b_i'$ by (12); also, a_{i+1}' occurs with a minus sign in (16) since a_{i+1}' occurs as $\pm\sigma(\lambda+\rho)_j$ where $j > i$, by (12), and $a_{i+1}' \geq b_{i+1}' \geq b_j'$. If $\varepsilon_i = 1$, then $\sigma(\lambda+\rho)_i = a_{i+1}' \leq b_i'$ by (12) and our assumption that $A_i \geq 0$ (see above), and so $b_i' - a_{i+1}'$ occurs with a minus sign in (16). Thus if $a_i \geq b_i$ and $a_{i+1} \geq b_{i+1}$, we have that $A_i+1 = \min(a_i', b_i') - \max(a_{i+1}', b_{i+1}')$ occurs with a plus sign in (16) if τ_i does not occur in $\sigma \in \tilde{W}$, and A_i+1 occurs with a minus sign in (16) if τ_i does occur in σ.

With i as above, suppose $a_i \geq b_i$ and $a_{i+1} < b_{i+1}$. Then every element $\sigma \in \tilde{W}$ is of the form $a\tau_i^{\varepsilon_i} v \tau_{i+1}^{\varepsilon_{i+1}} b$ (same notation as in the last paragraph). If $\varepsilon_i = 0$, then b_i' occurs with a plus sign in (16) since $\sigma(\lambda+\rho)_i \geq a_i' \geq b_i'$ by (12'); also, b_{i+1}' occurs with a minus sign in (16) since $\sigma(\lambda+\rho)_{i+1} \leq a_{i+1}' \leq b_{i+1}'$ by (12'). If $\varepsilon_i = 1$, then b_i' occurs with a minus sign in (16) since $\sigma(\lambda+\rho)_i \leq a_{i+1}' \leq b_{i+1}' \leq b_i'$ by (12'); also, b_{i+1}' occurs with a plus sign in (16) since $\sigma(\lambda+\rho)_{i+1} \geq$ $\geq a_i' \geq b_i' \geq b_{i+1}'$ by (12'). Thus if $a_i \geq b_i$ and $a_{i+1} < b_{i+1}$, we again have that A_i+1 occurs with a plus sign in (16) if τ_i does not occur in

$\sigma \in \tilde{W}$, and A_i+1 occurs with a minus sign in (16) if τ_i does occur in σ.

Next suppose $a_i < b_i$ and $a_{i+1} \geq b_{i+1}$. Then every element $\sigma \in \tilde{W}$ is of the form $a\tau_{i+1}^{\varepsilon_{i+1}} v\ \tau_i^{\varepsilon_i} b$ (same notation as above). If $\varepsilon_i=0$, then a_i' occurs with a plus sign in (16) since $a_i' = \sigma(\lambda+\rho)_j$ where $j \leq i$, by (12), and $a_i' \leq b_i' \leq b_j'$ if $j \geq 2$; also, a_{i+1}' occurs with a minus sign in (16) since $a_{i+1}' = \pm\sigma(\lambda+\rho)_j$ where $j > i$ by (12), and $a_{i+1}' \geq b_{i+1}' \geq b_j'$. If $\varepsilon_i=1$, then a_i' occurs with a minus sign in (16) since $a_i' = \pm\sigma(\lambda+\rho)_j$ where $j > i$ by (12), and $a_i' \geq a'_{i+1} \geq$ $\geq b_{i+1}' \geq b_j'$; also, a_{i+1}' occurs with a plus sign in (16) since $a_{i+1}' = \sigma(\lambda+\rho)_j$ where $j \leq i$, by (12), and $a_{i+1}' \leq a_i' \leq b_i' \leq b_j'$ if $j \geq 2$. Thus if $a_i < b_i$ and $a_{i+1} \geq b_{i+1}$, we again have that A_i+1 occurs with a plus sign in (16) if τ_i does not occur in $\sigma \in \tilde{W}$, and A_i+1 occurs with a minus sign in (16) if τ_i occurs in σ.

Finally, suppose $a_i < b_i$ and $a_{i+1} < b_{i+1}$. Then every element $\sigma \in \tilde{W}$ is of the form $av\ \tau_i^{\varepsilon_i}\ \tau_{i+1}^{\varepsilon_{i+1}} b$ (same notation as above). If $\varepsilon_i=0$, then a_i' occurs with a plus sign in (16) since a_i' occurs as $\sigma(\lambda+\rho)_j$ where $j \leq i$, by (12'), and $a_i' \leq b_i' \leq b_j'$ if $j \geq 2$; also, b_{i+1}' occurs with a minus sign in (16) since $\sigma(\lambda+\rho)_{i+1} \leq a_{i+1}' \leq b_{i+1}'$ by (12'). If $\varepsilon_i=1$, then $\sigma(\lambda+\rho)_{i+1} = a_i' \geq b_{i+1}'$ by (12') and our assumption that $A_i \geq 0$, and so $a_i'-b_{i+1}'$ occurs with a minus sign in (16).

Thus the last four paragraphs have shown that if $2 \leq i \leq n-1$, and if $A_i \geq 0$ (as we have assumed), then A_i+1 occurs with a plus sign in (16) if τ_i does not occur in $\sigma \in \tilde{W}$, and A_i+1 occurs with a minus sign in (16) if τ_i occurs in σ.

If $a_2 \geq b_2$, all the elements $\sigma \in \tilde{W}$ are of the form $a\tau_2^{\varepsilon_2}\tau_1^{\varepsilon_1}b$ where the ε's are 0 or 1, and $a,b \in W_3$. If $\varepsilon_1=0$, then a_1' occurs with a plus sign in (16), since $\sigma(\lambda+\rho)_1 = a_1'$ by (12); also, a_2' occurs with a minus sign in (16), since a_2' occurs as $\pm\sigma(\lambda+\rho)_j$ where $j \geq 2$, by (12), and $a_2' \geq b_2' \geq b_j'$. If $\varepsilon_1=1$, a_1' occurs with a minus sign in (16), since a_1' occurs as $\pm\sigma(\lambda+\rho)_j$ where $j \geq 2$, by (12), and $a_1' \geq a_2' \geq b_2' \geq b_j'$; also, a_2' occurs with a plus sign in (16), since $\sigma(\lambda+\rho)_1 = a_2'$ by (12).

If $a_2 < b_2$, all the elements $\sigma \in \tilde{W}$ are of the form $a\tau_1^{\varepsilon_1}\tau_2^{\varepsilon_2}b$ (same notation as above). If $\varepsilon_1=0$, then a_1' occurs with a plus sign in (16), since $\sigma(\lambda+\rho)_1 = a_1'$ by (12'); also, b_2' occurs with a minus sign in (16) since $\sigma(\lambda+\rho)_2 \leq a_2' \leq b_2'$ by (12'). If $\varepsilon_1=1$, a_1' occurs with a minus sign in (16), since a_1' occurs as $\pm\sigma(\lambda+\rho)_j$ where $j \geq 2$ by (12'), and $a_1' \geq b_2' \geq b_j'$ by our assumption that $A_1 \geq 0$; also, b_2' occurs with a plus sign in (16) since $\sigma(\lambda+\rho)_2 = a_1' \geq b_2'$, by (12') and our assumption that $A_1 \geq 0$. Hence under this assumption, we have that $A_1 + 1 = a_1' - \max(a_2', b_2')$ occurs with a plus sign in (16) if τ_1 does not occur in $\sigma \in \tilde{W}$, and A_1+1 occurs with a minus sign in (16) if τ_1 occurs in σ.

The conclusion is that in view of the preceding paragraphs, and by virtue of the assumptions $A_i \geq 0$ $(1 \leq i \leq n-1)$, we have that (16) can be rewritten as:

$$\ell_\sigma = \frac{1}{2}\left(-b_1 - n + \sum_{i=1}^{n} \delta_i (A_i+1)\right) \tag{17}$$

where $\delta_i = 1$ if and only if τ_i does not occur in σ, and $\delta_i = -1$ otherwise. If σ contains exactly r of the reflections $\tau_1, \tau_2, \ldots \tau_n$, the numerical summand in (17) is exactly $-r$ and $\det \sigma = (-1)^r$. But $P_\sigma = \binom{n-2+\ell_\sigma}{n-2}$ and $m_\lambda(\mu) = \sum_{\sigma \in \tilde{W}} (\det \sigma) P_\sigma$. Thus we have precisely the expression (8) for $m_\lambda(\mu)$ (see the statement of the theorem), and Theorem 6 is proved.

§ 2e Formulas for (F_4, Spin (9))

Let $U = F_4$, $K = \text{Spin}(9)$, Let $\mathfrak{h}$ be a Cartan subalgebra of the complexified Lie algebra $\mathfrak{k}$ of K, so that $\mathfrak{h}$ is also a Cartan subalgebra of the complexified Lie algebra $\mathfrak{g}$ of U. We shall describe the root structure of the complex simple Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ with respect to $\mathfrak{h}$.

We may choose a basis $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ of the dual of $\mathfrak{h}$ satisfying the following conditions: The λ_i are orthogonal and of the same length, with respect to the bilinear form on the dual of $\mathfrak{h}$ induced by the Killing form of $\mathfrak{g}$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are $\pm\lambda_i$ $(1 \leq i \leq 4)$, $\pm\lambda_i \pm\lambda_j$ $(1 \leq i < j \leq 4)$, and $\frac{1}{2}\sum_{i=1}^{4} \varepsilon_i\lambda_i$, where $\varepsilon_i = \pm 1$ $(1 \leq i \leq 4)$; we shall denote this last root by $\lambda(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$. The roots of $\mathfrak{k}$ with respect to $\mathfrak{h}$ are $\pm\lambda_i \pm\lambda_j$ $(1 \leq i < j \leq 4)$ and the roots $\lambda(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ such that an <u>odd</u> number of ε_i's are +1.

$\{\lambda_2-\lambda_3,\lambda_3-\lambda_4,\lambda_4, \lambda(1,-1,-1,-1)\}$ is a simple system for $\mathfrak{g}$. The corresponding positive system is $\{\lambda_i$ $(1 \leq i \leq 4)$, $\lambda_j \pm\lambda_k$ $(1 \leq j < k \leq 4)$, $\lambda(1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ $(\varepsilon_2,\varepsilon_3,\varepsilon_4 = \pm 1)\}$. The unique positive system for $\mathfrak{k}$ contained in this positive system for $\mathfrak{g}$ is $\{\lambda_i \pm\lambda_j$ $(1 \leq i < j \leq 4)$, $\lambda(1,1,1,-1)$, $\lambda(1,1,-1,1)$, $\lambda(1,-1,1,1)$, $\lambda(1,-1,-1,-1)\}$. We take these to be the positive systems for $\mathfrak{g}$ and $\mathfrak{k}$, respectively.

The dominant linear forms for $\mathfrak{g}$ are the forms $\sum_{i=1}^{4} a_i\lambda_i$ where $a_i \in \mathbb{R}$, $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$ and $a_1 \geq a_2 + a_3 + a_4$. The dominant linear forms for $\mathfrak{k}$ are the forms $\sum_{i=1}^{4} a_i\lambda_i$ where $a_i \in \mathbb{R}$, $a_1 \geq a_2 \geq a_3 \geq |a_4|$ and $a_1 \geq a_2 + a_3 + a_4$. The integral linear forms for the simply connected adjoint group U, and hence also for K, are the integral linear combinations of the roots of $\mathfrak{g}$, that is, the linear forms $\sum_{i=1}^{4} a_i\lambda_i$ such that either $a_i \in \mathbb{Z}$ $(1 \leq i \leq 4)$ or $a_i \in \mathbb{Z} + \frac{1}{2}$ $(1 \leq i \leq 4)$. Let D_U and D_K denote the sets of dominant integral linear forms for U and K, respectively.

It is not hard to show, using a (complexified) Iwasawa decomposition of $\mathfrak{g}$, that the highest weight of any class 1 finite dimensional irreducible representation of U is of the form $a\lambda_1$ for some $a \in \mathbb{Z}_+$. We can now state two multiplicity theorems for the pair (U, K). Theorem 7 deals with the class 1 representations, and Theorem 8 will be used in Chapter III. We only give the proof of Theorem 7, however, for reasons mentioned in the Introduction.

<u>Theorem 7</u>. Let $\lambda \in D_U$, $\mu = \sum_{i=1}^{4} b_i\lambda_i \in D_K$. Suppose λ is the highest weight of a class 1 finite dimensional irreducible representation of U, so that $\lambda = a\lambda_1$ $(a \in \mathbb{Z}_+)$. Then $m_\lambda(\mu) = 1 \Longleftrightarrow$ $b_2 = b_3 = -b_4$ and $b_1 + b_2 \leq a$; otherwise, $m_\lambda(\mu) = 0$.

<u>Theorem 8</u>. Let $\lambda = \sum_{i=1}^{4} a_i\lambda_i \in D_U$. Then $\mu = a_2\lambda_1 + a_3\lambda_2 + $ $+ a_4\lambda_3 - a_4\lambda_4 \in D_K$, and $m_\lambda(\mu) = 1$.

<u>Proof of Theorem 7</u>. We shall apply Kostant's formula to the pair (U,K).

Let ρ be half the sum of the positive roots of $\mathfrak{g}$. Then $\rho = \frac{11}{2}\lambda_1 + \frac{5}{2}\lambda_2 + \frac{3}{2}\lambda_3 + \frac{1}{2}\lambda_4$.

If x and y are two real numbers, we shall write $x \textcircled{\le} y$ or $y \textcircled{\ge} x$ for the relation $y-x \in \mathbb{Z}_+$, and we shall write $x \textcircled{<} y$ or $y \textcircled{>} x$ for the relation $x \neq y$ and $x \textcircled{\le} y$. The following lemma gives a description of the partition function P:

<u>Lemma 11</u>. Let $\nu = \sum_{i=1}^{4} x_i \lambda_i$ be an integral linear form. Then $P(\nu)$ is the number of real quadruples (p_1,p_2,p_3,p_4) satisfying the conditions

$$(18)\quad \begin{cases} p_1 + p_2 + p_3 + p_4 \geq 0 \\ p_1 + p_2 - p_3 - p_4 \geq 0 \\ p_1 - p_2 + p_3 - p_4 \geq 0 \\ p_1 - p_2 - p_3 + p_4 \geq 0, \end{cases}$$

$$(19)\quad \sum_{i=1}^{4} p_i \in 2\mathbb{Z},$$

$$(20)\quad p_i \textcircled{\le} x_i \quad (1 \leq i \leq 4).$$

<u>Proof</u>. Let $\alpha_1 = \lambda(1,1,1,1)$, $\alpha_2 = \lambda(1,1,-1,-1)$, $\alpha_3 = \lambda(1,-1,1,-1)$, $\alpha_4 = \lambda(1,-1,-1,1)$. The positive weights of the canonical representation of $\mathfrak{k}$ on $\mathfrak{g}/\mathfrak{k}$ are λ_i $(1 \leq i \leq 4)$ and α_j $(1 \leq j \leq 4)$.

Let $\mathcal{A}$ be the matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Now since $\mathcal{A}$ is orthogonal, det $\mathcal{A} \neq 0$. Also, $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\mathcal{A} = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. Hence α_1, α_2, α_3 and α_4 are linearly independent, and so $P(\nu)$ is the number of non-negative integral linear combinations $\sum_{i=1}^{4} p_i\lambda_i$ of α_1, α_2, α_3 and α_4 such that $p_i \leq x_i$ $(1 \leq i \leq 4)$. Hence it suffices to show that the non-negative integral linear combinations of α_1, α_2, α_3 and α_4 are precisely the linear forms $\sum_{i=1}^{4} p_i\lambda_i$ such that the p_i satisfy conditions (18) and (19), and such that either $p_i \in \mathbb{Z}$ $(1 \leq i \leq 4)$ or $p_i \in \mathbb{Z} + \frac{1}{2}$ $(1 \leq i \leq 4)$.

We note that if p_i, $q_i \in \mathbb{R}$ $(1 \leq i \leq 4)$, then $\sum_{i=1}^{4} p_i\lambda_i = \sum_{i=1}^{4} q_i\alpha_i \Longleftrightarrow (p_1,p_2,p_3,p_4) = (q_1,q_2,q_3,q_4)\mathcal{A}$. But since $\mathcal{A}^2$ is the identity, this is equivalent to the condition that $(p_1,p_2,p_3,p_4)\mathcal{A} = (q_1,q_2,q_3,q_4)$. Hence the non-negative integral linear combinations of α_1, α_2, α_3 and α_4 are the linear forms $\sum_{i=1}^{4} p_i\lambda_i$ such that

$$p_1 + p_2 + p_3 + p_4 \in 2\mathbb{Z}_+,$$

$$p_1 + p_2 - p_3 - p_4 \in 2\mathbb{Z}_+,$$

$$p_1 - p_2 + p_3 - p_4 \in 2\mathbb{Z}_+,$$

$$p_1 - p_2 - p_3 + p_4 \in 2\mathbb{Z}_+.$$

This proves the lemma.

We now describe the Weyl group W of $\mathfrak{g}$, regarded as a group of linear transformations of the dual of $\mathfrak{h}$. It is not hard to show that each Weyl chamber for $\mathfrak{k}$ splits into exactly 3 Weyl chambers for $\mathfrak{g}$. Hence W is the disjoint union of 3 right cosets of the Weyl group of $\mathfrak{k}$; in particular, W has $3 \cdot 384 = 1152$ elements. Let e denote its identity element.

For every root α of $\mathfrak{g}$, let $R_\alpha \in W$ be the Weyl reflection with respect to α. Then

$$(21) \quad R_{\lambda(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)}\Big(\sum_{i=1}^{4} x_i\lambda_i\Big) = \frac{1}{2}\sum_{i=1}^{4} \Big(x_i - \varepsilon_i \sum_{j\neq i} x_j\varepsilon_j\Big)\lambda_i$$

for all $\varepsilon_i = \pm 1$ and $x_i \in \mathbb{C}$. Indeed, the transformation defined by the right-hand side of (21) reverses $\sum_{i=1}^{4} \varepsilon_i\lambda_i$ and leaves fixed any linear form $\sum_{i=1}^{4} x_i\lambda_i$ such that $\sum_{i=1}^{4} x_i\varepsilon_i = 0$. It is easy to see from

(21) that $R_{\lambda(1,1,1,1)}$ takes the set of roots $S = \{\pm\lambda_i\ (1 \leq i \leq 4),$ $\pm\lambda_j \pm\lambda_k\ (1 \leq j < k \leq 4)\}$ onto the set of roots of $\mathfrak{k}$. Hence W is the disjoint union of 3 right cosets of the group W' generated by $\{R_\alpha \mid \alpha \in S\}$.

Now W' is the semidirect product of a normal subgroup $\mathcal{U}$ and a subgroup $\mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are described as follows: $\mathcal{U}$ is the 16-element group of transformations σ: $\sum_{i=1}^{4} x_i\lambda_i \to \sum_{i=1}^{4} (-1)^{\delta_i} x_i\lambda_i$, where $\delta_i = \delta_i(\sigma) = 0$ or 1 $(1 \leq i \leq 4)$; the $\delta_i(\sigma)$ are uniquely determined by σ, and $\det \sigma = (-1)^{\Sigma\delta_i(\sigma)}$. $\mathcal{V}$ is the 24-element group of transformations τ: $\sum_{i=1}^{4} x_i\lambda_i \to \sum_{i=1}^{4} x_{\Pi(i)}\lambda_i$, where $\Pi = \Pi(\tau)$ is a permutation of the set $\{1,2,3,4\}$; $\Pi(\tau)$ is uniquely determined by τ, and $\det \tau = \operatorname{sign} \Pi(\tau)$. In particular, (21) shows that the 3 cosets W', $W'R_{\lambda(1,-1,-1,-1)}$ and $W'R_{\lambda(1,-1,-1,1)}$ of W' are distinct, and so

$$W = W' \cup W'R_{\lambda(1,-1,-1,-1)} \cup W'R_{\lambda(1,-1,-1,1)} \qquad \text{(disjoint union).}$$

It will be convenient to write this decomposition in the form

$$W = W' \cup W'R_{\lambda(1,-1,-1,-1)} \cup W'R_{\lambda_4}R_{\lambda(1,-1,-1,1)} \qquad \text{(disjoint union).} \tag{22}$$

Now $m_\lambda(\mu) = \sum_{\sigma \in W} (\det \sigma)\, P(\sigma(\lambda+\rho) - (\mu+\rho))$, by Kostant's formula. The next two lemmas will be used to simplify this sum of 1152 terms. For all integral linear forms ν and ξ, and all subsets X of W, let $M_\nu^X(\xi) = \sum_{\sigma \in X} (\det \sigma) P(\sigma(\nu)-\xi)$.

<u>Lemma 12</u>. Let $\nu = \sum_{i=1}^{4} x_i \lambda_i$ and $\xi = \sum_{i=1}^{4} y_i \lambda_i$ be integral linear forms, and assume that $x_i > 0$ $(1 \leq i \leq 4)$. Then $M_\nu^{\mathcal{U}}(\xi)$ is the number of real quadruples (p_1, p_2, p_3, p_4) satisfying (18), (19) and the conditions

$$(23) \qquad -x_i - y_i < p_i \leq x_i - y_i \qquad (1 \leq i \leq 4).$$

<u>Proof</u>. For all $\sigma \in \mathcal{U}$, let $\mathcal{S}_\sigma$ denote the set of real quadruples (p_1, p_2, p_3, p_4) satisfying (18), (19) and the conditions $p_i \leq (-1)^{\delta_i(\sigma)} x_i - y_i$ $(1 \leq i \leq 4)$; then $P(\sigma(\nu)-\xi)$ is the number of elements in $\mathcal{S}_\sigma$, by Lemma 11. If (p_1, p_2, p_3, p_4) satisfies (18), (19) and (23), then it lies in $\mathcal{S}_e$, but not in $\mathcal{S}_\sigma$ for any $\sigma \in \mathcal{U}$ with $\sigma \neq e$. If (p_1, p_2, p_3, p_4) does not satisfy (18), (19) and (23), then it either lies in no $\mathcal{S}_\sigma$ $(\sigma \in \mathcal{U})$, or else $\{\sigma \in \mathcal{U} \mid (p_1, p_2, p_3, p_4) \in \mathcal{S}_\sigma\}$ has the same number of elements with determinant +1 as with determinant -1. The lemma now follows from the definition of $M_\nu^{\mathcal{U}}(\xi)$.

Let T_1, T_2, T_3 and T_4 be finite sets of real numbers, and let $\xi = \sum_{i=1}^{4} y_i\lambda_i$ be an integral linear form. For all $\tau \in \mathcal{V}$, let $\mathcal{J}_\tau^T(\xi)$ denote the set of real quadruples (p_1,p_2,p_3,p_4) satisfying (18), (19) and the conditions

$$p_i \in T_{\Pi(\tau)(i)} - y_i \quad (1 \leq i \leq 4), \tag{24}$$

and let $Q_\tau^T(\xi)$ denote the number of elements in $\mathcal{J}_\tau^T(\xi)$. Define $N_T(\xi) = \sum_{\tau \in \mathcal{V}} (\det \tau)\, Q_\tau^T(\xi)$.

Lemma 13. Let $\nu = \sum_{i=1}^{4} x_i\lambda_i$ and $\xi = \sum_{i=1}^{4} y_i\lambda_i$ be integral linear forms, and assume that $x_1 > x_2 > x_3 > x_4 > 0$. Define

$$\left\{\begin{array}{l} T(\nu)_1 = \{t \in \mathbb{R} \mid -x_1 < t \leq -x_2 \quad \text{or} \quad x_2 < t \leq x_1\} \\ T(\nu)_2 = \{t \in \mathbb{R} \mid -x_2 < t \leq -x_3 \quad \text{or} \quad x_3 < t \leq x_2\} \\ T(\nu)_3 = \{t \in \mathbb{R} \mid -x_3 < t \leq -x_4 \quad \text{or} \quad x_4 < t \leq x_3\} \\ T(\nu)_4 = \{t \in \mathbb{R} \mid -x_4 < t \leq x_4\}. \end{array}\right. \tag{25}$$

Then $M_\nu^{W'}(\xi) = N_{T(\nu)}(\xi)$.

Proof. We have

$$\begin{aligned} M_\nu^{W'}(\xi) &= M_\nu^{\mathcal{U}\mathcal{V}}(\xi) \\ &= \sum_{\substack{\sigma\in\mathcal{U} \\ \tau\in\mathcal{V}}} (\det\sigma)(\det\tau)P(\sigma\tau(\nu)-\xi) \\ &= \sum_{\tau\in\mathcal{V}} (\det\tau)M_{\tau(\nu)}^{\mathcal{U}}(\xi). \end{aligned}$$

For all $\tau\in\mathcal{V}$, let $\mathcal{R}_\tau$ denote the set of real quadruples (p_1,p_2,p_3,p_4) satisfying (18), (19) and the conditions $-x_{\Pi(\tau)(i)}-y_i \,\textcircled{<}\, p_i \,\textcircled{\le}\, x_{\Pi(\tau)(i)}-y_i$ $(1\le i\le 4)$; then $M_{\tau(\nu)}^{\mathcal{U}}(\xi)$ is the number of elements in $\mathcal{R}_\tau$, by Lemma 12. Hence we must show that the alternating sum of the numbers of elements in the sets $\mathcal{R}_\tau$ is the alternating sum of the numbers of elements in the sets $\mathcal{J}_\tau = \mathcal{J}_\tau^{T(\nu)}(\xi)$. But if $(p_1,p_2,p_3,p_4)\in\mathcal{J}_{\tau_o}$ for some $\tau_o\in\mathcal{V}$, then (p_1,p_2,p_3,p_4) lies in $\mathcal{R}_{\tau_o}$, but not in $\mathcal{R}_\tau$ for any $\tau\in\mathcal{V}$ with $\tau\ne\tau_o$. If $(p_1,p_2,p_3,p_4)\notin\mathcal{J}_\tau$ for any $\tau\in\mathcal{V}$, then either $\{\tau\in\mathcal{V}\mid(p_1,p_2,p_3,p_4)\in\mathcal{R}_\tau\}$ is empty, or else it has the same number of elements with determinant +1 as with determinant -1. This proves the lemma.

Now

(26) $$\lambda+\rho = (a+\tfrac{11}{2})\lambda_1 + \tfrac{5}{2}\lambda_2 + \tfrac{3}{2}\lambda_3 + \tfrac{1}{2}\lambda_4$$

and

$$(27) \qquad \mu+\rho = (b_1 + \tfrac{11}{2})\lambda_1 + (b_2 + \tfrac{5}{2})\lambda_2 + (b_3 + \tfrac{3}{2})\lambda_3 + (b_4 + \tfrac{1}{2})\lambda_4.$$

From (21), we have

$$(28) \qquad R_{\lambda(1,-1,-1,-1)}(\lambda+\rho) = (\tfrac{a}{2} + 5)\lambda_1 + (\tfrac{a}{2} + 3)\lambda_2 + (\tfrac{a}{2} + 2)\lambda_3 + {} + (\tfrac{a}{2} + 1)\lambda_4$$

and

$$(29) \qquad R_{\lambda_4} R_{\lambda(1,-1,-1,1)}(\lambda+\rho) = (\tfrac{a}{2} + \tfrac{9}{2})\lambda_1 + (\tfrac{a}{2} + \tfrac{7}{2})\lambda_2 + (\tfrac{a}{2} + \tfrac{5}{2})\lambda_3 + {} + (\tfrac{a}{2} + \tfrac{1}{2})\lambda_4.$$

Let $(\lambda+\rho)^* = R_{\lambda(1,-1,-1,-1)}(\lambda+\rho)$ and $(\lambda+\rho)^{**} = R_{\lambda_4} R_{\lambda(1,-1,-1,1)}(\lambda+\rho)$.

Then

$$(30) \qquad m_\lambda(\mu) = M^{W'}_{\lambda+\rho}(\mu+\rho) - M^{W'}_{(\lambda+\rho)^*}(\mu+\rho) + M^{W'}_{(\lambda+\rho)^{**}}(\mu+\rho),$$

from (22). By (30) and Lemma 13, we have

$$(31) \qquad m_\lambda(\mu) = N_{T(\lambda+\rho)}(\mu+\rho) - N_{T(\lambda+\rho)^*}(\mu+\rho) + N_{T(\lambda+\rho)^{**}}(\mu+\rho).$$

<u>Lemma 14</u>. If (p_1,p_2,p_3,p_4) is a real quadruple satisfying (18), then $p_1 \geq |p_2|$, $p_1 \geq |p_3|$ and $p_1 \geq |p_4|$.

<u>Proof</u>. We have $p_1 + p_2 \geq |p_3 + p_4| \geq 0$, and $p_1 - p_2 \geq |p_3 - p_4| \geq 0$, so that $p_1 \geq |p_2|$. Similarly, $p_1 \geq |p_3|$ and $p_1 \geq |p_4|$.

Lemma 15. We have

$$m_\lambda(\mu) = \sum_{\substack{\tau\in\mathcal{V} \\ \Pi(\tau)(1)=1}} (\det \tau) Q_\tau^{T(\lambda+\rho)}(\mu+\rho).$$

Proof. By (31), it suffices to show that $Q_\tau^{T(\lambda+\rho)*}(\mu+\rho) = Q_\tau^{T(\lambda+\rho)**}(\mu+\rho) = 0$ for all $\tau\in\mathcal{V}$, and that $Q_\tau^{T(\lambda+\rho)}(\mu+\rho) = 0$ for all $\tau\in\mathcal{V}$ such that $\Pi(\tau)(1) \neq 1$. Hence it suffices to show that the corresponding sets $\mathcal{J}_\tau^T(\mu+\rho)$ are empty. Suppose (p_1,p_2,p_3,p_4) lies in one of these sets. In view of (24) and (25), and the explicit expressions (26), (27), (28) and (29), we have the following: In the case $T = T(\lambda+\rho)$, $p_1 < 0$. In the cases $T = T(\lambda+\rho)*$ and $T = T(\lambda+\rho)**$, we have that $p_1 < \frac{a}{2} - b_1$, that $|p_{\Pi(\tau)^{-1}(1)}| > \frac{a}{2} - b_1$ if $\Pi(\tau)^{-1}(1) > 1$, and that $|p_{\Pi(\tau)^{-1}(2)}| > \frac{a}{2} - b_1$ if $\Pi(\tau)^{-1}(2) > 1$. Hence in all cases, $p_1 < |p_j|$ for some $j=2,3,4$. This contradicts Lemma 14, and so Lemma 15 is proved.

We shall use the notation τ_1, τ_2, τ_3, τ_4, τ_5 to denote the elements $\tau\neq e$ of $\mathcal{V}$ such that $\Pi(\tau)(1) = 1$, in the order determined by the following conditions:
$\Pi(\tau_1)$ is the transposition of 3 and 4; $\Pi(\tau_2)(2) = 4$, $\Pi(\tau_2)(3) = 2$, $\Pi(\tau_2)(4) = 3$; $\Pi(\tau_3)$ is the transposition of 2 and 4; $\Pi(\tau_4)(2) = 3$, $\Pi(\tau_4)(3) = 4$, $\Pi(\tau_4)(4) = 2$; $\Pi(\tau_5)$ is the transposition of 2 and 3.

For every real triple (p_2, p_3, p_4), let

$$\mathcal{M}(p) = \max(-p_2-p_3-p_4,\ -p_2+p_3+p_4,\ p_2-p_3+p_4,\ p_2+p_3-p_4),$$

and let

$$\mathcal{N}(p) = p_2+p_3+p_4.$$

Then (p_1, p_2, p_3, p_4) satisfies (18) if and only if $p_1 \geq \mathcal{M}(p)$, and it satisfies (19) if and only if $p_1 + \mathcal{N}(p) \in 2\mathbb{Z}$; here $\mathcal{M}$ and $\mathcal{N}$ are regarded as functions of the last 3 variables.

Now $Q_{\tau_2}^{T(\lambda+\rho)}(\mu+\rho) = Q_{\tau_3}^{T(\lambda+\rho)}(\mu+\rho)$. To prove this, it suffices to show that $\mathcal{J}_2 = \mathcal{J}_{\tau_2}^{T(\lambda+\rho)}(\mu+\rho)$ and $\mathcal{J}_3 = \mathcal{J}_{\tau_3}^{T(\lambda+\rho)}(\mu+\rho)$ are in 1 - 1 correspondence. For a quadruple $(p_1, p_2, p_3, p_4) \in \mathcal{J}_2$, (24) and (25) give 4 possible choices for (p_2, p_3, p_4) (see (26) and (27)), namely,

$$(-2-b_2,\ -3-b_3,\ -1-b_4),$$

$$(-2-b_2,\ -3-b_3,\ 1-b_4),$$

$$(-2-b_2,\ 1-b_3,\ -1-b_4),$$

$$(-2-b_2,\ 1-b_3,\ 1-b_4).$$

The respective values of $\mathcal{M}(p)$ are:

$$(32)\quad \begin{cases} 6 + b_2 + b_3 + b_4, \\ 4 + b_2 + b_3 + b_4, \\ 2 + b_2 + b_3 + b_4, \\ \max(b_2 + b_3 + b_4,\ 4 + b_2 - b_3 - b_4), \end{cases}$$

and the respective values of $\mathcal{N}(p)$ are:

$$(33)\quad \begin{cases} -\ (6 + b_2 + b_3 + b_4), \\ -\ (4 + b_2 + b_3 + b_4), \\ -\ (2 + b_2 + b_3 + b_4), \\ -\ (b_2 + b_3 + b_4). \end{cases}$$

For a quadruple $(p_1,p_2,p_3,p_4) \in \mathcal{J}_3$, (24) and (25) also give 4 possible choices for (p_2,p_3,p_4), namely,

$$(-2-b_2,\ -2-b_3,\ -2-b_4),$$
$$(-2-b_2,\ -b_3,\ -2-b_4),$$
$$(-2-b_2,\ -2-b_3,\ 2-b_4),$$
$$(-2-b_2,\ -b_3,\ 2-b_4).$$

The respective values of $\mathcal{M}(p)$ and $\mathcal{N}(p)$ are exactly the same as in (32) and (33). Since the values of p_1 determined by (24) and (25) are exactly the same for $\mathcal{J}_2$ and $\mathcal{J}_3$, we have that $Q_{\tau_2}^{T(\lambda+\rho)}(\mu+\rho) = Q_{\tau_3}^{T(\lambda+\rho)}(\mu+\rho)$.

The same argument shows that $Q_{\tau_4}^{T(\lambda+\rho)}(\mu+\rho) = Q_{\tau_5}^{T(\lambda+\rho)}(\mu+\rho)$.

Indeed, for a quadruple $(p_1,p_2,p_3,p_4) \in \mathcal{J}_{\tau_4}^{T(\lambda+\rho)}(\mu+\rho)$, (24) and (25) give the following choices for (p_2,p_3,p_4):

$$(-3-b_2,\ -1-b_3,\ -2-b_4),$$

$$(-1-b_2,\ -1-b_3,\ -2-b_4),$$

$$(-3-b_2,\ -1-b_3,\ \ 2-b_4),$$

$$(-1-b_2,\ -1-b_3,\ \ 2-b_4).$$

The choices for $\mathcal{J}_{\tau_5}^{T(\lambda+\rho)}(\mu+\rho)$ are:

$$(-3-b_2,\ -3-b_3,\ -b_4),$$

$$(-1-b_2,\ -3-b_3,\ -b_4),$$

$$(-3-b_2,\ \ 1-b_3,\ -b_4),$$

$$(-1-b_2,\ \ 1-b_3,\ -b_4).$$

The respective values of $\mathcal{M}(p)$ in both cases are:

$$6 + b_2 + b_3 + b_4,$$

$$4 + b_2 + b_3 + b_4,$$

$$\max(2 + b_2 + b_3 + b_4,\ 4 + b_2 - b_3 - b_4),$$

$$\max(b_2 + b_3 + b_4,\ 2 + b_2 - b_3 - b_4).$$

Since the respective values of $\mathcal{N}(p)$ are also the same in the two cases, we have that $Q_{\tau_4}^{T(\lambda+\rho)}(\mu+\rho) = Q_{\tau_5}^{T(\lambda+\rho)}(\mu+\rho)$.

From Lemma 15 and the last two paragraphs, we have

$$m_\lambda(\mu) = Q_e^{T(\lambda+\rho)}(\mu+\rho) - Q_{\tau_1}^{T(\lambda+\rho)}(\mu+\rho). \tag{34}$$

Let $\mathcal{J}_e = \mathcal{J}_e^{T(\lambda+\rho)}(\mu+\rho)$ and $\mathcal{J}_1 = \mathcal{J}_{\tau_1}^{T(\lambda+\rho)}(\mu+\rho)$. For a quadruple $(p_1,p_2,p_3,p_4) \in \mathcal{J}_e$, (24) and (25) give the following choices for (p_2,p_3,p_4):

$$\tag{35} \left\{ \begin{array}{l} (-4-b_2,\ -2-b_3,\ -b_4), \\ (-4-b_2,\ -b_3,\ -b_4), \\ (-b_2,\ -2-b_3,\ -b_4), \\ (-b_2,\ -b_3,\ -b_4). \end{array} \right.$$

The choices for $\mathcal{J}_1$ are:

$$\tag{36} \left\{ \begin{array}{l} (-4-b_2,\ -1-b_3,\ -1-b_4), \\ (-4-b_2,\ -1-b_3,\ 1-b_4), \\ (-b_2,\ -1-b_3,\ -1-b_4), \\ (-b_2,\ -1-b_3,\ 1-b_4). \end{array} \right.$$

The respective values of $\mathcal{M}(p)$ for $\mathcal{J}_e$ are:

$$6 + b_2 + b_3 + b_4,$$

$$4 + b_2 + b_3 + b_4,$$

$$2 + b_2 + b_3 + b_4,$$

$$b_2 + b_3 + b_4,$$

and the respective values of $\mathcal{M}(p)$ for $\mathcal{J}_1$ are

$$6 + b_2 + b_3 + b_4,$$

$$4 + b_2 + b_3 + b_4,$$

$$2 + b_2 + b_3 + b_4,$$

$$\max(b_2 + b_3 + b_4,\ 2 - b_2 + b_3 - b_4).$$

The respective values of $\mathcal{N}(p)$ are the same for $\mathcal{J}_e$ and $\mathcal{J}_1$. The values of p_1 determined by (24) and (25) constitute the set $T(\lambda+\rho)_1 - (b_1 + \frac{11}{2})$. Let $|A|$ be the number of elements in the set

$$A = \{p \in T(\lambda+\rho)_1 - (b_1 + \tfrac{11}{2}) \mid p \geq b_2 + b_3 + b_4,$$
$$p - b_2 - b_3 - b_4 \in 2\mathbb{Z}\},$$

and let $|B|$ be the number of elements in the set

$$B = \{p \in T(\lambda+\rho)_1 - (b_1 + \tfrac{11}{2}) \mid p \geq \max(b_2 + b_3 + b_4,$$
$$2 - b_2 + b_3 - b_4),\ p - b_2 - b_3 - b_4 \in 2\mathbb{Z}\}.$$

The first 3 rows of (35) and (36) give equal contributions to

$Q_e^{T(\lambda+\rho)}(\mu+\rho)$ and $Q_{\tau_1}^{T(\lambda+\rho)}(\mu+\rho)$, respectively, and so (34) reduces to:

(37) $$m_\lambda(\mu) = |A| - |B|.$$

If $b_2+b_4 \geq 1$, then $A=B$, so that $m_\lambda(\mu) = 0$ by (37). Suppose now that $b_2+b_4 < 1$, so that $b_2+b_4 = 0$, and $b_2 = b_3 = -b_4$. Then

$$A = \{p \in T(\lambda+\rho)_1 - (b_1 + \tfrac{11}{2}) \mid p \geq b_2,\ p - b_2 \in 2\mathbb{Z}\}$$

and

$$B = \{p \in T(\lambda+\rho)_1 - (b_1 + \tfrac{11}{2}) \mid p \geq b_2 + 2,\ p - b_2 \in 2\mathbb{Z}\}.$$

Now $B \subset A$, and the complement of B in A is the set

$$C = \{p \in T(\lambda+\rho)_1 - (b_1 + \tfrac{11}{2}) \mid b_2 \leq p < b_2 + 2,\ p - b_2 \in 2\mathbb{Z}\}.$$

$$= \{p \in T(\lambda+\rho)_1 - (b_1 + \tfrac{11}{2}) \mid p = b_2\}.$$

Hence C has exactly one element if and only if $b_2 \in T(\lambda+\rho)_1 - (b_1 + \frac{11}{2})$; otherwise, C is empty. But $b_2 \in T(\lambda+\rho)_1 - (b_1 + \frac{11}{2})$ if and only if $b_1 + b_2 \leq a$, that is, if and only if $b_1 + b_2 \leq a$. Since by (37) $m_\lambda(\mu)$ is the number of elements in C, $m_\lambda(\mu) = 0$ unless $b_1 + b_2 \leq a$, in which case $m_\lambda(\mu) = 1$. This completes the proof of Theorem 7.

Chapter III. Systems of minimal types and rank 1 groups

§1 Systems of minimal types

In this chapter, we use the notation of Chapter I, not Chapter II. In particular, G is a connected real semisimple Lie group with finite center.

Let $\mathfrak{m}_o$ be the Lie subalgebra of $\mathfrak{g}_o$ corresponding to M, and let $\mathfrak{m}$ be the complex subspace of $\mathfrak{g}$ generated by $\mathfrak{m}_o$. Let $\mathfrak{h}_{\mathfrak{m}}$ be a Cartan subalgebra of the complex reductive Lie algebra $\mathfrak{m}$, and fix a system $\Delta_+^{\mathfrak{m}}$ of positive roots of $\mathfrak{m}$ with respect to $\mathfrak{h}_{\mathfrak{m}}$. Let $\mathcal{H}_{\mathfrak{m}}$ denote the subalgebra of $\mathfrak{U}$ generated by $\mathfrak{h}_{\mathfrak{m}}$ and 1, so that $\mathcal{H}_{\mathfrak{m}}$ may be regarded as the algebra of polynomial functions on the dual of $\mathfrak{h}_{\mathfrak{m}}$. For all $\gamma \in \hat{M}$, let $\mu(\gamma)$ denote the highest weight (with respect to $\Delta_+^{\mathfrak{m}}$) of the representation of $\mathfrak{m}$ induced by any member of γ. We recall (cf. Chapter I, §5) that if $\beta \in \hat{K}$ and $\gamma \in \hat{M}$ are such that $m(\beta,\gamma) = 1$, then $\chi_{\zeta(\beta,\gamma)}$ is a homomorphism from $\mathcal{K}^M$ into $\mathbb{C}$. We recall the definition of $\gamma(\alpha) \in \hat{M}$ $(\alpha \in \hat{G})$ and of $s_o \in W$ from Chapter I, §3.

Definition 1 (see the Introduction for motivation). A system of minimal types for G is a family $(\mathcal{C}_i, f_i)_{i \in I}$ where I is a finite set, each $\mathcal{C}_i$ is a map of a subset of $\hat{M}$ into $\hat{K}$, and each f_i is a homomorphism from $\mathcal{K}^M$ into $\mathcal{H}_{\mathfrak{m}}$, such that the following conditions hold:

(1) $\bigcup_{i \in I} (\text{domain } \mathcal{C}_i) = \hat{M}$,

(2) for all $i \in I$ and $\gamma \in$ domain $\mathcal{C}_i$, $m(\mathcal{C}_i(\gamma),\gamma) = 1$,

(3) for all $i \in I$, $\gamma \in$ domain $\mathcal{C}_i$ and $x \in \mathcal{K}^M$,

$$\chi_{\zeta(\mathcal{C}_i(\gamma),\gamma)}(x) = f_i(x)(\mu(\gamma)),$$

(4) for all $\alpha \in \hat{G}$, $m(\alpha, \mathcal{C}_i(s_o\gamma(\alpha))) = 1$ for all $i \in I$ such that $s_o\gamma(\alpha) \in$ domain $\mathcal{C}_i$.

Let $\mathfrak{h} = \mathfrak{a} + \mathfrak{h}_m$, so that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ_+ be the system of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ determined by the following conditions: The set of non-zero restrictions to $\mathfrak{a}_o$ of the elements of Δ_+ is the set $-\Sigma_+$ (we recall the definition of Σ_+ from Chapter I, § 1), and the set of restrictions to $\mathfrak{h}_m$ of the elements of Δ_+ vanishing on $\mathfrak{a}_o$ is the set Δ_+^m. If $\alpha \in \hat{G}$ and λ is the highest weight of any member of α with respect to Δ_+, then

(1) $$\lambda|_{\mathfrak{a}} = s_o\lambda(\alpha) \quad \text{and} \quad \lambda|_{\mathfrak{h}_m} = \mu(s_o\gamma(\alpha)).$$

Let $\mathcal{H}$ denote the subalgebra of $\mathcal{G}$ generated by $\mathfrak{h}$ and 1. Then $\mathcal{H}$ may be regarded as the algebra of polynomial functions on the dual of $\mathfrak{h}$. Also, $\mathcal{H}$ is naturally isomorphic to $\mathfrak{A} \otimes \mathcal{H}_m$.

Suppose G has a system $\mathcal{S} = (\mathcal{C}_i, f_i)_{i \in I}$ of minimal types. Then

$$1 \otimes f_i \colon \mathfrak{A} \otimes \mathcal{K}^M \to \mathfrak{A} \otimes \mathcal{H}_m = \mathcal{H}$$

is a homomorphism for all $i \in I$. Let $F_i^{\mathcal{S}} : \mathcal{Y}^K \to \mathcal{H}$ be the linear map $(1 \otimes f_i) \circ q$ for all $i \in I$ (we recall the definition of q from Proposition 1 of Chapter I (§1)).

If $\gamma \in \hat{M}$ and $\nu \in \hat{A}$, then $\alpha_{\gamma,\nu}$ is defined, and if in addition $\beta \in \hat{K}$ is such that $m(\beta,\gamma) = 1$, then $\hat{\alpha}_{\gamma,\nu}^{\beta}$ is defined (Chapter I, §4).

We can now state:

Theorem 1. Suppose G has a system $\mathcal{S} = (\mathcal{C}_i, f_i)_{i \in I}$ of minimal types. Then $F_i^{\mathcal{S}} : \mathcal{Y}^K \to \mathcal{H}$ is a homomorphism for all $i \in I$.

Let $\alpha \in \hat{G}$, and let λ be the highest weight (with respect to Δ_+) of any member of α. Choose $i \in I$ such that $s_o\gamma(\alpha) \in$ domain $\mathcal{C}_i$. Then $m(\alpha, \mathcal{C}_i(s_o\gamma(\alpha))) = 1$ and

(2) $$\chi_{\eta(\alpha,\, \mathcal{C}_i(s_o\gamma(\alpha)))}(u) = F_i^{\mathcal{S}}(u)(\lambda) \quad \text{for all } u \in \mathcal{Y}^K.$$

Let $\gamma \in \hat{M}$ and $\nu \in \hat{A}$. Let Λ be the linear form on $\mathfrak{h}$ such that $\Lambda|_{\mathfrak{a}} = -\nu + \rho$ and $\Lambda|_{\mathfrak{h}_m} = \mu(\gamma)$. Choose $i \in I$ such that $\gamma \in$ domain $\mathcal{C}_i$. Then $m(\alpha_{\gamma,\nu}, \mathcal{C}_i(\gamma)) = 1$ and

(3) $$\chi_{\eta(\alpha_{\gamma,\nu},\, \mathcal{C}_i(\gamma))}(u) = F_i^{\mathcal{S}}(u)(\Lambda) \quad \text{for all } u \in \mathcal{Y}^K.$$

Moreover, $\hat{\alpha}_{\gamma,\nu}^{\mathcal{C}_i(\gamma)}$ is defined, and is an infinitesimal equivalence class of irreducible quasi-simple Banach space representations of G such that

(4) $$\chi_{\eta(\hat{\alpha}_{\gamma,\nu}^{\mathcal{C}_i(\gamma)},\, \mathcal{C}_i(\gamma))}(u) = F_i^{\mathcal{S}}(u)(\Lambda) \quad \text{for all } u \in \mathcal{Y}^K.$$

<u>Proof</u>. The first statement follows from the above remarks and from Proposition 1 of Chapter 1 (§1), together with the fact that $\mathcal{H}$ is commutative. To prove (2), we note that

$$p_{\mathcal{C}_i(s_o\gamma(\alpha)),s_o\gamma(\alpha)} = (1 \otimes \chi_{\zeta(\mathcal{C}_i(s_o\gamma(\alpha)),s_o\gamma(\alpha))})\circ q.$$

Hence from Theorem 1' of Chapter I (§ 5), and from Definition 1(3), we get

$$\begin{aligned}\chi_{\eta(\alpha,\mathcal{C}_i(s_o\gamma(\alpha)))}(u) &= \{[1 \otimes f_i(\cdot)(\mu(s_o\gamma(\alpha)))]q(u)\}\ (s_o\lambda(\alpha)) \\ &= ((1 \otimes f_i)q(u))(\lambda) \\ &\quad (\text{by } (1)) \\ &= F_i^{\flat}(u)(\lambda)\end{aligned}$$

for all $u \in \mathcal{G}^K$, and (2) is proved. (3) follows in exactly the same way from Theorem 2 of Chapter I (§ 4). ~~Finally, (4) follows from (3) in the same way that Theorem 2 of Chapter I (§ 4)~~. Finally, (4) follows from (3) in the same way that Theorem 3 of Chapter I (§ 4) follows from Theorem 2 of Chapter I (§ 4). This establishes the theorem.

Remark. As explained in the Appendix to § 1, Theorem 1 can be viewed as a generalization of Theorem 2.2 and Lemma 2.5 of [11].

Definition 2 (see the Introduction for motivation). A strong system of minimal types for G is a system $(\mathcal{C}_i, f_i)_{i \in I}$ of minimal types for G satisfying the following conditions:

(1) $\mathcal{C}_i(\gamma) = \mathcal{C}_j(s\gamma)$ for all $i,j \in I$, $\gamma \in \hat{M}$ and $s \in W$ such that $\gamma \in$ domain $\mathcal{C}_i$ and $s\gamma \in$ domain $\mathcal{C}_j$ (this allows us to define $\mathcal{C}(\gamma)$ $(\gamma \in \hat{M})$ as $\mathcal{C}_i(\gamma)$, where $\gamma \in$ domain $\mathcal{C}_i$),

(2) for all $\gamma_1, \gamma_2 \in \hat{M}$ such that γ_1 and γ_2 are not conjugate under W, either $m(\mathcal{C}(\gamma_1), \gamma_2) = 0$ or $m(\mathcal{C}(\gamma_2), \gamma_1) = 0$.

Definition 2 leads to the following theorem:

Theorem 2. Suppose G has a strong system $\mathcal{S} = (\mathcal{C}_i, f_i)_{i \in I}$ of minimal types. Let $i, j \in I$, $\gamma \in \hat{M}$ and $s \in W$ be such that $\gamma \in$ domain $\mathcal{C}_i$ and $s\gamma \in$ domain $\mathcal{C}_j$. Then

$$F_i^{\mathcal{S}}(u)(\nu, \mu(\gamma)) = F_j^{\mathcal{S}}(u)(s^A\nu, \mu(s\gamma)) \tag{5}$$

for all $u \in \mathfrak{U}^K$ and all linear forms ν on $\mathfrak{a}$ (in (5), elements of $\mathcal{H} = \mathfrak{A} \otimes \mathcal{H}_m$ are evaluated at ordered pairs of forms on $\mathfrak{a}$ and $\mathfrak{h}_m$). Moreover,

$$\hat{\alpha}_{\gamma,\nu}^{\mathcal{C}(\gamma)} = \hat{\alpha}_{s\gamma,s\nu}^{\mathcal{C}(\gamma)} \tag{6}$$

for all $\gamma \in \hat{M}$, $\nu \in \hat{A}$ and $s \in W$. Finally, if $\gamma_1, \gamma_2 \in \hat{M}$ are not conjugate under W, then

$$\hat{\alpha}_{\gamma_1,\nu_1}^{\mathcal{C}(\gamma_1)} \neq \hat{\alpha}_{\gamma_2,\nu_2}^{\mathcal{C}(\gamma_2)} \tag{7}$$

for any $\nu_1, \nu_2 \in \hat{A}$.

Proof. (5) follows easily from Theorem 4 of Chapter I (§ 6) and Definition 2(1), and (6) is contained in Theorem 4 of Chapter I. (7) follows immediately from Definition 2(2), and Theorem 2 is established.

Remark. Strong systems of minimal types exist for complex groups G (see the Appendix to §1) and for rank 1 real groups (see §2). Hence Theorems 1 and 2 apply to these groups. As explained in the Appendix to §1, Theorem 2 can be regarded as a generalization of Theorem 2.3 of [11].

Remark. We conjecture that (6) gives all possible infinitesimal equivalences between representations of the form $\hat{\Pi}_{\gamma,\nu}^{\mathcal{C}(\gamma)}$, when G has a strong system of minimal types as in Theorem 2. (As explained in the Appendix to §1, this would generalize Theorem 3.2 of [11].) We have proved this conjecture for many of the rank 1 real groups G. Specifically, whenever $\mathfrak{g}_o$ is full rank (cf. §2), $s\gamma = \gamma$ for all $\gamma \in \hat{M}$ and $s \in W$. Proposition 2 of Chapter I (§6) then implies the desired result. The only rank 1 groups G for which $\mathfrak{g}_o$ is not full rank are those for which the ideal of $\mathfrak{g}_o$ generated by $\mathfrak{a}_o$ is one of the Lorentz algebras $\mathfrak{so}(1,2n-1)$ where $n \geq 2$ (cf. §2), and it may be easy to verify our conjecture directly in this case by computing the image of the mapping $p_{\mathcal{C}(\gamma),\gamma}$ (cf. the Remark following Proposition 2 of Chapter I (§6)).

Remark. Suppose G has a system $\mathcal{S} = (\mathcal{C}_i, f_i)_{i \in I}$ of minimal types. Then $\mathcal{C}_i(\gamma_o) = \beta_o$ for all $i \in I$ such that $\gamma_o \in$ domain $\mathcal{C}_i$ (where β_o and γ_o are defined as in Chapter I, §5), in view of Definition 1(4), applied to the class of the trivial one-dimensional representation of G. Thus if $\gamma_o \in$ domain $\mathcal{C}_i$, we have

$$p_{\beta_o,\gamma_o}(u)(\nu) = F_i^{\mathcal{J}}(u)(\nu,0)$$

for all $u \in \mathfrak{Z}^K$ and linear forms ν on $\mathfrak{a}$; here 0 denotes the zero linear form on $\mathfrak{h}_{\mathfrak{m}}$. Hence our mappings $F_i^{\mathcal{J}}$ can be regarded as extensions of the classical mapping p_{β_o,γ_o} (cf. Chapter I, §5).

Appendix to §1: The complex case

Theorem 1 may be regarded as a generalization of Theorem 2.2 and Lemma 2.5 of [11], which deals with the case of complex G. When applied to the complex case, Theorem 1 and its proof also provide a clarification and simplification of these two results of [11].

Specifically, we first note that the example G = SU(1,1), as well as other examples of rank 1 real simple groups, show that the definition of "minimal type" given in [11, p. 390] is not appropriate for real semisimple Lie groups. Even if "weight" is replaced by "element of M" in this definition (these two concepts essentially coincide if G is complex), the same examples show that we still have no unique notion of minimal type. Hence we have not attempted to define "minimal type" in general.

On the other hand, Definition 1 above includes the complex case, in the following sense: Let G be complex. Then Lemma 1.1 of [11] shows that $\mathcal{J} = (\mathcal{C}_Q, f_Q)_{Q \in I}$ satisfies the third part of Definition 1, where I is the set of systems of positive roots, $f_Q = \beta^Q$ (in the terminology of [11]), and $\mathcal{C}_Q$ is the map which associates to each integral linear form μ in the Weyl chamber defined by Q the element of $\hat{K}$ such that any of its members has μ as an extremal weight. Also, Corollary 1 to Theorem 2.1 of [11] shows that the last part of Definition 1 is satisfied, so that $\mathcal{S}$ is a system of minimal types for G.

If $\alpha \in \hat{G}$ (G is still complex) and $\Pi \in \alpha$, then the minimal type of Π in the sense of [11] can be described in our terminology as the element of $\hat{K}$ such that any of its members contains $\gamma(\alpha)$ as an extremal weight. This shows that the minimal type of Π is precisely $\mathcal{C}_Q(\gamma(\alpha))$ (or $\mathcal{C}_Q(s_o\gamma(\alpha))$) for all $Q \in I$ such that $\gamma(\alpha) \in$ domain $\mathcal{C}_Q$ (or $s_o\gamma(\alpha) \in$ domain $\mathcal{C}_Q$). Moreover, if $\gamma \in \hat{M}$ and $\nu \in \hat{A}$, then for all $Q \in I$ such that $\gamma \in$ domain $\mathcal{C}_Q$, $\hat{\Pi}_{\gamma,\nu}^{\mathcal{C}_Q(\gamma)}$ is defined, and $\mathcal{C}_Q(\gamma)$ is the minimal type of $\Pi_{\gamma,\nu}$ and of $\hat{\Pi}_{\gamma,\nu}^{\mathcal{C}_Q(\gamma)}$. We note that the representations $\hat{\Pi}_{\gamma,\nu}^{\mathcal{C}_Q(\gamma)}$ are precisely the representations denoted $\hat{\Pi}_{\lambda,\nu}$ in [11].

Now the homomorphisms $F_Q^{\mathcal{S}}$ are almost the same as the homomorphisms h^Q of [11]. The difference is as follows: The $F_Q^{\mathcal{S}}$ are defined by means of the Iwasawa decomposition $\mathfrak{G} = \mathfrak{N}\mathfrak{A}\mathfrak{K}$, in place of the decomposition $\mathfrak{G} = \mathfrak{K}\mathfrak{A}'\mathfrak{N}$ (our terminology) used in [11] to define the h^Q; here $\mathfrak{A}'$ is the enveloping algebra of a subalgebra of $\mathfrak{g}$ which is defined in the complex case, but which has no natural meaning in the general case. With this understanding, and in view of the above remarks, we now see that Theorem 1 is a generalization of Theorem 2.2 and Lemma 2.5 of [11].

We also see that Theorem 1 and its proof yield a clarification and simplification of these two results. Indeed, the use of the Iwasawa decomposition gives the right-hand sides of formulas (2), (3) and (4) more natural form than those of their counterparts (2.34) and (2.49) in [11]; for example, the linear form λ in the right-hand side of (2) is the highest weight of the representation in question. Also, we have avoided the difficulties in passing from the finite to infinite dimensional case encountered in § 2.4 of [11], by using the "opposite" Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}\mathfrak{a}\mathfrak{k}$ and by using the contragredient module to prove formula (2) for the finite dimensional case (see the proof of Theorem 1' of Chapter I (§ 5) and the Remark following Theorem 1 of Chapter I (§ 3)). Moreover, by using Proposition 1 of Chapter I (§ 1), we have proved directly that the $F_Q^{\mathcal{S}}$ are homomorphisms, instead of having to rely on formula (2) and the fact that the highest weights λ are Zariski dense in the dual of $\mathfrak{h}$.

Finally, we note that $\mathcal{S}$ is a strong system of minimal types for G, that Theorem 2 is a generalization of Theorem 2.3 of [11], and that the second Remark following Theorem 2 indicates a partial generalization of Theorem 3.2 of [11].

§2 Systems of minimal types for rank 1 groups

In this section, we shall show that strong systems of minimal types exist for connected real semisimple Lie groups with finite center and real rank 1. We use the notation of §1 and of Chapter I.

Let M_o be the identity component of M. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}_m$. The Lie algebra $\mathfrak{g}_o$ is called <u>full rank</u> if $\mathfrak{k}$ contains a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}_o$ is called <u>split rank</u> if $\mathfrak{m}$ contains a Cartan subalgebra of $\mathfrak{k}$.

We assume that $\mathfrak{g}_o$ has real rank 1. Then $\mathfrak{g}_o$ is either full rank or split rank.

<u>Lemma 1</u>. No root of $\mathfrak{k}$ with respect to $\mathfrak{t}$ vanishes on $\mathfrak{h}_m$.

<u>Proof</u>. If $\mathfrak{g}_o$ is split rank, then $\mathfrak{t} = \mathfrak{h}_m$, and so the lemma is clear. Hence we may assume that $\mathfrak{g}_o$ is full rank. In this case, there is a root α of $\mathfrak{g}$ with respect to $\mathfrak{h}$ $(= \mathfrak{a} + \mathfrak{h}_m)$ which vanishes on $\mathfrak{h}_m$. Let X_α be a non-zero root vector for α, and let θ denote the Cartan involution corresponding to the complexified Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$. Now the centralizer of $\mathfrak{h}_m$ in $\mathfrak{g}$ is

$$\mathfrak{h} + \mathbb{C}\, X_\alpha + \mathbb{C}\, \theta X_\alpha = \mathfrak{a} + \mathfrak{h}_m + \mathbb{C}\,(X_\alpha + \theta X_\alpha) + \mathbb{C}\,(X_\alpha - \theta X_\alpha).$$

But $\mathfrak{a} + \mathbb{C}(X_\alpha - \theta X_\alpha) \subset \mathfrak{p}$ and $\mathfrak{h}_m + \mathbb{C}(X_\alpha + \theta X_\alpha) \subset \mathfrak{k}$, so that the centralizer $\mathfrak{t}'$ of $\mathfrak{h}_m$ in $\mathfrak{k}$ is $\mathfrak{h}_m + \mathbb{C}(X_\alpha + \theta X_\alpha)$. Since $\mathfrak{t} \subset \mathfrak{t}'$, we must have $\mathfrak{t} = \mathfrak{t}'$ by dimensionality, and this proves the lemma.

Lemma 1 implies the existence of an element $H_* \in \mathfrak{h}_{\mathfrak{m}}$ such that $\alpha(H_*) > 0$ for all $\alpha \in \Delta_+^{\mathfrak{m}}$, and such that all the roots of $\mathfrak{k}$ with respect to $\mathfrak{t}$ are real and non-zero on H_*. Let $\Delta_+^{\mathfrak{k}}(H_*)$ denote the set of roots of $\mathfrak{k}$ with respect to $\mathfrak{t}$ which are positive on H_*, so that $\Delta_+^{\mathfrak{k}}(H_*)$ is a system of positive roots for $\mathfrak{k}$.

We recall that for all $\gamma \in \hat{M}$, $\mu(\gamma)$ denotes the highest weight (with respect to $\Delta_+^{\mathfrak{m}}$) of the representation of $\mathfrak{m}$ induced by any member of γ. For all $\gamma_o \in \hat{M}_o$, let $\mu_o(\gamma_o)$ denote the highest weight (with respect to $\Delta_+^{\mathfrak{m}}$) of any member of γ_o. For all $\beta \in \hat{K}$, let $\kappa(\beta)$ denote the highest weight (with respect to $\Delta_+^{\mathfrak{k}}(H_*)$) of any member of β. For every linear form ν on $\mathfrak{t}$, let $\nu^{\natural}$ denote its restriction to $\mathfrak{h}_{\mathfrak{m}}$.

<u>Lemma 2</u>. Let $\gamma_o \in \hat{M}_o$ and $\beta \in \hat{K}$ be such that $\kappa(\beta)^{\natural} = \mu_o(\gamma_o)$. Then $m(\beta,\gamma_o) = 1$. Moreover, $\mu_o(\gamma_o)$ occurs with multiplicity exactly 1 as a weight of the representation of $\mathfrak{m}$ induced by any member of β.

<u>Proof</u>. $\mu_o(\gamma_o)$ clearly occurs with multiplicity ≥ 1 as such a weight. If it occurs with multiplicity > 1, or if $m(\beta,\gamma_o) > 1$, then any member of β has a weight $\nu \neq \kappa(\beta)$ such that $(\kappa(\beta)-\nu)^{\natural} = 0$. But $\kappa(\beta)-\nu$ is a non-zero non-negative integral linear combination of the members of $\Delta_+^{\mathfrak{k}}(H_*)$, and so $(\kappa(\beta)-\nu)(H_*) > 0$, a contradiction.

Suppose $m(\beta,\gamma_o) = 0$. Then there exists $\gamma_o' \in \hat{M}_o$ such that $m(\beta,\gamma_o') \geq 1$ and $\mu_o(\gamma_o)$ is a weight of any member of γ_o'. In particular, any member of β has a weight ν' such that $(\nu')^{\natural} = \mu_o(\gamma_o')$. Now $\mu_o(\gamma_o')-\mu_o(\gamma_o)$ is a non-zero non-negative integral linear combination of the members of $\Delta_+^{\mathfrak{m}}$, so that $(\mu_o(\gamma_o')-\mu_o(\gamma_o))(H_*) > 0$. On the other hand, $\kappa(\beta)-\nu'$ is a non-negative integral linear combination of the members of $\Delta_+^{\mathfrak{k}}(H_*)$, so that

$$(\mu_o(\gamma_o)-\mu_o(\gamma_o'))(H_*) = (\kappa(\beta)-\nu')(H_*) \geq 0,$$

a contradiction. Thus $m(\beta,\gamma_o) \geq 1$, and the lemma is proved.

Let $\{H_1,\ldots,H_r\}$ be a basis of $\mathfrak{t}$. Let $\phi_1,\ldots,\phi_s$ be the elements of $\Delta_+^{\mathfrak{k}}(H_*)$. For all $i=1,\ldots,s$, let X_i be a non-zero root vector for $\mathfrak{k}$ for the root ϕ_i, and let Y_i be a non-zero root vector for $\mathfrak{k}$ for the root $-\phi_i$. Then

$$\{Y_1,\ldots Y_s, H_1,\ldots H_r, X_1,\ldots X_s\}$$

is a basis of $\mathfrak{k}$. For all s-tuples $(q) = (q_1,\ldots q_s)$ and $(p) = (p_1,\ldots p_s)$, and all r-tuples $(\ell) = (\ell_1,\ldots \ell_r)$, of non-negative integers, let

$$Z_{(q),(\ell),(p)} = Y_1^{q_1} \ldots Y_s^{q_s} H_1^{\ell_1} \ldots H_r^{\ell_r} X_1^{p_1} \ldots X_s^{p_s} \in \mathcal{K}.$$

Then $\{Z_{(q),(\ell),(p)}\}$ is a basis of $\mathcal{K}$. For all $x \in \mathcal{K}$, let $a_{(q),(\ell),(p)}(x) \in \mathbb{C}$ be determined by the condition

$$x = \sum_{(q),(\ell),(p)} a_{(q),(\ell),(p)}(x)\, Z_{(q),(\ell),(p)}.$$

<u>Lemma 3</u>. Let $x \in \mathcal{K}^{M_o}$. Then $a_{(q),(\ell),(p)}(x) \neq 0 \implies (q)$ and (p) are simultaneously zero or non-zero.

Proof. Let ad denote the representation of $\mathfrak{g}$ by derivations of $\mathfrak{U}$ which uniquely extends the adjoint representation of $\mathfrak{g}$ on itself. Then

$$0 = (\mathrm{ad}\, H_*)(x)$$
$$= \sum_{(q),(\ell),(p)} a_{(q),(\ell),(p)}(x)(\mathrm{ad}\, H_*)Z_{(q),(\ell),(p)}$$
$$= \sum_{(q),(\ell),(p)} \left(\sum_{i=1}^{s} (p_i-q_i)\phi_i(H_*)\right) a_{(q),(\ell),(p)}(x)Z_{(q),(\ell),(p)}.$$

Thus

$$a_{(q),(\ell),(p)}(x) \neq 0 \Longrightarrow \sum_{i=1}^{s} (p_i-q_i)\phi_i(H_*) = 0,$$

and this proves the lemma since $\phi_i(H_*) > 0$ for all $i=1,\ldots s$.

Let $\mathcal{J}$ be the subalgebra of $\mathfrak{U}$ generated by $\mathfrak{t}$ and 1.

Lemma 4. For every $x \in \mathcal{K}^{M_o}$, there is a unique element $g(x) \in \mathcal{J}$ such that $x-g(x) \in \sum_{i=1}^{s} \mathcal{K}\, X_i$.

Proof. The existence is clear from Lemma 3, and the uniqueness follows immediately from the fact that $\mathcal{J} \cap \sum_{i=1}^{s} \mathcal{K}\, X_i = \{0\}$.

Lemma 5. The linear map $g\colon \mathcal{K}^{M_o} \to \mathcal{J}$ defined by Lemma 4 is a homomorphism.

Proof. Let $x, y \in \mathcal{K}^{M}$. Then

$$xy - g(x)g(y) = x(y-g(y)) + (x-g(x))g(y).$$

Thus

$$xy-g(x)g(y) \in \sum_{i=1}^{s} \mathcal{K} X_i + \sum_{i=1}^{s} \mathcal{K} X_i \mathcal{J}$$

$$\subset \sum_{i=1}^{s} \mathcal{K} X_i,$$

and the lemma is proved.

For all $\gamma_o \in \hat{M}_o$ and $\beta \in \hat{K}$ such that $m(\beta,\gamma_o) = 1$, let $\chi_{\zeta_o(\beta,\gamma_o)}$ denote the homomorphism from $\mathcal{K}^{M_o}$ into $\mathbb{C}$ associated with the γ_o-primary part of any member of β. If M is connected, so that $M = M_o$, then $\chi_{\zeta_o(\beta,\gamma_o)}$ is the same as the previously defined $\chi_{\zeta(\beta,\gamma_o)}$. We regard $\mathcal{J}$ as the algebra of polynomial functions on the dual of $\mathfrak{t}$.

<u>Lemma 6</u>. Let $\gamma_o \in \hat{M}_o$ and $\beta \in \hat{K}$ be such that $\kappa(\beta)^{\natural} = \mu_o(\gamma_o)$. Then $\chi_{\zeta_o(\beta,\gamma_o)}$ is defined, and $\chi_{\zeta_o(\beta,\gamma_o)}(x) = g(x)(\kappa(\beta))$ for all $x \in \mathcal{K}^{M_o}$.

<u>Proof</u>. $m(\beta,\gamma_o) = 1$ by Lemma 2, so that $\chi_{\zeta_o(\beta,\gamma_o)}$ is defined. Let w be a non-zero highest weight vector for any member of β. Lemma 2 implies that w transforms under M_o according to γ_o. Lemma 6 now follows immediately from Lemma 4.

<u>Remark</u>. Lemmas 2, 3, 4, 5 and 6 hold for all G (rank 1 or not) for which Lemma 1 holds - for example, whenever $\mathfrak{g}_o$ is complex, or, more generally, split rank. For G complex, Lemmas 3, 4, 5 and 6 correspond to Lemma 1.1 of [11], due to Harish-Chandra.

We continue to assume that $\mathfrak{g}_o$ has rank 1. We now assume in addition that $\mathfrak{g}_o$ is split rank. Then it is known that M is connected.

Let I be the set of systems $\Delta_+^{\mathfrak{k}}(H_*)$ of positive roots of $\mathfrak{k}$, as H_* varies. Thus I is the set of systems of positive roots of $\mathfrak{k}$ containing $\Delta_+^{\mathfrak{m}}$. For all $Q \in I$, let D_Q denote the set of dominant linear forms for $\mathfrak{k}$ with respect to Q. Let $\mathcal{C}_Q$ be the map which associates to each element $\gamma \in \hat{M}$ such that $\mu(\gamma) \in D_Q$ the element $\beta \in \hat{K}$ defined by the condition that $\kappa(\beta) = \mu(\gamma)$ (where κ is defined with respect to Q). Let $f_Q = g$ (where g is defined with respect to Q). Then Lemmas 2 and 6 show that $\mathcal{S} = (\mathcal{C}_Q, f_Q)_{Q \in I}$ satisfies the first three parts of Definition 1 (§ 1).

At this point we invoke the classification of the rank 1 real semisimple Lie groups (see [8, Chapter IX]). The ideal of $\mathfrak{g}_o$ generated by $\mathfrak{a}_o$ is a rank 1 split rank real simple Lie algebra, and so must be isomorphic to one of the Lorentz algebras $\mathfrak{so}(1,2n-1)$ $(n \geq 2)$. Direct computation using Theorem 5 of Chapter II (§ 2c) shows easily that $\mathcal{S}$ satisfies the fourth part of Definition 1. Also, $\mathcal{S}$ is a strong system of minimal types for G (see Definition 2 (§ 1)).

Suppose now that $\mathfrak{g}_o$ is full rank (and rank 1). Then the ideal $\mathfrak{g}'_o$ of $\mathfrak{g}_o$ generated by $\mathfrak{a}_o$ is a rank 1 full rank real simple Lie algebra.

Let us assume that $\mathfrak{g}_0' \neq \mathfrak{su}(1,1)$. It is known then that M is connected, and that $\mathfrak{g}_0'$ must be isomorphic to one of the following: $\mathfrak{su}(1,n)$ $(n \geq 2)$, $\mathfrak{so}(1,2n)$ $(n \geq 2)$, $\mathfrak{sp}(1,n-1)$ $(n \geq 2)$, or the rank 1 real form of $\mathfrak{f}_4$.

Case-by-case computation shows that there is a system $\Delta_+^{\mathfrak{k}}(H_*)$ of positive roots for $\mathfrak{k}$ and a wall $\mathcal{W}$ of the corresponding Weyl chamber for $\mathfrak{k}$ in the dual of $\mathfrak{t}$, such that if G is simply connected (without necessarily having finite center), then $(\hat{\mathcal{W}})^{\natural} = \hat{D}_M$; here $\hat{\mathcal{W}}$ is the set of integral elements of $\mathcal{W}$, and $\hat{D}_M$ is the set of dominant integral linear forms for M with respect to $\Delta_+^{\mathfrak{m}}$. It follows that $(\hat{\mathcal{W}})^{\natural} = \hat{D}_M$ even if G is not necessarily simply connected. Moreover, the inverse of the restriction map from $\hat{\mathcal{W}}$ to $\hat{D}_M$ extends to a linear map L from the dual of $\mathfrak{h}_{\mathfrak{m}}$ into the dual of $\mathfrak{t}$, and $L(\hat{D}_M) = \hat{\mathcal{W}}$. L gives rise to a homomorphism $g': \mathcal{J} \to \mathcal{H}_{\mathfrak{m}}$ such that $g'(y)(\nu) = y(L(\nu))$ for all $y \in \mathcal{J}$ and all linear forms ν on $\mathfrak{h}_{\mathfrak{m}}$. Let $f: \mathcal{K}^M \to \mathcal{H}_{\mathfrak{m}}$ be the homomorphism $g' \circ g$ (where g is defined with respect to $\Delta_+^{\mathfrak{k}}(H_*)$) and let $\mathcal{C}: \hat{M} \to \hat{K}$ be determined by the condition that for all $\gamma \in \hat{M}$, $\mathcal{C}(\gamma)$ is that element $\beta \in \hat{K}$ such that $L(\mu(\gamma)) = \mathrm{K}(\beta)$ (where K is defined with respect to $\Delta_+^{\mathfrak{k}}(H_*)$). Then Lemmas 2 and 6 show that $\mathcal{S} = (\mathcal{C}, f)$ satisfies the first three parts of Definition 1. The fact that $\mathcal{S}$ satisfies the fourth part of Definition 1 follows from Theorems 3, 4, 6 and 8 of Chapter II (§§ 2a, 2b, 2d and 2e, respectively).

$\mathcal{S}$ is in fact a strong system of minimal types for G. Indeed, the first part of Definition 2 is trivially satisfied, and the second part follows easily by the method of proof of Lemma 2.

Finally, let us assume that $\mathfrak{g}_0' = \mathfrak{su}(1,1)$. First suppose that $\mathfrak{g}_0$ is simple (so that $\mathfrak{g}_0 = \mathfrak{g}_0'$) and that G is simply connected (G has infinite center in this case). Then G is isomorphic to the simply connected covering group of SU(1,1), K is isomorphic to the additive group $\mathbb{R}$ of real numbers, and M is isomorphic to the additive group $\mathbb{Z}$ of integers. For all $z \in \mathbb{C}$, let $\alpha_{\mathbb{R}}(z) \in \hat{\mathbb{R}}$ be the class defined by the homomorphism $t \to e^{tz}$ from $\mathbb{R}$ into the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers, so that we may identify $\hat{K}$ with $\hat{\mathbb{R}} = \{\alpha_{\mathbb{R}}(z) | z \in \mathbb{C}\}$. Also, for all $z^* \in \mathbb{C}^*$, let $\alpha_{\mathbb{Z}}(z^*) \in \hat{\mathbb{Z}}$ be the class defined by the homomorphism $n \to (z^*)^n$ from $\mathbb{Z}$ into $\mathbb{C}^*$, so that we may identify $\hat{M}$ with $\hat{\mathbb{Z}} = \{\alpha_{\mathbb{Z}}(z^*) | z^* \in \mathbb{C}^*\}$.

Now for all $z^* \in \mathbb{C}^*$, let $\mathcal{C}_{z^*}$ be the map from $\{\alpha_{\mathbb{Z}}(z^*)\} \subset \hat{M}$ into $\hat{K}$ which takes $\alpha_{\mathbb{Z}}(z^*)$ into $\alpha_{\mathbb{R}}(\log z^*)$, where log denotes any single-valued inverse of the exponential function such that $\log 1 = 0$ and $\log -1 = \pm \underline{i}\Pi$ (here $\underline{i}$ and Π have their usual meanings as complex numbers). Also, let f_{z^*} be the homomorphism from $\mathcal{J} = \mathcal{K}^M$ into $\mathbb{C} = \mathcal{H}_m$ which takes any polynomial function on the dual of $\mathfrak{t} = \mathfrak{k}$ into its value at the differential of $\alpha_{\mathbb{R}}(\log z^*)$, where this differential is regarded as a linear form on $\mathfrak{k}$. Then except for the fact that $\mathbb{C}^*$ is infinite, we have that $\mathcal{S} = (\mathcal{C}_{z^*}, f_{z^*})_{z^* \in \mathbb{C}^*}$ is a system of minimal types for G. Indeed, the first three parts of Definition 1 are clear; the fourth part follows from the case n=1 of Theorem 3 of Chapter II (§ 2a), together with our choice of the log function (we note that every finite dimensional irreducible representation of G factors through SU(1,1)). It is also clear that $\mathcal{S}$ is a strong system of minimal types for G (except for the fact that $\mathbb{C}^*$ is infinite).

Using a similar method, it is now easy to construct a strong system of minimal types (except for a finite index set) for G when G is simply connected but $\mathfrak{g}_0$ is not necessarily simple. Finally, it is easy to see that such a system immediately yields a strong system of minimal types (with finite index set) for G, if the simple connectivity assumption is replaced by the assumption that G have finite center.

Summarizing, we have:

Theorem 3. Every connected real semisimple Lie group which has real rank 1 and finite center admits a strong system of minimal types.

It would be interesting to determine whether all connected real semisimple Lie groups with finite center admit strong systems of minimal types.

Remark. The case dealt with above in which G is the simply connected covering group of SU(1,1) shows that the finite center hypothesis in Theorem 3 is needed if we want I to be finite in Definition 1.

Remark. We have obtained an alternate description of the elements $\mathcal{C}(\gamma(\alpha)) \in \hat{K}$ $(\alpha \in \hat{G})$ when G has real rank 1 and $\mathfrak{g}_0' \neq \mathfrak{su}(1,1)$, or when G is complex. Specifically, there is a system $S_+^{\mathfrak{g}}$ of positive roots for $\mathfrak{g}$ and a system $S_+^{\mathfrak{k}}$ of positive roots for $\mathfrak{k}$ determined naturally by $S_+^{\mathfrak{g}}$, such that for all $\alpha \in \hat{G}$, the element $\beta \in \hat{K}$ whose highest weight μ is the lowest such that $m(\alpha,\beta) \neq 0$ actually satisfies the condition $m(\alpha,\beta) = 1$, and in fact coincides with $\mathcal{C}(\gamma(\alpha))$ (here μ is the highest weight of β with respect to $S_+^{\mathfrak{k}}$, and "lowest" refers to an ordering defined in terms of $S_+^{\mathfrak{g}}$). Moreover, if μ' is the highest weight of any $\beta' \in \hat{K}$ such that $m(\alpha,\beta') \neq 0$, then $\mu'-\mu$ lies in a certain "positive cone" determined naturally by $S_+^{\mathfrak{g}}$. Hence $\mathcal{C}(\gamma(\alpha))$ is characterized by a property closely analogous to that of highest weight. If $\mathfrak{g}_0$ is full rank, the above holds for all possible choices of the wall $\mathcal{W}$ used in the description of the family of minimal types, and the positive systems $S_+^{\mathfrak{g}}$ for $\mathfrak{g}$ which correspond to all such choices can be characterized among all positive systems for $\mathfrak{g}$ by a natural geometric property.

Remark. If G is rank 1 split rank, then the element $\mathcal{C}(\gamma(\alpha)) \in \hat{K}$ $(\alpha \in \hat{G})$ satisfies the definition of "minimal type" given in [11, p. 390], and the notion of minimal type is appropriate for these groups. However, as was noted in the Appendix to §1, this notion is not appropriate in general.

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Biographical note

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