DEVELOPMENT OF AN ANGULAR ACCELEROMETER

by

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1937

Submitted in partial fulfillment of the

requirements for the degree of

MASTER OF SCIENCE

from the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1939

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Signature of Chairman of Dept.
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Contents

Acknowledgement

I want to express my thanks to Prof. H. E. Edgerton for his help and suggestions throughout the progress of the thesis, to Prof. W. V. Lyon for the use of his derivation of the expression for the torque developed by a short-circuited induction motor, and to Mr. H. J. Lawrence and Mr. J. F. Gotimer for their help and advice in the construction of the accelerometer.

Introduction

At present there are several devices which, although not necessary to engineering advancement, would find much use in the engineering field. One such device is an instrument to measure angular acceleration. One possible use of angular accelerometers is in the routine testing of motors; some simple acceleration curve---starting of the unloaded motor, for example---could be the test. Any serious defect in the winding, alignment, or magnetic circuit could be readily detected by comparison of the acceleration characteristic of the motor being tested with some standard acceleration characteristic. Other uses to which angular accelerometers could be put are: determining the accelerations in existing machines to serve as a guide for designs of future machines, and checking theoretical calculations of transient torques by actual measurement of the accelerations they produce.

In general, three methods can be used to measure angular acceleration. The oldest and most fundamental method is that used in retardation runs and in stroboscopic determination of acceleration. Here either the change in angular displacement or the change in angular velocity occurring within a known time interval is measured. The chief advantage of this method is the simplicity of the apparatus necessary, and the chief disadvantage is the low accuracy produced by the necessary numerical differentiation.

A second method of measurement of angular acceleration is that of measuring the torque necessary to accelerate a mass having a known moment of inertia and determining the acceleration from the relation between torque, moment of inertia, and angular acceleration. The application of this principle requires more complicated apparatus than the first method, but its chief disadvantage is that all applicable methods of measuring torque require relative motion between parts of the accelerometer. If the torque is measured by means of a torsion meter or a strain gage, the necessary motion is so great that springs and masses producing a low natural frequency of mechanical vibration must be used. Angular accelerometers depending solely upon this principle have been built[1,2]. They consisted of a number of electrically insulated discs mounted on a rigid shaft; the discs were free to rotate on the shaft but were normally held by springs so that electrical contacts between the shaft and the discs were open. When the acceleration of the shaft became high enough to make the torque produced by accelerating one of the discs greater than the torque produced by the spring, the contact on that particular disc would close. The exact instant at which each of the contacts closed was recorded by means of an oscillograph.

The third method of measuring acceleration is that of measuring a variable which, because of the physical arrangement of the apparatus, is a function of the angular accelera-

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1. See bibliography for numbered references.

tion of some part of the apparatus. This principle may or may not involve one of the preceding methods. One of the earliest applications of this method was made in 1927 [1]. Two coils and their associated magnetic circuits were rotated by a shaft whose acceleration was to be measured. The magnetic circuits were so arranged that tangential motion of masses of iron mounted on springs varied the lengths of air-gaps in the circuits. When the shaft was accelerated, the iron masses shifted their positions by distorting the springs, and the lengths of the air-gaps were changed. By measuring the currents flowing through the coils when an alternating voltage was impressed upon them, the change in the air-gaps could be determined. As the constants of the springs and the moment of inertia of the masses were known, the relation between acceleration and deflection could be determined.

The preceding method depends upon the relation between torque, moment of inertia, and angular acceleration, and has the undesirable characteristics mentioned in connection with torsion meters. Another method, which is independent of this relation, has been used in England [4] and has received some attention at M. I. T. [5,6,7,8] If the armature of a separately-excited d-c generator, having negligible ripple voltage and internal resistance, is rotated, the generated voltage is proportional to the angular velocity of the rotor. If a condenser is connected to armature terminals, the current flowing into the condenser is proportional to the time rate of change of terminal voltage, which in turn is proportional to the time rate of change of rotor angular velocity, or angular acceleration. The condenser current and armature voltage can be measured by means of an oscillograph, and the angular acceleration and velocity determined. If the angular acceleration is high, the current will be great enough to actuate the oscillograph directly. A vacuum-tube amplifier can be used to measure low accelerations.

In the application of this method the generator shaft must be rigidly coupled to the shaft whose acceleration is to be measured; hence, the disadvantages of low natural frequency of mechanical vibration and damping do not appear. The generator can be made small in size and light in weight, and consequently, it can be moved easily from one machine to another.

As the d-c generator type of accelerometer has the advantage of being capable of measuring a wide range of accelerations, and of producing a continuous record throughout a transient process, this type of accelerometer was chosen to be used in this thesis.

Theory

Assume a separately-excited, d-c generator with negligible internal resistance and ripple voltage connected as shown in Fig. 1. As the generator is separately excited, the generated voltage e will be proportional to the angular velocity w of the rotor, or

$$w = ke \text{ ------- } (1)$$

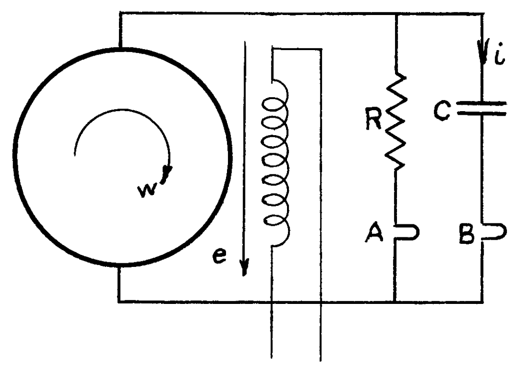


Fig. 1

The voltage will produce a deflection of element A, and the angular velocity can be determined from this deflection. The current i flowing to the condenser c will be

$$i = c\frac{de}{dt} \text{ ------------------- } (2)$$

Differentiating (1) with respect to time,

$$\frac{dw}{dt} = k\frac{de}{dt} .$$

Solving (2) for $\frac{de}{dt}$ and substituting, the angular acceleration becomes

$$a = \frac{dw}{dt}$$

$$= \frac{k}{c} i .$$

The current i can be determined from the deflection of element B.

When measuring acceleration by the d-c generator method the most important step is obtaining a generator having low ripple voltage. Calculations of the rates of change of voltage during ordinary transients show that the ripple voltage should be less than 0.1% of the average generated voltage, if the current charging a condenser is to have a negligible ripple. To reduce the ripple to this low value a generator of special construction is necessary. It has been shown by previous research [5,6] that the grain direction in the steel laminations of rotors of the customary construction produces appreciable ripple, even if the laminations are carefully stacked in such a fashion as to produce equal magnetic reluctance through the rotor iron along all diameters. Hence, a rotor of non-magnetic material is desirable. Other factors which help keep ripple low are: a large number of commutator segments, a small number of poles, high ratio of pole arc to pole pitch, and good contact between commutator and brushes. The current carried by the armature of this type of generator is too low to produce any shift in the field, and consequently the problem of commutation does not appear.

By making the rotor of fabric-base bakelite, the possibility of ripple produced by grain direction was eliminated, and at the same time the construction of a laminated armature was avoided. The field structure was made with four poles, as this was the minimum number which made a simple arrangement of armature conductors possible. The ratio of pole arc to pole pitch was approximately one to two; higher values increased field leakage flux to a prohibitive value. The commutator was made of copper, and the brushes which finally proved successful were of copper-graphite. This brush-commutator combination produced a much lower brush drop than would have existed if carbon brushes had been used.

When measuring transient accelerations, which alter magnitudes rapidly, it is necessary that the rotor of the generator have a natural frequency of mechanical vibration well above the frequency of the acceleration fluctuations. If the two are approximately equal, the value of acceleration obtained from the measurement will be much larger than the actual acceleration of the shaft driving the accelerometer. Itwas not expected that accelerations alternating at frequencies over 60 cps would be measured; so a natural frequency of approximately 100 cps or more was considered acceptable. To obtain this natural frequency a comparatively heavy shaft was necessary.

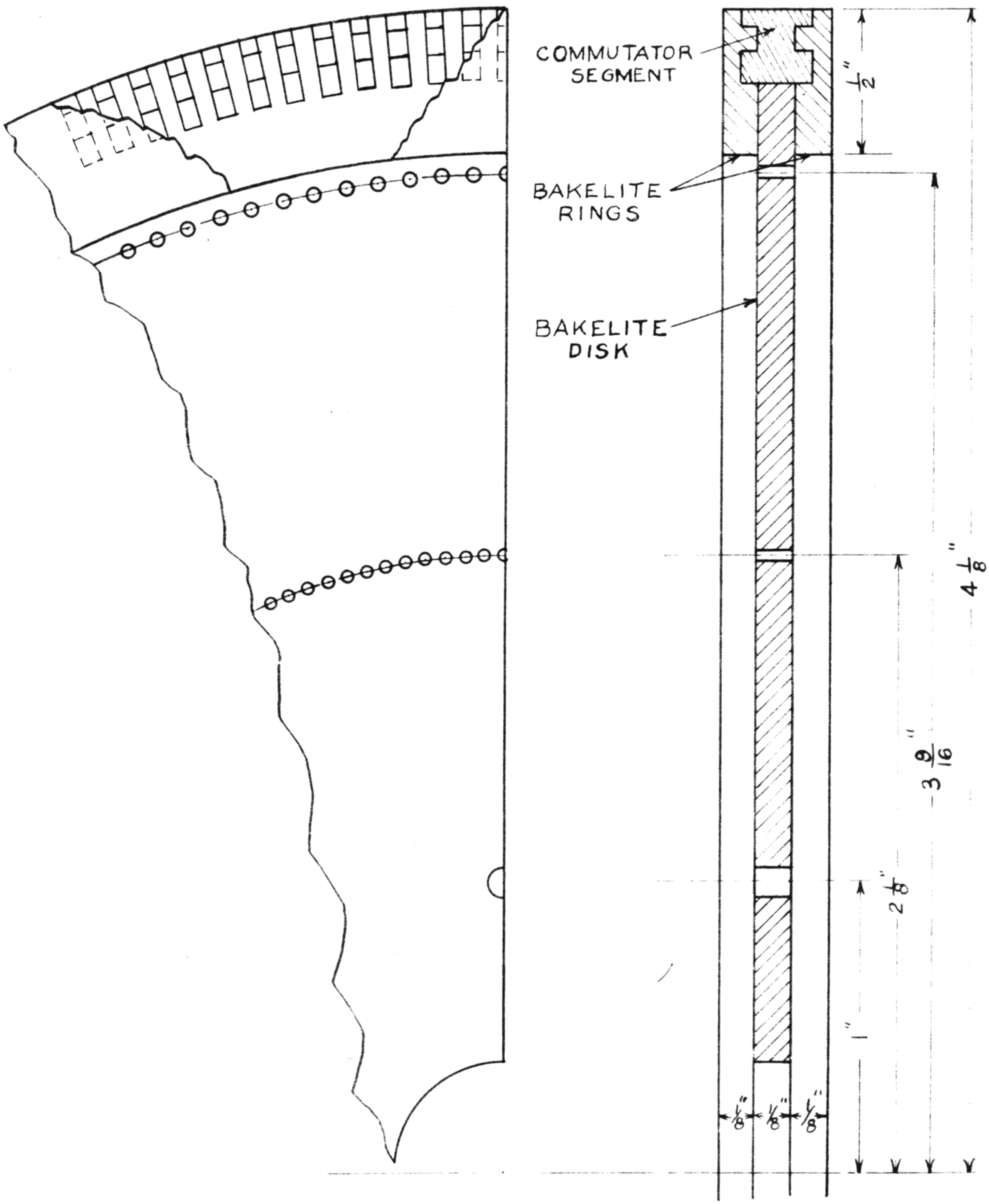
The constants of the circuit in which the current is measured are also quite important. The condenser, of course, must have little leakage current at the voltage impressed upon it. The time constant of the measuring circuit should be as low as practicable. If there were no resistance in either the armature or the oscillograph element, any acceleration could be measured, regardless of the value of the normal voltage of the generator, by using the proper capacitance. However, the armature and oscillograph element have a finite resistance, and hence the measuring circuit has a finite time-constant. If the time-constant of this resistance-capacitance circuit is large, much error will occur in the measurement of rapidly fluctuating accelerations, because of the time required to charge the condenser through the resistance. For this reason the normal voltage of the generator should be high (to permit the use of small capacitance), and the armature resistance should be low. In the case of a generator having no iron in the rotor the problem of a low time-constant is particularly important, because the long air gaps limit the flux, making the generated voltage per turn of armature conductors much lower than is usual for machines with iron rotors.

The generator constructed was wave-wound and designed to have a terminal voltage of 10 volts at a speed of 1800 rpm. The armature winding was designed to have approximately 3.5 ohms resistance, which would produce approximately 7 per cent voltage drop when the output current was 0.2 amperes (enough current to produce approximately 0.7 inch deflection simultaneously on each of two standard oscillograph elements.)

Construction

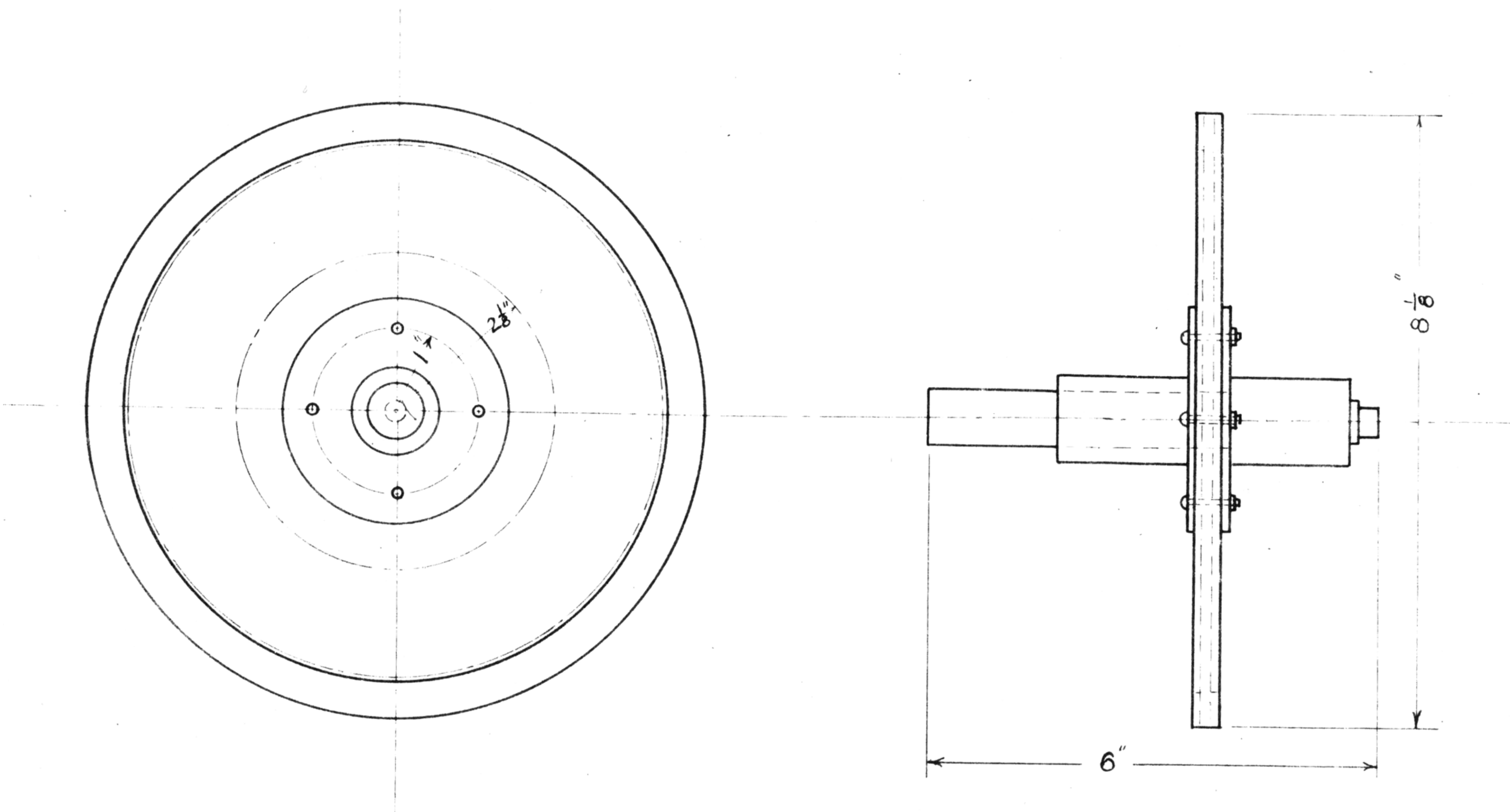
In order to make the air-gaps short and to simplify mechanical construction, a disc type of rotor was used. The commutator was built as an integral part of the rotor, the segments being mounted around the armature periphery. The individual segments were fitted into slots in the rotor, and held in place by bakelite rings, as is shown in Fig. 2. Commutator segments were made by cutting pieces of the proper length from a copper bar, which had been slotted on two opposite sides. Contacts between the commutator segments and the windings were made by short wires soldered to the segments and passing through radial grooves in the bakelite rings.

The armature windings were held on the disc by passing the wires through holes in the disc at every point at which a conductor changed its direction, and by giving the windings and the disc a thick coat of cement. The holes through which the windings passed were drilled on a drill press, equal spacing between holes being obtained by the use of a dividing head. Winding the armature proved to be quite a task. In order to keep the disc as rigid as possible, small holes were made, and hence wire with thin insulation had to be used. Drawing the wire through the holes tended to break the insulation and short circuit some of the turns. The complete rotor is



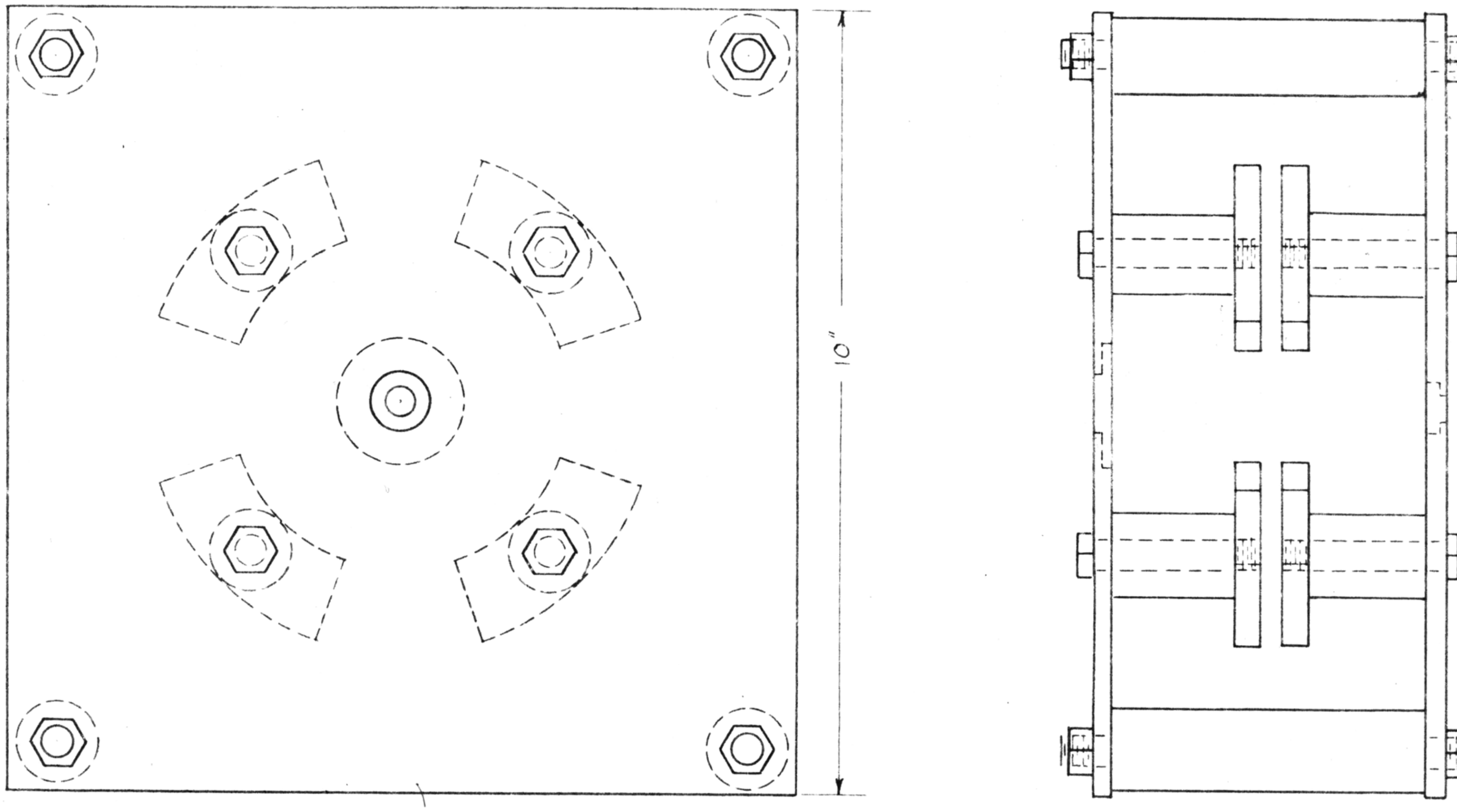
ARMATURE DETAILS

Fig. 2



ROTOR ASSEMBLY

Fig. 3



FRAME AND FIELD ASSEMBLY

Fig. 4

is shown in Fig. 3.

The field structure (see Fig. 4) was made by the Electrical Engineering Department. The only point worthy of mention in the field structure is that the pole shoes were prevented from rotating, with respect to the end plates, by pins through the pole shoes and other pins through the end plates.

The brush structure can be seen in Fig. 5. Four rectangular brushes, each wide enough to cover two commutator segments, were used. The brush-holders were box-shaped, and the brushes were pushed radially against the commutator by helical springs.

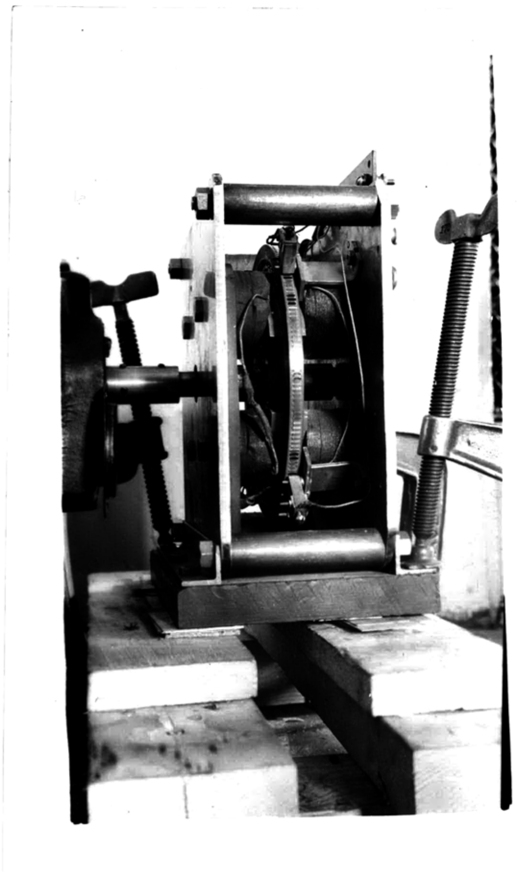


Fig. 5

Experiments and Results

The voltage of the generator was approximately 15 volts for the normal field current and rated speed. This was 50 per cent higher than was expected. The frequency components of the generator ripple voltage were measured by the use of a General Radio wave analizer, and it was found that the ripple voltage was approximately 0.2 per cent of the average voltage, the ripple having its rms value. The per cent ripple was independent of speed and field excitation, and the largest component of ripple had a frequency of four times the rotor speed in rps. Hence, the ripple must have been produced either by some lack of symmetry in the rotor winding or by successive changes in the length of the air gap produced by bending of the end plates. It is the personal opinion of the author that this ripple was produced chiefly by a slight eccentricity of the rotor disc. The effect of eccentricity would be to induce higher voltage in the conductors farther from the center of rotation because of their higher speed. This higher voltage would occur each time the eccentric section passed a pole, thus producing the observed frequency.

The ripple produced by the commutator was found to be approximately 0.02 per cent, which was not excessive. Experiments proved, however, that if the current output became high when the commutator was dirty or coated with oxide, the changing contact resistance between commutator and brushes produced high-frequency ripple. This ripple could be greatly reduced by sanding the commutator.

A flexible coupling was used at first, but was later replaced by a rigid coupling. In measuring rapidly changing accelerations it was found that if the flexible coupling were used, the acceleration curves obtained were inconsistent. The rigid couplings used consisted of collars which fitted over the two shafts, and were provided with set-screws to prevent relative rotation between the shafts.

As a test of the accelerometer , the accelerations of two motors during starting and during full-speed reversal were measured. Typical oscillograms of these processes are shown in Figures 6 to 9, inclusive. Figures 6 and 7 are from a one-tenth horse-power, d-c shunt motor, and Figures 8 and 9 are for a 15 kw, three phase induction motor. The flexible coupling was used in taking the oscillogram shown in Fig. 6. Rigid couplings were used in taking the others. Acceleration curves of the d-c motor showed more inconsistency than those of the induction motor. This effect was probably produced by the commutator of the d-c machine.

As there was no standard from which the accelerometer could be calibrated, some other method had to be devised to check the extent of error in the acceleration measurements. The check selected depended upon the relation

$$\int_{t_1}^{t_2} a \, dt = w_2 - w_1$$

where

a = instantaneous acceleration

$w_2 - w_1$ = change in angular velocity

t = time

$t_2 - t_1$ = time required to produce the change in angular velocity.

The integral was obtained by measuring the area between the acceleration curve and the zero acceleration axis with a planimeter.

The errors in the change in speed as obtained from the area measurements are given in Table I.

Table I

Fig	Process	Motor	$\int_{t_1}^{t_2} a\,dt$	Actual Change in Speed	Error
6	Line Start	1/10 hp, d-c Shunt	1410 rpm	1340 rpm	5.2%
7	Full Speed Reversal	1/10 hp, d-c Shunt	2750 rpm	2620 rpm	4.9%
8	Line Start	15 kw, Three-phase Induction	1820 rpm	1790 rpm	2.2%
9	Full Speed Reversal	15 kw, Three-phase Induction	3840 rpm	3420 rpm	6.6%

This method of measuring error does not give values of error for the instantaneous values of acceleration, but is more or less an average error for the complete transient process.

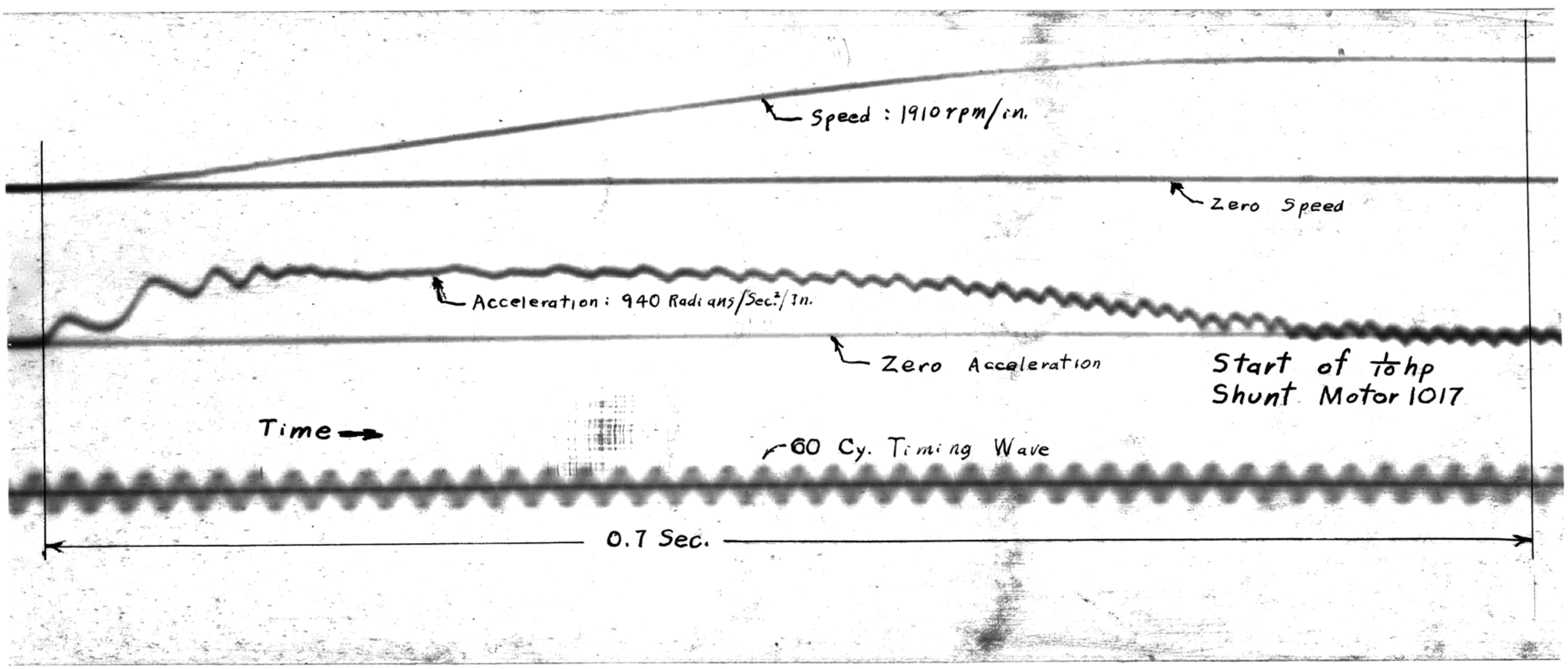

Speed : 1910 rpm/in.
Zero Speed
Acceleration : 940 Radians/Sec.²/In.
Zero Acceleration
Start of $\frac{1}{10}$ hp
Shunt Motor 1017
Time
60 Cy. Timing Wave
0.7 Sec.

Fig. 6

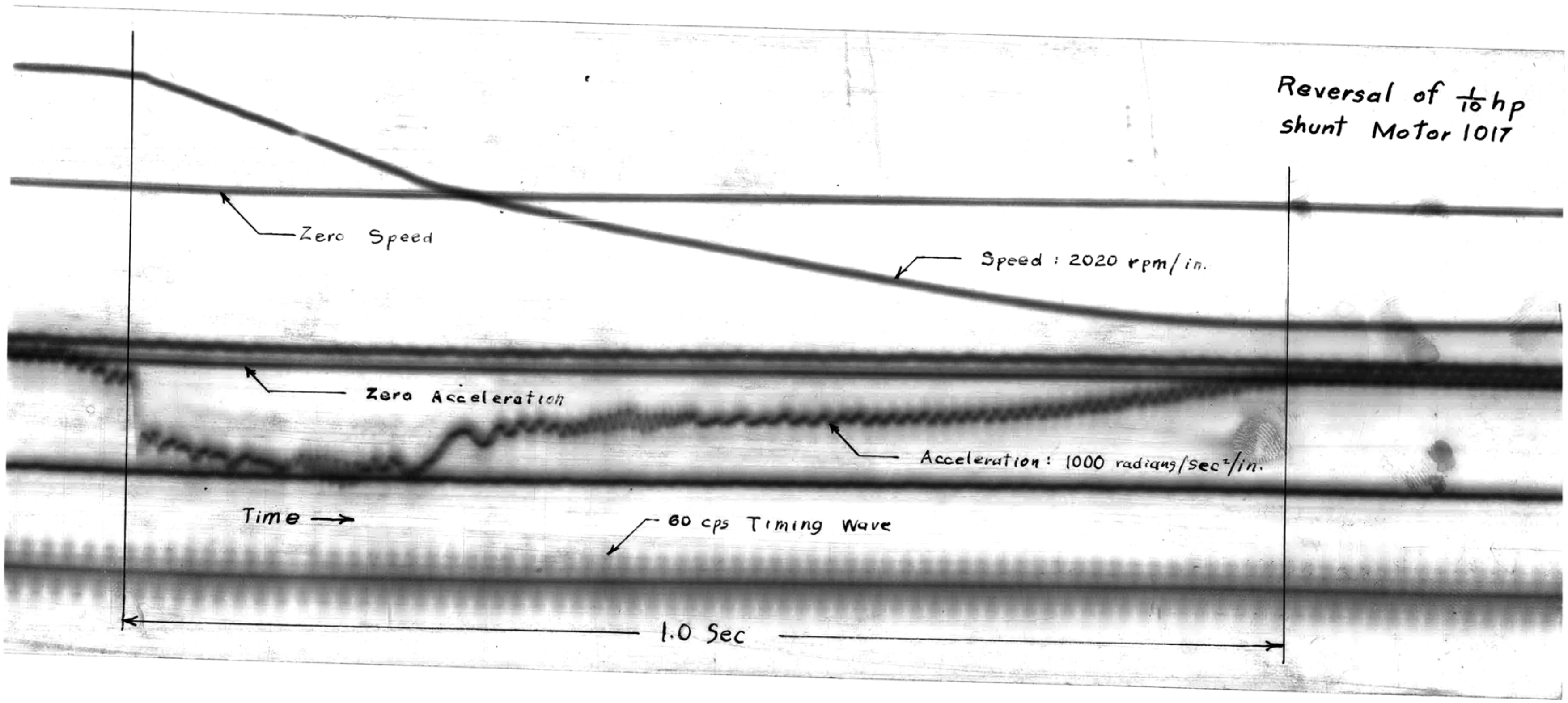

Reversal of 1/10 hp shunt Motor 1017
Zero Speed
Speed : 2020 rpm/in.
Zero Acceleration
Acceleration : 1000 radians/sec²/in.
Time →
60 cps Timing Wave
1.0 Sec

Fig. 7

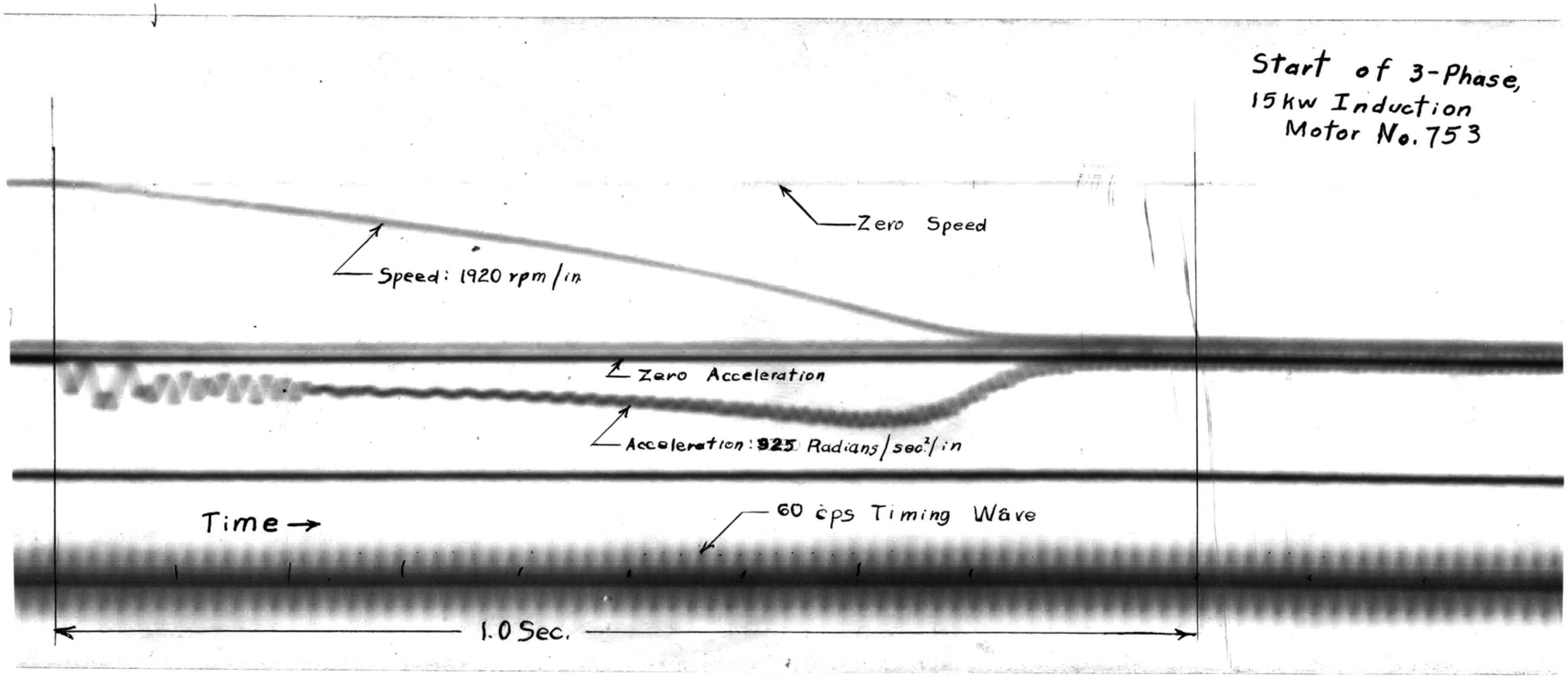

Start of 3-Phase,
15 kw Induction
Motor No. 753
Zero Speed
Speed: 1920 rpm/in
Zero Acceleration
Acceleration: 925 Radians/sec²/in
60 cps Timing Wave
Time →
1.0 Sec.

Fig. 8

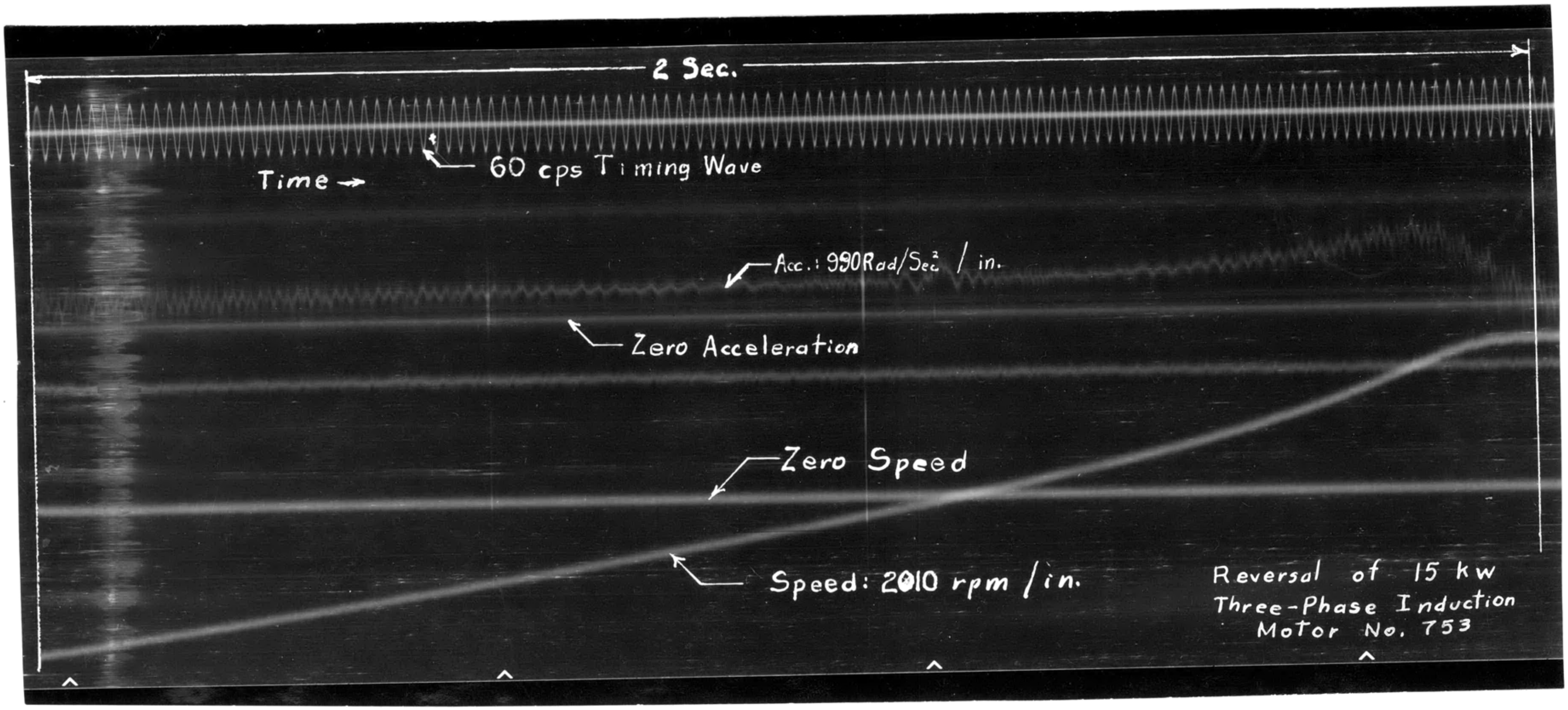

2 Sec.
Time →
60 cps Timing Wave
Acc.: 990 Rad/Sec² / in.
Zero Acceleration
Zero Speed
Speed: 2010 rpm / in.
Reversal of 15 kw
Three-Phase Induction
Motor No. 753

Fig. 9

It was thought that the comparison of calculated and measured accelerations during some transient condition of electrical machinery would be a valuable test of the performance of the accelerometer. The transient selected was that occurring immediately after all of the lines feeding a three-phase induction motor are short-circuited, the short-circuit occurring while the motor was unloaded. Reactors were placed in the lines feeding the motor, as shown in Fig. 10, to limit the line currents. The effect of these reactors upon the transient was to decrease the motor current and torque slightly.

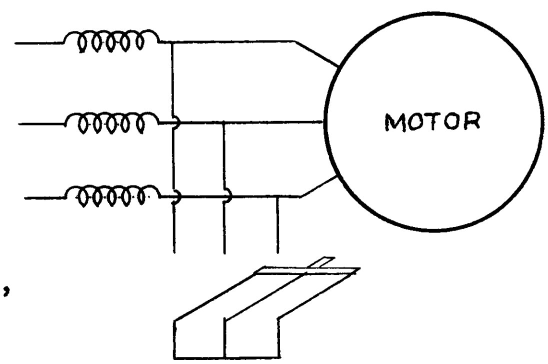


Fig. 10

If the windage and friction of the motor are neglected, the torque T, moment of inertia J, and angular acceleration a, of the rotor will have the relation

$$a = \frac{T}{J}$$

The approximate expression for acceleration following short-circuit of the terminals of a three-phase induction motor is derived in Appendix II, and is

$$a = 0.738\,\frac{9Pr_d I_n^2}{4J\sigma^2}\,\epsilon^{-\frac{2k}{\sigma}t}\left[\frac{n}{\frac{k^2}{\sigma^2}+n^2}\,(1-\cos nt) + \frac{n^2}{\frac{k}{\sigma}\left(\frac{k^2}{\sigma^2}+n^2\right)}\,\sin nt\right],$$

where the symbols and their values for the 15 kw, 220 volt, three-phase, 1800 rpm induction motor used in the test are:

a = acceleration in radians per sec per sec

p = number of poles = 4

J = moment of inertia of rotor in lb - ft^2 = 15.75

g = acceleration due to gravity = 32.2 ft per sec per sec

n = rotor speed in electrical radians per sec = 371

r_1 = stator resistance in ohms = 0.0492

r_2 = rotor resistance in ohms = 0.0851

L_1 = stator inductance in henrys = 0.0142

L_2 = rotor inductance in henrys = 0.022

$$k = \frac{r_1}{L_1} \cong \frac{r_2}{L_2} = 3.45$$

M = mutual inductance between stator and rotor in henrys = 0.0167

$$\sigma = 1 - \frac{M^2}{L_1 L_2} = 0.167$$

Substituting these values in the equation for acceleration,

$$a = [58.3 - 58.3 \cos 371t + 649 \sin 371t] \, \mathcal{E}^{-66.8t}$$

radians per sec per sec.

The acceleration was measured; the oscillogram is shown in Fig. 11. The ripple, which was quite regular both before and after the transient occurred, was subtracted from the transient acceleration, and the measured

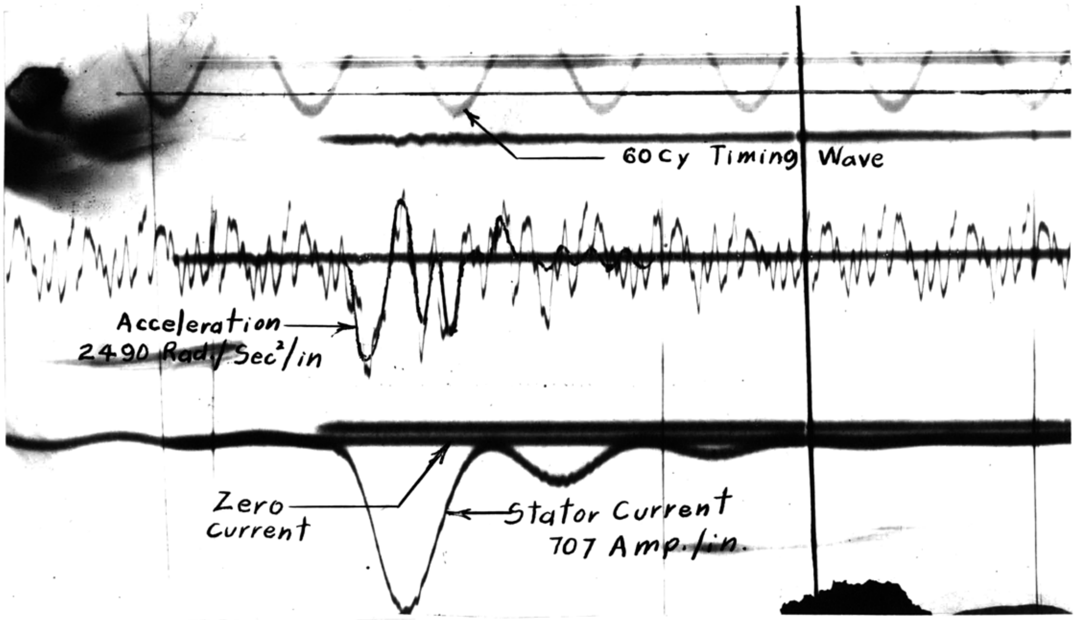


Fig. 11

acceleration plotted with the calculated acceleration in Fig. 12.

As can be seen from Fig. 12, the agreement between calculated and measured accelerations was very poor. This lack of agreement was produced by mechanical vibrations of the rotor itself. The reasons for the apparently random nature of the measured acceleration were the non-uniform damping of mechanical oscillations and the number of frequencies with which the various parts of the disc can vibrate. Each part of the rotor has a spring constant and a moment of inertia. The interaction of these parts after a sudden impulse is applied produces a motion having a large number of frequency components.

As mentioned before, a natural frequency of mechanical vibration of the rotor of over 100 cps was considered high enough. For the measurement of rapidly damped 60 cps

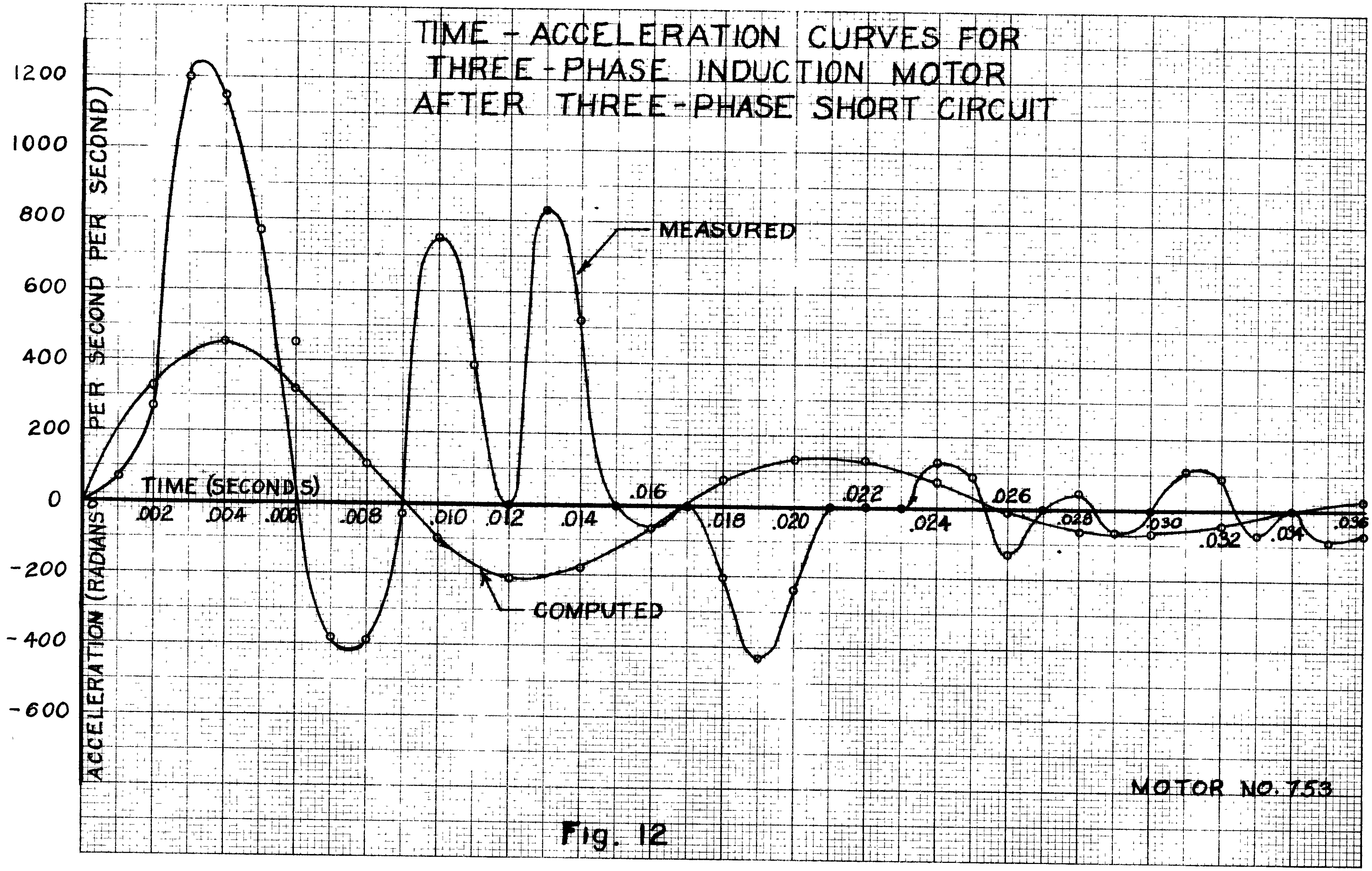


Fig. 12

FORM 4 T

oscillations, however, a natural frequency of at least 500 cps is necessary. The initial shaft of the accelerometer was replaced with a heavier one in an effort to raise the natural frequency. The second shaft made the natural frequency of the rotor, when considered as an inertia at the end of a spring, well above 1000 cps. As the lowest frequency observed in the oscillogram shown in Fig. 11 was approximately 140 cps, the vibration was evidently not due entirely to the shaft. Unsuccessful attempts were made to measure the natural frequency of mechanical vibration directly. To obtain an approximate value of the natural frequency, the spring constant of the rotor was measured by holding the shaft rigidly in a lathe and applying a force tangentially to the armature disc. The moment of inertia of the outer part of the disc was calculated and the natural frequency computed. The frequency found by this method was approximately 240 cps. This value is higher than the actual natural frequency because of several reasons: The moment of inertia of the windings and cement was neglected; the mass distributed in the disc would tend to lower the natural frequency of the disc; the long shaft of the motor would lower the spring constant to a value below that measured. The effect of having the end of the shaft connected to a rotor with finite inertia would be to lower the natural frequency, but because of the large inertia of the rotor of the motor, compared with the inertia of the rotor of the accelerometer, the natural frequency would be approximately the same as it would be if the rotor of the motor were infinite.

Conclusions

The angular accelerometer as it now stands will measure accelerations of over 100 radians per second per second with less than eight per cent error if the acceleration has no very sudden changes in magnitude. The accelerometer is unable to measure rapidly changing accelerations because of its low natural frequency of mechanical vibration.

-- Suggestions for further study --

The development of an angular accelerometer which will measure accelerations of the type produced by an inductive motor immediately after short circuit should be an interesting problem. If a continuous record of the acceleration is to be obtained, the d-c generator type of accelerometer will probably give the best results because it is the only type of accelerometer which gives a continuous record and does not have an inherently low natural frequency of vibration. The lowest natural frequency of the rotor should be at least 500 cps. As the natural frequency of a massless disc having a heavy rim varies directly as the square-root of the disc thickness (See calculations of natural frequency in Appendix I.), a thick disc does not help much. The only practical way of increasing the natural frequency of the disc is to make the disc with a larger diameter than was used in this thesis, and to couple the disc to the shaft at a distance of three or four inches from the center of the shaft. The shaft, of course, should have enough rigidity to keep the natural frequency of the rotor high, and should be coupled to the shaft of the motor whose acceleration is to be measured as rigidly as is practicable.

APPENDIX I

Design

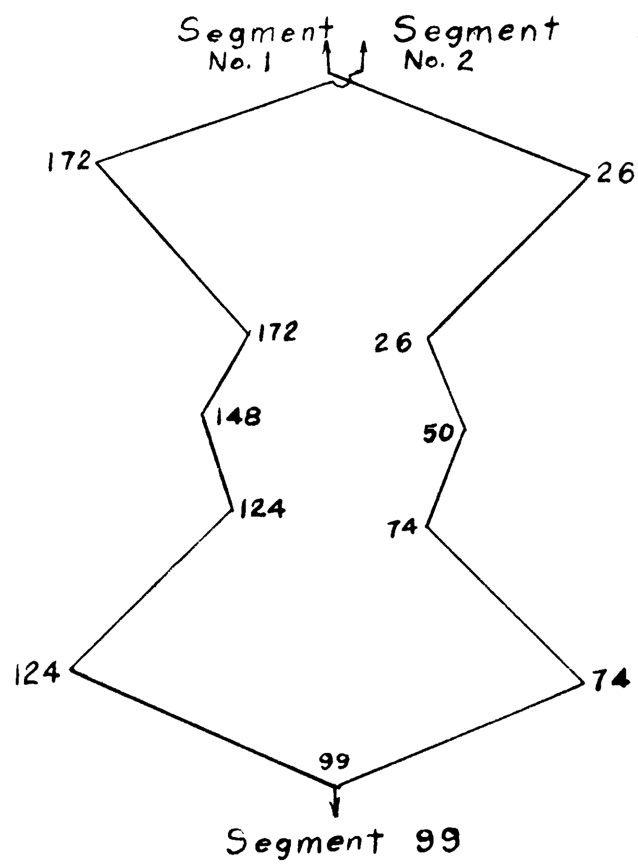
Dimensions of disc

Assume: number of conductors = 195

number of commutator segments = 195

conductors are of no.28 double-silk-covered wire

connections as shown in wiring diagram



WIRING DIAGRAM

Number of wires passing through each hole = 4.

Necessary hole diameter = $D(1 + \sqrt{2})$,

where D = wire diameter = 0.0166 in.

Hole diameter = 0.0166(1 + 1.414) = 0.0355 in.

Angle between holes = $\frac{360}{195}$ = 1.85 deg.

Assume inner circle of holes drilled on a circle of 2 1/8 in. radius.

Distance between hole centers = 2(2.125) sin (0.5)(1.85)

= 0.0686 in.

Width of material between holes = 0.0686-0.0355

= 0.0331 in.

With 2 1/8 in. radius for the smaller circle of holes, the pole shoe will fit between the end conductors of the coils if the radial width of pole shoes is one in. and the outer circle of holes has a radius of 3 9/16 in. Let the inside diameter of the ring holding the commutator be 1/4 in. greater than the diameter of the outer circle of holes, and let the ring holding the commutator be 1/2 in. wide.

Outside diameter of disc = 2(3.563) + 0.125 + 2(0.5)

= 8.25 in.

Air-gap mmf

Inside radius of pole shoe = 2 3/16 in. = r_1.

Outside radius of pole shoe = 3 3/16 in. = r_2.

Assume pole arc = θ = 52 deg.

$$\text{Area of pole shoe} = \frac{\theta}{360}\pi(r_2^2 - r_1^2)$$

$$= \frac{52}{360}\pi\left[(3.188)^2 - (2.188)^2\right]$$

$= 2.45$ sq in.

$= 15.8$ sq cm.

Let 10 volts = generated voltage.

Assume four poles, wave winding, and 1800 rpm.

$$\phi = \frac{60\ a\ E\ (10)^8}{n\ p\ z}, \text{ where}$$

ϕ = flux per pole

n = speed in rpm

z = total number of armature conductors

p = number of poles

a = number of armature parallels.

$$\phi = \frac{60(2)(10)(10)^8}{1800(4)(195)}$$

$= 85{,}300$ lines per pole.

$$\text{Air-gap flux density} = \frac{85{,}300}{15.8}$$

$= 5{,}400$ gauss

Let air-gap length = 0.25 in.

$= 0.635$ cm.

$$\text{Air-gap mmf} = \frac{B\ l}{0.4\pi}, \text{ where}$$

B = flux density in gauss

l = air-gap length in cm.

$$\text{Air-gap mmf} = \frac{5{,}400\ (0.635)}{0.4\ \pi}$$

$= 2{,}720$ amp turns.

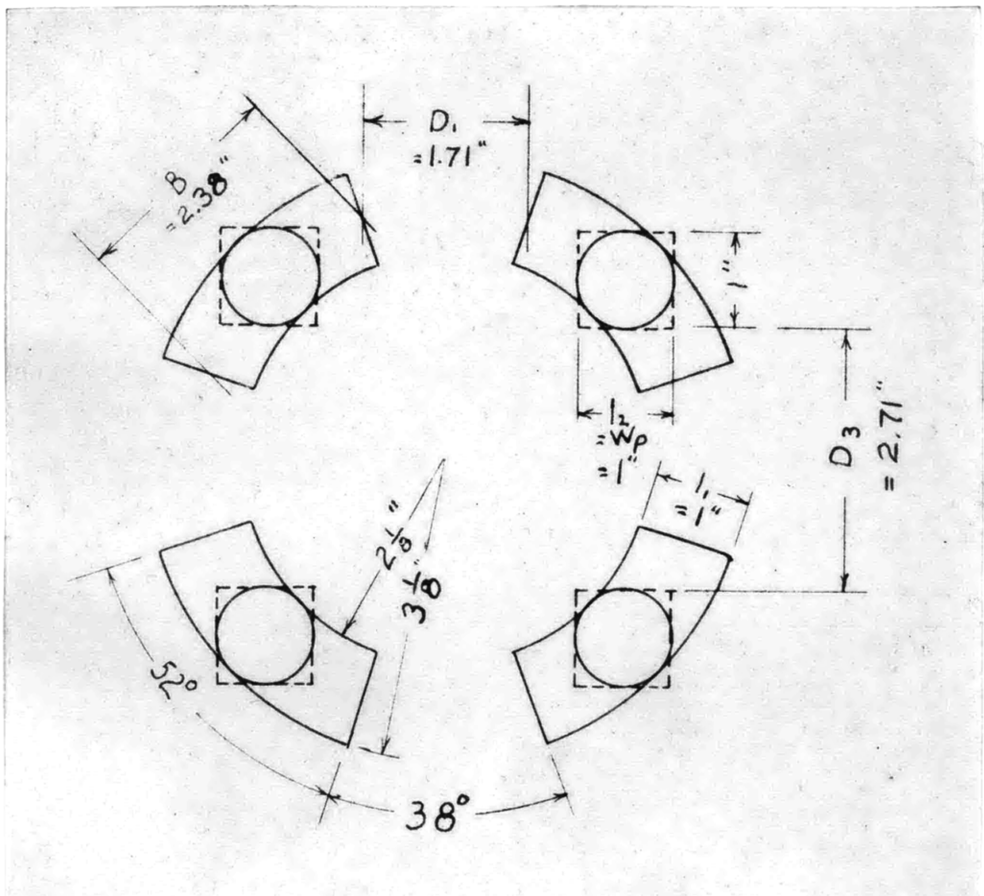
Leakage flux

Assume the dimensions given in the diagram and that the leakage is from parallel surfaces as indicated by the dotted lines. Let the total mmf of the field be absorbed in the air-gap.

Let: pole shoe thickness = h_s = 0.375 in.

pole piece length = h_p = 1.5 in.

Using the leakage flux formulas from "Machine Design" by A. D. Moore,



ϕ_1 = flux between inner pole-shoe surfaces

$$= 2(3.2)(2X)\frac{l_1 h_s}{D_1}\text{ , where}$$

X = amp turns per pole

= 1360 amp turns

$$\phi_1 = 2(3.2)(2720)\frac{(1)(0.375)}{1.71}$$

$= 3,820$ lines per pole

ϕ_2 = flux between lateral pole surfaces

$= 8.15 \times h_s \ln \left[1 + \frac{\pi B}{2 D_1}\right]$

$= 8.15(1360)(0.375) \ln \left[1 \quad \frac{\pi(2.38)}{2(1.71)}\right]$

$= 4,780$ lines per pole.

ϕ_3 = flux between inner pole piece surfaces

$= 3.2(2X) \frac{h_p l_2}{D_3}$

$= 3.2(2720)\frac{(1)(1.5)}{2.71}$

$= 4810$ lines per pole

ϕ_4 = flux between lateral pole piece surfaces

$4.08\ h_p\ X \ln \left[1 + \frac{\pi W_p}{2D_3}\right]$

$4.08(1.5)(1360) \ln \left[1 + \frac{\pi(1)}{2(2.71)}\right]$

$3,770$

Total leakage flux $= \phi_L = \phi_1 + \phi_2 + \phi_3 + \phi_4$

$= 17,180$ lines per pole.

<u>Flux densities</u>

Total flux per pole $= \phi + \phi_L$

$= 85,300 + 17,180$

$= 102,480$ lines.

Cross-sectional area of pole piece $= \frac{\pi(1)}{4}$

$= 0.785$ sq in.

Pole piece flux density $= \frac{102,500}{0.785}$

$= 130,600$ lines per sq. in.

This is too high for cold-rolled steel; hence Norway iron should be used.

Pole shoe flux $= \phi + \phi_1 + \phi_2$

$= 93{,}900$ lines

Pole shoe cross-sectional area $= 1(0.375)$

$= 0.375$ sq in.

Maximum pole shoe flux density $= \frac{93{,}900}{2(0.375)}$

$= 125{,}000$ lines per sq in.

Assume that the flux in the end plates is evenly distributed in a radial direction around the pole pieces.

Area crossed by end-plate flux $= \pi(1)(.25)$

$= 0.785$ sq in.

End-plate flux density $= \frac{102{,}480}{0.785}$

$= 130{,}600$ lines per sq in.

This is a high value of flux density, but it falls off rapidly as the flux distributes itself in the plates.

Field winding

Mmf required per air-gap $= 2{,}720$ amp turns.

Assume 360 turns per coil and two coils per air-gap.

Turns per gap $= 2(360) = 720$ turns.

Field current $= \frac{2{,}720}{720} = 3.78$ amp.

Cross-section area of no. 20 wire $= 0.000805$ sq in.

Current density in field winding $= \frac{3.78}{0.000805}$

$= 4{,}700$ amp per sq in.

Generator voltage regulation

From measurement, the length of wire per conductor is 11.1 in.

Resistance of no. 28 copper wire = 0.06617 ohms per ft.

Total resistance of armature winding

$$= \frac{195(11.1)(.06617)}{12}$$

$$= 12.45 \text{ ohms.}$$

Resistance of rotor when connected in

two parallels $= \frac{12.45}{4}$

3.11 ohms.

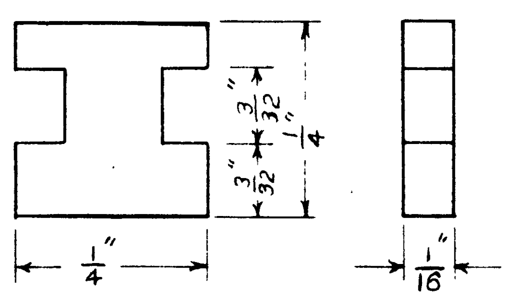
With 0.2 amp current, voltage drop = (0.2)(3.11)

= 0.622 volts.

Voltage regulation $= \frac{0.62(100)}{10}$

= 6.22%

Rotor stresses



COMMUTATOR SEGMENT

Density of copper = 0.318 lb per cu in.

Mass of one commutator bar

$$= 0.318(.0625)\left[(0.25)(0.25) - 2(.0625)(.0938)\right]$$

$$= 0.00122 \text{ lb.}$$

Centrifugal force on one commutator bar is

$$f = \frac{M\,r\,w^2}{g}, \text{ where}$$

f = force in lb

r = distance from center of gravity to center of rotation in ft

w = angular velocity in radians per second

g = gravitational acceleration

= 32.2 ft per sec per sec.

When the rotor speed is 1800 rpm,

$$f = \frac{0.00122\ (4/12)\ (189)^2}{32.2}$$

$$= 0.45 \text{ lb.}$$

The total hoop tension in the rings holding the commutator can be found from the breadth - factor formula used in a-c machinery analysis. The total tension becomes

$$T_1 = f\ n\ \frac{\sin \frac{n\theta}{2}}{n \sin \frac{\theta}{2}}$$

$$= f\ \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}, \text{ where}$$

θ = angle between bars

n = number of commutator bars in one half of the disc

=97.

$$T_1 = 0.45\ \frac{\sin \frac{97(1.85)}{2}}{\sin \frac{1.85}{2}}$$

$= 27.8$ lb.

A = cross-sectional area of both rings

$= 2\left[(0.125)(0.5) - (.0938)(.0625)\right]$

$= 0.1132$ sq in.

W = weight of one-half of both rings

$= \pi\ r\ A\ D$, where

r = mean radius of rings

$= 3.875$ in.

D = density of bakelite

$= 0.05$ lb per cu in.

$W = \pi(3.875)(0.113)(0.05)$

$= 0.685$ lb.

T_2 = hoop tension in rings due to their own mass

$$= \frac{2\ W\ r\ w^2}{g}$$

$$= \frac{2(0.0687)(0.12)(189)^2}{32.2}$$

$= 5.82$ lb.

Total ring tension = T

$= T_1 + T_2$

$= 33.62$ lb.

$$\text{Stress in rings} = \frac{T}{A}$$

$$= \frac{33.62}{0.1132}$$

$= 297$ lb. per sq in.

Tensile strength of bakelite = 5,500 lb per sq in.

See Fig. 2 for dimensions of the disc and rings.

Moment of inertia of a wide ring is

$$J = \frac{\pi t D(B^4 - A^4)}{2}$$, where

t = thickness of disc

D = density of material

A = inside radius of hoop

B = outside radius of hoop

Moment of inertia of disc, from inside circle of holes to the rim =

$$J_D = \frac{\pi(.125)(.05)}{2}\left[(4.125)^4 - (2.125)^4\right]$$

$$= 2.64 \text{ lb in.}^2$$

Moment inertia of the two rings =

$$J_R = \frac{\pi(2)(.125)(.05)}{2}\left[(4.125)^4 - (3.625)^4\right]$$

$$= 2.29 \text{ lb in.}^2$$

Moment of inertia of the commutator =

$J_c = n\, r^2 M$, where

n = number of commutator segments

r = radius to the center of gravity of a segment

M = mass of one segment

$$J_c = (195)(4)^2\,(0.00122)$$

$$= 3.81 \text{ lb in.}^2$$

Moment of inertia of disc, commutator, and rings =

$$J = J_D + J_R + J_c$$

$$= 8.72 \text{ lb in.}^2$$

Assume acceleration is ten times as large as that necessary to bring the rotor speed up to 1800 rpm in one sec, or

$$a = 1890 \text{ radians per sec per sec.}$$

$$\text{Torque} = T = \frac{J\,a}{g}$$

$$= \frac{8.74(1890)}{32.2}$$

$$= 513 \text{ lb-in.}$$

Shearing force at smaller circle of holes =

$$\frac{513}{2.125} = 241 \text{ lb.}$$

Material remaining after holes are drilled

$$= (0.125)(195)(0.0331)$$

$$= 0.806 \text{ sq in.}$$

$$\text{Shearing stress} = \frac{241}{0.806}$$

$$= 299 \text{ lb per sq in.}$$

Strength of bakelite in shear = 13,000 lb per sq in.

<u>Natural frequency of mechanical vibration</u>

Moment of inertia of the complete disc

$$J'_D = \frac{(0.125)(0.05)\pi}{2}\left[(4.125)^4 - (0.375)^4\right]$$

$$= 2.83 \text{ lb in.}^2$$

Moment of inertia of the armature disc

$$= J'_D + J_c + J_R$$

$$= 8.93 \text{ lb in.}^2$$

Moment of inertia of the brass discs holding the bakelite disc $= (0.5)(0.125)(0.318)(\pi)(1.5)^4$

$$= 0.316 \text{ lb in.}^2$$

Moment of inertia of complete armature

$$J_A = 8.93 + 0.316$$

$$= 9.246 \text{ lb in.}^2$$

Spring constant of a round shaft

$$K_s = \frac{G\,I}{L}\text{ , where}$$

G = shear modulus

$= 12\ (10)^6$ for steel

I = polar moment of inertia of cross-section area of the shaft

L = shaft length.

For the first shaft, which had a diameter of 3/8 in,

$$I = (0.5)\ \pi\ (.375)^4$$

$$= 0.030 \text{ in.}^4$$

$$L = 2 \text{ in.}$$

$$K = \frac{12(10)^6\ (0.03)}{2}$$

$$= 1.8\ (10)^5 \text{ lb in. per radian.}$$

Natural frequency of the 3/8 in. shaft and the disc

$$= \frac{1}{2\pi}\sqrt{\frac{g\,K_{s1}}{J_A}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{(1.8)(10)^5(32.2)(12)}{9.256}}$$

437 cps.

For the second shaft, which had a diameter of 3/4 in,

$$I = (0.5)(\pi)(0.75)^4$$

$$= 0.495 \text{ in.}^4$$

$$L = 2.5 \text{ in.}$$

$$K_{s2} = \frac{12(10)^6(0.495)}{2.5}$$

$$= 2.38(10)^6 \text{ lb in per radian.}$$

$$\text{Natural frequency} = \frac{1}{2\pi}\sqrt{\frac{(2.38)(10)^6(32.2)(12)}{9.246}}$$

$$= 1{,}590 \text{ cps.}$$

It can be shown that the spring constant of a massless disc having chickness t, outside radius b, and inside radius a, and made of material having a shear modulus of G is

$$K = \frac{4\pi tG}{\frac{1}{a^2} - \frac{1}{b^2}}$$

For bakelite, G is approximately 214,000 lb per sq in.

To find the spring constant of the disc used in the accelerometer, the disc must be considered as three sections. For the portion between the holding bolts and the inner circle of holes

$t = 0.125$

$a = 1.00$ in.

$b = 2.125$ in.

$$K_1 = \frac{4\pi\ (0.125)^2(214{,}000)}{\frac{1}{1} - \frac{1}{2.125}}$$

432,000 lb-in. per radian.

For the portion between the inner circle of holes and the outside edge,

$t = 0.125$

$a = 2.125$

$$b = 4.125$$

$$K_2 = \frac{4\pi(0.125)(214,000)}{\frac{1}{2.125} - \frac{1}{4.125}}$$

1,950,000 lb in. per radian.

Assume that the material between the holes of the inner circle of holes is rectangular and of width equal to the minimum distance between holes and of length equal to the diameter of the holes. It can be shown that under these assumptions the spring constant of the material between holes becomes

$$K = \frac{G\,t\,r\,(2\pi r - nd)}{D}$$, where

G = shear modulus = 214,000 lb per sq in.

t = disc thickness = 0.125 in.

r = radius of circle of holes = 2.125 in.

n = number of holes = 195

D = diameter of holes = 0.0355 in.

Substituting in the formula,

$K_3 = 1.03 \times 10^7$ lb in. per radian.

The spring constant of the complete disc becomes

$$K_r = \frac{1}{\frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}}$$

$$= \frac{1}{\frac{1}{4.32(10)} + \frac{1}{1.95(10)} + \frac{1}{1.03(10)}}$$

$= 3.41(10)^5$ lb - in. per radian.

Natural frequency of the disc alone

$$= \frac{1}{2\pi}\sqrt{\frac{KTg}{J}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{3.41(10)^5(32.2)(12)}{8.72}}$$

= 236 cps.

APPENDIX II

Torque developed by a short-circuited three-phase induction motor

It can be shown that the force between two eleetric circuits which have constant self inductances and a mutual inductance which is a function of displacement between circuits is

$F_x = i_a i_d \frac{dM}{dx}$, where

F is the force

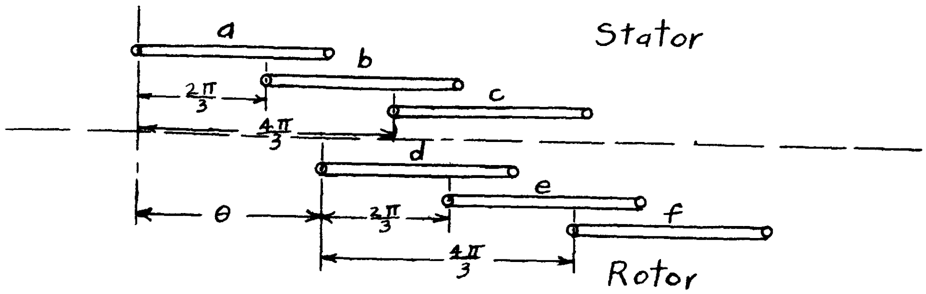
i_a and i_d are the currents in the two circuits

M is the mutual inductance between circuits when the displacement is x.

If this force acts on a rotor coil at a distance R from the center of rotation, the torque produced is

$T = \frac{P}{2} i_a \; i_d \frac{dM}{d\theta}$, where θ is the electrical angular displacement between coils and P is the number of poles.

In a wound-rotor, three-phase induction motor the coil arrangement will be as shown in the diagram on the next page. Assume that the mutual inductance between coils varies as the cosine of the electrical angle between coils.



The mutual inductances between coils become

$$M_{ad} = M\cos\theta = \frac{M}{2}\left[\varepsilon^{j\theta} + \varepsilon^{-j\theta}\right]$$

$$M_{ae} = M\cos\left(\theta + \tfrac{2\pi}{3}\right) = \frac{M}{2}\left[\varepsilon^{j(\theta + \frac{2\pi}{3})} + \varepsilon^{j(-\theta - \frac{2\pi}{3})}\right]$$

$$M_{af} = M\cos\left(\theta + \tfrac{4\pi}{3}\right) = \frac{M}{2}\left[\varepsilon^{j(\theta + \frac{4\pi}{3})} + \varepsilon^{j(-\theta - \frac{4\pi}{3})}\right]$$

$$M_{bd} = M\cos\left(\theta - \tfrac{2\pi}{3}\right) = \frac{M}{2}\left[\varepsilon^{j(\theta - \frac{2\pi}{3})} + \varepsilon^{j(-\theta + \frac{2\pi}{3})}\right]$$

$$M_{be} = M\cos\left(\theta \quad\right) = \frac{M}{2}\left[\varepsilon^{j(\theta)} + \varepsilon^{j(-\theta)}\right]$$

$$M_{bf} = M\cos\left(\theta + \tfrac{2\pi}{3}\right) = \frac{M}{2}\left[\varepsilon^{j(\theta + \frac{2\pi}{3})} + \varepsilon^{j(-\theta - \frac{2\pi}{3})}\right]$$

$$M_{cd} = M\cos\left(\theta - \tfrac{4\pi}{3}\right) = \frac{M}{2}\left[\varepsilon^{j(\theta - \frac{4\pi}{3})} + \varepsilon^{j(-\theta + \frac{4\pi}{3})}\right]$$

$$M_{ce} = M\cos\left(\theta - \tfrac{2\pi}{3}\right) = \frac{M}{2}\left[\varepsilon^{j(\theta - \frac{2\pi}{3})} + \varepsilon^{j(-\theta + \frac{2\pi}{3})}\right]$$

$$M_{cf} = M\cos\left(\theta \quad\right) = \frac{M}{2}\left[\varepsilon^{j\theta} + \varepsilon^{j(-\theta)}\right]$$

The currents can be resolved into components which rotate in opposite directions. If this is done, the currents become

$$i_a = i_{a_1} + i_{a_2}$$

$$i_b = i_{a_1}\varepsilon^{-j\frac{2\pi}{3}} + i_{a_2}\varepsilon^{j\frac{2\pi}{3}}$$

$$i_c = i_{a_1}\varepsilon^{-j\frac{4\pi}{3}} + i_{a_2}\varepsilon^{j\frac{4\pi}{3}}$$

$$i_d = i_{d_1}\ i_{d_2}$$

$$i_e = i_{d_1}\varepsilon^{-j\frac{2\pi}{3}} + i_{d_2}\varepsilon^{j\frac{2\pi}{3}}$$

$$i_f = i_{d_1}\varepsilon^{-j\frac{4\pi}{3}} + i_{d_2}\varepsilon^{j\frac{4\pi}{3}}$$

The total torque will be

$$T = \frac{P}{2} \sum i_s \, i_r \frac{dM_{sr}}{d\theta} \quad .$$

Substituting the values of current and mutual inductance and collecting terms, the total torque becomes

$$T = j \, \frac{9}{4} PM \left[i_{a_2} i_{d_1} \mathcal{E}^{j\theta} - i_{a_1} i_{d_2} \mathcal{E}^{-j\theta} \right]$$

The voltage equations for coil d of the rotor, in operational form, are

$$(r_d + L_d \, p) \, i_{d_1} + M \, p(i_{a_1} \mathcal{E}^{-j\theta}) = 0$$

$$(r_d + L_d \, p) \, i_{d_2} + M \, p(i_{a_2} \mathcal{E}^{j\theta}) = 0 \qquad \text{, where}$$

r_d and L_d are the phase rotor resistance and self inductance respectively. The operators $\mathcal{E}^{j\theta}$ and $\mathcal{E}^{-j\theta}$ are introduced to produce the proper phase relation between stator and rotor voltages. Solving for the stator currents,

$$i_{a_1} = -\frac{(r_d \, p^{-1} + L_d) i_{d_1}}{M \, \mathcal{E}^{-j\theta}}$$

$$i_{a_2} = -\frac{(r_d \, p^{-1} + L_d) i_{d_2}}{M \, \mathcal{E}^{j\theta}}$$

Substituting these expressions in the equation for torque,

$$T = \frac{9}{4} P \, r_d \left[i_{d_2} \frac{i_{d_1}}{p} - i_{d_1} \frac{i_{d_2}}{p} \right]$$

It has been shown[11] that the rotor currents flowing immediately after three-phase short-circuit of an induction is

$$i_d = \sqrt{2} \, \frac{I_n}{\sigma} \, \mathcal{E}^{-kt} \left[\cos(nt + \theta) - \cos \theta \right] \quad \text{, where}$$

I_n = rms no load stator current

σ = leakage factor = $1 - \frac{M^2}{L_1 L_2}$

M = mutual inductance per phase between stator and rotor coils

L_1 = self inductance per phase of stator winding

L_2 = self inductance per phase of rotor winding

$$k = \frac{r_a}{L_1}$$

r_a = stator resistance per phase

n = rotor velocity in electrical radians per second.

Resolving the current into vectors which rotate in opposite directions,

$$i_d = -\frac{\overline{I_n}}{\sigma\sqrt{2}}\varepsilon^{-\frac{k}{\sigma}t} + \frac{\overline{I_n}}{\sigma\sqrt{2}}\varepsilon^{-(\frac{k}{\sigma}-jn)t} - \frac{\widehat{I_n}}{\sigma\sqrt{2}}\varepsilon^{-\frac{k}{\sigma}t} + \frac{\widehat{I_n}}{\sigma\sqrt{2}}\varepsilon^{-(\frac{k}{\sigma}+jn)t}.$$

Hence,

$$i_{d_1} = \frac{\overline{I_n}}{\sigma\sqrt{2}}\left[-\varepsilon^{-\frac{k}{\sigma}t} + \varepsilon^{-(\frac{k}{\sigma}-jn)t}\right]$$

$$I_{d_2} = \frac{\widehat{I_n}}{\sigma\sqrt{2}}\left[-\varepsilon^{-\frac{k}{\sigma}t} + \varepsilon^{-(\frac{k}{\sigma}+jn)t}\right]$$

$$\frac{i_{d_1}}{p} = \frac{\overline{I_n}}{\sigma\sqrt{2}}\left[\frac{\varepsilon^{-\frac{k}{\sigma}t}}{\frac{k}{\sigma}} + \frac{\varepsilon^{-\left(\frac{k}{\sigma}-jn\right)t}}{-\frac{k}{\sigma}+jn}\right]$$

$$\frac{i_{d_2}}{p} = \frac{\widehat{I_n}}{\sigma\sqrt{2}}\left[\frac{\varepsilon^{-\frac{k}{\sigma}t}}{\frac{k}{\sigma}} - \frac{\varepsilon^{-\left(\frac{k}{\sigma}+jn\right)t}}{-\frac{k}{\sigma}-jn}\right].$$

Substituting these values in the equation for torque,

$$T = \frac{9Pr_dI_n^2}{4\sigma^2}\left[\frac{n}{\frac{k^2}{\sigma^2}+n^2}(1-\cos nt) + \frac{n^2}{\frac{k}{\sigma}\left(\frac{k^2}{\sigma^2}+n^2\right)}\sin nt\right]\varepsilon^{-\frac{2k}{\sigma}t}.$$

If the motor is not loaded and if friction and if friction and windage are neglected, the torque is

$0.738\ T = \frac{J}{g}a$, where

T = torque in the units of the above equation if current is in amperes and resistance is in ohms and inductance is in henrys.

J = moment of inertia of rotor in lb-ft.

a = rotor acceleration in radians per sec.

g = gravitational acceleration in ft per sec.

Solving for a and substituting the expression for torque, the acceleration becomes

$$a = \frac{0.738(9)\ g\ P\ r_d\ I_n}{4J\sigma^2}\left[\frac{n}{\frac{k^2}{\sigma^2}+n^2}(1-\cos nt) \quad \frac{n^2}{\frac{k}{\sigma}\left(\frac{k^2}{\sigma^2}+n^2\right)}\sin nt\right]\varepsilon^{-\frac{2k}{\sigma}t}.$$

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