

DISTORTION IN PULSE-DURATION MODULATION

E. R. KRETZMER

TECHNICAL REPORT NO. 11

RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

(reprinted from the Proceedings of the Institute of Radio Engineers, volume 35, number 11, pp. 1230-1235, November, 1947)

Distortion in Pulse-Duration Modulation*

ERNEST R. KRETZMER†, STUDENT, I.R.E.

Summary—Pulse-duration modulation inherently gives rise to a certain amount of audio distortion. The analysis presented in this paper relates the distortion to system parameters. The method of analysis is exact, and therefore correct for any degree of modulation. It does not, however, lend itself to periodic sampling. The results are applied to three specific cases.

I. INTRODUCTION

A SPECTRUM analysis of duration-modulated pulses may be of interest in systems where pulse-duration modulation is used directly, or where

pulse-position modulation is converted to pulse-duration modulation for decoding.¹

The problem investigated here may be classed under the general heading of the analysis of waves derived by sampling a signal wave at discrete intervals. This general problem exists in one form or another in all pulse communication systems, where it is a well-known principle that the sampling rate should be at least twice the highest frequency to be transmitted for faithful reproduction.²

Several papers including analyses of pulses with duration or position modulation, as well as comments on these, have recently been published. Most of these,

* Decimal classification: R148.6. Original manuscript received by the Institute, September 30, 1946; revised manuscript received, March 20, 1947. The research reported in this document was made possible through support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the Army Signal Corps, the Navy Department (Office of Naval Research), and the Army Air Forces (Air Matériel Command), under the Signal Corps Contract No. W-36-039 sc-32037.

† Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge 39, Mass.

¹ "Pulse position modulation technic," *Electronic Ind.*, vol. 4, pp. 82-87, 180-190; December, 1945.

² W. R. Bennett, "Time division multiplex systems," *Bell Sys. Tech. Jour.*, vol. 20, pp. 199-221; April, 1941.

however, do not take into account the precise law of modulation, which determines the time instant at which the signal is sampled and used to produce a time shift.³⁻⁶ The analysis presented here applies to that law of modulation which is believed to be most practical.

II. TERMINOLOGY AND NOTATION

The time function to be analyzed is a sequence of rectangular pulses, for convenience chosen to be of unit amplitude.

The notation used is as follows:

p = angular pulse-repetition frequency
 q = angular frequency of modulating signal
 d_0 = average or unmodulated pulse duration
 d = variable pulse duration

$$k = \text{modulation index} = \frac{d_{\max} - d_{\min}}{d_{\max} + d_{\min}}$$

n = harmonic index number of p

m = harmonic index number of q

A_{np+mq} = amplitude of a sinusoidal component of angular frequency $np + mq$

U_{np+mq} = relative intermodulation distortion due to

$$A_{np+mq}; U_{np+mq} = \frac{A_{np+mq}}{A_q}$$

Two types of pulse-duration modulation are considered: (a) "symmetrical," and (b) asymmetrical. The former has application primarily to pulse-duration modulation as such^{3,7}; the latter has direct application to some present-day pulse-position-modulation systems. In asymmetrical pulse-duration modulation only one of the two pulse edges is time-modulated, while the other one is fixed. In "symmetrical" pulse-duration modulation both edges are modulated. The word "symmetrical" is enclosed in quotation marks because, although both edges move, they do not, in general, move by equal and opposite amounts.

III. ANALYSIS OF ACTUAL MODULATION PROCESS

In position or duration modulation the pulses or pulse edges are shifted by amounts proportional to the instantaneous signal values sampled at certain instants at approximately the time of the pulse or pulse edge. Before any analysis is made, one must first determine exactly what these certain instants are. In a few instances in the

³ G. L. Fredendall, K. Schlesinger, and A. C. Schroeder, "Transmission of television sound on the picture carrier," *Proc. I.R.E.*, vol. 34, pp. 49-61; February, 1946.

⁴ F. F. Roberts and J. C. Simmonds, "Multichannel communications systems. Preliminary investigation based upon modulated pulses," *Wireless Eng.*, vol. 22, pp. 538-549; November, 1945.

⁵ R. B. Shepherd, "Letter to the Editor," *Wireless Eng.*, vol. 23, pp. 114-115; April, 1946.

⁶ F. F. Roberts and J. C. Simmonds, "Letter to the Editor," *Wireless Eng.*, vol. 23, p. 204; July, 1946.

⁷ W. A. Beatty, "Proposals for television and broadcasting systems," *Jour. Instn. Radio Eng.*, vol. 5, pp. 54-78; March-April, 1945.

literature³⁻⁶ these instants were assumed to be fixed and equally spaced along the time axis. In other cases, no specification was made.

Most time modulators are based on the principle that the sum of the signal voltage and a linearly rising or falling voltage crosses a given reference voltage at an instant of time which is a function of the signal, a pulse edge being produced at that instant.^{4,8-10} It is shown in Fig. 1 that the instant of crossing is a function of the

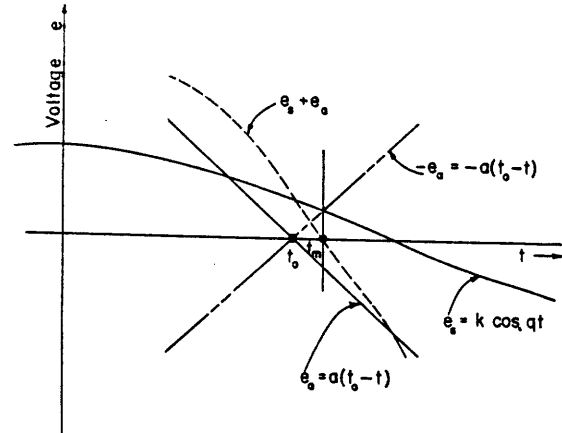


Fig. 1—Illustration of time-position-modulation process.

signal value at the instant of crossing only. The linearly changing voltage is here represented by $e_a = a(t_0 - t)$, the signal by $e_s = k \cos qt$, and the reference line by $e = 0$. The time t_m at which the total voltage crosses this line is given implicitly by $[e_a + e_s]_{t=t_m} = 0$:

$$k \cos qt_m + a(t_0 - t_m) = 0 \quad (1a)$$

$$t_m = t_0 + \frac{k}{a} \cos qt_m \quad (1b)$$

Stated in words, the time shift ($t_m - t_0$) of a given pulse edge is proportional to the instantaneous modulating signal at the instant t_m at which the pulse edge actually occurs. It should be noted that the condition (1a) can also be written in the form

$$k \cos (qt_m) = -a(t_0 - t_m) \quad (1c)$$

This shows that the pulse edge, i.e., the instant t_m , occurs when the signal voltage and the negative of the linear voltage (indicated by the dot-dash line in Fig. 1) intersect. This idea is useful for graphical construction of modulated pulses (see Figs. 2 and 3).

IV. "SYMMETRICAL" PULSE-DURATION MODULATION

Consider first the case of "symmetrical" pulse-duration modulation, shown graphically in Fig. 2. The pulses

⁸ R. D. Kell, U. S. Patent No. 2,061,734.

⁹ W. A. Beatty, British Patent No. 523,575.

¹⁰ D. D. Grieg and A. M. Levine, "Pulse-time-modulated multiplex radio relay system-terminal equipment," *Elec. Commun.*, vol. 23, pp. 159-178; June, 1946.

are so phased that one pulse, in the absence of modulation, is centered at zero time. By ordinary Fourier analysis it is found that the series

$$\frac{pd}{2\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin \frac{npd}{2} \right] \cos npt \quad (2)$$

represents an unmodulated pulse train with pulse duration d , and with a pulse centered at the origin. Although

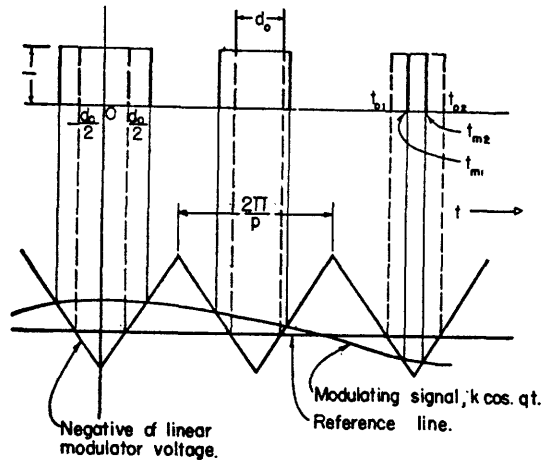


Fig. 2—“Symmetrical” pulse-duration-modulation process.

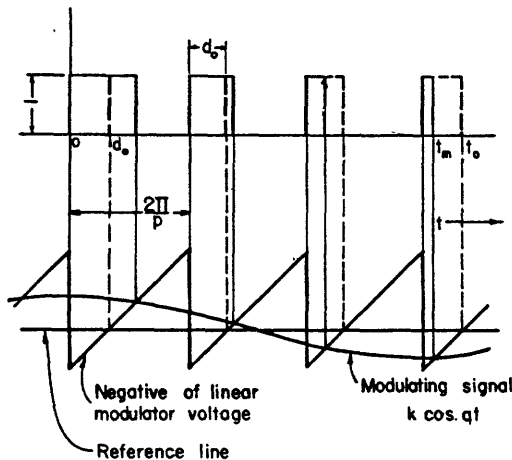


Fig. 3—Asymmetrical pulse-duration-modulation process.

this has been derived for d constant, d may be made variable in accordance with the modulating signal, and the series will then represent the function generated in the actual modulation process described above.¹¹ To facilitate an understanding of this, it is helpful to think of d , not as the pulse duration, but as a parameter which determines independently the instants at which the pulse edges occur; the value of d is of importance only at these instants.

V. ASYMMETRICAL PULSE-DURATION MODULATION

In analyzing asymmetrical modulation, the leading edges are assumed fixed and the trailing edges modu-

lated as before, which automatically covers also the case of fixed trailing edges. The picture of the modulation process is shown in Fig. 3. One of the fixed leading edges is chosen to coincide with zero time. The Fourier series for the unmodulated pulse train with pulse duration d , phased in this manner, is

$$\begin{aligned} \frac{pd}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin npd \right] \cos npt \\ + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} (1 - \cos npd) \right] \sin npt. \end{aligned} \quad (3)$$

As in the case of (2), the desired expression for the modulated pulse train is obtained by letting the parameter d vary with the instantaneous signal. This time, the value of d matters only at the instants at which the trailing pulse edges occur.

VI. SPECTRUM ANALYSES

(A) “Symmetrical” Modulation¹²

By ordinary Fourier analysis of the wave shown in Fig. 4,

$$f(t) = \frac{pd}{2\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin \frac{npd}{2} \right] \cos npt. \quad (4)$$

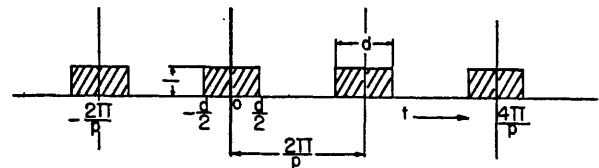


Fig. 4—Pulse train to be modulated “symmetrically.”

Letting d become a function of time proportional to the modulating signal, one has

$$d = d_0(1 + k \cos qt), \quad (5)$$

where a cosine wave represents the signal. This step has been discussed in Section IV. The relative phase of the modulating signal affects only the relative phases of the components of the spectrum, not their magnitudes, which are of chief interest here. The relative phases may be important only in the special degenerate cases where p and q are commensurable. Hence, for present purposes, little generality is lost by choosing a fixed-phase sinusoid for modulation. Substitution of (5) into (4) results in

$$\begin{aligned} f(t) = \frac{pd_0}{2\pi} (1 + k \cos qt) \\ + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin \frac{npd_0}{2} (1 + k \cos qt) \right] \cos npt. \end{aligned} \quad (6)$$

¹² A similar analysis, but more restricted in scope, has been presented by J. L. Callahan, J. N. Whitaker, and H. Shore, “Photo-radio apparatus and operating technique improvements,” Proc. I.R.E., vol. 23, pp. 1441-1483; December, 1935.

¹¹ E. R. Kretzmer, “Letter to the Editor,” *Wireless Eng.*, vol. 23, pp. 232-233; August, 1946.

Using a trigonometric identity, one obtains

$$f(t) = \frac{pd_0}{2\pi} (1 + k \cos qt) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \sin \frac{npd_0}{2} \cos \left[\frac{nkpd_0}{2} \cos qt \right] + \cos \frac{npd_0}{2} \sin \left[\frac{nkpd_0}{2} \cos qt \right] \right\} \cos npt. \quad (7)$$

But

$$\cos(A \cos qt) = J_0(A) + 2 \sum_{m=2,4,\dots}^{\infty} (-1)^{m/2} J_m(A) \cos mqt$$

$$\sin(A \cos qt) = 2 \sum_{m=1,3,\dots}^{\infty} (-1)^{(m-1)/2} J_m(A) \cos mqt.$$

Substituting these relations in (7) and applying an identity for $(\cos mqt) (\cos npt)$ yields

$$f(t) = \frac{pd_0}{2\pi} (1 + k \cos qt) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \left[J_0 \left(\frac{nkpd_0}{2} \right) \sin \frac{npd_0}{2} \right] \cos npt + \sum_{m=1}^{\infty} J_m \left(\frac{nkpd_0}{2} \right) \sin \left(\frac{npd_0}{2} + \frac{m\pi}{2} \right) \cdot [\cos (np + mq)t + \cos (np - mq)t] \right\}. \quad (8)$$

(B) Asymmetrical Modulation

By ordinary Fourier analysis of the wave shown in Fig. 5,

$$f(t) = \frac{pd}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin npd \right] \cos npt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} (1 - \cos npd) \right] \sin npt. \quad (10)$$

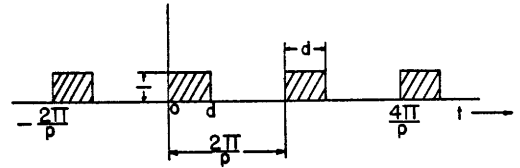


Fig. 5—Pulse train to be modulated asymmetrically.

Letting a cosine wave represent the signal, as before,

$$d = d_0(1 + k \cos qt).$$

Substitution of this relation into (10) yields

$$f(t) = \frac{pd_0}{2\pi} (1 + k \cos qt) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [\sin npd_0(1 + k \cos qt)] \cos npt + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - \cos npd_0(1 + k \cos qt)] \sin npt. \quad (11)$$

By means of the same identities used to obtain (8) from (6), as well as an identity for $(\cos mqt) (\sin npt)$, one obtains (12) from (11).

$$f(t) = \frac{pd_0}{2\pi} (1 + k \cos qt) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ [J_0(nkpd_0) \sin npd_0] \cos npt + [1 - J_0(nkpd_0) \cos npd_0] \sin npt + \sum_{m=2,4,\dots}^{\infty} (-1)^{m/2} J_m(nkpd_0) \sin npd_0 [\cos (np + mq)t + \cos (np - mq)t] + \sum_{m=1,3,\dots}^{\infty} (-1)^{(m-1)/2} J_m(nkpd_0) \cos npd_0 [\cos (np + mq)t + \cos (np - mq)t] - \sum_{m=2,4,\dots}^{\infty} (-1)^{m/2} J_m(nkpd_0) \cos npd_0 [\sin (np + mq)t + \sin (np - mq)t] + \sum_{m=1,3,\dots}^{\infty} (-1)^{(m-1)/2} J_m(nkpd_0) \sin npd_0 [\sin (np + mq)t + \sin (np - mq)t] \right\}. \quad (12)$$

If the summation over m is extended to cover zero and negative values of m , all components may be covered by $\cos(np + mq)t$ alone. This change requires that the integer m be replaced by its absolute value wherever it appears in the coefficients, as indicated by the magnitude signs in (9).

Finally, this expression can be written more compactly, as follows:

$$f(t) = \frac{pd_0}{2\pi} (1 + k \cos qt) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [\sin npt]$$

$$f(t) = \frac{pd_0}{2\pi} (1 + k \cos qt) + \left[\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{n} J_{|m|} \left(\frac{nkpd_0}{2} \right) \sin \left(\frac{npd_0}{2} + \frac{|m|\pi}{2} \right) \right] \cos (np + mq)t. \quad (9)$$

$$\begin{aligned}
& + \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{n} \left[J_{|m|}(nkp d_0) \sin \left(np d_0 + |m| \frac{\pi}{2} \right) \right] [\cos (np + mq)t] \\
& - \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{n} \left[J_{|m|}(nkp d_0) \cos \left(np d_0 + |m| \frac{\pi}{2} \right) \right] [\sin (np + mq)t]. \quad (13)
\end{aligned}$$

Several interesting facts are to be noted. The magnitude of a given component, of frequency $np+mq$, is totally independent of p and q , and also of the algebraic sign of m . Further, there are no components of frequency mq ,¹³ showing that harmonic distortion is not inherent in the modulation process. Finally, an exactly linear relationship exists between the amplitude of the signal-frequency component and the product of duty cycle and modulation index.

These facts follow from the law of modulation assumed, which, as has been shown, corresponds to the law actually governing the modulation process analyzed in Section III.

VII. NUMERICAL RESULTS

It is readily seen that the components of angular frequencies q and $p+mq$ ($n=1, m=-1, -2, -3, \dots$) are of greatest interest from the point of view of audio fidelity. The former is the desired signal and the latter are undesired intermodulation products which may fall within the pass band. The values given by (9) and (13) are peak amplitudes relative to the unit height of the pulses. It is convenient to define a quantity $U_{p+mq} = A_{p+mq}/A_q$, which is the ratio of the undesired component of frequency $p+mq$ to the signal component. The signal amplitude is $A_q = kp d_0/2\pi$ in both cases; the undesired beat amplitudes, divided by A_q , give the following:

"Symmetrical" modulation:

$$\begin{aligned}
U_{p+mq} &= \frac{4}{kp d_0} J_{|m|} \left(\frac{kp d_0}{2} \right) \cos \frac{p d_0}{2} \quad (\text{for } m \text{ odd}) \\
J_{p+mq} &= \frac{4}{kp d_0} J_{|m|} \left(\frac{kp d_0}{2} \right) \sin \frac{p d_0}{2} \quad (\text{for } m \text{ even})
\end{aligned} \quad (14)$$

Asymmetrical modulation:

$$\begin{aligned}
U_{p+mq} &= \frac{2}{kp d_0} \sqrt{J_{|m|}^2(kp d_0) \sin^2 \left(p d_0 + \frac{|m| \pi}{2} \right) + J_{|m|}^2(kp d_0) \cos^2 \left(p d_0 + \frac{|m| \pi}{2} \right)} \\
&= \frac{2}{kp d_0} J_{|m|}(kp d_0) \quad (m \neq 0). \quad (15)
\end{aligned}$$

These expressions contain the essential information for establishing the relations between intermodulation distortion, degree of modulation, and highest signal-to-pulse-frequency ratio.¹⁴ For large degrees of modulation with relatively large average pulse duration, (14) and

(15) should be used directly. However, for small degrees of modulation and also for small pulse durations the expressions may be simplified by approximations to the Bessel and trigonometric functions.

Three different cases will be briefly considered:

- (1) Average pulse duration equals the average time between pulses; high degree of modulation.
- (2) Average pulse duration in the order of 3 per cent of the pulse-repetition period; high degree of modulation.
- (3) Average pulse duration anything from 5 to 95 per cent of the pulse-repetition period. Duration variation small in all cases, in the order of 1 per cent of the pulse repetition period. (Asymmetrical modulation only.)

Case 1

The first case is chiefly of academic interest, especially with regard to a comparison between "symmetrical" and asymmetrical modulation. Since $d_0 = \pi/p$, U_{p+mq} is zero

TABLE I
PER CENT DISTORTION (100 U_{np+mq}) (CASE 1)
 $d_0 = \pi/p$; $k=1$ or as stated, so as to make U_{np+mq} a maximum

Angular Frequency of Undesired Component	"Symmetrical" Modulation Per Cent Distortion	Asymmetrical Modulation Per Cent Distortion
$p-q$	0	65 ($k=0.57$)
$p-2q$	32	32 ($k=0.95$)
$p-3q$	0	21
$p-4q$	1.9	9.5
$p-5q$	0	3.4
$p-6q$	0.04	1*
$2p-3q$	21	21 ($k=0.68$)
$2p-5q$	3.4	12*
$3p-4q$	15.7	15* ($k \approx 0.5$)

* Order of magnitude only.

for m odd in the "symmetrical" case, since $\cos p d_0/2$ in (14) is zero; more generally, for $d_0 = \pi/p$, all $np+mq$ components with n odd and m even, or n even and m odd, are zero. No such phenomenon exists in the asymmetrical case. Table I gives the distortion U_{p+mq} expressed in per cent, for $k \leq 1$ chosen so as to give maximum distortion for each component. Maximum distortion does not always occur for $k=1$. It is true, in the

¹³ Except, of course, for $m=1$.

¹⁴ For components with n other than 1, n will multiply the letter p wherever it appears in the above equations.

present case of deep modulation, that components with n other than 1, e.g., $2p-3q$ and $3p-4q$, are also very large; but if the audio pass band does not extend to over half the pulse frequency these components will fall outside the pass band, as will the $p-q$ component. On the other hand a component such as that of frequency $2p-5q$ will fall within the audio pass band, but will be completely masked by the $p-2q$ component.

Table I shows superiority on the part of "symmetrical" pulse-duration modulation. It should be remembered, however, that the zero values in the symmetrical case hold only for the particular case where $d_0 = \pi/p$ precisely, and cannot exactly be attained in practice.

Case 2

The second case may be of interest because of its proposed use in television sound channels.³ The average pulse duration d_0 is chosen 3.0 per cent, and d will be varied from 0.5 to 5.5 per cent of a period, corresponding to $k=0.83$. Since d_0 is small, the trigonometric and Bessel functions in (14) and (15) may be approximated as follows with not more than 1 per cent error.

$$\begin{aligned}
 J_1(x) &\approx 0.50x \\
 J_2(x) &\approx 0.125x^2 \quad \cos\left(\frac{pd_0}{2}\right) \approx 1 \\
 J_3(x) &\approx 0.021x^3 \quad \sin\left(\frac{pd_0}{2}\right) \approx \frac{pd_0}{2}
 \end{aligned}$$

If these approximations are substituted, (14) and (15) become

<i>"Symmetrical"</i>	<i>Asymmetrical</i>	
$U_{p-q} = 1.0$	$U_{p-q} = 1.0$	
$U_{p-2q} = 0.062k(pd_0)^2$	$U_{p-2q} = 0.25(kpd_0)$	(16)
$U_{p-3q} = 0.010(kpd_0)^2$	$U_{p-3q} = 0.042(kpd_0)^2$	

TABLE II
PER CENT DISTORTION ($100 U_{p+mq}$)—(CASE 2)

Angular Frequency of Unwanted Component	"Symmetrical" Modulation Per Cent Distortion	Asymmetrical Modulation Per Cent Distortion
$p-q$	100	100
$p-2q$	0.18	3.9
$p-3q$	0.024	0.10

These are general relations which hold whenever the above approximations are justified. For the specific nu-

merical example chosen, the duty cycle $pd_0/2\pi=0.03$, and the modulation index $k=0.83$. Substituting these values in (16) yields the results listed in Table II.

"Symmetrical" pulse duration modulation is seen again to be superior in this case. The ratio of p to the highest value of q should be at least two in order to prevent the $p-q$ component from falling within the pass band, or three if the $p-2q$ component is also to be excluded.

Case 3

The third case may have direct bearing on multichannel pulse-position-modulation systems. This is so because a conventional method of demodulation consists of generating asymmetrically duration-modulated pulses, one edge being formed by the synchronizing pulse and the other by the information-carrying pulse of the channel in question.

The quantity $kpd_0/2\pi$ is the ratio of maximum time shift to pulse-repetition period and may be about 0.01 or 0.02, regardless of the average pulse duration. The same approximation to the Bessel functions as in Case 2 may be used here, and the results are identical to those given by (16) for the asymmetrical case.

For a ten-channel system, using "1 per cent" pulses, a reasonable time-shift amplitude might be 2 per cent of the repetition period, so that each channel pulse covers a range of 5 per cent. Table III gives the inherent distortion for all channels.

TABLE III
PER CENT DISTORTION ($100 U_{p+mq}$)—(CASE 3)

Angular Frequency of Undesired Component	$100 U_{p+mq}$
$p-q$	100
$p-2q$	3.15
$p-3q$	0.067

These results lead to the following conclusions, if one assumes that ideal low-pass audio filters are used. In Case 2 (asymmetrical) and in Case 3, the ratio of pulse-repetition frequency to the highest audio frequency can be as low as two for low distortion, and three for negligible distortion. In Case 2 ("symmetrical"), the distortion is negligible even for a ratio of only two, the theoretical limit mentioned in the introduction. Case 1, on the other hand, may call for somewhat higher ratios. Some of these results have been checked experimentally.

