Nonarchimedean Differential Modules and Ramification Theory

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by

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Bachelor of Science, Peking University (2005)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

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Abstract

In this thesis, I first systematically develop the theory of nonarchimedean differential modules, deducing fundamental theorems about the variation of generic radii of convergence for differential modules over polyannuli. The theorems assert that the log of subsidiary radii of convergence are convex, continuous, and piecewise affine functions of the log of the radii of the polyannuli.

Then I apply these results to the ramification theory and deduce the fundamental result, Hasse-Arf theorem, for ramification filtrations defined by Abbes and Saito. Also, we include a comparison theorem to differential conductors and Borger's conductors in the equal characteristic case.

Finally, I globalize this construction and give a new understanding of the ramification theory for smooth varieties, which provides some new insight to the global class field theory. We end the thesis with a series of conjectures as a starting point of a long going project on understanding global ramification.

Thesis Supervisor: Kiran S. Kedlaya
Title: Associate Professor of Mathematics

Acknowledgments

In this thesis, we give a more detailed and integrated presentation of [KX08+, Xia08+a, Xia08+b]. We also include some side results that are relevant to the topic and outline the big picture of the ramification theory.

Many thanks are due to my advisor, Kiran S. Kedlaya, for introducing me to the problems, for generating some crucial ideas, for answering my stupid questions, and for spending many hours reviewing early drafts.

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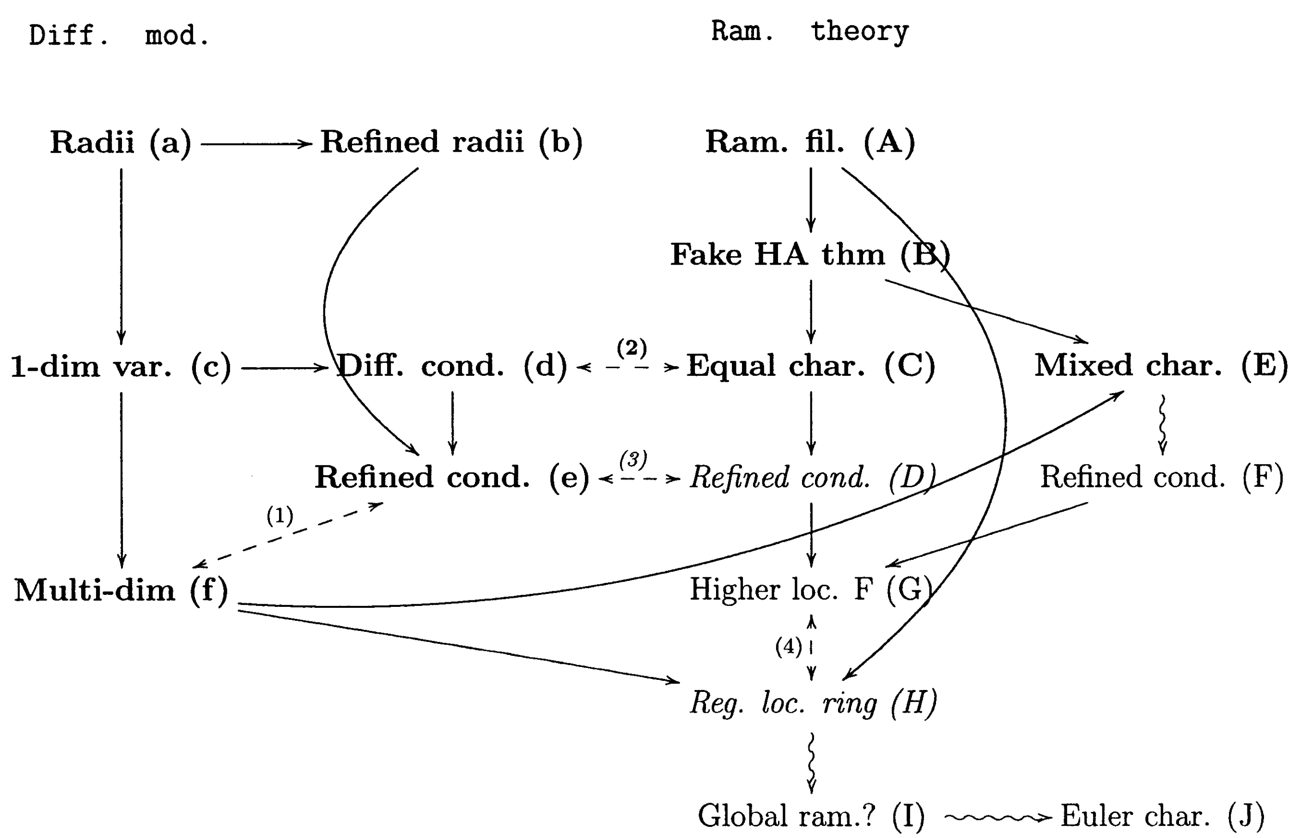
Reader's Guide

This is a guide to the project which applies the theory of nonarchimedean differential modules to the ramification theory. We try to explain the structure of the project so that people who are interested specific topics can take a fast-pass.

There are two main topics of this project: the nonarchimedean differential modules and the ramification theory. Each of them has several levels.

The following diagram indicates the relation between the main components of two topics. Topics in bold letters are covered by this thesis; topics in slanted letters are partially solved in this thesis; others are mostly conjectural. Direct arrows indicate prerequisites, dashed two-sided arrows refer to relations, and curly arrows are expectations or conjectures. Letters or numbers in the parentheses will be explained later, where lowercases are on the differential module side, capital letters are on the ramification theory side, and numbers are links for the corresponding two objects.



(a) Radii of convergence: This consists of Subsections 1.1.2, 1.1.3, 1.1.4, and 1.1.6. We define the notion of radii of convergence in Definitions 1.1.2.6, 1.1.2.8, and 1.1.6.3. Also, derivation of rational type is a key concept, defined in Definition 1.1.4.1 (see also Situation 1.1.6.7.) Theorem 1.1.4.27 on decomposition by radii is the key theorem on this topic. Remark 1.1.3.5 also provides some point of view.

(b) Refined radii: This is defined in Subsection 1.1.5 and the later part of Subsection 1.1.6. The key results is Theorem 1.1.5.22. Also, Example 1.1.5.24 and Lemma 1.1.5.26 are very interesting and are the motivation of Definition 1.1.5.20.

(c) 1-dimensional variation of subsidiary radii: This is discussed in Subsections 1.2.2–1.2.7. The key results are Theorem 1.2.4.4 and theorems in Subsection 1.2.5, where the latter are all of the same flavor.

(d) Differential conductors: This basically consists of Subsections 1.2.7 and 1.2.8, where the former is just a repeat of the variation and decomposition results in (c) (suggest to ignore). One need to read all of Subsection 1.2.8 to understand the definition of differential conductors.

(e) Refined differential conductors: This is discussed in Subsection 1.2.6. The key is Theorem 1.2.6.7, which relies on the very explicit Example 1.2.6.1.

(f) Multi-dimensional variation of intrinsic subsidiary radii: This is nothing but Section 1.3. Subsection 1.3.2 mainly deals with a simple mathematical analysis exercise, which is not of much interest if one does not care about the proof.

(A) Ramification filtration: We review the definition of Abbes-Saito's ramification filtrations in Subsection 2.2.2. Just before that, Proposition 2.2.1.7 is very illustrative.

(B) Fake proof of Hasse-Arf theorem: A fake proof of Hasse-Arf Conjecture 2.2.2.17 is presented in Section 2.3; along the way, we point out the remedy to the gaps and will give the correct proof in Chapters 3 and 4. Two direct applications of the Hasse-Arf theorem are comparison with Borger's conductor and a Hasse-Arf theorem for the ramification filtration on finite flat group schemes; they are discussed in Section 2.4 and 2.5.

(C) Equal characteristic Hasse-Arf theorem: This is done in a way trying to link (d) and (C) directly and transfer all the results of (d) to the ramification side. The only purpose of Section 3.1 is to verify Theorem 3.4.1.3. Subsection 3.2.1 is dedicated to the construction of differential modules. Subsection 3.2.2 meant to prove Proposition 3.2.2.4, which enables us to apply results in Subsection 1.2.8 as discussed in Subsection 3.2.3. Sections 3.3 and 3.4 basically follows the description in Subsections 2.3.5 and 2.3.6. Slightly different from what is stated in Section 2.3, we prove the Hasse-Arf theorem by comparison to differential conductors, which have Hasse-Arf properties by the result in Subsection 1.2.8.

(D) Refined Swan conductors in the equal characteristic case: Saito [Sai07+] defined refined Swan conductors in the equal characteristic case. We have a definition of differential Swan conductors in Subsection 3.2.5. However, we do not know yet if the two refined Swan conductors agree.

(E) Mixed characteristic Hasse-Arf theorem: This basically follows the line drawn in Section 2.3. However, it is more technically involved. Theorem 4.2.1.7 is the technical core of the proof.

(F) Refined Swan conductors in the mixed characteristic case: We except a similar theory as in (D); however, there are technical difficulties which we do not know how to solve.

(G) Multi-indexed ramification filtration for higher local fields: This should follow from the refined Swan conductors fairly easily (if we know how to define (F)).

(H) Ramification for regular local rings: In the equal characteristic case, this is discussed in Section 5.2. We do not know how to deal with the mixed characteristic case.

(I) Global ramification theory: Some expectation is discussed in Section 5.1.

(J) Euler characteristic: We only have Conjecture 5.1.3.6.

(1) relation between (e) and (f): This should be easy and straightforward but has not been carried out yet.

(2) relation between (d) and (C): This is discussed in Sections 3.3 and 3.4; it mainly consists of Theorems 3.3.4.6 and 3.4.2.2.

(3) relation between (e) and (D): This refers the comparison between Saito's re-

fined Swan conductors with differential refined Swan conductors. We do not know this yet.

(4) relation between (G) and (H): This is expected to be easy and follows from (1) immediately in the equal characteristic case. We do not know this in the mixed characteristic case.

Contents

Chapter 1

Nonarchimedean Differential Modules

The study of p-adic differential modules is initiated by Dwork, in his groundbreaking paper [Dwo60] on rationality of Weil's zeta functions. After that, Christol, Dwork, Matsuda, Mebkhout, Robba, and many other mathematicians have devoted to the study of the behavior of solutions of p-adic differential modules. They discovered that the differential modules have a nasty habit of failing to admit global solutions even in the absence of singularities; for instance, the exponential series fails to be entire. To measure this, Dwork and his collaborators introduced a very important notion of the *generic radius of convergence* of a p-adic differential module over a 1-dimensional space (for simplicity, we restrict attention here to discs and annuli). The modern definition of this concept was given and studied in depth by Christol and Dwork [CD94]. A further refinement, the collection of *subsidiary generic radii of convergence*, was introduced (under different terminology) by Young [You92]. Matsuda [Mat95] pointed out the mysterious analogy between this notion and Malgrange's irregularities [Mal74] of differential modules over the complex number. Precisely speaking, when the differential module comes from a Galois representation (will be discussed in Chapter 3), the variation of the generic radius of convergence is related to the Swan conductor of the representation. We will get back to this point in Chapter 3.

Given a differential module over a p-adic disc or annulus of the form $\alpha \leq |t| \leq \beta$, one obtains a generic radius of convergence and some subsidiary radii for each radius $\rho \in [\alpha, \beta]$, and one would like to be able to say something about how these quantities vary with ρ. (In fact, one also obtains these data for each point of the Berkovich analytic space; this is the point of view adopted in ongoing work of Baldassarri and di Vizio, starting with [BdV08+].) Many partial results involving heavy computation have been achieved by Christol, Dwork, and Robba, but no systematic and conceptual approach is available for a long time. By pulling together techniques from the literature and adding one or two new ideas, Kedlaya gave fairly definitive statements about the nature of this variation; this was done in a course given in fall 2007, whose compiled notes constitute the volume [Ked**].

The course [Ked**] was deliberately restricted to the study of p-adic *ordinary* differential equations. One could view the extension of the variational results to higher-dimensional spaces as an implied exercise in [Ked**]. This chapter constitutes a partial solution of this implied exercise, in which we obtain variational properties for differential modules over certain higher-dimensional p-adic analytic spaces. Most part of this chapter is taken from the joint paper [KX08+] of Kedlaya and the author.

Structure of the Chapter

In Section 1.1, we consider differential modules over a field. In particular, we introduce the notion of derivations of rational type, which characterizes the nature of "reasonable" differential operators. Also, we interpret Frobenius homomorphism as a inclusion of subfield determined by derivation and a rational parameter. Another new feature, refined intrinsic radii, is discussed in Subsections 1.1.5 and in the later part of Subsection 1.1.6; this will be related to the refined Swan conductors discussed in later Chapters.

In Section 1.2, we study the variation of subsidiary radii of convergence over a 1-dimensional disc or annulus. This is done in several steps. First, we consider the variation properties and decompositions for each single derivations; this consists of Subsections 1.2.2 and 1.2.3. In the multi-derivation case, we use a rotation technique to reduce the problem to single derivation case; this is carried out in Subsections 1.2.4

and 1.2.5. After that, we discuss the variation of refined intrinsic radii in Subsection 1.2.6. We also insert a short discussion in Subsection 1.2.8 of application to Artin and Swan conductors, which requires Subsection 1.2.7 as a preparation. Finally, we discuss briefly in Subsection 1.2.9 the subharmonicity in the characteristic zero case.

In Section 1.3, we first do some simple mathematical analysis on piecewise linear and convex functions (see Subsections 1.3.1 and 1.3.2). Then, we apply this result to reduce the higher-dimensional variation problem to the 1-dimensional variation problem.

1.1 Differential modules over a field

In this section, we assemble a comprehensive collection of definitions and basic results concerning differential modules over a field. We adopt the point of view to build up everything from differential operators.

1.1.1 Setup

The setup of this subsection applies throughout the thesis.

Notation 1.1.1.1. Let $f^* : R_1 \to R_2$ be a homomorphism of rings. For an R_1-module M_1, we write f^*M_1 to denote the extension of scalars $M_1 \otimes_{R_1, f^*} R_2$. For an R_2-module M_2 we write f_*M_2 to mean M_2 viewed as an R_1-module via f^* (i.e., the restriction of scalars).

Notation 1.1.1.2. The *lexicographic* order on $\mathbb{Z}^n$ is that for $(i_1, \dots, i_n), (i'_1, \dots, i'_n) \in \mathbb{Z}^n$, we have $(i_1, \dots, i_n) \succeq (i'_1, \dots, i'_n)$ if there exists some $j \in \{1, \dots, n\}$ such that $i_1 = i'_1, \dots, i_{j-1} = i'_{j-1}$ and $i_j \succeq i'_j$.

Notation 1.1.1.3. By a *multiset* S, we mean a set where we allow elements to have multiplicity. For $s \in S$, the *multiplicity* of s in S is denoted by $\mathrm{multi}_s(S)$.

Notation 1.1.1.4. For K a field, we use $\mathrm{char}\, k$ to denote its characteristic. We fix an algebraic closure of K^{alg} and let K^{sep} denote the separable closure inside K^{alg}.

Denote $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. For a finite Galois extension L/K, we denote the Galois group by $G_{L/K} = \mathrm{Gal}(L/K)$. For a finite extension L/K, denote the norm map by $\mathrm{N}_{L/K} : L \to K$.

Notation 1.1.1.5. By a *nonarchimedean* field, we mean a field K equipped with a nonarchimedean norm $|\cdot| = |\cdot|_K : K^\times \to \mathbb{R}_+^\times$. A subring of K (with the induced norm and topology) is called a *nonarchimedean ring.* If K is complete for the nonarchimedean norm and L/K is a finite extension, the norm $|\cdot|_K$ extends uniquely to a norm $|\cdot|_L$ on L.

Let K be a nonarchimedean field. Denote the ring of integers and the maximal ideal of K by $\mathcal{O}_K = \{x \in K \| x| \leq 1\}$ and $\mathfrak{m}_K = \{x \in K \| x| < 1\}$, respectively; the residue field of K is denoted by $\kappa_K = \mathcal{O}_K/\mathfrak{m}_K$. We reserve the letter p for the characteristic of κ_K. If $\operatorname{char} \kappa_K = p > 0$ and $\operatorname{char} K = 0$, we normalize the norm on K so that $|p| = 1/p$. For an element $a \in \mathcal{O}_K$, we denote its reduction in κ_K by $\bar{a}$. In case K is discretely valued, let π_K denote a uniformizer of $\mathcal{O}_K$ and let $v_K(\cdot)$ be the corresponding valuation on K, normalized so that $v_K(\pi_K) = 1$.

More generally, for $s \in \mathbb{R}$, we set

$$\mathfrak{m}_K^{(s)} = \{x \in K \mid |x| \leq e^{-s}\}, \quad \mathfrak{m}_K^{(s)+} = \{x \in K \mid |x| < e^{-s}\}, \quad \kappa_K^{(s)} = \mathfrak{m}_K^{(s)}/\mathfrak{m}_K^{(s)+}.$$

If $s \in -\log|K^\times|$, we have a non-canonical isomorphism $\kappa_K \simeq \kappa_K^{(s)}$. For $a \in K$ with $|a| \leq e^{-s}$, we sometimes denote its image in $\kappa_K^{(s)}$ by $\bar{a}^{(s)}$. In particular, $\kappa_K^{(0)} = \kappa_K$ and $\bar{a}^{(0)} = \bar{a}$ if $v(a) \geq 0$.

Notation 1.1.1.6. Let K be a nonarchimedean field. We say K is of *equal characteristic*, if $\operatorname{char} K = \operatorname{char} k = p$. We say K is of *mixed characteristic* if $\operatorname{char} K = 0$ and $\operatorname{char} k = p > 0$.

Definition 1.1.1.7. Let K be a complete discretely valued field. For L a finite extension of K, the *(naïve) ramification degree* of L/K, denoted by $e_{L/K}$ is the index of the value group of K in that of L. If L is the completion of an infinite algebraic extension of K, we define the ramification degree to be the supremum of the ramification degrees of the finite subextensions.

Convention 1.1.1.8. A finite separable extension L of a complete nonarchimedean field K is *unramified* if L and K have the same value group, and the residue field extension is separable of degree $[L : K]$. It is *tamely ramified* if $p \nmid e_{L/K}$ and the residue field extension is separable of degree $[L : K]/e_{L/K}$. If $\operatorname{char} \kappa_K = p = 0$ (and hence K is of characteristic zero), any finite extension of K is tamely ramified, by a theorem of Ostrowski (see [Rib99, Chapter 6]). For L the completion of an infinite algebraic extension of K, we say that L is unramified or tamely ramified if the same is true of each finite subextension of L over K.

Notation 1.1.1.9. Let J be a finite index set. We will write e_J for a tuple $(e_j)_{j\in J}$. For another tuple u_J, write $u_J^{e_J} = \prod_{j\in J} u_j^{e_j}$. We also use $\sum_{e_J=0}^{n}$ to mean the sum over $e_j \in \{0, 1, \ldots, n\}$ for each $j \in J$; for notational simplicity, we may suppress the range of the summation when it is clear. Write $|e_J| = \sum_{j\in J} |e_j|$ and $(e_J)!$ for $\prod_{j\in J}(e_j)!$.

Convention 1.1.1.10. Throughout this paper, all derivations on topological modules will be assumed to be continuous; in particular, Ω^1_R will denote the continuous differentials. We may suppress the base ring from the module of continuous differentials when it is $\mathbb{F}_p$, $\mathbb{Z}$ or $\mathbb{Z}_p$.

Moreover, all derivations considered on nonarchimedean rings will be assumed to be bounded (i.e., to have bounded operator norms). All connections considered will be assumed to be integrable.

Definition 1.1.1.11. For ℓ/κ a (not necessarily finite) extension of fields of characteristic $p > 0$, we say the extension is *separable* if ℓ is geometrically reduced over κ, that is $\ell \otimes_\kappa \kappa'$ is reduced for any finite extension of κ. A *p-basis* of ℓ over κ is a set $\{c_j\}_{j\in J} \subset \ell$ such that the products $c_J^{e_J}$, where $e_j \in \{0, 1, \ldots, p-1\}$ for all $j \in J$ and $e_j = 0$ for all but finitely many j, form a basis of the vector space ℓ over $\kappa\ell^p$. By a *p-basis* of ℓ we mean a p-basis of ℓ over ℓ^p. (For more details, see [Eis95, p. 565] or [EGAIV1, Ch.0, §21].)

For K a complete nonarchimedean field with residual characteristic $p > 0$, a *lifted p-basis* of K will mean a set of elements $b_J \subset \mathcal{O}_L^\times$ whose images $\bar{b}_J \subset \kappa_L$ form a p-basis of κ_K.

Remark 1.1.1.12. When K is of equal characteristic p, a lifted p-basis together with a uniformizer π_K form a p-basis of K.

Remark 1.1.1.13. Let κ be a field of characteristic $p > 0$. For a p-basis $\bar{b}_J \subset \kappa$, $d\bar{b}_J$ form a basis for the differentials Ω^1_κ as an κ-vector space.

Let K be a complete discretely valued field. If K is of mixed characteristic, then for a lifted p-basis $b_J \in \mathcal{O}_K$, db_J form a basis for the differentials Ω^1_K as a K-vector space. If K is of equal characteristic $p > 0$, then for a lifted p-basis $b_J \in \mathcal{O}_K$ together with a uniformizer π_K, $db_J, d\pi_K$ from a basis for the differentials $\Omega^1_{\mathcal{O}_K}$ as a free $\mathcal{O}_K$-module.

Convention 1.1.1.14. For a matrix $A = (A_{ij})$ with coefficients in a nonarchimedean ring, we use $|A|$ to denote the supremum norm over entries.

Hypothesis 1.1.1.15. For the rest of this subsection, we assume that K is a complete nonarchimedean field.

Notation 1.1.1.16. Let $I \subset [0, +\infty)$ be an interval and let $n \in \mathbb{N}$. Let

$$A^n_K(I) = \{(x_1, \ldots, x_n) \in K^{\mathrm{alg}} \mid |x_i| \in I \text{ for } i = 1, \ldots, n\}$$

denote the polyannulus of dimension n with radii in I. (We do not impose any rationality condition on the endpoints of I, so this space should be viewed as an analytic space in the sense of Berkovich [Berk90].) If I is written explicitly in terms of its endpoints (e.g., $[\alpha, \beta]$), we suppress the parentheses around I (e.g., $A^n_K[\alpha, \beta]$).

Remark 1.1.1.17. Throughout this paper, we will implicit use Berkovich spaces, except at only two places (See Remarks 3.1.2.13 and 3.4.1.4) where we have to shift back to the classical rigid analytic setting to talk about (geometric) connected components [BGR84, 9.1.4/8] by imposing some rationality on the radii of discs or annuli.

Notation 1.1.1.18. For $0 \leq \alpha \leq \beta < \infty$, we have the ring of analytic functions on $A^1_K[\alpha, \beta]$, denoted by

$$K\langle \alpha/t, t/\beta \rangle = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \in K[\![t]\!] : \lim_{i \to +\infty} \{|a_i| \beta^i\} = 0, \lim_{i \to -\infty} \{|a_i| \alpha^i\} = 0 \right\}.$$

If $\alpha = 0$, we have the ring of analytic function on the disc $A^1_K[0, \beta]$, denoted b

$$K\langle t/\beta\rangle = \left\{ \sum_{i=0}^{\infty} a_i t^i \in K[\![t]\!] : \lim_{i\to\infty}\{|a_i|\beta^i\} = 0 \right\}.$$

Definition 1.1.1.19. We have the ring of *series with bounded coefficients*

$$K[\![t/\beta]\!]_0 = \left\{ \sum_{i=0}^{\infty} a_i t^i \in K[\![t]\!] : \sup_i\{|a_i|\beta^i\} < \infty \right\};$$

these are the power series which converge and take bounded values on the open disc $|t| < \beta$. Note that for any $\delta \in (0, \beta)$,

$$K\langle t/\beta\rangle \subset K[\![t/\beta]\!]_0 \subset K\langle t/\delta\rangle.$$

In particular, when $\beta = 1$, we have

$$K[\![t]\!]_0 = \mathcal{O}_K[\![t]\!] \otimes_{\mathcal{O}_K} K.$$

An analogue of this construction for an annulus is

$$K\langle \alpha/t, t/\beta]\!]_0 = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i : a_i \in K, \lim_{i\to-\infty} |a_i|\alpha^i = 0, \sup_i\{|a_i|\beta^i\} < \infty \right\};$$

these are the Laurent series which converge and take bounded values on the half-open annulus $\alpha \le |t| < \beta$. For any $\delta \in [\alpha, \beta)$, this ring satisfies

$$K\langle \alpha/t, t/\beta\rangle \subset K\langle \alpha/t, t/\beta]\!]_0 \subset K\langle \alpha/t, t/\delta\rangle.$$

Definition 1.1.1.20. Define the ring

$$K\{\{t/\beta\}\} = \bigcap_{\delta\in(0,\beta)} K\langle t/\delta\rangle = \left\{ \sum_{i=0}^{\infty} a_i t^i : a_i \in K, \lim_{i\to\infty} |a_i|\rho^i = 0 \text{ for all } \rho \in (0, \beta) \right\};$$

these are the power series convergent on the open disc $|t| < \beta$, with no boundedness restriction. In particular, for any $\delta \in (0, \beta)$,

$$K[\![t/\beta]\!]_0 \subset K\{\{t/\beta\}\} \subset K\langle t/\delta\rangle.$$

An analogue of the previous construction for an annulus is

$$K\{\{\alpha/t, t/\beta\}\} = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i : a_i \in K,\ \lim_{i\to\pm\infty} |a_i|\eta^i = 0 \text{ for all } \eta \in (\alpha, \beta) \right\};$$

these are the Laurent series convergent on the open annulus $\alpha < |t| < \beta$.

Definition 1.1.1.21. Put $I = \{1, \dots, n\}$. For $(\eta_i)_{i\in I} \in (0, +\infty)^n$, the η_I-*Gauss norm* on $K[t_I]$ is the norm $|\cdot|_{\eta_I}$ given by

$$\left|\sum_{e_I} a_{e_I} t_I^{e_I}\right|_{\eta_I} = \max_{e_I} \{|a_{e_I}| \cdot \eta_I^{e_I}\};$$

this norm extends uniquely to $K(t_I)$.

For $\eta \in [\alpha, \beta]$ and $\eta \neq 0$, let $x = \sum_{i\in\mathbb{Z}} a_i t^i$ be an element of $K\langle\alpha/t, t/\beta\rangle$, $K\langle\alpha/t, t/\beta]\!]_0$, or (if $\eta \neq \alpha, \beta$) $K\{\{\alpha/t, t/\beta\}\}$. We define the η-*Gauss norm* of x to be

$$|x|_\eta = \sup_{i\in\mathbb{Z}} \left\{|a_i| \cdot \eta^i\right\}.$$

Notation 1.1.1.22. For a nonarchimedean ring R, we use $R\langle u_1, \dots, u_n\rangle$ to denote the Tate algebra, consisting of formal power series $\sum_{i_1,\dots,i_n\in\mathbb{Z}_{\geq 0}} f_{i_1,\dots,i_n} u_1^{i_1} \cdots u_n^{i_n}$ with $f_{i_1,\dots,i_n} \in R$ and $|f_{i_1,\dots,i_n}| \to 0$ as $i_1 + \cdots + i_n \to +\infty$. For $\eta_1, \dots, \eta_n \in (0, 1]$, the ring admits a $(\eta_1, \dots, \eta_n)$-*Gauss norm* given by

$$\left|\sum_{i_1,\dots,i_n\in\mathbb{Z}_{\geq 0}} f_{i_1,\dots,i_n} u_1^{i_1} \cdots u_n^{i_n}\right|_{\eta_1,\dots,\eta_n} = \max_{i_1,\dots,i_n} \left\{|f_{i_1,\dots,i_n}|\eta_1^{i_1} \cdots \eta_n^{i_n}\right\}.$$

Convention 1.1.1.23. By a *G-map*, we will mean a morphism of affinoid (K-analytic) spaces or (direct) limits of them with G-topology, which need not respect the K-space structure. This amounts to a homomorphism between the corresponding rings of global sections, which need not be K-linear. For example, the homomorphism

f^*_{gen} defined in Lemma 1.1.2.16 below gives rise to a G-map $f_{\text{gen}} : A^1_K[0, R_\partial(K)) \to \text{Spm}(K)$, where $\text{Spm}(K)$ denotes the rigid space associated to K.

Definition 1.1.1.24. A *differential module* or ∇*-module* over a ring R is a locally free R-module equipped with an integrable connection, i.e. a map $\nabla : M \to M \otimes \Omega^1_R$ subject to Leibniz rule (that is $\nabla(am) = a\nabla(m) + m \otimes da$ for all $a \in R$ and $m \in M$) and $\nabla(\nabla(m)) = 0$ for all $m \in M$. Any homomorphism $\Omega^1_R \to R$ gives rise to a *derivation* ∂ on R by sending $a \in R$ to the image of da in R; it satisfies the Leibniz rule (that is $\partial(ab) = a\partial b + b\partial a$ for all $a, b \in R$).

Remark 1.1.1.25. When Ω^1_R is a free R-module, we may read off the information of differential modules over R by looking at the derivatives. This is the point of view we will be taking for the rest of this chapter.

Be caution that it is not true that all the information in Ω^1_R can be detected using derivations. In particular, the torsion part of Ω^1_R usually does not admit a homomorphism to R and hence may not be seen using derivations.

1.1.2 Differential fields and differential modules

Definition 1.1.2.1. Let K be a differential ring of order 1, i.e., a ring equipped with a derivation ∂. Let $K\{T\}$ denote the (noncommutative) ring of twisted polynomials over K [Ore33]; its elements are finite formal sums $\sum_{i \geq 0} a_i T^i$ with $a_i \in K$, multiplied according to the rule $Ta = aT + \partial(a)$ for $a \in K$.

Definition 1.1.2.2. A ∂*-differential module* over K is a finite projective K-module V equipped with an action of ∂ (subject to the Leibniz rule); any ∂-differential module over K inherits a left action of $K\{T\}$ where T acts via ∂. The rank of V is the rank of V as a K-module. The module dual $V^\vee = \text{Hom}_K(V, K)$ of V may be viewed as a ∂-differential module by setting $(\partial f)(\mathbf{v}) = \partial(f(\mathbf{v})) - f(\partial(\mathbf{v}))$. We say V is *free* if V as a module is free over K. We say V is *trivial* if it is free and there exists a K-basis $\mathbf{v}_1, \ldots, \mathbf{v}_d \in V$ such that $\partial(\mathbf{v}_i) = 0$ for $i = 1, \ldots, d$, where $d = \text{rank}\,(V)$.

For V a ∂-differential module free of rank d over K, we say $\mathbf{v} \in V$ is a *cyclic vector* if $\mathbf{v}, \partial\mathbf{v}, \ldots, \partial^{d-1}\mathbf{v}$ form a basis of V. A cyclic vector defines an isomorphism

$V \simeq K\{T\}/K\{T\}P$ of ∂-differential modules for some twisted polynomial $P \in K\{T\}$ of degree d, where the ∂-action on $K\{T\}/K\{T\}P$ is the left multiplication by T.

Definition 1.1.2.3. For a ∂-differential module V over K, define

$$H^0_\partial(V) = \operatorname{Ker} \partial, \quad H^1_\partial(V) = \operatorname{Coker} \partial = V/\partial(V).$$

The latter computes Yoneda extensions; see, e.g., [Ked**, Lemma 5.3.3].

Lemma 1.1.2.4. *If K is a differential field , every ∂-differential module over K contains a cyclic vector.*

Proof. See, e.g., [DGS94, Theorem III.4.2] or [Ked**, Theorem 5.4.2]. □

Hypothesis 1.1.2.5. For the rest of Subsection 1.1.2, we assume that K is a complete nonarchimedean field of characteristic zero, equipped with a derivation ∂ with operator norm $|\partial|_K < \infty$, and that V is a nonzero ∂-differential module over K.

Definition 1.1.2.6. The *spectral norm of ∂ on V* is defined to be

$$|\partial|_{\mathrm{sp},V} = \lim_{n\to\infty} |\partial^n|_V^{1/n}$$

for any fixed K-compatible norm $|\cdot|_V$ on V. Any two such norms on V are equivalent [Sch02, Proposition 4.13], so the spectral norm does not depend on the choice [Ked**, Proposition 6.1.5]. One can show that $|\partial|_{\mathrm{sp},V} \geq |\partial|_{\mathrm{sp},K}$ [Ked**, Lemma 6.2.4].

Explicitly, if one chooses a basis of V and uses the matrix D_n to denote the action of ∂^n on this basis, then

$$|\partial|_{\mathrm{sp},V} = \max\{|\partial|_{\mathrm{sp},K}, \lim_{n\to\infty} |D_n|^{1/n}\}.$$

Remark 1.1.2.7. If $K \to K'$ is an isometric embedding of complete nonarchimdedean differential fields, then for a ∂-differential module V over K, $V' = V \otimes_K K'$ is a ∂-differential module over K', and $|\partial|_{\mathrm{sp},V'} = \max\{|\partial|_{\mathrm{sp},K'}, |\partial|_{\mathrm{sp},V}\}$.

Definition 1.1.2.8. Let p denote the residual characteristic of K; we conventionally write

$$\omega = \begin{cases} 1 & p = 0 \\ p^{-1/(p-1)} & p > 0 \end{cases}.$$

Define the *generic ∂-radius of convergence* (or for short, the *generic ∂-radius*) of V to be

$$R_\partial(V) = \omega|\partial|_{\mathrm{sp},V}^{-1};$$

note that $R_\partial(V) > 0$. We will see later (Proposition 1.1.2.18) that this indeed computes the radius of convergence of Taylor series on a "generic disc". In some situations, it is more natural to consider the *intrinsic generic ∂-radius of convergence*, or for short the *intrinsic ∂-radius*, defined as

$$IR_\partial(V) = \frac{|\partial|_{\mathrm{sp},K}}{|\partial|_{\mathrm{sp},V}};$$

note that this is a number in $(0, 1]$ by [Ked**, Lemma 6.2.4].

Let $V_1, \dots, V_d$ be the Jordan-Hölder constituents of V as $K\{T\}$ modules. We define the *(extrinsic) subsidiary generic ∂-radii of convergence*, or for short the *subsidiary ∂-radii*, to be the multiset $\mathfrak{R}_\partial(V)$ consisting of $R_\partial(V_i)$ with multiplicity $\dim V_i$ for $i = 1, \dots, d$. Let $R_\partial(V; 1) \leq \cdots \leq R_\partial(V; \dim V)$ denote the elements in $\mathfrak{R}_\partial(V)$ in increasing order. We similarly define *intrinsic subsidiary (generic) ∂-radii of convergence* $\mathfrak{IR}_\partial(V)$, or for short *intrinsic subsidiary ∂-radii*, by aggregating the intrinsic ∂-radii of V_i for $i = 1, \dots, d$. Let $IR_\partial(V; 1) \leq \cdots \leq IR_\partial(V; \dim V)$ denote the elements in $\mathfrak{IR}_\partial(V)$ in increasing order.

We say that V has *pure (intrinsic) ∂-radii* if $\mathfrak{R}(V)$ consists of d copies of $R_\partial(V)$.

Lemma 1.1.2.9. *Let V, V_1, V_2 be nonzero ∂-differential modules over K.*

(a) For $0 \to V_1 \to V \to V_2 \to 0$ exact,

$$R_\partial(V) = \min\{R_\partial(V_1),\ R_\partial(V_2)\}; \quad IR_\partial(V) = \min\{IR_\partial(V_1),\ IR_\partial(V_2)\}.$$

More precisely,

$$\mathfrak{R}_\partial(V) = \mathfrak{R}_\partial(V_1) \cup \mathfrak{R}_\partial(V_2); \quad \mathfrak{IR}_\partial(V) = \mathfrak{IR}_\partial(V_1) \cup \mathfrak{IR}_\partial(V_2).$$

(b) We have

$$R_\partial(V^\vee) = R_\partial(V); \qquad IR_\partial(V^\vee) = IR_\partial(V);$$
$$\mathfrak{R}_\partial(V^\vee) = \mathfrak{R}_\partial(V); \qquad \mathfrak{IR}_\partial(V^\vee) = \mathfrak{IR}_\partial(V);$$

(c) We have

$$R_\partial(V_1 \otimes V_2) \geq \min\{R_\partial(V_1),\ R_\partial(V_2)\}; \quad IR_\partial(V_1 \otimes V_2) \geq \min\{IR_\partial(V_1),\ IR_\partial(V_2)\}. \tag{1.1.2.10}$$

Moreover, if $R_\partial(V_1) \neq R_\partial(V_2)$, or equivalently, if $IR_\partial(V_1) \neq IR_\partial(V_2)$, we have equalities in (1.1.2.10).

(d) If V_1 and V_2 are irreducible and $IR_\partial(V_1) \neq IR_\partial(V_2)$, then $\mathfrak{IR}_\partial(V_1 \otimes V_2)$ is just $\dim V_1 \cdot \dim V_2$ *copies of* $\min\{IR_\partial(V_1), IR_\partial(V_2)\}$.

Proof. As in [Ked**, Lemma 6.2.8] and [Ked**, Corollary 6.2.9]. □

Definition 1.1.2.11. Let R be a complete K-algebra. For $\mathbf{v} \in V$ and $T \in R$, define the *∂-Taylor series* to be

$$\mathbb{T}(\mathbf{v}; \partial, T) = \sum_{n=0}^{\infty} \frac{\partial^n(\mathbf{v})}{n!} T^n \in V \otimes_K R$$

in case this series converges.

We include some formal properties of Taylors series as follows.

Proposition 1.1.2.12. *Keep the notation as above.*

(a) *If $V = K$, the ∂-Taylor series $x \mapsto \mathbb{T}(x; \partial, T)$ for fixed ∂ and $T \in R$ gives a ring homomorphism $K \to R$ if it converges; that is in particular saying that for $x_1, x_2 \in K$, $\mathbb{T}(x_1 x_2; \partial, T) = \mathbb{T}(x_1; \partial, T)\mathbb{T}(x_2; \partial, T)$. For general V, the ∂-Taylor series gives a* *homomorphism* $\mathbf{v} \mapsto \mathbb{T}(\mathbf{v}; \partial, T)$ *of modules* $V \to V \otimes_K R$ *via the aforementioned ring homomorphism, if it converges.*

(b) *If* $\mathbf{v} \in V$, $x \in K$, *and* $T \in R$, *we have*

$$\mathbb{T}\big(\mathbb{T}(\mathbf{v}; \partial, x); \partial, T\big) = \mathbb{T}\big(\mathbf{v}; \partial, T + \mathbb{T}(x; \partial, T)\big), \tag{1.1.2.13}$$

if the ∂*-Taylor series involved all converge.*

Proof. Statement (a) follows from formal properties of Taylor series immediately. Statement (b) also follows but less trivially. We include the deduction here.

We expand the left hand side of (1.1.2.13) to be

$$\begin{aligned}
\mathbb{T}\big(\mathbb{T}(\mathbf{v}; \partial, x); \partial, T\big) &= \sum_{m=0}^{\infty} \frac{T^m}{m!} \partial^m \Big(\sum_{n=0}^{\infty} \frac{x^n \partial^n(\mathbf{v})}{n!} \Big) \\
&= \sum_{m=0}^{\infty} \frac{T^m}{m!} \sum_{i=0}^{m} \binom{m}{i} \sum_{n=0}^{\infty} \frac{\partial^{m-i}(x^n)}{n!} \cdot \partial^{n+i}(\mathbf{v}) \\
&= \sum_{m,n=0}^{\infty} \sum_{i=0}^{m} \frac{T^m \partial^{m-i}(x^n)}{(m-i)! i! n!} \cdot \partial^{n+i}(\mathbf{v}).
\end{aligned} \tag{1.1.2.14}$$

Similarly, we expand the right hand side of (1.1.2.13) to be

$$\begin{aligned}
\mathbb{T}\big(\mathbf{v}; \partial, T + \mathbb{T}(x; \partial, T)\big) &= \sum_{\alpha=0}^{\infty} \frac{\big(u + \mathbb{T}(x; \partial, T)\big)^{\alpha}}{\alpha!} \cdot \partial^{\alpha}(\mathbf{v}) \\
&= \sum_{\alpha=0}^{\infty} \frac{\sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} T^{\gamma} \mathbb{T}(x; \partial, T)^{\alpha-\gamma}}{\alpha!} \cdot \partial^{\alpha}(\mathbf{v}) \\
&= \sum_{\alpha=0}^{\infty} \sum_{\gamma=0}^{\alpha} \frac{T^{\gamma} \mathbb{T}(x^{\alpha-\gamma}; \partial, T)}{(\alpha-\gamma)! \gamma!} \cdot \partial^{\alpha}(\mathbf{v}) \quad \text{by (a)} \\
&= \sum_{\alpha=0}^{\infty} \sum_{\gamma=0}^{\alpha} \sum_{\beta=0}^{\infty} \frac{T^{\beta+\gamma} \partial^{\beta}(x^{\alpha-\gamma})}{(\alpha-\gamma)! \gamma! \beta!} \cdot \partial^{\alpha}(\mathbf{v}).
\end{aligned} \tag{1.1.2.15}$$

One checks that (1.1.2.14) and (1.1.2.15) match if we set $m = \beta + \gamma$, $n = \alpha - \gamma$, and $i = \gamma$. □

Lemma 1.1.2.16. *The Taylor series* $x \mapsto \mathbb{T}(x; \partial, T)$ *uives a continuous homomorphism* $f_{\mathrm{gen}}^* : K \to K[\![T/R_\partial(K)]\!]_0$, *which induces a G-map* $f_{\mathrm{gen}} : A_K^1[0, R_\partial(K)) \to$

Spm(K). *Moreover, for $\eta \in [0, R_\partial(K)]$, f^*_{gen} is isometric for the η-Gauss norm on the target.*

Proof. It is straightforward to check that f^*_{gen} is bounded for the η-Gauss norm for any $\eta \in [0, R_\partial(K))$; that is, there exists $c > 0$ such that for all $x \in K$, $|f^*_{\text{gen}}(x)|_\eta \leq c|x|$. For any positive integer n, we can plug x^n into the previous inequality to deduce $|f^*_{\text{gen}}(x)|_\eta \leq c^{1/n}|x|$. Consequently, $|f^*_{\text{gen}}(x)|_\eta \leq |x|$ for any $\eta \in [0, R_\partial(K))$, and by continuity also for $\eta = R_\partial(K)$. We also have $|f^*_{\text{gen}}(x)|_{R_\partial(K)} \geq |x|$ because the first term in $\mathbb{T}(x; \partial, T)$ is x itself which contributes to the Gauss norm. □

Corollary 1.1.2.17. *For each positive integer n, we have $|\partial^n/n!|_K \leq R_\partial(K)^{-n} = \omega^{-n}|\partial|^n_{\text{sp},K}$. In particular (by taking $n = 1$), $|\partial|_{\text{sp},K} \geq \omega|\partial|_K$.*

We have the following geometric interpretation of generic radii. This is slightly different from, but essentially equivalent to, the treatments in [Ked07a, Section 2.2] and [Ked**, Section 9.7].

Proposition 1.1.2.18. *With notation as in Lemma 1.1.2.16, the pullback $f^*_{\text{gen}}V$ becomes a ∂_T-differential module over $A^1_K[0, R_\partial(K))$, where $\partial_T = \frac{d}{dT}$. Then for any $r \in (0, R_\partial(K)]$, $R_\partial(V) \geq r$ if and only if $f^*_{\text{gen}}V$ restricts to a trivial ∂_T-differential module over $A^1_K[0, r)$.*

Proof. Since f^*_{gen} is an isometry and $|\partial_T|_{K[\![T/R_\partial(K)]\!]_0} = R_\partial(K)^{-1}$, we have $R_\partial(V) = R_{\partial_T}(f^*_{\text{gen}}V \otimes \operatorname{Frac}K[\![T/R_\partial(K)]\!]_0)$. It then suffices to check that $R_{\partial_T}(f^*_{\text{gen}}V) \geq r$ if and only if $f^*_{\text{gen}}V$ restricts to a trivial ∂_T-differential module over $A^1_K[0, r)$; this is the content of Dwork's transfer theorem [Ked**, Theorem 9.6.1]. □

1.1.3 Newton polygons

In this subsection, we summarize some results in [Ked**, Chapter 5 and 6] and [Ked07a, Section 1]. Throughout this subsection, let K be a complete nonarchimedean differential field of characteristic zero.

Definition 1.1.3.1. For $P(T) = \sum_i a_i T^i \in K[T]$ or $K\{T\}$ a nonzero (twisted) polynomial, define the *Newton polygon* of P as the lower convex hull of the set

$\{(-i, -\log|a_i|)\} \subset \mathbb{R}^2$. The Newton polygons for twisted polynomials obey the usual additivity rules only for slopes less than $-\log|\partial|_K$.

Proposition 1.1.3.2 (Christol-Dwork). *Suppose that $V \simeq K\{T\}/K\{T\}P$, and let s be the lesser of $-\log|\partial|_K$ and the least slope of P. Then*

$$\max\{|\partial|_K, |\partial|_{\mathrm{sp},V}\} = e^{-s}.$$

Proof. See [CD94, Théorème 1.5] or [Ked**, Theorem 6.5.3]. □

Proposition 1.1.3.3 (Robba). *Any monic twisted polynomial $P \in K\{T\}$ admits a unique factorization*

$$P = P_+ P_n \cdots P_1$$

such that for some $s_1 < \cdots < s_n < -\log|\partial|_K$, each P_i is monic with all slopes equal to s_i, and P_+ is monic with all slopes at least $-\log|\partial|_K$.

Proof. See [Ked07a, Proposition 1.1.10] or [Ked08+a, Corollary 3.2.4]. □

Proposition 1.1.3.4. *Suppose that $\omega \cdot |\partial|_K^{-1} = r_0$. Then there is a unique decomposition*

$$V = V_+ \oplus \bigoplus_{r<r_0} V_r$$

of ∂-differential modules, such that V_r has pure ∂-radii r, and the subsidiary ∂-radii of V_+ are all at least r_0.

Proof. Apply Lemma 1.1.2.4 to write $V \simeq K\{T\}/K\{T\}P$ for P a twisted polynomial. Then the statement may be deduced from Proposition 1.1.3.3, applied first to P in $K\{T\}$ and then to P in the opposite ring. For more details, one may consult [Ked**, Theorem 6.6.1]. □

Remark 1.1.3.5. If $V \simeq K\{T\}/K\{T\}P$ for P a twisted polynomial, then Propositions 1.1.3.2 and 1.1.3.3 imply that the multiplicity of any $s < -\log|\partial|_K$ as a slope of the Newton polygon of P coincides with the multiplicity of ωe^s in $\mathcal{R}_\partial(V)$.

Notation 1.1.3.6. Keep the notation as in Proposition 1.1.3.4. We call $\bigoplus_{r<r_0} V_r$ the *visible part* of V; its subsidiary radii are called the *visible radii* and the corresponding set of spectral norms is called the *visible spectrum.* If $V_+ = 0$, we say that V *has visible ∂-radii.*

1.1.4 Moving along Frobenius

As discovered originally by Christol-Dwork [CD94], and amplified by Kedlaya [Ked**], in the situation of Definition 1.1.4.1, one can overcome the limitation on subsidiary radii imposed by Proposition 1.1.3.2 by using the pushforward along the Frobenius. In this subsection, we imitate the techniques in [Ked**, Chapter 10] and obtain Theorems 1.1.4.25 and 1.1.4.27 as analogues of [Ked**, Theorems 10.5.1 and 10.6.2].

Definition 1.1.4.1. Let K be a complete nonarchimedean differential field of characteristic zero and residual characteristic p. The derivation ∂ on K is of *rational type* if there exists $u \in K$ such that the following conditions hold. (If these hold, we call u a *rational parameter* for ∂.)

(a) We have $\partial(u) = 1$ and $|\partial|_K = |u|^{-1}$.

(b) For each positive integer n, $|\partial^n/n!|_K \leq |\partial|_K^n$.

It is equivalent to formulate (b) as follows.

(b′) We have $|\partial|_{\mathrm{sp},K} \leq \omega|\partial|_K$.

(It is clear that (b) implies (b′); the reverse implication holds by Corollary 1.1.2.17.) For $p > 0$, in the presence of (a), yet another equivalent formulation of (b) is as follows.

(b″) For each polynomial $P \in \mathbb{Q}_p[T]$ such that $P(\mathbb{Z}_p) \subseteq \mathbb{Z}_p$, $|P(u\partial)|_K \leq 1$.

This relies on the fact that the $\mathbb{Z}_p$-module of such P is freely generated by the binomial polynomials

$$\binom{T}{n} = \frac{T(T-1)\cdots(T-n+1)}{n!} \qquad (n = 0, 1, \dots).$$

Remark 1.1.4.2. Note that in Definition 1.1.4.1, the inequality in (b′) is forced to be an equality by Corollary 1.1.2.17, while the inequality in (b) is forced to be an equality if (a) holds because then $(\partial^n/n!)(u^n) = 1$. In particular, for any nonzero ∂-differential module V, $IR_\partial(V) = |u| \cdot R_\partial(V)$. Similarly, if (a) holds and $p > 0$, then the inequality in (b″) becomes an equality whenever $P(\mathbb{Z}_p) \not\subset p\mathbb{Z}_p$.

Remark 1.1.4.3. If u' is a second rational parameter for ∂, then $u - u' \in \ker(\partial)$ and $|u' - u| \leq |u|$. The converse is also true; that is, if u is a rational parameter, $u - u' \in \ker(\partial)$, and $|u' - u| \leq |u|$, then u' is also a rational parameter. The only nonobvious part of this statement is the fact that these two conditions imply $|u'| = |u|$. This is because $\partial(u') = 1$ implies $1 \leq |\partial|_K |u'| = |u'|/|u|$, so $|u'| \geq |u|$.

Remark 1.1.4.4. The simplest case of Definition 1.1.4.1 is the derivation d/dt on the completion of the rational function field $\mathbb{Q}_p(t)$ for any Gauss norm if $p > 0$, or on the ring of Laurent series $\mathbb{C}((t))$ if $p = 0$. For more cases, see Situation 1.1.6.7 and the following remarks.

Remark 1.1.4.5. We will often run into the case when we need to enlarge the valuation of K. We can achieve this via replacing K by K' the completion of $K(z)$ with respect to η-Gauss norm for some $\eta \in \mathbb{R}_{>0}$ and setting $\partial(z) = 0$, where z is some transcendental element over K. We have $|\partial|_K = |\partial|_{K'}$ and $|\partial|_{\mathrm{sp},K} = |\partial|_{\mathrm{sp},K'}$, and hence ∂ will continue to be of rational type (with respect to u) over K'.

Lemma 1.1.4.6. *Let K be a complete nonarchimedean field of characteristic zero, equipped with a differential operator ∂ of rational type with respect to u. Let L be a complete tamely ramified extension of K. Then the unique extension of ∂ to L is of rational type (with u again as rational parameter).*

Proof. We reduce immediately to the case of a finite tamely ramified extension. The extension of ∂ to L is obtained from the isomorphism $\Omega^1_L \cong L \otimes_K \Omega^1_K$. We need to prove that for each positive integer n and each $x \in L$, $|u^n \partial^n(x)/n!| \leq |x|$. We point out a useful fact that, for $x, y \in L$, to check the above property for xy, it is enough to check it for x and y separately; this is because

$$\frac{\partial^n(xy)}{n!} = \sum_{i=0}^{n} \binom{n}{i} \cdot \frac{\partial^i(x)}{i!} \cdot \frac{\partial^{n-i}(y)}{(n-i)!}.$$

We may consider the unramified extension and the totally tamely ramified extension separately.

Suppose first that L/K is unramified. Since every element of L equals an element of K times an element of $\mathcal{O}_L^\times$, we need only check the inequality $|u^n\partial^n(x)/n!| \leq |x|$ for $x \in \mathcal{O}_L^\times$. We do this by induction on n. Let $h(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0 \in \mathcal{O}_K[T]$ be the minimal polynomial of x; thus $h'(x) \in \mathcal{O}_L^\times$. For the base case $n = 1$ of the induction, applying $u\partial$ to the equation $h(x) = 0$ gives

$$u\partial(x) = -\frac{u\partial(a_{d-1})x^{d-1} + \cdots + u\partial(a_0)}{h'(x)} \in \mathcal{O}_L.$$

Assume the statement is proved for $n-1$. Applying $u^n\partial^n/n!$ to the equation $h(x) = 0$ gives

$$\sum_{i=0}^{d} \sum_{\lambda_0+\cdots+\lambda_i=n} \frac{u^{\lambda_0}\partial^{\lambda_0}}{\lambda_0!}(a_i)\frac{u^{\lambda_1}\partial^{\lambda_1}}{\lambda_1!}(x)\cdots\frac{u^{\lambda_i}\partial^{\lambda_i}}{\lambda_i!}(x) = 0,$$

where $a_d = 1$ by convention. Each summand belongs to $\mathcal{O}_L$ by the induction hypothesis except for those in which $\lambda_j = n$ for some $j > 0$; those terms add up to $h'(x)u^n\partial^n(x)/n!$. Therefore $u^n\partial^n(x)/n! \in \mathcal{O}_L$, completing the induction.

Now suppose that L/K is totally tamely ramified. We induct on $[L : K]$, which we may assume is greater than 1. (We need the induction because we did not assume that K is discretely valued and hence the group $L^\times/K^\times$ may not be cyclic.) Then we can find $d > 1$ and $x_0 \in \mathcal{O}_L$ such that $|x_0^i| \notin |K^\times|$ for $i = 1, \dots, d-1$. Choose an element $y \in \mathcal{O}_K$ with $|y - x_0^d| < |x_0^d|$. By Hensel's lemma, y has a d-th root z in L. Let K' be the completion of $K(t)$ for the $|y|^{1/d}$-Gauss norm, and extend ∂ to K' by setting $\partial(t) = 0$. The residue field of K' is $\kappa_K(y/t^d)$. Put $L' = K' \otimes_K K(z)$; then $L' = K'(z) = K'(z/t)$. Now z/t is a d-th root of the quantity $y/t^d \in \mathcal{O}_{K'}$, whose image in the residue field has no i-th root for any $i > 1$ dividing d. Hence L'/K' is unramified, so by the previous paragraph, ∂ extends to L' and is of rational type with respect to u. We may then read off the same conclusion for $K(z)$; applying the induction hypothesis to $L/K(z)$ yields the claim. □

Corollary 1.1.4.7. *Let K be a complete nonarchimedean field of characteristic zero, equipped with a differential operator ∂. Let L be a complete tamely ramified extension of K. Then ∂ extends to L and $|\partial|_L = |\partial|_K$.*

Proof. It follows from the proof of Lemma 1.1.4.6 above. □

Hypothesis 1.1.4.8. For the rest of this subsection, we assume that K is a complete nonarchimedean field of characteristic zero and residual characteristic p, equipped with a differential operator ∂ of rational type with respect to the rational parameter u. We also assume $p > 0$ unless otherwise specified.

Construction 1.1.4.9. If K contains a primitive p-th root of unity ζ_p, we may define an action of the group $\mathbb{Z}/p\mathbb{Z}$ on K using ∂-Taylor series:

$$x^{(i)} = \mathbb{T}(x; \partial, (\zeta_p^i - 1)u), \qquad (i \in \mathbb{Z}/p\mathbb{Z}, x \in K);$$

in particular, $u^{(i)} = \zeta_p^i u$. Indeed, it is a field homomorphism by Proposition 1.1.2.12(a), and it gives an action because, by Proposition 1.1.2.12(b),

$$\left(x^{(i)}\right)^{(j)} = \mathbb{T}\big(x; \partial, (\zeta_p^j - 1)u + (\zeta^i - 1)u^{(j)}\big) = \mathbb{T}(x; \partial, (\zeta_p^j - 1)u + (\zeta^i - 1)\zeta_p^j u) = x^{(i+j)}.$$

Since $|(\zeta_p^i - 1)^n u^n \partial^n / n!|_K < |u^n \partial^n / n!|_K \leq 1$ for $i \in \mathbb{Z}/p\mathbb{Z}$, we have $|x^{(i)}| = |x|$, in other words, the action of $\mathbb{Z}/p\mathbb{Z}$ on K is isometric. Let $K^{(\partial)}$ be the fixed subfield of K under this action; in particular, $u^p \in K^{(\partial)}$. By simple Galois theory, K is the Galois extension of $K^{(\partial)}$ generated by u with Galois group $\mathbb{Z}/p\mathbb{Z}$. Moreover, $K^{(\partial)}$ is stable under the action of $u\partial$ because $(u\partial x)^{(i)} = u\partial(x^{(i)})$ for $x \in K$. (If K does not contain a primitive p-th root of unity, we may still define $K^{(\partial)}$ using Galois descent.)

We call the inclusion $\varphi^{(\partial)*} : K^{(\partial)} \hookrightarrow K$ the *∂-Frobenius morphism.* We view $K^{(\partial)}$ as being equipped with the derivation $\partial' = \partial/(pu^{p-1})$; we will see below (Lemma 1.1.4.14) that ∂' is of rational type with parameter u^p. ($u^p \in K^{(\partial)}$ because $(u^p)^{(i)} = (\zeta_p^i u)^p = u^p$ for $i \in \mathbb{Z}/p\mathbb{Z}$.)

It is worthwhile to point out that $K^{(\partial)}$ depends on the choice of the rational parameter u, not just the derivation ∂.

Occasionally, we use $\varphi^{(\partial,n)} : K^{(\partial,n)} \hookrightarrow K$ to denote the p^n-th ∂-Frobenius obtained by applying the above construction n times; if K contains a primitive p^n-th root of unity ζ_{p^n}, this is the same as the fixed field for the natural action of $\mathbb{Z}/p^n\mathbb{Z}$ on K given by $x^{(i)} = \mathbb{T}(x; \partial, (\zeta_{p^n}^i - 1)u)$ for $i \in \mathbb{Z}/p^n\mathbb{Z}$.

Example 1.1.4.10. Let K_0 be a complete nonarchimedean field of characteristic zero. If K is the completion of $K_0(u)$ with respect to the η-Gauss norm and $\partial = \frac{d}{du}$, then $K^{(\partial)}$ is the completion of $K_0(u^p)$ with respect to the η^p-Gauss norm and $\varphi^{(\partial)}$ sends u^p to u^p.

Lemma 1.1.4.11. *The residue field $\kappa_{K^{(\partial)}}$ contains κ_K^p.*

Proof. We know that K is generated by u over $K^{(\partial)}$. If $|u| \notin |K^{(\partial)\times}|$, K would have same residue field as K does. If $|u| \in |K^{(\partial)\times}|$, let $x \in K^{(\partial)}$ be an element with $|x| = |u|$. Then κ_K is generated over $\kappa_{K^{(\partial)}}$ by $\overline{u/x}$, whose p-th power lies in $\kappa_{K^{(\partial)}}$. The statement follows. □

Lemma 1.1.4.12. *We have $|\partial'|_{K^{(\partial)}} = |u|^{-p}$.*

Proof. We may assume that K contains a primitive p-th root of unity ζ_p. We need only show that $u^p\partial'$ preserves $\mathcal{O}_{K^{(\partial)}}$. For any $x \in \mathcal{O}_{K^{(\partial)}}$, we have

$$x = \frac{1}{p}(x + x^{(1)} + \cdots + x^{(p-1)}) = \frac{1}{p}\sum_{n=0}^{\infty} \frac{\partial^n(x)}{n!} u^n \sum_{i=0}^{p-1} (\zeta_p^i - 1)^n.$$

Applying $u^p\partial' = u\partial/p$ gives

$$\begin{aligned} u^p\partial'(x) &= \frac{u}{p^2}\sum_{n=0}^{\infty}\left(\frac{\partial^{n+1}(x)}{n!}u^n\sum_{i=0}^{p-1}(\zeta_p^i-1)^n + \frac{\partial^n(x)}{(n-1)!}u^{n-1}\sum_{i=0}^{p-1}(\zeta_p^i-1)^n\right) \\ &= \frac{u}{p^2}\sum_{n=0}^{\infty}\frac{\partial^{n+1}(x)}{n!}u^n\sum_{i=0}^{p-1}(\zeta_p^i-1)^n\zeta_p^i. \end{aligned} \tag{1.1.4.13}$$

The sum $\sum_{i=0}^{p-1}(\zeta_p^i - 1)^n \zeta_p^i$ equals 0 for $n = 0, \ldots, p-2$; it equals p for $n = p-1$; and it is a multiple of p^2 for any $n \geq p$ (because the quantity belongs both to $\mathbb{Z}$ and to the ideal

$(\zeta_p - 1)^p$ in $\mathbb{Z}[\zeta_p]$). Hence by (1.1.4.13) and the fact that $u^{n+1}\partial^{n+1}(x)/(n+1)! \in \mathcal{O}_K$ from Definition 1.1.4.1, $u^p\partial'(x)$ equals $u^p\partial^p(x)/p!$ plus an element of $\mathcal{O}_K$, yielding $u^p\partial'(x) \in \mathcal{O}_K \cap K^{(\partial)} = \mathcal{O}_{K^{(\partial)}}$. □

Lemma 1.1.4.14. *The differential operator ∂' on $K^{(\partial)}$ is of rational type, with parameter u^p.*

Proof. Condition (a) in Definition 1.1.4.1 follows from Lemma 1.1.4.12 above and the simple fact that $\partial'(u^p) = \partial(u^p)/pu^{p-1} = 1$. Now, we check Condition (b) in Definition 1.1.4.1. Write

$$\begin{aligned}\frac{u^{pn}\partial'^n}{n!}(x) &= \frac{(u^p\partial')(u^p\partial' - 1)\cdots(u^p\partial' - (n-1))}{n!}(x) \\ &= \frac{(u\partial)(u\partial - p)\cdots(u\partial - (n-1)p)}{n!\cdot p^n}(x)\end{aligned}$$

As a corollary of Lemma 1.1.4.12, for any element $x \in K^{(\partial)}$ and $i \in \mathbb{Z}\setminus p\mathbb{Z}$, $|(u\partial - i)(x)| = |x|$. Since $u\partial$ maps $K^{(\partial)}$ to itself, applying differential operators $u\partial - i$ for $i \in \mathbb{Z}\setminus p\mathbb{Z}$ to the result will not change the norm, so

$$\left|\frac{u^{pn}\partial'^n}{n!}(x)\right| = \left|\frac{(u\partial)(u\partial-1)\cdots(u\partial-(np-1))}{n!\cdot p^n}(x)\right| = \left|\frac{u^{np}\partial^{np}}{(np)!}(x)\right|.$$

□

Definition 1.1.4.15. Given a ∂'-differential module V' over $K^{(\partial)}$, we may view $\varphi^{(\partial)*}V' = V' \otimes_{K^{(\partial)}} K$ as a ∂-differential module over K by setting

$$\partial(\mathbf{v}'\otimes x) = pu^{p-1}\partial'(\mathbf{v}')\otimes x + \mathbf{v}'\otimes\partial(x) \qquad (\mathbf{v}' \in V', x \in K).$$

Lemma 1.1.4.16. *Let V' be a ∂'-differential module over $K^{(\partial)}$. Then*

$$IR_\partial(\varphi^{(\partial)*}V') \geq \min\{IR_{\partial'}(V')^{1/p}, p\, IR_{\partial'}(V')\}.$$

Proof. This is essentially [Ked**, Lemma 10.3.2]. Consider the diagram

$$\begin{array}{ccc} K^{(\partial)} & \xrightarrow{f'^*_{\mathrm{gen}}} & K^{(\partial)}[\![T'/u^p]\!]_0 \\ \downarrow{\scriptstyle \varphi^{(\partial)*}} & & \downarrow{\scriptstyle \tilde{\varphi}^{(\partial)*}} \\ K & \xrightarrow{f^*_{\mathrm{gen}}} & K[\![T/u]\!]_0 \end{array}$$

where $\tilde{\varphi}^{(\partial)*}$ is a $K^{(\partial)}$-homomorphism extending $\varphi^{(\partial)*}$ by $\tilde{\varphi}^{(\partial)*}(T') = (u+T)^p - u^p$. The diagram commutes because formally

$$\begin{aligned}
(\tilde{\varphi}^{(\partial)*} \circ f'^*_{\mathrm{gen}})(x) &= \tilde{\varphi}^{(\partial)*}\left(\sum_{n=0}^{\infty} \binom{u^p\partial'}{n}(x)\left(\frac{T'}{u^p}\right)^n\right) \\
&= \sum_{n=0}^{\infty} \binom{u\partial/p}{n}(\varphi^{(\partial)*}(x))\left(\left(1+\frac{T}{u}\right)^p - 1\right)^n \\
&= \left(1 + \left(1+\frac{T}{u}\right)^p - 1\right)^{u\partial/p}(\varphi^{(\partial)*}(x)) \\
&= \left(\left(1+\frac{T}{u}\right)^p\right)^{u\partial/p}(\varphi^{(\partial)*}(x)) = \left(1+\frac{T}{u}\right)^{u\partial}(\varphi^{(\partial)*}(x)) \\
&= \sum_{n=0}^{\infty} \binom{u\partial}{n}(\varphi^{(\partial)*}(x))\left(\frac{T}{u}\right)^n = (f^*_{\mathrm{gen}} \circ \varphi^{(\partial)*})(x).
\end{aligned}$$

For $x \in K^{(\partial)}$, all of the series in this formal equation converge, and we obtain correct equalities.

For $r' \in [0,1)$, set $r = \min\{(r')^{1/p}, pr'\}$, or equivalently, $r' = \max\{r^p, p^{-1}r\}$. By Proposition 1.1.2.18,

$$\begin{aligned}
& R_{\partial'}(V') \geq r'|u|^p \\
\Leftrightarrow\quad & f'^*_{\mathrm{gen}}V' \text{ is a trivial } \partial_{T'}\text{-differential module over } A^1_{K^{(\partial)}}[0, r'|u|^p) \\
\Rightarrow\quad & \tilde{\varphi}^{(\partial)*}f'^*_{\mathrm{gen}}V' = f^*_{\mathrm{gen}}\varphi^{(\partial)*}V' \text{ is a trivial } \partial_T\text{-differential module over } A^1_K[0, r|u|) \\
\Leftrightarrow\quad & R_\partial(\varphi^{(\partial)*}V') \geq r|u|,
\end{aligned}$$

where the second implication is a direct corollary of Lemma 1.1.4.17 below. The statement follows. □

Lemma 1.1.4.17. [Ked**, Lemma 10.2.2] *Let K be a nonarchimedean field of residual characteristic $p > 0$. For $u, T \in K$ and $r \in (0,1)$, if $|u - T| \leq r|u|$, then*

$$|u^p - T^p| \leq \max\{r^p|u|^p, p^{-1}r|u|^p\}.$$

Example 1.1.4.18. We give a rank 1 example to see the spectral norm change after pulling back along $\varphi^{(\partial)}$. Let K be as above and let $x \in K$ be an element such that $\partial x = 0$ and $|x| \in (|u|^{-p}, p|u|^{-p})$. Then the rank 1 ∂-differential module $\mathcal{L}'_x$ over $K^{(\partial)}$ defined by $\partial' \mathbf{v} = x\mathbf{v}$ has visible intrinsic ∂'-radii $IR'_\partial(\mathcal{L}'_x) = \omega|u|^{-p}/|x| \in (p^{-p/(p-1)}, p^{-1/(p-1)})$. By Lemma 1.1.4.16, $IR_\partial(\varphi^{(\partial)*}\mathcal{L}'_x) > \omega$ is expected to be not visible.

The ∂-differential module $\varphi^{(\partial)*}\mathcal{L}'_x$ is generated by $\mathbf{v}$ and the derivation acts on it via $\partial\mathbf{v} = pu^{p-1}\mathbf{v}$. Then

$$\begin{aligned}
\partial^2\mathbf{v} &= \big((pu^{p-1}x)^2 + p(p-1)u^{p-2}x\big)\mathbf{v},\\
\partial^3\mathbf{v} &= \big((pu^{p-1}x)^3 + p^2(2p-2)u^{2p-1}x^2 + p(p-1)(p-2)u^{p-3}x\big)\mathbf{v},
\end{aligned}$$

and so on. Since $|x| < p|u|^{-p}$, we have $|pu^{p-1}x| < |u|^{-1}$. The dominant term is the last term (but not the first term) in the above equations. More generally, the dominant term in the expansion of $\partial^{np}\mathbf{v}$ is $\big(\partial^{p-1}(pu^{p-1}x)\big)^n\mathbf{v} = (p!)^n x^n \mathbf{v}$; it has norm $|px|^n|\mathbf{v}|$. Hence, $IR_\partial(\varphi^{(\partial')*}\mathcal{L}'_x) = \omega|u|^{-1}|px|^{-1/p} = IR_{\partial'}(\mathcal{L}'_x)^{1/p}$; this justifies Lemma 1.1.4.16.

Definition 1.1.4.19. Keep Hypothesis 1.1.4.8. For a ∂-differential module V over K, define the *∂-Frobenius descendant* of V as the $K^{(\partial)}$-module $\varphi^{(\partial)}_* V$ obtained from V by restriction along $\varphi^{(\partial)*} : K^{(\partial)} \to K$, viewed as a ∂'-differential module over $K^{(\partial)}$ with differential $\partial' = \frac{1}{pu^{p-1}}\partial$. Note that this operation commutes with duals.

Definition 1.1.4.20. For $n = 0, \ldots, p-1$, let $W_n^{(\partial)}$ be the ∂'-differential module over $K^{(\partial)}$ with one generator $\mathbf{v}$, such that

$$\partial'(\mathbf{v}) = \frac{n}{p}u^{-p}\mathbf{v}.$$

From the Newton polynomial associated to $\mathbf{v}$, we read off $IR_{\partial'}(W_n^{(\partial)}) = p^{-p/(p-1)}$ for $n \neq 0$. (One may view the generator $\mathbf{v}$ as a proxy for u^n.)

Lemma 1.1.4.21. *We have the following relations between ∂-Frobenius pullbacks and ∂-Frobenius descendants.*

(a) *For V a ∂-differential module over K, there are canonical isomorphisms*

$$\iota_n : (\varphi_*^{(\partial)} V) \otimes W_n^{(\partial)} \simeq \varphi_*^{(\partial)} V \qquad (n = 0, \dots, p-1).$$

(b) *For V a ∂-differential module over K, a submodule U of $\varphi_*^{(\partial)} V$ is itself the ∂-Frobenius descendant of a submodule of V if and only if $\iota_n(U \otimes W_n^{(\partial)}) = U$ for $n = 0, \dots, p-1$.*

(c) *For V a ∂-differential module over K, there is a canonical isomorphism*

$$\varphi^{(\partial)*} \varphi_*^{(\partial)} V \simeq V^{\oplus p}.$$

(d) *For V' a ∂'-differential module over $K^{(\partial)}$, there is a canonical isomorphism*

$$\varphi_*^{(\partial)} \varphi^{(\partial)*} V' \simeq \bigoplus_{n=0}^{p-1} (V' \otimes W_n^{(\partial)}).$$

(e) *For V_1, V_2 ∂-differential modules over K, there is a canonical isomorphism*

$$\varphi_*^{(\partial)} V_1 \otimes \varphi_*^{(\partial)} V_2 \simeq \bigoplus_{n=0}^{p-1} W_n^{(\partial)} \otimes \varphi_*^{(\partial)} (V_1 \otimes V_2).$$

(f) *For V a ∂-differential module over K, there are canonical bijections*

$$H_\partial^i(V) \simeq H_{\partial'}^i(\varphi_*^{(\partial)} V) \qquad (i = 0, 1).$$

Proof. Straightforward. □

Definition 1.1.4.22. Let V be a ∂-differential module over K such that $IR_\partial(V) > p^{-1/(p-1)}$. A *$\partial$-Frobenius antecedent* of V is a ∂'-differential module V' over $K^{(\partial)}$ such that $V \simeq \varphi^{(\partial)*}V'$ and $IR_{\partial'}(V') > p^{-p/(p-1)}$.

Proposition 1.1.4.23 (Christol-Dwork). *Let V be a ∂-differential module over K such that $IR_\partial(V) > p^{-1/(p-1)}$. Then there exists a unique ∂-Frobenius antecedent V' of V. Moreover, $IR_{\partial'}(V') = IR_\partial(V)^p$.*

Proof. As in [Ked**, Theorem 10.4.2]. □

Remark 1.1.4.24. As in [Ked**, Theorem 10.4.4], one can form a version of Proposition 1.1.4.23 for differential modules over discs and annuli.

Theorem 1.1.4.25. *Let V be a ∂-differential module over K. Then*

$$\mathfrak{IR}_{\partial'}(\varphi_*^{(\partial)}V) = \bigcup_{r\in\mathfrak{IR}_\partial(V)} \begin{cases} \{r^p,\ p^{-p/(p-1)}\ (p-1\ \textit{times})\} & r > p^{-1/(p-1)} \\ \{p^{-1}r\ (p\ \textit{times})\} & r \le p^{-1/(p-1)}. \end{cases}$$

In particular, $IR_{\partial'}(\varphi_*^{(\partial)}V) = \min\{p^{-1}IR_\partial(V),\ p^{-p/(p-1)}\}$.

Proof. The proof is identical to that of [Ked**, Theorem 10.5.1]. □

Corollary 1.1.4.26. *Let V' be a ∂'-differential module over $K^{(\partial)}$ such that $IR_{\partial'}(V') \neq p^{-p/(p-1)}$. Then $IR_\partial(\varphi^{(\partial)*}V') = \min\{IR_{\partial'}(V')^{1/p}, p\,IR_{\partial'}(V')\}$.*

Proof. In case $IR_{\partial'}(V') > p^{-p/(p-1)}$, this holds by [Ked**, Corollary 10.4.3]. Otherwise, by Lemma 1.1.4.21(d), $\varphi_*^{(\partial)}\varphi^{(\partial)*}V' \cong \oplus_{n=0}^{p-1}(V' \otimes W_n^{(\partial)})$ and $IR_{\partial'}(V' \otimes W_n^{(\partial)}) = IR_{\partial'}(V')$ since $IR_{\partial'}(V') < IR_{\partial'}(W_n^{(\partial)})$. Hence by Theorem 1.1.4.25,

$$IR_{\partial'}(V') = IR_{\partial'}(\varphi_*^{(\partial)}\varphi^{(\partial)*}V') = \min\{p^{-1}IR_\partial(\varphi^{(\partial)*}V'), p^{-p/(p-1)}\}.$$

We get a contradiction if the right side equals $p^{-p/(p-1)}$, so we must have $IR_{\partial'}(V') = p^{-1}IR_\partial(\varphi^{(\partial)*}V') \le p^{-p/(p-1)}$, proving the claim. □

For the following theorem, we do not assume $p > 0$.

Theorem 1.1.4.27. *Let V be a ∂-differential module over K. Then there exists a decomposition*

$$V = \bigoplus_{r \in (0,1]} V_r,$$

where every subquotient of V_r has pure intrinsic ∂-radii r. Moreover, if $p = 0$, then $r^{\dim V_r} \in |K^\times|$; if $p > 0$, then for any nonnegative integer h, we have

$$r < p^{-p^{-h}/(p-1)} \quad \Longrightarrow \quad r^{\dim V_r} \in |(K^{(\partial,h)})^\times|^{p^{-h}}.$$

Proof. The proof is similar to those of [Ked**, Theorem 10.6.2] and [Ked**, Theorem 10.7.1]. □

Remark 1.1.4.28. In the case of Example 1.1.4.10, $K^{(\partial,h)}$ is the completion of $K_0(u^{p^h})$ with respect to the η^{p^h}-Gauss norm. We deduce thus from Theorem 1.1.4.27 that $r^{\dim V_r} \in |K_0^\times|^{p^{-h}}\eta^{\mathbb{Z}}$.

Remark 1.1.4.29. Let K' be a complete extension of K equipped with an extension of ∂ which is again of rational type with parameter u. Then the intrinsic ∂-radii of a ∂-differential module over K are the same as that of its base extension to K': namely, this is clear from Remark 1.1.3.5 for those ∂-radii less than ω, but we can reduce to this case using Theorem 1.1.4.25.

1.1.5 Refined radii

In this subsection, we discuss the refined radii, which is a secondary information attached to a differential module. This will lead to the construction of refined Swan conductors later in Subsection 3.2.5.

Hypothesis 1.1.5.1. Until Hypothesis 1.1.5.19, let K be a complete nonarchimedean field of characteristic zero, equipped with a derivation ∂. Let V be a ∂-differential module of rank d over K of *pure* ∂-radii $R_\partial(V)$. Denote $s = -\log(\omega R_\partial(V)^{-1}) = -\log|\partial|_{\mathrm{sp},V}$.

Definition 1.1.5.2. A norm $|\cdot|_V$ on V is called *good* if it has an orthogonal basis and $|\partial|_V \leq \max\{|\partial|_K, |\partial|_{\mathrm{sp},V}\}$; it is forced to have equality when $|\partial|_{\mathrm{sp},V} \geq |\partial|_K$. We will see in Lemma 1.1.5.6 that such a good norm always exists.

Notation 1.1.5.3. Let $P(T) = T^d + a_1T^{d-1} + \cdots + a_d$ be a (twisted) polynomial with coefficients in K, whose Newton polygon has pure slope s (Definition 1.1.3.1). The *reduced roots* of P are the reductions of the roots in $\kappa^{(s)}_{K^{\mathrm{alg}}}$. If the characteristic polynomial P of a matrix A has pure slope s, we call the reduced roots of P *the reduced eigenvalues* of A.

Definition 1.1.5.4. Assume that V has (pure) visible ∂-radii. Let $|\cdot|_V$ be a good norm on V. Enlarge the value group of K in the sense of Remark 1.1.4.5 so that V admits an orthonormal basis. Let N be the matrix of ∂ acting on the chosen basis. Define the *refined ∂-radii* of V, denoted by $\Theta_\partial(V, |\cdot|) \subset \kappa^{(s)}_{K^{\mathrm{alg}}}$ to be the multiset of reduced eigenvalues of N. (In fact, we define the refined ∂-radii after we enlarged K, but this will not matter as we will explain in Remark 1.1.5.5 below.) We will see in Lemmas 1.1.5.7 and 1.1.5.8 that the refined ∂-radii are independent of the choices of good norm and orthonormal basis of V. After these lemmas, we will abbreviate $\Theta_\partial(V)$ for $\Theta_\partial(V, |\cdot|)$.

Remark 1.1.5.5. In the definition of refined ∂-radii, we first enlarged K to K' the completion of $K(u_1, \dots, u_r)$ for some $(\eta_1, \dots, \eta_r)$-Gauss norm. However, the refined ∂-radii $\Theta_\partial(V, |\cdot|)$ is still a (multi)subset of $\kappa^{(s)}_{K^{\mathrm{alg}}}$. Indeed, since the construction is canonical, for any $\theta \in \Theta_\partial(V, |\cdot|)$, $g\theta \in \Theta_\partial(V, |\cdot|)$ for any automorphism g of K' fixing K. But $\Theta_\partial(V, |\cdot|)$ is a finite set. So it can consist only of elements in $\kappa^{(s)}_{K^{\mathrm{alg}}}$. Alternatively, we can work out this more carefully in the computation of reduced eigenvalues to cancel the new variables we introduced.

Lemma 1.1.5.6. *For any V in Hypothesis 1.1.5.1, it has a good norm.*

Proof. By Lemma 1.1.2.4, there exists a cyclic vector $\mathbf{v} \in V$. We use P to denote the associated twisted polynomial. Let s be the lesser of $-\log|\partial|_K$ and the least slope of the Newton polygon of P. Then we can define a good norm on V by taking the orthogonal basis to be $\mathbf{v}, \partial\mathbf{v}, \dots, \partial^{d-1}\mathbf{v}$ and set $|\partial^i\mathbf{v}| = e^{-is}$ for $i = 0, \dots, d-1$. □

Lemma 1.1.5.7. *Assume that V has visible ∂-radii. Let $|\cdot|$ be a good norm on V. Then the refined ∂-radii $\Theta_\partial(V,|\cdot|)$ are well-defined.*

Proof. We may enlarge K as in Remark 1.1.4.5 so that we can find orthonormal bases for $|\cdot|$. Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ be two orthonormal bases for $|\cdot|$ and hence the transition matrix $A \in \mathrm{GL}_d(\mathcal{O}_K)$. Denote N for the matrix of ∂ acting on $\{\mathbf{e}_i\}$ and thus $|N| \leq |\partial|_V$. Since $|\det N| = |\partial|^d_{\mathrm{sp},V} = |\partial|^d_V$, the Newton and Hodge polygons of V coincide and have pure slope $-\log|\partial|_V$. The same is true for $A^{-1}NA$ since $A \in \mathrm{GL}_d(\mathcal{O}_K)$. On the other hand, $|A^{-1}\partial A| \leq |\partial|_K$ and hence all the singular values are not larger than $|\partial|_K < |\partial|_{\mathrm{sp},V}$. By [Ked**, Theorem 4.4.2], the reduced eigenvalues of N coincide with those of $A^{-1}NA + A^{-1}\partial A$. □

Lemma 1.1.5.8. *Let V be as above and let $|\cdot|_1$ and $|\cdot|_2$ be two good norms on V. Then, we have $\Theta_\partial(V,|\cdot|_1) = \Theta_\partial(V,|\cdot|_2)$.*

Proof. We may enlarge K as in Remark 1.1.4.5 so that $|\cdot|_1$ and $\cdot|_2$ both have orthonormal bases $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ and $\{\mathbf{f}_1,\dots,\mathbf{f}_n\}$, respectively. Let A be the matrix for which $(\mathbf{e}_1,\dots,\mathbf{e}_d)A = (\mathbf{f}_1,\dots,\mathbf{f}_d)$. By Gaussian elimination, we can find $P, Q \in \mathrm{GL}_d(\mathcal{O}_K)$ such that PAQ is a diagonal matrix. By Lemma 1.1.5.7, we may change bases $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_i\}$ so that A is in fact a diagonal matrix $\mathrm{Diag}\{a_{11},\dots,a_{dd}\}$.

Let N be the matrix of ∂ acting on the basis $\{\mathbf{e}_i\}$, then the matrix of ∂ acting on the basis $\{\mathbf{f}_i\}$ is given by $A^{-1}NA + A^{-1}\partial(A) = A^{-1}(N + \partial(A)A^{-1})A$. It suffices to show that N has the same reduced eigenvalues as $N + \partial(A)A^{-1}$. This is true by [Ked**, Theorem 4.4.2] because $\left|\partial(A)A^{-1}\right| = \left|\mathrm{Diag}\left(a_{11}^{-1}\partial(a_{11}),\dots,a_{dd}^{-1}\partial(a_{dd})\right)\right| \leq |\partial|_K$. □

Corollary 1.1.5.9. *Let V be as above. For* any *cyclic vector $\mathbf{v} \in V$, the reduced roots of the twisted polynomial associated to $\mathbf{v}$ are exactly the refined ∂-radii of V. In particular, they are not zero in $\kappa^{(s)}_{K^{\mathrm{alg}}}$.*

We would like to obtain a decomposition by refined ∂-radii as in Theorem 1.1.5.16 in the visible range and as in Theorem 1.1.5.22 when ∂ is of rational type.

Lemma 1.1.5.10. *Let V be as above. Then $\Theta_\partial(V^\vee) = -\Theta_\partial(V) = \{-\theta \mid \theta \in \Theta_\partial(V)\}$.*

Proof. Straightforward. □

Lemma 1.1.5.11. *Let V and W be two ∂-differential modules over K of pure and visible ∂-radii $R_\partial(V) = R_\partial(W)$. The following two statements are equivalent.*

(a) *The refined ∂-radii of V and W are distinct, i.e., $\Theta_\partial(V) \cap \Theta_\partial(W) = \emptyset$.*

(b) *The tensor product $V \otimes W^\vee$ has pure ∂-radii $R_\partial(V)$.*

Moreover, if either of the statements holds, we have $\Theta_\partial(V \otimes W^\vee) = \{\theta_1 - \theta_2 | \theta_1 \in \Theta_\partial(V), \theta_2 \in \Theta_\partial(W)\}$ as multisets.

As a corollary, we have

(1) *If $\Theta_\partial(V) \cap \Theta_\partial(W) = \emptyset$, any homomorphism $f : W \to V$ of ∂-differential modules is zero.*

(2) *If $\Theta_\partial(W)$ consists of only one element $\theta \in \kappa^{(s)}_{K^{\mathrm{alg}}}$ (with multiplicity), then $\theta \in \Theta_\partial(V)$ if and only if $V \otimes W^\vee$ is* not *of pure ∂-radii $R_\partial(V)$.*

Proof. By Lemma 1.1.5.10 above, $\Theta_\partial(W^\vee) = -\Theta_\partial(W)$. We may enlarge K as in Remark 1.1.4.5 so that we have good norms on V and $W^\vee$ given by orthonormal bases. Consider $V \otimes W^\vee$ with the norm given by the tensors of elements in the bases from two modules. Let $N_0, N_1 \in \mathrm{Mat}(\mathfrak{m}^{(s)}_{K^{\mathrm{alg}}})$ be the corresponding matrices of ∂ on V and $W^\vee$, respectively. Since N_0 has reduced eigenvalues $\Theta_\partial(V)$ and N_1 has reduced eigenvalues $-\Theta_\partial(W)$, the mamodultrix $N = N_0 \otimes 1 + 1 \otimes N_1$ would have reduced eigenvalues exactly the same as $\{\theta_1 - \theta_2 | \theta_1 \in \Theta_\partial(V), \theta_2 \in \Theta_\partial(W)\}$.

If (a) holds, N has nonzero reduced eigenvalues and hence $|N^n| = e^{-ns}$ for all $n \in \mathbb{N}$ with full rank when working modulo $\mathfrak{m}^{(ns)+}_{K^{\mathrm{alg}}}$ (and when identifying $\kappa^{(ns)}_{K^{\mathrm{alg}}}$ with $\kappa_{K^{\mathrm{alg}}}$). Therefore, $R_\partial(V \otimes W^\vee) = R_\partial(V)$.

If (b) holds, we in fact have a good norm on $V \otimes W^\vee$ already and the reduced eigenvalues of N should give the refined ∂-radii of $V \otimes W^\vee$. By Corollary 1.1.5.9, $0 \notin \Theta_\partial(V \otimes W^\vee)$. This implies (a).

Now, we prove (1). Since $R_\partial(V \otimes W^\vee) = R_\partial(V)$, $H^0_\partial(V \otimes W^\vee) = 0$, which parametrizes all homomorphisms of ∂-differential modules from W to V.

(2) is just the inverse statement of $(a) \Leftrightarrow (b)$. □

Lemma 1.1.5.12. *Let V and W be two ∂-differential modules over K. Assume that V has pure and visible ∂-radii and $R_\partial(V) < R_\partial(W)$. Then $R_\partial(V \otimes W^\vee) = R_\partial(V)$ and $\Theta_\partial(V \otimes W^\vee)$ is just $\Theta_\partial(V)$ with the multiplicity of each element multiplied by* $\dim W$.

Proof. By Proposition 1.1.3.4, we may assume that W has pure visible ∂-radii or non-visible radii. By Lemma 1.1.5.6, we may find a good norm on W. We proceed as in Lemma 1.1.5.11. Now, if $N_0, N_1 \in \mathrm{Mat}(K^{\mathrm{alg}})$ denote the matrices of ∂ on V and $W^\vee$, respectively, then $N_1 \in \mathrm{Mat}(\mathfrak{m}_K^{(s)+})$ and N_0 has reduced eigenvalues $\Theta_\partial(V)$. Hence $N_0 \otimes 1 + 1 \otimes N_1$ has the same reduced eigenvalues as N_0 but with multiplicity multiplied by $\dim W$. The lemma follows. □

Lemma 1.1.5.13. *Keep the notation as in Lemma 1.1.5.11(2) above. If, moreover, $\Theta_\partial(V)$ also consists only of $\theta \in \kappa_{K^{\mathrm{alg}}}^{(s)}$ (with multiplicity), then $R_\partial(V \otimes W^\vee) > R_\partial(V)$.*

Proof. We proceed similarly as in Lemma 1.1.5.11. Now N_0 and N_1 have pure reduced eigenvalues θ and $-\theta$, respectively. Hence $N = N_0 \otimes 1 + 1 \otimes N_1$ reduced to a matrix in $\kappa_{K^{\mathrm{alg}}}^{(s)}$ with zero eigenvalues (if we identify $\kappa_{k^{\mathrm{alg}}}^{(s)}$ with $\kappa_{K^{\mathrm{alg}}}$). It is then nilpotent, i.e., $N^n \in \mathrm{Mat}\big(\mathfrak{m}_{K^{\mathrm{alg}}}^{(ns)+}\big)$ for $n \gg 0$. This implies that $R_\partial(V \otimes W) > R_\partial(V)$. □

Remark 1.1.5.14. An alternative way to think about refined ∂-radii is the following. We call a ∂-differential module over K absolutely indecomposable if it is indecomposable over any finite tamely ramified extension K' of K. Lemma 1.1.5.13 implies that we can define an equivalence relation between all absolutely irreducible ∂-differential modules over finite tamely ramified extensions of K with same pure ∂-radii e^{-r} as follows: $V \sim W$ if and only if $R_\partial(V \otimes K'' \otimes W^\vee) > e^{-r}$ for some finite tamely ramified extension K'' on which V and W are both defined. Then the refined ∂-radii give a parameterization of this equivalence relation.

Lemma 1.1.5.15. *Let $\theta \in \kappa_{K^{\mathrm{alg}}}^{(s)} \backslash \{0\}$, where $s < -\log|\partial|_K$ and $s \in -\log|K^\times|^{\mathbb{Q}}$. Then we have the following.*

(a) *If $p = 0$, then $s \in -\log|(K')^\times|$ and $\theta \in \kappa_{K'}^{(s)}$ for some finite* tamely ramified *extension K'/K. Let $x \in \mathfrak{m}_{K'}^{(s)}$ be a lift of θ. Also, we set $n = 0$ in this case and $p^n = 1$ by convention.*

(b) *If $p > 0$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\theta^{p^n} \in \kappa_{K'}^{(p^n s)}$ with $p^n s \in -\log|(K')^\times|$ for some finite* tamely ramified *extension K'/K. Let $x \in \mathfrak{m}_{K'}^{(p^n s)}$ be a lift of θ^{p^n}.*

Define $\mathcal{L}_{x,(n)}$ to be the ∂-differential module over K' of rank p^n with basis $\{\mathbf{e}_1, \dots, \mathbf{e}_{p^n}\}$, on which ∂ acts as $\partial \mathbf{e}_i = \mathbf{e}_{i+1}$ for $i = 1, \dots, p^n - 1$ and $\partial \mathbf{e}_{p^n} = x\mathbf{e}_1$. Then $\mathcal{L}_{x,(n)}$ has pure ∂-radii ωe^s and $\Theta_\partial(\mathcal{L}_{x,(n)})$ consists of only θ with multiplicity p^n.

Proof. The existence of x is obvious. For the calculation of $\Theta_\partial(V)$, we may replace K by the completion of $K(z)$ with respect to the e^{-s}-Gauss norm (and set $\partial z = 0$). Then $\mathbf{e}_1, z^{-1}\mathbf{e}_2, \dots, z^{-(p^n-1)}\mathbf{e}_{p^n}$ gives a good norm on $\mathcal{L}_{x,(n)}$, for which the refined ∂-radii can be easily computed to be as stated. □

Using $\mathcal{L}_{x,(n)}$, we can obtain a decomposition by refined ∂-radii as follows.

Theorem 1.1.5.16. *Let K be a complete nonarchimedean field of characteristic zero, equipped with a derivation ∂. Let V be a ∂-differential module over K with pure and visible ∂-radii $R_\partial(V)$. Denote $s = -\log(\omega R_\partial(V)^{-1})$. Then V admits a canonical decomposition by refined ∂-radii as follows.*

$$V = \bigoplus_{\{\theta\} \subset \kappa_{K^{\mathrm{alg}}}^{(s)}} V_{\{\theta\}}, \tag{1.1.5.17}$$

where the direct sum runs through all Galois conjugacy classes in $\kappa_{K^{\mathrm{alg}}}^{(s)}$ and the refined ∂-radii of $V_{\{\theta\}}$ are exactly the Galois conjugacy class $\{\theta\}$ with same multiplicity on each element.

After making a finite tamely ramified *extension K' of K, one can obtain the canonical decomposition* (1.1.5.17) *without taking the conjugacy classes. In particular, $\Theta_\partial(V) \subset \cup_n \big(\kappa_{K'}^{(p^n s)}\big)^{1/p^n}$.*

Proof. By making a finite tamely ramified extension K' of K, we may assume that $\Theta_\partial(V) \subset \cup_n \big(\kappa_{K'}^{(p^n s)}\big)^{1/p^n}$. (By Corollary 1.1.4.7, doing so will not change the visible range.) For each $\theta \in \Theta_\partial(V)$, we construct $\mathcal{L}_{x,(n)}$ as in Lemma 1.1.5.15, which is a ∂-differential module of pure ∂-radii $R_\partial(V)$ and pure refined radii θ. By Lemma 1.1.5.11(2),

$V \otimes \mathcal{L}^\vee_{x,(n)}$ is not of pure radii $R_\partial(V)$. By Proposition 1.1.3.4, we get a decomposition $V \otimes \mathcal{L}^\vee_{x,(n)} = W_0 \oplus W_1$, where $R_\partial(W_0) > R_\partial(V)$ and W_1 is of pure ∂-radii $R_\partial(V)$.

Denote $\widetilde{W_0} = W_0 \otimes \mathcal{L}_{x,(n)}$ and $\widetilde{W_1} = W_1 \otimes \mathcal{L}_{x,(n)}$. Now consider the following homomorphisms of ∂-differential modules

$$V \underset{j}{\overset{i}{\rightleftarrows}} V \otimes \mathcal{L}^\vee_{x,(n)} \otimes \mathcal{L}_{x,(n)} \xrightarrow{\sim} \widetilde{W_0} \oplus \widetilde{W_1},$$

where i is the diagonal embedding and j is the projection so that $ji = \mathrm{id}$. Let p_0 and p_1 be the projection from $V \otimes \mathcal{L}^\vee_{x,(n)} \otimes \mathcal{L}_{x,(n)}$ to the factors $\widetilde{W_0}$ and $\widetilde{W_1}$, respectively, viewed as submodules of the source. Hence $p_0^2 = p_0$, $p_1^2 = p_1$, and $p_0 + p_1 = 1$. We claim that jp_0i and jp_1i are projections on V which give the desired decomposition.

By Lemma 1.1.5.13, $R_\partial(\mathcal{L}^\vee_{x,(n)} \otimes \mathcal{L}_{x,(n)}) > R_\partial(V)$. By Lemma 1.1.5.12, $V \otimes \mathcal{L}^\vee_{x,(n)} \otimes \mathcal{L}_{x,(n)}$ and hence $\widetilde{W_0}$ and $\widetilde{W_1}$ have pure ∂-radii $R_\partial(V)$. Also, by Lemma 1.1.5.12, $\Theta_\partial(\widetilde{W_0})$ consists of solely θ, and by Lemma 1.1.5.11,

$$\Theta_\partial(\widetilde{W_1}) = \big\{\theta_1 + \theta \text{ (with multiplicity } p^n) \mid \theta_1 \in \Theta_\partial(W_1)\big\}.$$

In particular, $\theta \notin \Theta_\partial(\widetilde{W_1})$. Hence any homomorphism of ∂-differential modules between $\widetilde{W_0}$ and $\widetilde{W_1}$ has to be zero by Lemma 1.1.5.11(1). In particular, $p_1ijp_0 = p_0ijp_1 = 0$. Thus, we have

$$\begin{aligned}
&(jp_0i)(jp_0i) = jp_0ij(1-p_1)i = jp_0i(ji) - j(p_0ijp_1)i = jp_0i\\
&(jp_1i)(jp_1i) = jp_1ij(1-p_0)i = jp_1i(ji) - j(p_1ijp_0)i = jp_1i\\
&jp_0i + jp_1i = j(p_0+p_1)i = ji = 1.
\end{aligned}$$

This proves that $V = jp_0i(V) \oplus jp_1i(V)$. Moreover, $\Theta_\partial(jp_0i(V))$ consists of only θ since it is a quotient of $\widetilde{W_0}$, and $\Theta_\partial(jp_1i(V))$ does not contain θ since it is a quotient of $\widetilde{W_1}$. Applying this process to each of $\theta \in \Theta_\partial(V)$ gives the desired decomposition (1.1.5.17).

The decomposition (over K') is canonical because it is characterized by the way that ∂ acts on direct summands. By Galois descent, we can easily get the decomposition over K as stated in the theorem. $\square$

Corollary 1.1.5.18. *Let V and W be two ∂-differential modules over K of pure and visible ∂-radii $R_\partial(V) = R_\partial(W)$. If we use U to denote the maximal ∂-differential submodule of $V \otimes W^\vee$ whose ∂-radius is larger than $R_\partial(V)$, we have*

$$\dim U = \sum_{\theta \in \kappa^{(s)}_{K^{\mathrm{alg}}}} \mathrm{multi}_\theta(\Theta_\partial(V)) \cdot \mathrm{multi}_\theta(\Theta_\partial(W)).$$

Proof. We may replace K by a finite tamely ramified extension. By Theorem 1.1.5.16, we may assume that V and W both have pure refined ∂-radii. If V and W have the same refined ∂-radii, the statement follows from Lemma 1.1.5.13; if they are different, the statement follows from Lemma 1.1.5.11. $\square$

We now extend the definition of refined ∂-radii to the non-visible case when ∂ is a derivation of rational type.

Hypothesis 1.1.5.19. For the rest of this subsection, we abandon Hypothesis 1.1.5.1. Let K be a complete nonarchimedean field of characteristic zero equipped with a derivation ∂ of rational type with respect to u. Let V be a ∂-differential module of pure intrinsic ∂-radii $IR_\partial(V) < 1$. Denote $s = -\log(\omega IR_\partial(V)^{-1})$.

Definition 1.1.5.20. Let $\varphi^{(\partial)} : K^{(\partial)} \to K$ be the ∂-Frobenius. We define the *refined intrinsic ∂-radii*, denoted by $\mathcal{I}\Theta_\partial(V)$ as follows.

(a) If $IR_\partial(V) < \omega$, define $\mathcal{I}\Theta_\partial(V) = u \cdot \Theta_\partial(V)$.

(b) If $p > 0$ and $IR_\partial(V) = \omega$, $\varphi^{(\partial)}_*(V)$ has pure ∂'-radii $p^{-p(p-1)}$. By Lemma 1.1.4.21(a) and Lemma 1.1.5.11, the element in $\mathcal{I}\Theta_{\partial'}(\varphi^{(\partial)}_*(V))$ can be grouped into p-tuples $\left(\theta, \theta + \frac{1}{p}, \dots, \theta + \frac{p-1}{p}\right)$ (with some multiplicity), where $\theta \in \kappa^{(-\log p)}_{K^{\mathrm{alg}}}$. Define $\mathcal{I}\Theta_\partial(V)$ to be the multiset consisting of $(p^p\theta^p - p\theta)^{1/p} \in \kappa_{K^{\mathrm{alg}}}$ for each p-tuple in $\mathcal{I}\Theta_{\partial'}(\varphi^{(\partial)}_*(V))$.

(c) If $p > 0$ and $IR_\partial(V) > \omega$, by Proposition 1.1.4.23, $V = \varphi^{(\partial)*}W$ for some ∂'-differential module W on $K^{(\partial)}$ such that $IR_{\partial'}(W) = IR_\partial(V)^p$. We define

$$\mathcal{I}\Theta_\partial(V) = \left\{(p\theta')^{1/p} \mid \theta' \in \mathcal{I}\Theta_{\partial'}(W)\right\} \subset \kappa^{(s)}_{K^{\mathrm{alg}}}.$$

If $IR_\partial(V)$ is large, we need to iteratively apply (c) to seek for its higher ∂-Frobenius antecedents until we arrive at case (a) or (b), and then we solve all the way back to define $\mathcal{I}\Theta_\partial(V)$.

Remark 1.1.5.21. An alternative way to see the Frobenius antecedent in Definition 1.1.5.20 is to consider the decomposition of $\varphi^{(\partial)}_* V$ given by Theorem 1.1.4.27. The Frobenius antecedent is exactly the submodule in this decomposition which has intrinsic ∂-radii $IR_\partial(V)^{1/p}$.

Theorem 1.1.5.22. *Let K be a complete nonarchimedean field of characteristic zero equipped with a derivation ∂ of rational type. Let V be a ∂-differential module over K with pure intrinsic ∂-radii $IR_\partial(V) < 1$. Denote $s = -\log(\omega IR_\partial(V)^{-1})$. Then V admits a canonical decomposition by refined radii as follows.*

$$V = \bigoplus_{\{\theta\} \subset \kappa^{(s)}_{K^{\mathrm{alg}}}} V_{\{\theta\}}, \tag{1.1.5.23}$$

where the direct sum runs through all Galois conjugacy classes in $\kappa^{(s)}_{K^{\mathrm{alg}}}$ and the refined ∂-radii of $V_{\{\theta\}}$ are exactly the Galois conjugacy class $\{\theta\}$ with same multiplicity on each element.

After making a finite tamely ramified *extension K' of K, one can obtain the canonical decomposition* (1.1.5.23) *without taking the conjugacy classes. In particular, $\Theta_\partial(V) \subset \cup_n \left(\kappa^{(p^n s)}_{K'}\right)^{1/p^n}$.*

Proof. If $IR_\partial(V) < \omega$, this is just Theorem 1.1.5.16. If $p > 0$ and $IR_\partial(V) = \omega$, applying Theorem 1.1.5.16 to $\varphi^{(\partial)}_* V$ as a ∂'-differential module gives a decomposition (1.1.5.17). If we group the direct summands according the refined ∂'-radii as in

Definition 1.1.5.20(b), i.e., we write

$$V = \bigoplus_{\{(\theta,\dots,\theta+\frac{p-1}{p})\}\subset\kappa_{K^{\mathrm{alg}}}/\mathbb{F}_p} V_{\{(\theta,\dots,\theta+\frac{p-1}{p})\}},$$

where the refined radii of $V_{\{(\theta,\dots,\theta+\frac{p-1}{p})\}}$ are exactly the Galois conjugates of p-tuples $(\theta,\dots,\theta+\frac{p-1}{p})$ with same multiplicity on each element. By Lemma 1.1.5.11, the decomposition is invariant under twisting by $W_n^{(\partial)}$ and hence it descends to a decomposition of ∂-differential modules over K.

If $p>0$ and $IR_\partial(V)>\omega$, the decomposition (1.1.5.23) comes from the decomposition of its ∂-Frobenius antecedent. □

Example 1.1.5.24. Let $p>0$. We extend the example in Lemma 1.1.5.15 to the non-visible case as follows, giving examples with pure intrinsic refined ∂-radii. Let $\theta\in\kappa_{K^{\mathrm{alg}}}^{(s)}\backslash\{0\}$, where $s\in[0,\frac{1}{p}\log p)$ and $s\in-\log|K^\times|^{\mathbb{Q}}$. Similar to Lemma 1.1.5.15, there exists $n\in\mathbb{N}$ such that $\theta^{p^n}\in(\kappa_{K'}^{(p^{n-1}s)})^p$ with $p^{n-1}s\in-\log|(K')^\times|$ for some finite *tamely ramified* extension K'/K. Let $x\in\mathfrak{m}_{K'^{(\partial)}}^{(p^ns)}$ be a lift of θ^{p^n}; we may find a lift in $K'^{(\partial)}$ because of Lemma 1.1.4.11.

Let $\mathcal{L}_{x,(n)}$ denote be the ∂-differential module over K' of rank p^n with basis $\{\mathbf{e}_1,\dots,\mathbf{e}_{p^n}\}$, on which ∂ acts as $\partial\mathbf{e}_i=\mathbf{e}_{i+1}$ for $i=1,\dots,p^n-1$ and $\partial\mathbf{e}_{p^n}=xu^{-p^n}\mathbf{e}_1$, where the extra u^{-p^n} compare to Lemma 1.1.5.15 reflects the different normalizations of refined intrinsic ∂-radii and refined ∂-radii.

Remark 1.1.5.25. The restriction $s\in[0,\frac{1}{p}\log p)$ in Example 1.1.5.24 is linked with the choice $x\in\mathfrak{m}_{K'^{(\partial)}}^{p^ns}$. In order to extend s to the radius $[0,(\frac{1}{p-1}-\frac{1}{p^r(p-1)})\log p)$ we need to be able take $x\in\mathfrak{m}_{K'^{(\partial,r)}}^{p^ns}$ lifting θ^{p^n} for some $n\in\mathbb{N}$ and some tamely ramified K'/K. However, as r gets larger, n might need to take a bigger value accordingly to guarantee the existence of the valid lift x. That is why we cannot essentially remove this restriction.

Lemma 1.1.5.26. *Keep the notation as in Example 1.1.5.24. Then $\mathcal{L}_{x,(n)}$ has pure non-visible intrinsic ∂-radii $IR_\partial(\mathcal{L}_{x,(n)})=\omega e^s$ and pure refined intrinsic ∂-radii θ.*

Proof. We need to consider $\varphi_*^{(\partial)}\mathcal{L}_{x,(n)}$, which has a basis given by

$$\{u^{i+l}\mathbf{e}_i \mid l = 0, \ldots, p-1;\ i = 1, \ldots, p^n\}.$$

The derivation $u^p\partial' = \frac{1}{p}u\partial$ acts on this basis as follows.

$$\begin{aligned} u^p\partial' u^{i+l}\mathbf{e}_i &= \tfrac{i+l}{p}u^{i+l}\mathbf{e}_i + \tfrac{1}{p}u^{i+l+1}\mathbf{e}_{i+1}, \quad l = 0, \ldots, p-1; i = 1, \ldots, p^n - 1, \\ u^p\partial' u^{p^n+l}\mathbf{e}_{p^n} &= \tfrac{p^n+l}{p}u^{p^n+l}\mathbf{e}_{p^n} + \tfrac{1}{p}xu^{l+1}\mathbf{e}_1, \quad l = 0, \ldots, p-1. \end{aligned}$$

We replace K by the completion of $K(z)$ with respect to the e^{-s}-Gauss norm and set $\partial(z) = 0$. Then, with respect to the basis given by

$$\begin{aligned} &u\mathbf{e}_1, z^{-1}u^2\mathbf{e}_2, \ldots, z^{-p^n+1}u^{p^n}\mathbf{e}_{p^n}, \\ &u^2\mathbf{e}_1, z^{-1}u^3\mathbf{e}_2, \ldots, z^{-p^n+1}u^{p^n+1}\mathbf{e}_{p^n}, \\ &\cdots \quad \cdots \\ &u^p\mathbf{e}_1, z^{-1}u^{p+1}\mathbf{e}_2, \ldots, z^{-p^n+1}u^{p^n+p-1}\mathbf{e}_{p^n}, \end{aligned}$$

the action of $u^p\partial'$ is given by a block diagonal matrix N, whose diagonal blocks are

$$N_l = \begin{pmatrix} \frac{l+1}{p} & 0 & \cdots & 0 & \frac{1}{p}xz^{-p^n+1} \\ \frac{z}{p} & \frac{l+2}{p} & \cdots & 0 & 0 \\ 0 & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{l+p^n-1}{p} & 0 \\ 0 & 0 & \cdots & \frac{z}{p} & \frac{l+p^n}{p} \end{pmatrix}$$

where $l = 0, \ldots, p-1$. (Here, we used the crucial fact that $x \in K'^{(\partial)}$.) Hence, $\varphi_*^{(\partial)}\mathcal{L}_{x,(n)} = \oplus_{l=0}^{p-1}V_l$, where V_l corresponds to the ∂-differential module on which ∂ acts via N_l, for $l = 0, \ldots, p-1$. It is easy to compute the characteristic polynomial of the matrix N_l, as

$$\prod_{i=1}^{p^n}(T - \frac{l+i}{p}) - \frac{x}{p^{p^n}}. \tag{1.1.5.27}$$

The Newton polygon for the product has slopes $r\log p$ with multiplicity $p^{n-r-2}(p-1)$ for $r = -1, 0, \ldots, n-2$ and a slope $(n-1)\log p$ with multiplicity 1. Since $|x| \in (p^{-p^{n-1}}, 1]$, $|\frac{x}{p^{p^n}}| \in (p^{p^{n-1}(p-1)}, p^{p^n}]$ and hence the Newton polygon of (1.1.5.27) has slope $-\log p$ with multiplicity $p^{n-1}(p-1)$ (from the product) and slope

$$\frac{-\log|\frac{x}{p^{p^n}}| - p^{n-1}(p-1)(-\log p)}{p^{n-1}} = \frac{p^n s - p^{n-1}\log p}{p^{n-1}} = ps - \log p,$$

with multiplicity p^{n-1} (from the new constant term $\frac{x}{p^{p^n}}$). The two kinds of slopes are the same if (and only if) $s = 0$.

If $s = 0$, the refined ∂'-radii of V_l are exactly the reduced roots of (1.1.5.27), which are $\frac{1}{p}$ times the reduced roots of $\prod_{i=1}^{p^n}(T' - (l+i)) = x$, that is, $\frac{1}{p}$ times the roots of $\prod_{i=0}^{p-1}(T' - \bar{i})^{p^{n-1}} = \theta^{p^n}$. By Definition 1.1.5.20(b), this says exactly the statement of the lemma.

Now, we assume that $s > 0$. The condition on slopes of (1.1.5.27) implies that the singular values of N_l match the norms of the eigenvalues of N_l. By [Ked**, Theorem 4.3.11], we can find a matrix $U \in \mathrm{GL}_{p^n}(\mathcal{O}_{K'(\partial)})$ such that $U^{-1}N_lU$ is block diagonal triangular, with the top left block accounting for the singular values p and the bottom right block accounting for the singular values pe^{-ps}. Then applying [Ked**, Lemma 6.7.1] to $U^{-1}N_lU + U^{-1}u^p\partial'(U)$ implies that we can find $V \in \mathrm{GL}_{p^n}(\mathcal{O}_{K'(\partial)})$ such that $V^{-1}N_lV + V^{-1}u^p\partial'(V)$ is block lower triangular.

The key point here is that the leading term for the characteristic polynomial of the lower block is $T^{p^{n-1}} - \frac{x}{p^{p^{n-1}}}$ because adding $V^{-1}u^p\partial'(V) \in \mathrm{Mat}_{p^n}(\mathcal{O}_{K'(\partial)})$ does not affect the leading term. Therefore, there is a unique submodule W_l of rank p^{n-1} of V_l of intrinsic ∂'-radii $p^{-p/(p-1)}e^{ps}$ and pure intrinsic ∂'-radii $\Theta_{\partial'}(W_l) = \overline{x^{1/p^{n-1}}/p}^{(ps-\log p)} = \theta^p/p$. Thus, by Definition 1.1.5.20(c), this implies the statement of the lemma. □

1.1.6 Multiple derivations

In this subsection, we introduce differential fields of higher order.

Notation 1.1.6.1. In this subsection, set $J = \{1, \ldots, m\}$ for notational convenience.

Definition 1.1.6.2. Let K denote a differential ring of order m, i.e., a ring K equipped with m commuting derivations $\partial_1, \dots, \partial_m$. For $j \in J$, a *∂_j-differential module* is a finite projective K-module V equipped with the action of ∂_j. In other words, we view K as a differential ring of order 1 by forgetting the derivations other than ∂_j. A *$(\partial_1, \dots, \partial_m)$-differential module* (or *∂_J-differential module*, or simply a *differential module*) is a finite projective K-module V equipped with commuting actions of $\partial_1, \dots, \partial_m$. We may apply the results above by singling out one of $\partial_1, \dots, \partial_m$.

Definition 1.1.6.3. Let K be a complete nonarchimedean differential field of order m and characteristic zero, and let V be a nonzero $(\partial_1, \dots, \partial_m)$-differential module over K. Define the *intrinsic generic radius of convergence*, or for short the *intrinsic radius*, of V to be

$$IR(V) = \min_{j \in J} \{IR_{\partial_j}(V)\} = \min_{j \in J} \{|\partial_j|_{\mathrm{sp},K} / |\partial_j|_{\mathrm{sp},V}\}.$$

For $j \in J$, we say ∂_j is *dominant* for V if $IR_{\partial_j}(V) = IR(V)$. We define the *intrinsic subsidiary radii* $\mathfrak{IR}(V) = \{IR(V;1), \dots, IR(V; \dim V)\}$ by collecting and ordering intrinsic radii from Jordan-Hölder factors, as in Definition 1.1.2.8. We again say that V has *pure intrinsic radii* if the elements of $\mathfrak{IR}(V)$ are all equal to $IR(V)$.

Similarly, we define the *extrinsic generic radius of convergence* (or *extrinsic radius*) $R(V)$ to be the minimum of $R_{\partial_j}(V)$ and *extrinsic subsidiary radii* $\mathfrak{R}(V) = \{R(V;1), \dots, R(V; \dim V)\}$ by collecting and ordering extrinsic radii from Jordan-Hölder factors. (This concept will not be used until Subsection 1.2.7).

Definition 1.1.6.4. Let K be a complete nonarchimedean differential field of order m and characteristic zero. We say that K is of *rational type* with respect to a set of parameters $\{u_j : j \in J\}$ if each ∂_j is of rational type with respect to u_j, and $\partial_i(u_j) = 0$ for $i \neq j$ in J.

Remark 1.1.6.5. Recall that if $p > 0$, we have a ∂_j-Frobenius $\varphi^{(\partial_j)*} : K^{(\partial_j)} \hookrightarrow K$ for $j \in J$. Since the elements $u_{J \setminus \{j\}}$ are killed by ∂_j, they are elements in $K^{(\partial_j)}$. Hence by Lemma 1.1.4.14, the differential operators $\partial_{J \setminus \{j\}}$ and ∂_j' are of rational type over $K^{(\partial_j)}$ with respect to the parameters $u_{J \setminus \{j\}}$ and u_j^p.

Theorem 1.1.6.6. *Let K be a complete nonarchimedean differential field of order m and characteristic zero, of rational type. Let V be a ∂_J-differential module over K. Then there exists a decomposition*

$$V = \bigoplus_{r \in (0,1]} V_r,$$

where every subquotient of V_r has pure intrinsic radii r. Moreover, if $p = 0$, then $r^{\dim V_r} \in |K^\times|$; if $p > 0$, then

$$r < p^{-p^{-h}/(p-1)} \implies r^{\dim V_r} \in |K^\times|^{1/p^h}.$$

Proof. Since the ∂_J commute with each other, the theorem follows by applying Theorem 1.1.4.27 to each ∂_j and forming a common refinement of the resulting decompositions. □

One important instance of Definition 1.1.6.4 is the following.

Situation 1.1.6.7. Let F be a complete discretely valued field of characteristic zero with residue field κ of characteristic $p > 0$. Let K_1 be a complete extension of F with the same value group and residue field k_1 separable over κ. Assume that $u_1, \ldots, u_m \in \mathcal{O}_{K_1}$ lifts a set of p-basis of κ_{K_1} over κ_F (Definition 1.1.1.11). Let F' be an extension of F complete for a (not necessarily discrete) nonarchimedean norm $|\cdot|$, with the same residue field κ. Let K_2 be the completion of $K_1 \otimes_F F'$. Let k be a (possibly infinite) separable algebraic extension of k_1, and let K be the completion of the unramified extension of K_2 with residue field k.

In this case, we call such K *standard differential fields.*

Lemma 1.1.6.8. *In Situation 1.1.6.7, the natural projection $\Omega^1_K \twoheadrightarrow K \otimes \Omega^1_{K_1/F} \simeq \bigoplus_{j=1}^m K \cdot du_j$ gives derivations $(\partial_j = \partial_{u_j})_{j \in J}$ of rational type with respect to $u_1, \ldots, u_m$.*

Proof. It is enough to check for K_1: it is clear that the same conclusion then holds for K_2, and then Lemma 1.1.4.6 implies the same conclusion for K. That is, we must check that $\mathcal{O}_{K_1}$ is stable under $\partial_j^n/n!$ for all nonnegative integers n and all $j \in J$. For each $n \in \mathbb{N}$, any element $x \in \mathcal{O}_{K_1}$ can be written (not uniquely) as

$$x = \sum_{i=0}^{+\infty} \sum_{e_J=0}^{p^n-1} \alpha_{n,i,e_J}^{p^n} u_J^{e_J} f_i,$$

where $\alpha_{n,i,e_J} \in \mathcal{O}_{K_1}^\times \cup \{0\}$ and $f_i \in \mathcal{O}_F$. Then for any $j_0 \in J$,

$$\frac{\partial_{j_0}^n}{n!}(x) = \sum_{i=0}^{+\infty} \sum_{e_J=0}^{p^n-1} \sum_{\beta=0}^{n} \frac{\partial_{j_0}^\beta}{\beta!} \left(\alpha_{n,i,e_J}^{p^n}\right) \frac{\partial_{j_0}^{n-\beta}}{(n-\beta)!} (u_J^{e_J})\, f_i \in \mathcal{O}_{K_1}.$$

The lemma follows. □

Remark 1.1.6.9. Situation 1.1.6.7 includes the two options in [Ked07a, Hypothesis 2.1.3]. (Note that [Ked07a, Hypothesis 2.1.3(b)] should require that l/k be separable.) We will see later (Theorem 1.2.8.2) that the results in [Ked07a] carry over to differential fields of rational type.

Next, we continue the discussion in Subsection 1.1.5 to the case with multiple derivations.

Definition 1.1.6.10. Let K be a complete nonarchimedean field of characteristic zero, equipped with m commuting derivations ∂_J. Let V be a ∂_J-differential module of pure ∂_j-radii for all $j \in J$. A norm $|\cdot|_V$ on V is *good* if it is good for each ∂_j.

Proposition 1.1.6.11. *Let K and V be as above. If K is discretely valued and $\max\{|\partial_j|_K, |\partial_j|_{\mathrm{sp},V}\} \in |K^\times|^{\mathbb{Q}}$ for all $j \in J$, then a good norm of V always exists.*

Proof. Denote $r_j = \max\{|\partial_j|_K, |\partial_j|_{\mathrm{sp},V}\}$ for $j \in J$. There is no harm to replace K by the completion of $K(x_J)$ with respect to the r_J-Gauss norm, where we set $\partial_j(x_{j'}) = 0$ for $j, j' \in J$. (In particular, K is still discretely valued.) It suffices to show that given any norm $|\cdot|_V$ with orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_d$, the submodule M of V generated by

$$\left\{x_J^{a_J} \partial_J^{a_J} \mathbf{e}_i \middle| a_j \in \mathbb{Z}_{\geq 0} \text{ for } j \in J; i \in \{1, \dots, d\}\right\}$$

over $\mathcal{O}_K$ is a finite $\mathcal{O}_K$-module; if so, M would give V a norm, under which $|\partial_j| \leq |x_j| = r_j$ verifies the condition of good norm in Definition 1.1.5.2. To prove that M

is a finite sub-$\mathcal{O}_K$-module, it suffices to prove that $|x_j^n\partial_j^n|_V$ is bounded for each j as $n \to +\infty$. (Here, we used the fact that K is discretely valued, otherwise, boundness may not imply finiteness.) It is enough to verify this boundness condition for any norm on V. In particular, for each of ∂_j, we can choose a good norm by Lemma 1.1.5.6, for which $|x_j^n\partial_j^n|_V \leq 1$. Thus, M is finite over $\mathcal{O}_K$ and the lemma follows. □

We now define the notion of refined radii for multi-dirivations.

Definition 1.1.6.12. Let K be a complete nonarchimedean field of characteristic zero, equipped with derivations ∂_J of rational type with respect to parameters u_J. Let V be a ∂_J-differential module of pure intrinsic radii $IR(V)$. (This does not mean that it is of pure ∂-radii for each j.) Denote $s = -\log(\omega IR(V)^{-1})$. By Theorem 1.1.5.22, we may replace K by a finite tamely ramified extension such that $V \otimes K$ admits a direct sum decomposition $V = \oplus V_{\theta_J}$ such that, for each direct summand V_{θ_J},

(a) it has pure refined intrinsic ∂_j-radius θ_j for any j such that $IR_{\partial_j}(V_{\theta_J}) = IR(V_{\theta_J})$,

(b) we set $\theta_j = 0$ for other j.

Define the *refined intrinsic radii* of V, denoted by $\mathcal{I}\Theta(V)$, to be $\vartheta = \sum_{j\in J}\theta_j\frac{du_j}{u_j}$ with multiplicity $\dim V_{\theta_J}$; it is an element in $\oplus_{j\in J}\kappa_{K^{\mathrm{alg}}}^{(s)}\frac{du_j}{u_j}$.

Remark 1.1.6.13. One may hope to find some analogue of Example 1.1.5.24 for ∂_J-differential modules. This, however, amounts to carefully choosing the x in Example 1.1.5.24 for each single differential operator so that the action of ∂_J *commutes.* This places some restriction on possible refined intrinsic radii. In other words, all possible intrinsic refined radii form only a subset of $\oplus_{j\in J}\kappa_{K^{\mathrm{alg}}}^{(s)}\frac{du_j}{u_j}$. Unfortunately, we do not know how to identify this subset in general. The following proposition partly answers this question.

Proposition 1.1.6.14. *Keep the notation as above. Assume that K is discretely valued and $s < 0$, i.e., V has visible intrinsic radii. Assume moreover that $p = 0$ or $d = \operatorname{rank} V = 1$. Note that the action of $u_j\partial_j$ on K induces a derivation on $\kappa_{K^{\mathrm{unr}}}^{(s)}$. If $\vartheta = \sum_{j\in J}\theta_j\frac{du_j}{u_j} \in \mathcal{I}\Theta(V)$, then for $i, j \in J$, we have $u_i\partial_i\theta_j = u_j\partial_j\theta_i$ in $\kappa_{K^{\mathrm{unr}}}^{(s)}$.*

Proof. By possibly replacing K by a finite tamely ramified extension, we reduce to the case when V is irreducible and has pure refined intrinsic radii $\sum_{j\in J}\theta_j\frac{du_j}{u_j}$. By Proposition 1.1.6.11, we can find a good norm $|\cdot|_V$, for which $u_j\partial_j$ acts as a matrix $N_j\in \mathrm{Mat}_{d\times d}(\mathfrak{m}_K^{(r)})$. Since ∂_i and ∂_j commute with each other for any $i,j\in J$, we have

$$N_iN_j+u_i\partial_i(N_j)=N_jN_i+u_j\partial_j(N_i). \qquad (1.1.6.15)$$

Taking the trace of (1.1.6.15), we have $d\cdot u_i\partial_i\theta_j=d\cdot u_j\partial_j\theta_i$. The proposition follows.

□

Proposition 1.1.6.16. *Let K be a complete discretely valued field of characteristic zero, equipped with derivations ∂_J of rational type with respect to parameters u_J. Let V be a ∂_J-differential module of pure* visible *intrinsic radii $IR(V)$. Denote $s=-\log(\omega IR(V)^{-1})$. Assume that $s\in -\log|K^\times|^{\mathbb{Q}}$. Assume moreover that $\partial=\alpha_1\frac{u_1}{u}\partial_1+\cdots+\alpha_m\frac{u_m}{u}\partial_m$ for some $\alpha_1,\dots,\alpha_m\in\mathcal{O}_K$ is another derivation on K of rational type with respect to u, and V is of pure intrinsic ∂-radii $IR_\partial(V)=IR(V)$. Then, we have*

$$\mathcal{I}\Theta_\partial(V)=\big\{\alpha_1\theta_1+\cdots+\alpha_m\theta_m \mid \vartheta=\theta_1\frac{du_1}{u_1}+\cdots+\theta_m\frac{du_m}{u_m}\in\mathcal{I}\Theta(V)\big\}.$$

Moreover, if V has pure refined intrinsic radii $\vartheta=\theta_1\frac{du_1}{u_1}+\cdots+\theta_m\frac{du_m}{u_m}$, $IR_\partial(V)=IR(V)$ if and only if $\alpha_1\theta_1+\cdots+\alpha_m\theta_m\neq 0$ in $\kappa_{K^{\mathrm{alg}}}^{(s)}$.

Proof. Let J_0 be those $j\in J$ for which $IR_{\partial_j}(V)=IR(V)$. We may assume that V has pure intrinsic ∂_j-radii for any $j\in J$. By Proposition 1.1.6.11 and by possibly enlarging K, we may assume that V admits a good norm given by some orthonormal basis and V has pure refined intrinsic ∂_j-radii θ_j for any $j\in J_0$. For $j\in J$, let N_j be the matrix of $u_j\partial_j$ acting on this basis. Then N_j has a unique reduced eigenvalue θ_j for $j\in J_0$ and $N_j\in\mathrm{Mat}(\mathfrak{m}_K^{(s)+})$ for $j\in J\backslash J_0$. The key point here is that ∂_i commutes with ∂_j for $i,j\in J_0$. Hence, (1.1.6.15) implies that $N_iN_j\equiv N_jN_i$ in $\mathrm{Mat}(\kappa_K^{(2s)})$. Therefore, N_{J_0} can be *uniformly* diagonalized in $\mathrm{Mat}(\kappa_{K^{\mathrm{alg}}}^{(s)})$ by a matrix in $\mathrm{GL}(\kappa_K^{\mathrm{alg}})$.

Therefore, if N is the matrix for $u\partial$ acting on this basis, then

$$N = \sum_{j \in J} \alpha_j N_j \equiv \sum_{j \in J_0} \alpha_j N_j$$

in $\mathrm{Mat}(\kappa_K^{(s)})$, which has all reduced eigenvalue equal to $\sum_{j \in J_0} \alpha_j \theta_j$. The proposition is proved. □

1.2 Differential modules on 1-dimensional spaces

Having considered differential modules over fields, we next consider differential modules on a disc or annulus over a differential field. This parallels [Ked**, Chapters 11 and 12].

1.2.1 Setup

Hypothesis 1.2.1.1. Throughout this section, we assume that K is a complete (not necessarily discretely valued) nonarchimedean differential field of order m, characteristic zero, and residual characteristic p (not necessarily positive). We also assume K is of rational type.

Notation 1.2.1.2. Let $\partial_1, \dots, \partial_m$ denote the derivations on K and let $u_1, \dots, u_m$ denote a set of corresponding rational parameters. Let $J = \{1, \dots, m\}$. We reserve j and J for indexing derivations.

Notation 1.2.1.3. For $\eta > 0$, let F_η be the completion of $K(t)$ under the η-Gauss norm $|\cdot|_\eta$. Put $\partial_0 = \frac{d}{dt}$ on F_η; by Remark 1.1.4.5, F_η is of rational type for the derivations ∂_{J^+}, where $J^+ = J \cup \{0\} = \{0, \dots, m\}$.

Remark 1.2.1.4. For $I \subseteq [0, +\infty)$ an interval and $j \in J^+$, we may refer to differential modules or ∂_j-differential modules over $A_K^1(I)$, meaning locally free coherent sheaves with the appropriate derivations. For $I = [\alpha, \beta]$ closed, these are just modules with appropriate derivations over the principal ideal domain $K\langle \alpha/t, t/\beta \rangle$; in particular, any ∂_j-differential module over a closed annulus is free by [Ked**, Proposition 9.1.2].

Caution 1.2.1.5. When $j \in J$, the category of ∂_j-differential modules over $A_K^1(I)$ is not abelian. In fact, the quotient of two ∂_j-differential modules may not be locally free. However, the category of ∂_0-differential modules over $A_K^1(I)$ is free because the existence of the derivation forces the modules to be free.

Remark 1.2.1.6. For $I \subseteq [0, +\infty)$ an interval, and M a nonzero ∂_j-differential module over $A_K^1(I)$, it is unambiguous to refer to the intrinsic ∂_j-radius $IR_{\partial_j}(M \otimes F_\eta)$ of M at $|t| = \eta$.

The intrinsic radii are stable under tame base change.

Proposition 1.2.1.7. *Let n be a (possibly negative) nonzero integer (coprime to p if $p > 0$), and let $f_n^* : F_\eta \to F_{\eta^{1/n}}$ be the map $t \to t^n$. Then for any $j \in J^+$, and for any ∂_j-differential module V over F_η, $IR_{\partial_j}(V) = IR_{\partial_j}(f_n^* V)$ and hence $\mathfrak{IR}_{\partial_j}(V) = \mathfrak{IR}_{\partial_j}(f_n^* V)$.*

Proof. The proof for $j = 0$ is in [Ked**, Proposition 9.7.6], and the proof for $j \in J$ is to apply Remark 1.1.2.7. □

Remark 1.2.1.8. One may also consider off-centered tame base change; see Subsection 4.2.2.

1.2.2 Variation of subsidiary radii

In this subsection, we prove slightly weakened analogues of some results in [Ked**, Chapter 11]. We begin by studying the variation of slopes of Newton polygons.

Notation 1.2.2.1. Let $P \in K\langle \alpha/t, t/\beta \rangle [T]$ be a polynomial of degree d. For $r \in [-\log\beta, -\log\alpha]$, let $\mathrm{NP}_r(P)$ denote the Newton polygon of P under $|\cdot|_{e^{-r}}$.

Proposition 1.2.2.2. *For $r \in [-\log\beta, -\log\alpha]$, let $f_1(P, r), \ldots, f_d(P, r)$ be the slopes of $\mathrm{NP}_r(P)$ in increasing order. For $i = 1, \ldots, d$, put $F_i(P, r) = f_1(P, r) + \cdots + f_i(P, r)$.*

(a) *(Linearity) For $i = 1, \ldots, d$, the functions $f_i(P, r)$ and $F_i(P, r)$ are continuous and piecewise affine in r.*

(b) *(Integrality) If $i = d$ or $f_i(r_0) < f_{i+1}(r_0)$, then the slopes of $F_i(P, r)$ in some neighborhood of $r = r_0$ belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i(P, r)$ and $F_i(P, r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{d}\mathbb{Z}$.*

(c) *(Monotonicity) Suppose that P is monic and $\alpha = 0$. For $i = 1, \ldots, d$, the slopes of $F_i(P, r)$ are nonnegative.*

(d) *(Concavity) Suppose that P is monic. For $i = 1, \ldots, d$, the function $F_i(P, r)$ is concave.*

(e) *(Truncation) For any fixed $a \in \mathbb{R}_{\geq 0}$ and $b \in \mathbb{R}$, the statements (a), (c), and (d) are also true if we replace $f_i(P, r)$ by $\min\{f_i(P, r), ar + b\}$ for all $i \in \{1, \ldots, d\}$.*

Proof. See [Ked**, Theorem 11.2.1] and [Ked**, Remark 11.2.4]. □

Lemma 1.2.2.3 (Lattice lemma). *Put $R = K\langle t \rangle$, $\cup_{\alpha<1} K\langle \alpha/t, t \rangle$, or $\cup_{\alpha<1<\beta} K\langle \alpha/t, t/\beta \rangle$, or (if K is discrete) $K[\![t]\!]_0$ or $\cup_{\alpha<1} K\langle \alpha/t, t]\!]_0$ equipped with the norm $|\cdot|_1$. Let M be a finite free R-module of rank n, and let $|\cdot|_M$ be a norm on M compatible with R. Assume that either:*

(a) *$c > 1$, and the value group of K is not discrete; or*

(b) *$c \geq 1$, and the value groups of K and M coincide and are discrete.*

Then there exists a basis of M defining a supremum norm $|\cdot|'_M$ for which $c^{-1}|m|_M \leq |m|'_M \leq c|m|_M$ for $m \in M$.

Proof. Let F be the completion of $\mathrm{Frac} R$ under $|\cdot|_1$. By [Ked**, Lemma 1.3.7], we can construct a basis of $M \otimes F$ defining a supremum norm $|\cdot|'_M$ for which $c^{-1}|m|_M \leq |m|'_M \leq c|m|_M$ for $m \in M$. If $R = K\langle t \rangle$, or K is discrete and $R = K[\![t]\!]_0$, then [Ked**, Lemma 8.6.1] gives a basis over M defining the same supremum norm $|\cdot|'_M$. If $R = \cup_{\alpha<1} K\langle \alpha/t, t \rangle$ or $\cup_{\alpha<1<\beta} K\langle \alpha/t, t/\beta \rangle$, then [Ked**, Lemma 8.6.1] gives a basis of $K\langle 1/t, t \rangle$ defining $|\cdot|'_M$. However, we can approximate that basis arbitrarily closely with a basis of M itself, because R is dense in $K\langle 1/t, t \rangle$ under $|\cdot|_1$, and any element of R with an inverse in $K\langle 1/t, t \rangle$ also has an inverse in R. Any sufficiently good approximation will define the same supremum norm. If K is discrete and $R = \cup_{\alpha<1} K\langle \alpha/t, t]\!]_0$, then R itself is a field, so we can approximate a basis of $M \otimes F$ with a basis of M defining the same supremum norm. $\square$

Notation 1.2.2.4. Fix $j \in J^+$. Let M be a ∂_j-differential module of rank d over $K\langle \alpha/t, t/\beta\rangle$. For $r \in [-\log\beta, -\log\alpha]$ and $i \in \{1, \dots, d\}$, define

$$f_i^{(j)}(M, r) = -\log R_{\partial_j}(M \otimes F_{e^{-r}}; i), \quad F_i^{(j)}(M, r) = f_1^{(j)}(M, r) + \cdots + f_i^{(j)}(M, r).$$

Theorem 1.2.2.5. [Ked**, Theorem 11.3.2] *Let M be a ∂_0-differential module of rank d over $K\langle \alpha/t, t/\beta\rangle$.*

(a) *(Linearity) For $i = 1, \dots, d$, the functions $f_i^{(0)}(M, r)$ and $F_i^{(0)}(M, r)$ are continuous and piecewise affine.*

(b) *(Integrality) If $i = d$ or $f_i^{(0)}(M, r_0) > f_{i+1}^{(0)}(M, r_0)$, then the slopes of $F_i^{(0)}(M, r)$ in some neighborhood of r_0 belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i^{(0)}(M, r)$ and $F_i^{(0)}(M, r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{d}\mathbb{Z}$.*

(c) *(Monotonicity) Suppose that $\alpha = 0$. For any point r_0 where $f_i^{(0)}(M, r_0) > r_0$, the slopes of $F_i^{(0)}(M, r)$ are nonpositive in some neighborhood of r_0. Also, $f_i^{(0)}(M, r_0) = r_0$ for r_0 sufficiently large.*

(d) *(Convexity) For $i = 1, \dots, d$, the function $F_i^{(0)}(M, r)$ is convex.*

We have a similar but slightly weaker result for ∂_j-differential modules when $j \in J$.

Theorem 1.2.2.6. *Fix $j \in J$. Let M be a ∂_j-differential module of rank d over $K\langle \alpha/t, t/\beta\rangle$.*

(a) *(Linearity) For $i = 1, \dots, d$, the functions $f_i^{(j)}(M, r)$ and $F_i^{(j)}(M, r)$ are continuous. They are piecewise affine in the locus where $f_i^{(j)}(M, r) > -\log|u_j|$; if $p = 0$, they are in fact piecewise affine everywhere.*

(b) *(Weak integrality)*

(i) *Suppose $p=0$. If $i=d$ or $f_i^{(j)}(M,r_0) > f_{i+1}^{(j)}(M,r_0)$, then the slopes of $F_i^{(j)}(M,r)$ in some neighborhood of r_0 belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i^{(j)}(M,r)$ and $F_i^{(j)}(M,r)$ at $r=r_0$ belong to $\frac{1}{1}\mathbb{Z}\cup\cdots\cup\frac{1}{d}\mathbb{Z}$.*

(ii) *Suppose $p>0$. If $i=d$ or $f_i^{(j)}(M,r_0) > f_{i+1}^{(j)}(M,r_0)$, and $f_i^{(j)}(M,r_0) > \frac{1}{p^n(p-1)}\log p - \log|u_j|$ for some $n\in\mathbb{Z}_{\geq 0}$, then the slopes of $F_i^{(j)}(M,r)$ in some neighborhood of r_0 belong to $\frac{1}{p^n}\mathbb{Z}$. Consequently, if $f_i^{(j)}(M,r_0) > \frac{1}{p^n(p-1)}\log p - \log|u_j|$ for some $n\in\mathbb{Z}_{\geq 0}$, the slopes of $f_i^{(j)}(M,r)$ and $F_i^{(j)}(M,r)$ at $r=r_0$ belong to $\frac{1}{p^n}\mathbb{Z}\cup\cdots\cup\frac{1}{p^n d}\mathbb{Z}$.*

(c) *(Monotonicity) Suppose that $\alpha=0$. For $i=1,\ldots,d$, the slopes of $F_i^{(j)}(M,r)$ are nonpositive.*

(d) *(Convexity) For $i=1,\ldots,d$, the function $F_i^{(j)}(M,r)$ is convex.*

Proof. We prove the theorem analogously to [Ked**, Theorem 11.3.2]. First of all, as in Remark 1.1.4.5, we may replace K by the completion of $K(x)$ with respect to the $|u_j|$-Gauss norm. We may then replace u_j by u_j/x and hence ∂_j by $x\partial_j$ to reduce to the case $|u_j|=1$.

We first show that the statements are true for $\tilde{f}_i^{(j)}(M,r)=\max\{f_i^{(j)}(M,r),\epsilon\}$ with $\epsilon > -\log\omega$ and $\widetilde{F}_i^{(j)}(M,r)=\tilde{f}_1^{(j)}(M,r)+\cdots+\tilde{f}_i^{(j)}(M,r)$. Let $F=\mathrm{Frac}K\langle\alpha/t,t/\beta\rangle$. Choose a cyclic vector for $M\otimes F$ to obtain an isomorphism $M\otimes F\cong F\{T\}/F\{T\}P$ for some monic twisted polynomial P over F. We may then apply Proposition 1.2.2.2 and Remark 1.1.3.5 to deduce (a) and (b), provided we omit the last assertion in (a) (in case $p=0$); for that, see below.

For (c) and (d), it suffices to work in a neighborhood of some r_0. Again by Remark 1.1.4.5, there is no harm in enlarging K so that $e^{-r_0}\in|K^\times|$. We may reduce to the case $r_0=0$ by replacing t by λt for some $\lambda\in K^\times$ with $|\lambda|=e^{-r_0}$. We then argue as in [Ked**, Lemma 11.5.1] and deduce (c) and (d) from Proposition 1.2.2.2, as follows. We may further enlarge K to include $\lambda_1,\ldots,\lambda_d\in\mathrm{Ker}(\partial_j)$ such that

$$-\log|\lambda_i| = \min\left\{-\log\omega - f_i^{(j)}(M,0), 0\right\} \qquad (i=1,\ldots,d).$$

Let B_0 be the basis of $M \otimes F_1$ given by

$$\lambda_{d-1}^{-1} \cdots \lambda_{d-i}^{-1} T^i \qquad (i = 0, \dots, d-1).$$

Let N_0 be the characteristic polynomial of the matrix of action of ∂_j on B_0. Let $\mu_1, \dots, \mu_d$ be the eigenvalues of N_0, labeled so that $|\mu_1| \geq \cdots \geq |\mu_d|$. By [Ked**, Proposition 4.3.10], we have $\max\{|\mu_i|, 1\} = \max\{\omega e^{f_i^{(j)}(M,0)}, 1\}$ for $i = 1, \dots, d$. By Lemma 1.2.2.3, for each $c > 1$, we may construct a basis B_c of M such that the supremum norms $|\cdot|_0, |\cdot|_c$ defined by B_0, B_c satisfy $c^{-1}|\cdot|_c \leq |\cdot|_0 \leq c|\cdot|_c$. Let N_c be the matrix of action of ∂_j on B_c. For $c > 1$ sufficiently small, [Ked**, Theorem 6.7.4] implies that for r close to 0, the visible spectrum of $M \otimes F_{e^{-r}}$ is the multiset of those norms of eigenvalues of the characteristic polynomial of N_c which exceed 1. We may then deduce (c) and (d) from Proposition 1.2.2.2(c) and (d).

We next relax the truncation condition that we have imposed; we may assume $p > 0$ as otherwise there is nothing to check. For each nonnegative integer n, we will prove the claim for $\tilde{f}_i^{(j)}(M,r) = \max\{f_i^{(j)}(M,r), \epsilon\}$ and $\widetilde{F}_i^{(j)}(M,r) = \tilde{f}_1^{(j)}(M,r) + \cdots + \tilde{f}_i^{(j)}(M,r)$ with $\epsilon \in \left(\frac{1}{p^n(p-1)}\log p, \frac{1}{p^{n-1}(p-1)}\log p\right]$, by induction on n; the base case $n = 0$ is proved already. As above, we may reduce to the case $r_0 = 0$.

Consider the ∂_j-Frobenius $\varphi^{(\partial_j)*} : F_{e^{-r}}^{(\partial_j)} \hookrightarrow F_{e^{-r}}$. Put $g_i^{(j)}(r) = -\log R_{\partial_j}(\varphi_*^{(\partial_j)} M \otimes F_{e^{-r}}^{(\partial_j)}; i)$ and $\tilde{g}_i^{(j)}(r) = \max\{g_i^{(j)}(r), p\epsilon\}$ for $i = 1, \dots, pd$. By Theorem 1.1.4.25, the list $\{g_1^{(j)}(r), \dots, g_{pd}^{(j)}(r)\}$ consists of

$$\bigcup_{i=1}^{d} \begin{cases} \{pf_i^{(j)}(M,r), \frac{p}{p-1}\log p \ (p-1 \text{ times})\} & f_i^{(j)}(M,r) \leq \frac{1}{p-1}\log p \\ \{\log p + f_i^{(j)}(M,r) \ (p \text{ times})\} & f_i^{(j)}(M,r) \geq \frac{1}{p-1}\log p. \end{cases}$$

Thus, the list $\tilde{g}_1^{(j)}(r), \dots, \tilde{g}_{pd}^{(j)}(r)$ consists of

$$\bigcup_{i=1}^{d} \begin{cases} \{p\tilde{f}_i^{(j)}(M,r), \frac{p}{p-1}\log p \ (p-1 \text{ times})\} & f_i^{(j)}(M,r) \leq \frac{1}{p-1}\log p \\ \{\log p + \tilde{f}_i^{(j)}(M,r) \ (p \text{ times})\} & f_i^{(j)}(M,r) \geq \frac{1}{p-1}\log p. \end{cases}$$

We may thus deduce (a) and (b) directly from the induction hypothesis. We similarly deduce (d) as in [Ked**, Lemma 11.6.1], except that we are considering $\tilde{g}_i^{(j)}(r)$ but not $\tilde{g}_i^{(j)}(pr)$; this explains the weakened integrality result. (See also Remark 1.1.4.28.) Also, we can luckily deduce (c) directly, because $\varphi^{(\partial_j)*}$ does not introduce a singularity on $A_K^1[0,\beta]$; by contrast, in the proof of [Ked**, Theorem 11.3.2], one must switch to an off-centered Frobenius to avoid a singularity at $t=0$.

We deduce that (a)-(d) hold for $\tilde{f}_i^{(j)}(M,r) = \max\{f_i^{(j)}(M,r),\epsilon\}$ and $\widetilde{F}_i^{(j)}(M,r) = \tilde{f}_1^{(j)}(M,r) + \cdots + \tilde{f}_i^{(j)}(M,r)$ with $\epsilon > 0$. The desired results hold by taking $\epsilon \to 0^+$.

This completes the proof except that if $p = 0$, we must still prove piecewise affinity everywhere. In this case, the integrality of (b) is not burdened with an extra denominator of p^n, so we may repeat the argument from [Ked**, Lemma 11.6.3]; see Step 3 of Theorem 1.2.4.4 for essentially the same argument. $\square$

Example 1.2.2.7. When $j \in J$, we do not expect an integrality result as in the $j=0$ case; see Remark 1.1.4.28. One can easily generate an example in which the strong integrality statement for ∂_j fails, as follows. Suppose $p > 0$, $\alpha \in (p^{-1/(p-1)}, 1)$, and $|u_j| = 1$. We take the rank one ∂_j-differential module M over $K\langle \alpha/t, t\rangle$ generated by $\mathbf{v}$ with $\partial_j(\mathbf{v}) = t^{-1}\mathbf{v}$. Thus, $f_1^{(j)}(M,r) = r$ for $r \in [0, -\log\alpha]$. By Corollary 1.1.4.26, $f_1^{(j)}(\varphi^{(\partial_j)*}M, r) = \frac{r}{p}$.

Remark 1.2.2.8. Besides the weakening of the integrality condition, there are some other aspects in which Theorem 1.2.2.6 is weaker than its counterpart [Ked**, Theorem 11.3.2] if $p > 0$. For one, the latter includes a subharmonicity assertion, which refers to the algebraic closure of the residue field of K. It is awkward to add a subharmonicity assertion here because the residue field of K is crucially imperfect, so that it can admit a nontrivial p-basis. (By contrast, if $p = 0$, we can achieve a subharmonicity result; see Theorem 1.2.9.6.) For another, Theorem 1.2.2.6(a) does not apply in a neighborhood of a point r_0 at which $f_i^{(j)}(M, r_0) = -\log|u_j|$. The argument in [Ked**, Lemma 11.6.3] does not extend to this case because the weak integrality result does not give a lower bound on slopes. On the other hand, we do not have a counterexample against the claim that $f_i^{(j)}(M,r)$ is everywhere piecewise affine.

1.2.3 Decomposition by subsidiary radii

In this subsection, we prove some decomposition theorems over annuli and discs, as in [Ked**, Chapter 12]. We start by a technical lemma, copied from [Ked07+b, Lemma 1.2.7].

Lemma 1.2.3.1. *Let*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

be a commuting diagram of inclusions of integral domains, such that the intersection $S \cap T$ *within* U *is equal to* R. *Let* M *be a finite locally free* R*-module. Then the intersection of* $M \otimes_R S$ *and* $M \otimes_R T$ *within* $M \otimes_R U$ *is equal to* M.

Proof. Choose $\mathbf{e}_1, \dots, \mathbf{e}_n \in M$ which form a basis of $M \otimes_R (\mathrm{Frac} R)$; then there exists $f \in R$ such that $fM \subseteq R\mathbf{e}_1 + \cdots + R\mathbf{e}_n$. Given $\mathbf{v} \in M \otimes_R U$ which belongs to both $M \otimes_R S$ and $M \otimes_R T$, we can uniquely write $f\mathbf{v} = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n$ with $c_i \in U$. From the intersection property, we have $c_i \in R$ for $i = 1, \dots, n$, whence $f\mathbf{v} \in M$.

Since M is locally free, as we vary the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, the values of f obtained generate the unit ideal of R. We thus have $\mathbf{v} \in M$, as desired. □

Lemma 1.2.3.2. *Retain notation as in Lemma 1.2.3.1. Then any direct sum decompositions of* $M \otimes_R S$ *and* $M \otimes_R T$ *which agree on* $M \otimes_R U$ *are induced by a unique direct sum decomposition of* M.

Proof. Apply Lemma 1.2.3.2 to the idempotents in $M^\vee \otimes M$ giving the projections onto the factors in the decompositions. □

Lemma 1.2.3.3. *Given* $\alpha < \beta$ *and* $x \in K\{\{\alpha/t, t/\beta\}\}$ *such that the function* $r \mapsto \log|x|_{e^{-r}}$ *is affine for* $r \in (-\log\beta, -\log\alpha)$, *then* x *is a unit in* $K\{\{\alpha/t, t/\beta\}\}$.

Proof. The condition is equivalent to saying that the Newton polygon of x does not have any slopes in $(-\log\beta, -\log\alpha)$. This immediately implies the claim. □

Lemma 1.2.3.4. *Let* $P = \sum_i P_i T^i$ *and* $Q = \sum_i Q_i T^i$ *be polynomials over* $K\langle\alpha/t, t/\beta\rangle$ *satisfying the following conditions.*

(a) *We have* $|P-1|_\gamma < 1$ *for all* $\gamma \in [\alpha, \beta]$.

(b) *For* $d = \deg(Q)$, Q_d *is a unit and* $|Q|_\gamma = |Q_d|_\gamma$ *for all* $\gamma \in [\alpha, \beta]$.

Then P *and* Q *generate the unit ideal in* $K\langle \alpha/t, t/\beta \rangle[T]$.

Proof. We may assume without loss of generality that $Q_d = 1$. The hypothesis that $|Q|_\gamma = |Q_d|_\gamma$ for all $\gamma \in [\alpha, \beta]$ implies that if S is the remainder upon dividing R by Q, then $|S|_\gamma \leq |R|_\gamma$ for all $\gamma \in [\alpha, \beta]$ (compare [Ked**, Lemma 2.3.1]). If we then let S_i denote the remainder upon dividing $(1-P)^i$ by Q, the series $\sum_{i=0}^{\infty} S_i$ converges in $K\langle \alpha/t, t/\beta \rangle[T]$ (since the degrees of the S_i are bounded by $d-1$) and its limit S satisfies $PS \equiv 1 \pmod{Q}$. □

Theorem 1.2.3.5. *Fix* $j \in J^+$. *Let* M *be a* ∂_j*-differential module of rank* d *on* $A_K^1(\alpha, \beta)$. *Suppose that the following conditions hold for some* $i \in \{1, \ldots, d-1\}$.

(a) *The function* $F_i^{(j)}(M, r)$ *is affine for* $-\log\beta < r < -\log\alpha$.

(b) *We have* $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ *for* $-\log\beta < r < -\log\alpha$.

Then M *admits a unique direct sum decomposition separating the first* i *subsidiary* ∂_j*-radii of* $M \otimes F_\eta$ *for any* $\eta \in (\alpha, \beta)$.

Proof. When $j = 0$, this is [Ked**, Theorem 12.4.2]; we thus assume hereafter that $j \in J$. The proof is similar to those of [Ked**, Theorems 12.2.2 and 12.3.1]; for the benefit of the reader, we fill in some of the key details.

By Lemma 1.2.3.2, we may enlarge K as needed; in particular, we may reduce to the case $|u_j| = 1$ as in the proof of Theorem 1.2.2.6. Since the decomposition is unique if it exists, it is sufficient to exhibit it on an open cover of (α, β) and then glue. That is, it suffices to work in a neighborhood of any fixed $\gamma \in (\alpha, \beta)$; again, we may enlarge K to reduce to the case $\gamma = 1$.

Suppose first that $f_i^{(j)}(M, 0) > -\log\omega$. Set notation as in the proof of Theorem 1.2.2.6. For some sufficiently small $c > 1$, we can choose $\gamma_1 \in (\alpha, 1)$ and $\gamma_2 \in (1, \beta)$ such that the coefficient of T^{d-i} in the characteristic polynomial $Q(T)$ of N_c computes $F_i^{(j)}(M, r)$ for $r \in [-\log\gamma_2, -\log\gamma_1]$; by (a), we may apply Lemma 1.2.3.3

(after changing γ_1, γ_2 slightly) to deduce that this coeffficient is a unit in $K\langle\gamma_1/t, t/\gamma_2\rangle$. By (b), we can apply [Ked**, Theorem 2.2.2] to factor $Q = Q_2Q_1$ so that the roots of Q_1 are the i largest roots of Q under $|\cdot|_\gamma$ for all $\gamma \in [\gamma_1, \gamma_2]$. (This is true for all γ simultaneously because the construction is purely algebraic and [Ked**, Theorem 2.2.2] takes care of convergence of the procedure.)

Use the basis B_c to identify M with $K\langle\gamma_1/t, t/\gamma_2\rangle^d$. Then we obtain a short exact sequence

$$0 \to \mathrm{Ker}\,(Q_1(N_c)) \to M \to \mathrm{Coker}\,(Q_2(N_c)) \to 0$$

of free modules over $K\langle\gamma_1/t, t/\gamma_2\rangle$. (The quotient is free because by Lemma 1.2.3.4 applied after rescaling, Q_1 and Q_2 generate the unit ideal in $K\langle\gamma_1/t, t/\gamma_2\rangle[T]$.) Applying Lemma 1.2.2.3 to both factors (again for $c > 1$ sufficiently small, and a choice of γ_1, γ_2 depending on c), we construct a basis of M on which ∂_j acts via a matrix

$$N'_c = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}$$

for which the following conditions hold.

(a) The matrix A_c is invertible and $|A_c^{-1}|_\gamma \cdot \max\{|\partial_j|_\gamma, |B_c|_\gamma, |C_c|_\gamma, |D_c|_\gamma\} < 1$ for all $\gamma \in [\gamma_1, \gamma_2]$.

(b) The Newton slopes of A_c under $|\cdot|_\gamma$ account for the first i subsidiary radii of $M \otimes F_\gamma$ for all $\gamma \in [\gamma_1, \gamma_2]$.

By [Ked**, Lemma 6.7.1], M admits a ∂_j-differential submodule accounting for the last $n - i$ subsidiary ∂_j-radii of $M \otimes F_\gamma$ for all $\gamma \in [\gamma_1, \gamma_2]$. By repeating this argument for $M^\vee$, we obtain the desired splitting.

To deduce the theorem in the case $p > 0$ without assuming that $f_i^{(j)}(M, 0) > \frac{1}{p-1}\log p$, we prove the theorem in the case when $f_i^{(j)}(M, 0) > \frac{1}{p^n(p-1)}\log p$ by induction on n, using ∂_j-Frobenius pushforward. This is sufficient because (b) forces $f_i^{(j)}(M, 0) > 0$, so there exists some n for which $f_i^{(j)}(M, 0) > \frac{1}{p^n(p-1)}\log p$. $\square$

Caution 1.2.3.6. In Theorem 1.2.3.5, M is only a locally free coherent sheaf and need not be free, because the annulus on which we are working is not closed. Even if M is free, the summands need not be free unless K is spherically complete, in which case any locally free coherent sheaf on $A^1_K(\alpha, \beta)$ is free.

Remark 1.2.3.7. In [Ked**, Chapter 12], the analogous development starts with a full decomposition theorem over a closed annulus [Ked**, Theorem 12.2.2]. We cannot do this here because we have not established an analogue of subharmonicity [Ked**, Theorem 11.3.2(c)] for ∂_j-differential modules, except in the case $p = 0$ (see Theorems 1.2.9.10 and 1.2.9.11). We can however recover partial decomposition theorems over a closed disc or annulus, analogous to [Ked**, Theorems 12.5.1 and 12.5.2], as follows.

Lemma 1.2.3.8. (a) *For $x \in K[\![t]\!]_0$ nonzero, x is a unit if and only if $|x|_{e^{-r}}$ is constant in a neighborhood of $r = 0$.*

(b) *For $x \in \cup_{\alpha \in (0,1)} K\langle \alpha/t, t]\!]_0$ nonzero, x is a unit if and only if the function $r \mapsto \log|x|_{e^{-r}}$ is affine in some neighborhood of 0.*

Proof. This is [Ked**, Exercise 12.3]; for the benefit of the reader, we sketch the proof here. We may assume that $|x|_1 = 1$. For (a), this means that $x \in \mathcal{O}_K[\![t]\!]$. Hence, $x = \sum_{i=0}^{\infty} a_i t^i$ is a unit if and only if a_0 is a unit in $\mathcal{O}_K$, which is equivalent to $|x|_{e^{-r}}$ being constant in a neighborhood of $r = 0$. For (b), by [Ked**, Lemma 8.2.6(c)], x is a unit if and only if its image modulo $\mathfrak{m}_K$ in $\kappa_K((t))$ is a unit or equivalently nonzero, which is equivalent to the function $r \mapsto \log|x|_{e^{-r}}$ being affine in some neighborhood of 0. $\square$

Theorem 1.2.3.9. *Fix $j \in J^+$. Let M be a ∂_j-differential module of rank d over $A^1_K(\alpha, \beta]$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

(a) *The function $F_i^{(j)}(M, r)$ is affine for $-\log\beta \leq r < -\log\alpha$.*

(b) *We have $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ for $-\log\beta \leq r < -\log\alpha$.*

Then $M \otimes K\{\{\alpha/t, t/\beta]\!]_0$ admits a direct sum decomposition separating the first i subsidiary ∂_j-radii of $M \otimes F_\eta$ for $\eta \in (\alpha, \beta)$.

Proof. We first obtain a decomposition of $M \otimes K\langle\delta/t, t/\beta]\!]_0$ for some uncontrolled $\delta \in (\alpha, \beta)$, by arguing as in Theorem 1.2.3.5, but using Lemma 1.2.3.8(b) instead of Lemma 1.2.3.3. (So far we have not used condition (a).) To get the desired result, we use the fact that the decomposition of M over $A^1_K(\alpha, \beta)$ given by Theorem 1.2.3.5 is unique, so we may thus glue together the decomposition of $M \otimes K\langle\delta/t, t/\beta]\!]_0$ with the decomposition from Theorem 1.2.3.5. More explicitly, this involves applying Lemma 1.2.3.2 to the following situation: for any $\epsilon \in (\delta, \beta)$, we have

$$K\{\{\alpha/t, t/\epsilon\rangle \cap K\langle\delta/t, t/\beta]\!]_0 = K\{\{\alpha/t, t/\beta]\!]_0$$

within $K\langle\delta/t, t/\epsilon\rangle$. □

Theorem 1.2.3.10. *Fix $j \in J^+$. Let M be a ∂_j-differential module of rank d over $K\langle t/\beta\rangle$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

(a) *The function $F^{(j)}_i(M, r)$ is constant in a neighborhood of $r = -\log\beta$.*

(b) *We have $f^{(j)}_i(M, -\log\beta) > f^{(j)}_{i+1}(M, -\log\beta)$.*

Then $M \otimes K[\![t/\beta]\!]_0$ admits a direct sum decomposition separating the first i subsidiary ∂_j-radii of $M \otimes F_\eta$ for $\eta \in (0, \beta)$.

Proof. Similar to Theorem 1.2.3.5, but using Lemma 1.2.3.8(a) instead of Lemma 1.2.3.3. □

Remark 1.2.3.11. In Theorems 1.2.3.9 and 1.2.3.10, if K is discrete and $\beta \in |K^\times|^{\mathbb{Q}}$, we can begin with free differential modules over the rings $K\langle\alpha/t, t/\beta]\!]_0$ and $K[\![t/\beta]\!]_0$, respectively. (The main reason for the restrictive hypotheses is to ensure that is that the resulting rings are noetherian; among other reasons, this is needed to ensure that we may freely pass between finite projective modules and finite locally free modules.) Also, we can extend the results to ring of analytic elements, which does not require the valuation on K to be discrete. Note that these statement require extending the definition of $f_i^{(j)}(M,r)$ to $r = -\log\beta$, using the completion of $\mathrm{Frac}K[\![t/\beta]\!]_0$ for the β-Gauss norm instead of F_β. (Compare [Ked**, Remark 12.5.3].)

1.2.4 Variation for multiple derivations

In this subsection, we study the variation of intrinsic generic radii of a differential module over a disc or annulus. The results here more closely match those of [Ked**] than in the case of a ∂_j-differential module with $j \in J$.

We first introduce a rotation construction, in the manner of [Ked07a].

Notation 1.2.4.1. Fix $\eta_+ \in \mathbb{R}_{>0}$. Assume that $|u_J| = 1$. Denote $\widetilde{K}$ to be the completion of $K(x_J)$ with respect to the $(\eta_+^{-1}, \dots, \eta_+^{-1})$-Gauss norm; view $\widetilde{K}$ as a differential field of order m with derivations $\partial_1, \dots, \partial_m$. We may use Taylor series (as in Lemma 1.1.2.16) to define, for any $\eta_- \in [0, \eta_+)$, an injective homomorphism $\tilde{f}^* : K\langle \eta_-/t, t/\eta_+\}\} \to \widetilde{K}\langle \eta_-/t, t/\eta_+\}\}$ such that $\tilde{f}^*(u_j) = u_j + x_j t$. More precisely, for $k \in K$

$$\tilde{f}^*(k) \stackrel{\text{def}}{=} \mathbb{T}(\cdots(\mathbb{T}(k;\partial_1, x_1 t);\partial_2, x_2 t)\cdots);\partial_m, x_m t) = \sum_{e_J} \frac{\partial_J^{e_J}(k)}{(e_J)!} x_J^{e_J} t^{|e_J|}.$$

In particular, $|\tilde{f}^*(k)|_\eta = |k|$ for any $\eta \in [\eta_-, \eta_+)$.

For $\eta \in [0, \eta_+)$, we use $\widetilde{F}_\eta$ to denote the completion of $\widetilde{K}(t)$ with respect to the η-Gauss norm. Then $\tilde{f}^*$ extends to an injective isometric homomorphism $\tilde{f}^* : F_\eta \hookrightarrow \widetilde{F}_\eta$.

Lemma 1.2.4.2. *For any subinterval I of $[0, \eta_+)$ and any ∂_{J^+}-differential module M on $A_K^1(I)$, $\tilde{f}^*M$ gives a ∂_0-differential module on $A_{\widetilde{K}}^1(I)$. Moreover, for $\eta \in I$,*

$$R_{\partial_0}(\tilde{f}^*M \otimes \widetilde{F}_\eta) = \min\left\{\eta IR_{\partial_0}(M \otimes F_\eta);\ \eta_+ IR_{\partial_j}(M \otimes F_\eta) \quad (j \in J)\right\}.$$

Proof. This follows from the fact that

$$\partial_0|_{\tilde{f}^*M} = \partial_0|_M + \sum_{j \in J} x_j \partial_j|_M,$$

after accounting for the different normalizations. □

Notation 1.2.4.3. Let M be a ∂_{J^+}-differential module of rank d on $K\langle \alpha/t, t/\beta \rangle$. For $r \in [-\log\beta, -\log\alpha]$ and $i \in \{1, \dots, d\}$, denote

$$f_i(M,r) = -\log IR(M \otimes F_{e^{-r}}; i), \quad F_i(M,r) = f_1(M,r) + \cdots + f_i(M,r).$$

Note that we have changed the normalization from Notation 1.2.2.4, as we are now using intrinsic rather than extrinsic radii.

Theorem 1.2.4.4. *Let M be a ∂_{J^+}-differential module of rank d on $A^1_K[\alpha, \beta]$.*

(a) *(Linearity) For $i = 1, \dots, d$, the functions $f_i(M,r)$ and $F_i(M,r)$ are continuous and piecewise affine.*

(b) *(Integrality) If $i = d$ or $f_i(M, r_0) > f_{i+1}(M, r_0)$, then the slopes of $F_i(M,r)$ in some neighborhood of r_0 belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i(M,r)$ and $F_i(M,r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{d}\mathbb{Z}$.*

(c) *(Monotonicity) Suppose that $\alpha = 0$. Then the slopes of $F_i(M,r)$ are nonpositive, and each $F_i(M,r)$ is constant for r sufficiently large.*

(d) *(Convexity) For $i = 1, \dots, d$, the function $F_i(M,r)$ is convex.*

Proof. Before proceeding, we reduce to the case $|u_J| = 1$ as in the proof of Theorem 1.2.2.6. (Note that when enlarging K, we do not retain the derivations with respect to any added parameters.)

Step 1: In this step, we prove that for $i = 1, \dots, d$, $f_i(M,r)$ and $F_i(M,r)$ are continuous at $r = -\log\beta$. Moreover, if $f_i(M, -\log\beta) > 0$, we show that there exists $\gamma \in [\alpha, \beta)$ such that (a) and (b) hold for $r \in [-\log\beta, -\log\gamma]$. As in the proof of Theorem 1.2.2.6, we may reduce to the case $\beta = 1$.

Let R denote the completion of $\mathcal{O}_K((t)) \otimes_{\mathcal{O}_K} K$ for the 1-Gauss norm; note that this contains both F_1 and $K\langle \gamma/t, t]\!]_0$ for any $\gamma \in [\alpha, 1)$. We first apply Theorem 1.2.2.5 (if $j = 0$) or Theorem 1.2.2.6 (if $j \in J$), and Theorem 1.2.3.9, to decompose

$$M \otimes K\langle \gamma/t, t]\!]_0 = \bigoplus_{\lambda=1}^{d'} M_\lambda^{[\gamma,1]}$$

for some $\gamma \in [\alpha, 1)$, in such a manner that the following conditions hold for $j \in J^+$ and $\lambda = 1, \dots, d'$.

(i) The module $M_\lambda^{[\gamma,1]} \otimes R$ is the base extension to R of a differential submodule M'_λ of $M \otimes F_1$ of pure intrinsic ∂_j-radii.

(ii) For $\mu = 1, \dots, \operatorname{rank} M_\lambda^{[\gamma,1]}$ the function $f_\mu^{(j)}(M_\lambda^{[\gamma,1]}, r)$ tends to $-\log IR_{\partial_j}(M'_\lambda)$ as $r \to 0^+$. If $j = 0$ or $IR_{\partial_j}(M'_\lambda) < 1$, then also $f_\mu^{(j)}(M_\lambda^{[\gamma,1]}, r)$ is affine for $r \in (0, -\log\gamma]$.

This alone suffices to imply continuity of $f_i(M, r)$ and $F_i(M, r)$ at $r = 0$.

Applying Theorem 1.2.3.5 after possibly making γ closer to 1, we get a further decomposition $M_\lambda^{[\gamma,1]} = \bigoplus_{\mu=1}^{d_\lambda} M_{\lambda,\mu}^{[\gamma,1)}$ over $A_K^1[\gamma, 1)$ such that the following conditions hold for $\lambda = 1, \dots, d'$.

(iii) For $j \in J^+$, $\mu = 1, \dots, d_\lambda$, if $IR_{\partial_j}(M'_\lambda) < 1$, then $M_{\lambda,\mu}^{[\gamma,1)} \otimes F_{e^{-r}}$ has pure intrinsic ∂_j-radii for $r \in (0, -\log\gamma]$.

(iv) If $IR(M'_\lambda) < 1$, then for $j \in J^+$, $\mu = 1, \dots, d_\lambda$, ∂_j is dominant for $M_{\lambda,\mu}^{[\gamma,1)} \otimes F_{e^{-r}}$ for some $r \in (0, -\log\gamma]$ if and only if the same holds for all $r \in (0, -\log\gamma]$.

(v) If $\lambda, \lambda' \in \{1, \dots, d'\}$ satisfy $IR(M'_\lambda) > IR(M'_{\lambda'})$, then $IR(M_{\lambda,\mu}^{[\gamma,1)} \otimes F_{e^{-r}}) > IR(M_{\lambda',\mu'}^{[\gamma,1)} \otimes F_{e^{-r}})$ for all $\mu \in \{1, \dots, d_\lambda\}$, $\mu' \in \{1, \dots, d_{\lambda'}\}$ and $r \in (0, -\log\gamma]$.

The piecewise affinity from (a) in the case $f_i(M, 0) > 0$ now follows from Theorems 1.2.2.5(a) and 1.2.2.6(a) applied to each $M_{\lambda,\mu}^{[\gamma,1)}$.

To check (b), it suffices to verify integrality of slope times rank for each component $M_{\lambda,\mu}^{[\gamma,1)}$ for which $IR(M'_\lambda) < 1$. If ∂_0 is dominant for $M_{\lambda,\mu}^{[\gamma,1)} \otimes F_{e^{-r}}$ for some (hence all) $r \in (0, -\log\gamma]$, (b) follows from Theorem 1.2.2.5(b). Otherwise, pick arbitrary $\eta_- < \eta_+ \in (\gamma, 1)$ such that for $\eta \in (\eta_-, \eta_+)$,

$$\eta_-/\eta_+ > IR(M_{\lambda,\mu}^{[\gamma,1)} \otimes F_\eta)/IR_{\partial_0}(M_{\lambda,\mu}^{[\gamma,1)} \otimes F_\eta).$$

Define $\widetilde{K}$ as in Notation 1.2.4.1. By Lemma 1.2.4.2, for $\eta \in (\eta_-, \eta_+)$, we have

$$\begin{aligned} R_{\partial_0}(\tilde{f}^* M_{\lambda,\mu}^{[\gamma,1)} \otimes \widetilde{F}_\eta) &= \min\left\{\eta IR_{\partial_0}(M_{\lambda,\mu}^{[\gamma,1)} \otimes F_\eta);\ \eta_+ IR_{\partial_j}(M_{\lambda,\mu}^{[\gamma,1)} \otimes F_\eta) \quad (j \in J)\right\} \\ &= \eta_+ IR(M_{\lambda,\mu}^{[\gamma,1)} \otimes F_\eta). \end{aligned}$$

In particular, $(f_1^{(0)})'(\tilde{f}^* M_{\lambda,\mu}^{[\gamma,1)}, -\log\eta) = f_1'(M_{\lambda,\mu}^{[\gamma,1)}, -\log\eta) = (f_1)'_-(M_{\lambda,\mu}^{[\gamma,1)}, 0)$ for $\eta \in (\eta_-, \eta_+)$. (Note that we showed in the proof of (a) that $f_1(M_{\lambda,\mu}^{[\gamma,1)}, r)$ extends continuously to $r = 0$, so its left derivative at 0 makes sense.) Thus, the statement (b) follows by applying Theorem 1.2.2.5(b) to $\tilde{f}^* M_{\lambda,\mu}^{[\gamma,1)}$.

<u>**Step 1′:**</u> As a corollary of step 1, we deduce that for any $r_0 \in [-\log\beta, -\log\alpha]$, $f_i(M, r)$ and $F_i(M, r)$ are continuous at r_0, and in case $f_i(M, r_0) > 0$ one also has (a) and (b) in a neighborhood of r_0. (In particular, we will then have continuity of $f_i(M, r)$ and $F_i(M, r)$ over all of $[-\log\beta, -\log\alpha]$.) To make this deduction, we first replace β by $\gamma = e^{-r_0}$ in case $r_0 < -\log\alpha$, to obtain all the desired assertions in a right neighborhood of r_0. By pulling back along $t \mapsto t^{-1}$ and then repeating the argument, we obtain the desired assertions in a left neighborhood of r_0.

<u>**Step 2:**</u> In this step, we prove that (d) holds in a neighborhood of each $r_0 \in (-\log\beta, -\log\alpha)$ for which $f_i(M, r_0) > 0$. It suffices to check in the case $f_i(M, r_0) > f_{i+1}(M, r_0)$, as the general case follows by interpolation.

At this point, we may reduce to the case $r_0 = 0$. As in Step 1, for some $\eta_- \in (\alpha, 1)$, we have a partial decomposition of M over $K\langle \eta_-/t, t]\!]_0$ as $M = \bigoplus_{\lambda_-=1}^{d_-} M_{\lambda_-}^{[\eta_-,1]}$ satisfying (i) and (ii). For some $\eta_+ \in (1, \beta)$, we also have a partial decomposition $M = \bigoplus_{\lambda_+=1}^{d_+} M_{\lambda_+}^{[1,\eta_+]}$ obtained by pulling back the decomposition over $K\langle \eta_+^{-1}/t, t]\!]_0$ along $t \mapsto t^{-1}$; it satisfies appropriate analogues of (i) and (ii). By making η_- and η_+ closer to 1, we may guarantee that for each index λ_- (resp. λ_+) for which the ratio $IR(M'_{\lambda_-})/IR_{\partial_0}(M'_{\lambda_-})$ (resp. $IR(M'_{\lambda_+})/IR_{\partial_0}(M'_{\lambda_+})$) is less than 1, this ratio is also less than η_-/η_+.

Use Notation 1.2.4.1; by Theorem 1.2.2.5, $F_i^{(0)}(\tilde{f}^* M, r)$ is convex at $r = 0$. In particular, $(F_i^{(0)})'_-(\tilde{f}^*M,0) \leq (F_i^{(0)})'_+(\tilde{f}^*M,0)$. It suffices to show that

$$(F_i^{(0)})'_+(\tilde{f}^*M,0) - \theta_i(M,0) \leq (F_i)'_+(M,0) \tag{1.2.4.5}$$

$$(F_i^{(0)})'_-(\tilde{f}^*M,0) - \theta_i(M,0) \geq (F_i)'_-(M,0), \tag{1.2.4.6}$$

where $\theta_i(M,0)$ denotes the sum of the dimensions of the constituents N of $M \otimes F_1$ for which ∂_0 is dominant and $f_1(N,0) \geq f_i(M,0)$.

The proofs of (1.2.4.5) and (1.2.4.6) are similar, so we focus on (1.2.4.5). Decompose M as in Step 1. For each λ such that ∂_0 is dominant for M'_λ, we have by Lemma 1.2.4.2 that in a punctured right neighborhood of $r = 0$,

$$F_1^{(0)}(\tilde{f}^*M_{\lambda,\mu}^{[\gamma,1)}, r) = F_1^{(0)}(M_{\lambda,\mu}^{[\gamma,1)}, r),$$

and so

$$(F_1^{(0)})'_+(\tilde{f}^*M_{\lambda,\mu}^{[\gamma,1)},0) - 1 = (F_1^{(0)})'_+(M_{\lambda,\mu}^{[\gamma,1)},0) - 1 \leq (F_1)'_+(M_{\lambda,\mu}^{[\gamma,1)},0).$$

(The term -1 comes from the change of normalization from Notation 1.2.2.4 to Notation 1.2.4.3. The inequality can be strict if ∂_j is also dominant for M'_λ for some $j > 0$.) For each λ such that ∂_0 is not dominant for M'_λ, we have by Lemma 1.2.4.2 (and the choice of η_+, η_-) that in a punctured right neighborhood of $r = 0$,

$$F_1^{(0)}(\tilde{f}^*M_{\lambda,\mu}^{[\gamma,1)}, r) = F_1^{(j)}(M_{\lambda,\mu}^{[\gamma,1)}, r) - \log\eta_+$$

and so

$$(F_1^{(0)})'_+(\tilde{f}^*M_{\lambda,\mu}^{[\gamma,1)},0) = (F_1)'_+(M_{\lambda,\mu}^{[\gamma,1)},0).$$

Summing over components yields (1.2.4.5).

<u>**Step 3:**</u> In this step, we prove (a), (b), (d) in general, by induction on i. Keep in mind that we already have the continuity aspect of (a) in general (by Step 1′), and all of (a), (b), (d) in a neighborhood of any $r_0 \in [-\log\beta, -\log\alpha]$ for which $f_i(M, r_0) > 0$

(by Steps 1, 1′, 2).

We first check the piecewise affinity aspect of (a) in a right neighborhood of some r_0 for which $f_i(M, r_0) = 0$. By the induction hypothesis, we can pick $r_1 > r_0$ such that $F_{i-1}(M, r)$ is affine on $[r_0, r_1]$. Suppose that $r_2 \in (r_0, r_1)$ is a value for which $f_i(M, r_2) > 0$. By continuity of f_i, there exists an open neighborhood of r_2 on which $f_i(M, r)$ is everywhere positive. Let U be the union of all such neighborhoods in $[r_0, r_1]$; then U is an open interval (r_3, r_4), and $f_i(M, r_3) = 0$. Since (a) and (d) hold in a neighborhood of each $r \in U$, $F_i(M, r)$ and hence $f_i(M, r)$ are piecewise affine and convex on U. In order for $f_i(M, r)$ to both be convex and to tend to 0 as $r \to r_3^+$, $f_i(M, r)$ must have no nonpositive slopes; that is, $f_i(M, r)$ is strictly increasing on U. However, we must also have $f_i(M, r_4) = 0$ unless $r_4 = r_1$. The former possibility leads to a contradiction, so we must have $r_4 = r_1$.

To sum up the previous paragraph, we now know that if there exists $r_2 \in (r_0, r_1]$ such that $f_i(M, r_2) > 0$, then $f_i(M, r) > 0$ for all $r \in [r_2, r_1]$. Consequently, on some right neighborhood of r_0, $f_i(M, r)$ is either everywhere zero or everywhere positive. In the former case, $f_i(M, r)$ is clearly affine on a right neighborhood of r_0. In the latter case, pick $r_2 \in (r_0, r_1]$ for which $f_i(M, r_2) > 0$; then the slopes of $f_i(M, r)$ on $(r_0, r_2]$ are nondecreasing, bounded below by 0, and (by (b)) confined to a discrete subset of $\mathbb{R}$. Consequently, there must be a least slope achieved, occurring on a right neighborhood of r_0. We thus deduce (a) in a right neighborhood of r_0. By symmetry, the same argument applies to left neighborhoods; we may thus deduce (a) in general.

Since (a) is now known, $f_i(M, r)$ takes only finitely many slopes on all of $[-\log\beta, -\log\alpha]$. Except possibly for the slope 0, each slope must occur at some r for which $f_i(M, r) > 0$; consequently, the knowledge of (b) at such points now implies (b) in general.

Finally, we still need to check (d) in a neighborhood of a point r_0 at which $f_i(M, r_0) = 0$. By (a), $f_i(M, r)$ is affine on a right neighborhood of r_0 and on a left neighborhood of r_0; since $f_i(M, r) \geq 0$ everywhere, the right slope of $f_i(M, r)$ at r_0 must be greater than or equal to the left slope of $f_i(M, r)$ at r_0. Since the same is true of $F_{i-1}(M, r)$ by the induction hypothesis, the same must also be true of $F_i(M, r)$. This yields (d).

<u>Step 4:</u> In this step, we prove (c). By Dwork's transfer theorem (see Proposition 1.1.2.18), for any $\eta < R_{\partial_0}(M \otimes F_\beta)$, $M \otimes K\langle t/\eta \rangle$ admits a basis in the kernel of ∂_0. In other words, $M \otimes K\langle t/\eta \rangle$ is isomorphic to the pullback of a (∂_J)-differential module over K. Consequently, $F_i(M, r)$ is constant for r sufficiently large; by (d), this implies that $F_i(M, r)$ has all slopes nonpositive. □

Remark 1.2.4.7. If $p = 0$, then the assertion that $r^{\dim V_r} \in |K^\times|$ in Theorem 1.1.6.6 implies that $d!F_i(M, r) \in \log|K^\times| + \mathbb{Z}r$. If $p > 0$, then we only deduce that for h a nonnegative integer,

$$f_i(M, r) > \frac{p^{-h}}{p-1}\log p \quad \Longrightarrow \quad d!F_i(M, r) \in p^{-h}\log|K^\times| + \mathbb{Z}r.$$

In either case, we may conclude that the values of r at which $F_i(M, r)$ changes slope must belong to $\mathbb{Q} \cdot \log|K^\times|$.

1.2.5 Decomposition for multiple variations

We now obtain decomposition theorems which allow for multiple derivations.

Theorem 1.2.5.1. *Let M be a ∂_{J^+}-differential module of rank d on $A_K^1(\alpha, \beta)$. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

(a) *The function $F_i(M, r)$ is affine for $-\log\beta < r < -\log\alpha$.*

(b) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $-\log\beta < r < -\log\alpha$.*

Then M admits a unique direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\eta$ for any $\eta \in (\alpha, \beta)$.

Proof. Before proceeding, we reduce to the case $|u_J| = 1$ as in the proof of Theorem 1.2.3.5. It suffices to prove the decomposition in a neighborhood of each $r_0 \in (-\log\beta, -\log\alpha)$. Again, we may assume $r_0 = 0$.

We continue with Step 2 in the proof of Theorem 1.2.4.4. We may further impose the auxiliary condition that

$$\log(\eta_+) < f_i(M, 0) - f_{i+1}(M, 0). \tag{1.2.5.2}$$

By (1.2.4.5) and the symmetric result, we have

$$(F_i)'_-(M,0) \le (F_i^{(0)})'_-(\tilde{f}^*M,0) - \theta_i(M,0) \le (F_i^{(0)})'_+(\tilde{f}^*M,0) - \theta_i(M,0) \le (F_i)'_+(M,0); \tag{1.2.5.3}$$

all the inequalities are forced to be equalities as $F_i(M,r)$ is affine in a neighborhood of $r=0$. In particular, $F_i^{(0)}(\tilde{f}^*M,r)$ is affine when $r \in (-\log\eta_+, -\log\eta_-]$. We would get the decomposition by Theorem 1.2.3.5 if we knew that $f_i^{(0)}(\tilde{f}^*M,r) > f_{i+1}^{(0)}(\tilde{f}^*M,r)$ for r in a neighborhood of $r=0$. Indeed, by our auxiliary condition (1.2.5.2) and Lemma 1.2.4.2,

$$f_i^{(0)}(\tilde{f}^*M,0) > -\log(\eta_+) + f_i(M,0) > f_{i+1}(M,0) \ge f_{i+1}^{(0)}(\tilde{f}^*M,0).$$

The theorem follows. □

Theorem 1.2.5.4. *Let M be a ∂_{J^+}-differential module of rank d on $A_K^1[0,\beta)$. Suppose that the following conditions hold for some $i \in \{1,\dots,d-1\}$.*

(a) *The function $F_i(M,r)$ is affine for $r > -\log\beta$. (This implies $F_i(M,r)$ is constant by Theorem 1.2.4.4(c).)*

(b) *We have $f_i(M,r) > f_{i+1}(M,r)$ for all (some) $r > -\log\beta$.*

Then M admits a unique direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\eta$ for any $\eta \in (0,\beta)$.

Proof. Before proceeding, we reduce to the case $|u_J| = 1$ as in the proof of Theorem 1.2.3.5. As noted in Step 4 of the proof of Theorem 1.2.4.4, there exists some $\eta \in (0,\beta)$ such that $M \otimes K\langle t/\eta\rangle$ is isomorphic to the pullback of a ∂_J-differential module M_0 over K. Consequently, we have the desired decomposition of M over $A_K^1[0,\eta]$ by pulling back the decomposition of M_0 in the sense of Theorem 1.1.4.27. The theorem follows by applying Theorem 1.2.5.1 to $A_K^1(\eta',\beta)$ for some $\eta' \in (0,\eta)$. □

Remark 1.2.5.5. Under condition (a), the condition (b) for some $r > -\log\beta$ implies that for all $r > -\log\beta$. Indeed, the affinity of $F_i(M,r)$ and the convexity of $F_{i-1}(M,r)$

and $F_{i+1}(M,r)$ implies that $f_i(M,r)$ is concave and $f_{i+1}(M,r)$ is convex. In particular, $f_{i+1}(M,r) - f_i(M,r)$ is concave and nonnegative. Hence, if it is zero once, it is zero identically.

Remark 1.2.5.6. We can sometimes verify the hypotheses of Theorem 1.2.5.4 using monotonicity and convexity (Theorem 1.2.4.4(c) and (d)). For example, if $F_i'(M,r_0) = 0$, then $F_i(M,r)$ is constant for $r \geq r_0$. Moreover, if we also have $f_i(M,r_0) > f_{i+1}(M,r_0)$, then condition (b) holds for $r \geq r_0$.

Remark 1.2.5.7. As in Remark 1.2.3.7, we cannot state a decomposition theorem over a closed annulus without assuming $p = 0$ (in which case see Theorems 1.2.9.12 and 1.2.9.13). However, we do get partial decomposition theorems analogous to Theorems 1.2.9.10 and 1.2.9.11, as follows.

Theorem 1.2.5.8. *Let M be a ∂_{J^+}-differential module of rank d on $A_K^1(\alpha,\beta]$. Suppose that the following conditions hold for some $i \in \{1,\dots,d-1\}$.*

(a) *The function $F_i(M,r)$ is affine for $-\log\beta \leq r < -\log\alpha$.*

(b) *We have $f_i(M,r) > f_{i+1}(M,r)$ for $-\log\beta \leq r < -\log\alpha$.*

Then $M \otimes K\{\{\alpha/t, t/\beta]\!]_0$ admits a unique direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\eta$ for any $\eta \in (\alpha,\beta)$.

Proof. The fact that this holds over $M \otimes K\langle\gamma/t, t/\beta]\!]_0$ for some $\gamma \in (\alpha,\beta)$, even without hypothesis (a), is a corollary of Step 1 of the proof of Theorem 1.2.4.4. The desired conclusion follows by combining this assertion with Theorem 1.2.5.1. □

Theorem 1.2.5.9. *Let M be a ∂_{J^+}-differential module of rank d on $A_K^1[0,\beta]$. Suppose that the following conditions hold for some $i \in \{1,\dots,d-1\}$.*

(a) *The function $F_i(M,r)$ is affine for $r \geq -\log\beta$.*

(b) *We have $f_i(M,-\log\beta) > f_{i+1}(M,-\log\beta)$.*

Then $M \otimes K[\![t/\beta]\!]_0$ admits a unique direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\eta$ for any $\eta \in (0,\beta)$.

Proof. This follows by combining Theorems 1.2.5.4 and 1.2.5.8. □

Remark 1.2.5.10. As in Remark 1.2.3.11, if K is discretely valued and $\beta \in |K^\times|^{\mathbb{Q}}$, we can admit modules in Theorems 1.2.5.8 and 1.2.5.9 defined directly over the corresponding rings of bounded functions, namely $K\langle\alpha/t, t/\beta]\!]_0$ and $K[\![t/\beta]\!]_0$.

Proposition 1.2.5.11. *Let M be an* indecomposable *∂_{J^+}-differential module over $A^1_K(\alpha,\beta)$, i.e., it is not the direct sum of two nonzero ∂_{J^+}-differential submodules. Assume that $f_1(M,r)$ is affine for $r \in (-\log\beta, -\log\alpha)$. Then, for each $j \in J^+$,*

(a) *either $M \otimes F_\eta$ has pure intrinsic ∂_j-radii the same as $IR(M \otimes F_\eta)$ for all $\eta \in (\alpha,\beta)$, or*

(b) *$IR_{\partial_j}(M \otimes F_\eta) > IR(M \otimes F_\eta)$ for all $\eta \in (\alpha,\beta)$.*

Proof. Assume that we are not in case (b). Then $IR_{\partial_j}(M\otimes F_\eta) = IR(M\otimes F_\eta)$ for some $\eta \in (\alpha,\beta)$. By Theorems 1.2.2.5(d) and 1.2.2.6(d), the convexity of $f_1^{(j)}(M,r)$ forces $IR_{\partial_j}(M\otimes F_\eta) = IR(M\otimes F_\eta)$ for all $\eta \in (\alpha,\beta)$. Now, if $IR_{\partial_j}(M\otimes F_\eta; 2) > IR(M\otimes F_\eta)$ for all $\eta \in (\alpha,\beta)$, the decomposition in Theorem 1.2.5.1 would imply that M is decomposable, which contradicts what we assumed earlier. Therefore $IR_{\partial_j}(M \otimes F_\eta; 2) = IR(M \otimes F_\eta)$ for some $\eta \in (\alpha,\beta)$. By Theorems 1.2.2.5(d) and 1.2.2.6(d) again, we have the equality for all $\eta \in (\alpha,\beta)$. Continue this argument will lead us to case (a). □

1.2.6 Variation of refined intrinsic radii

In this subsection, we discuss the variation of refined intrinsic radii of a ∂_{J^+}-differential module M when $f_1(M,r) = \cdots = f_{\dim M}(M,r)$ is *affine*.

We continue to assume Hypothesis 1.2.1.1 and keep the notation as in previous subsections.

Before proving general results, we first look at an example of pure refined radii. It is a 1-dimensional analogue of Lemma 1.1.5.15.

Example 1.2.6.1. Let $j \in J^+$ and let $(\alpha, \beta) \in (0, \infty)$ be an open interval. Let $\theta \in \kappa_{K^{\mathrm{alg}}}^{(b)}$ for some $b \in -\log|K^\times|^{\mathbb{Q}}$ and let $a \in \mathbb{Q}$. Assume that

$$e^{-b}\alpha^a, e^{-b}\beta^a > \begin{cases} 1 & \text{if } p = 0 \\ p^{-1/p} & \text{if } p > 0. \end{cases}$$

Note that this is actually incorporate some non-visible radii. For the reason of this restriction, see Remark 1.1.5.25.

Let d be the prime-to-p part of the denominator of a. Then we have the following.

(a) If $p = 0$, then $b \in -\log|(K')^\times|$ and $\theta \in \kappa_{K'}^{(b)}$ for some finite *tamely ramified* extension K'/K. Let $x \in \mathfrak{m}_{K'}^{(b)}$ be a lift of θ. We set $n = 0$ and $p^n = 1$ in this case.

(b0) If $p > 0$ and $j = 0$, there exists $n \in \mathbb{N}$ such that $\theta^{p^n} \in \kappa_{K'}^{(p^n b)}$ with $p^n s \in -\log|(K')^\times|$ and $p^n da \in p\mathbb{Z}$ is a multiple of p, for some finite *tamely ramified* extension K'/K. Let $x \in \mathfrak{m}_{K'}^{(p^n b)}$ be a lift of θ^{p^n}.

(b1) If $p > 0$ and $j \in J$, there exists $n \in \mathbb{N}$ such that $\theta^{p^n} \in (\kappa_{K'}^{(p^{n-1}s)})^p$ and $p^n da \in \mathbb{Z}$ with $p^{n-1}s \in -\log|(K')^\times|$ for some finite *tamely ramified* extension K'/K. Let $x \in \mathfrak{m}_{K'(\partial)}^{(p^n s)}$ be a lift of θ^{p^n} in the fixed field of ∂; this is possible by Lemma 1.1.4.11.

Let $A_{K'}^1(\alpha^{1/d}, \beta^{1/d})$ be the open annulus with coordinate $t^{1/d}$. Define $\mathcal{L}_{x,a,(n)}^{(j)}$ to be the ∂_j-differential module over $A_{K'}^1(\alpha^{1/d}, \beta^{1/d})$ of rank p^n with basis $\{\mathbf{e}_1, \dots, \mathbf{e}_{p^n}\}$, on which ∂ acts as $\partial \mathbf{e}_i = \mathbf{e}_{i+1}$ for $i = 1, \dots, p^n - 1$ and $\partial \mathbf{e}_{p^n} = xt^{p^n a}u_j^{-p^n}\mathbf{e}_1$, if $j \in J$ and $\partial \mathbf{e}_{p^n} = xt^{p^n(a-1)}\mathbf{e}_1$, if $j = 0$. (The added $u_j^{-p^n}$ and t^{-p^n} is to balance the different normalization on intrinsic ∂_j-radii and ∂_j-radii.)

Lemma 1.2.6.2. *Keep the notation as in Example 1.2.6.1. If we denote* $F'_{e^{-r}} = F_{e^{-r}}(t^{1/d})$*, then for all* $r \in (-\log\beta, -\log\alpha)$*,* $\mathcal{L}_{x,a,(n)}^{(j)} \otimes F'_{e^{-r}}$ *has pure intrinsic* ∂_j*-radius* ωe^{ar+b} *and* $\Theta_\partial(\mathcal{L}_{x,a,(n)}^{(j)} \otimes F'_{e^{-r}})$ *consists of only* θt^a *with multiplicity* p^n.

Proof. At the point $r \in (-\log\beta, -\log\alpha)$ for which $e^{-ra-b} > 1$, $\mathcal{L}^{(j)}_{x,a,(n)} \otimes F'_{e^{-r}}$ has visible ∂_j-radii and it follows from Lemma 1.1.5.15. If $p > 0$, at the point $r \in (-\log\beta, -\log\alpha)$ for which $e^{-ra-b} \in (p^{-1/p(p-1)}, 1]$, this follows from Lemma 1.1.5.26. $\square$

Theorem 1.2.6.3. *Fix $j \in J^+$. Let M be a ∂_j-differential module over an* open *annulus $A^1_K(\alpha, \beta)$ such that $M \otimes F_{e^{-r}}$ has pure intrinsic ∂_j-radii $\omega e^{ar+b} < 1$ for any $r \in (-\log\beta, -\log\alpha)$. (This is saying that $f^{(j)}_1(M, r) = \cdots = f^{(j)}_{\dim M}(M, r)$ is an affine function of slope $-a$.) Then there exists a canonical decomposition*

$$M = \bigoplus_{\{\theta\} \subset \kappa^{(b)}_{K^{\mathrm{alg}}}} M_{\{\theta\}}, \tag{1.2.6.4}$$

where the direct sum runs through all conjugacy classes of $\kappa^{(b)}_{K^{\mathrm{alg}}}$, such that $M_{\{\theta\}} \otimes F_\eta$ has refined intrinsic ∂-radii $\{\theta t^a\}$ with multiplicity $\operatorname{rank} M_{\{\theta\}} / \#\{\theta\}$ *for all $\{\theta\} \subset \kappa^{(b)}_{K^{\mathrm{alg}}}$ and $\eta \in I$.*

Let d be the prime-to-p part of the denominator of a and let $A^1_K(\alpha^{1/d}, \beta^{1/d})$ be the open annulus with coordinate $t^{1/d}$. Then, by replacing K by a finite tamely ramified extension, we may make the decomposition similar to (1.2.6.4) *but over $A^1_K(\alpha^{1/d}, \beta^{1/d})$ and with the sum over elements in $\kappa^{(b)}_K$ instead of conjugacy classes.*

Proof. Since the decomposition if it exists is determined by the decomposition at each radius $e^{-r} \in (\alpha, \beta)$, it is canonical. Thus, we may replace K by any finite tamely ramified extension when we need (and then obtain the decomposition over K by Galois descent). Also, it suffices to obtain the decomposition in a neighborhood of each radius in (α, β).

Let $r \in (-\log\beta, -\log\alpha)$ be a point. We first assume that $IR_\partial(M \otimes F_{e^{-r}}) < 1$ when $p = 0$ and $IR_\partial(M \otimes F_{e^{-r}}) < p^{-1/p(p-1)}$ when $p > 0$. (Note that this restriction still allows some non-visible radii.) Let $\theta t^a \in \mathcal{I}\Theta_\partial(M \otimes F_{e^{-r}})$ be an intrinsic refined ∂_j-radius. Since $M \otimes F_{e^{-r}}$ has pure intrinsic ∂_j-radii ωe^{ar+b}, we have $\theta \in \kappa^{(b)}_{K^{\mathrm{alg}}}$. Now, applying the construction in Example 1.2.6.1 gives a ∂_j-differential module $\mathcal{L}^{(j)}_{x,a,(n)}$ over $A^1_{K'}(\alpha'^{1/d}, \beta'^{1/d})$ of pure ∂_j-radii ω^{ar+b} and pure intrinsic ∂-radii θt^a at radius $e^{-r/d}$, where $(-\log\beta', -\log\alpha')$ is a neighborhood of r in $(-\log\beta, -\log\alpha)$ and the coordinate on the annulus is $t^{1/d}$.

Denote $N = M \otimes (\mathcal{L}^{(j)}_{x,a,(n)})^\vee$ and $F'_{e^{-r'}} = F_{e^{-r'}}(t^{1/d})$. Then $IR_\partial(N \otimes F'_{e^{-r'}}) \leq \omega a^{ar'+b}$ for $r' \in (-\log\beta', -\log\alpha)$. By Lemma 1.1.5.11(2), $N \otimes F'_{e^{-r}}$ is not of pure intrinsic ∂_j-radii ωe^{ar+b}. By Theorem 1.2.2.6(d), if $N \otimes F'_{e^{-r'}}$ were of pure intrinsic ∂_j-radii $\omega e^{ar'+b}$ for some $r' \in (-\log\beta', -\log\alpha')$, the same has to be true for all $r' \in (-\log\beta', -\log\alpha')$ because a convex function below a linear function is same as the linear function if the two functions touch at some point; this would contradict to what we just showed. Therefore, $N \otimes F'_{e^{-r'}}$ does not have pure intrinsic ∂_j-radii $\omega e^{ar'+b}$ for any $r' \in (-\log\beta', -\log\alpha')$. By Lemma 1.1.5.11(2), that is to say $\theta t^a \in \mathcal{I}\Theta_\partial(N \otimes F'_{e^{-r'}})$. Moreover, using the exact argument as in Theorem 1.1.5.16, invoking Theorem 1.2.9.10 in place of Proposition 1.1.3.4, we can obtain the decomposition (1.2.6.4).

Now, it suffices to deal with the case when $p > 0$ and $IR_{\partial_j}(M \otimes F_{e^{-r}}) \in [p^{-1/p(p-1)}, 1)$. In this case, we have ∂_j-Frobenius antecedent of M in a neighborhood of r. The decomposition follows from the decomposition of the ∂_j-Frobenius antecedent or, more generally, iterative ∂_j-Frobenius antecedent (until the intrinsic ∂_j-radii fall in the range above.) $\square$

Remark 1.2.6.5. The above theorem stays valid if we replace the rational type condition by visible condition.

Remark 1.2.6.6. We do not know if we can extend this result to bounded rigid analytic rings when K is discretely valued.

Theorem 1.2.6.7. *Let M be a ∂_{J^+}-differential module over an* open *annulus $A^1_K(\alpha, \beta)$ such that $M \otimes F_{e^{-r}}$ has pure intrinsic radii $\omega e^{ar+b} < 1$. Let d denote the prime-to-p part of the denominator of a. Then after a finite unramified extension K'/K, there exists a canonical decomposition*

$$M = \bigoplus_{\vartheta} M_\vartheta \tag{1.2.6.8}$$

over $A^1_{K'}(\alpha^{1/d}, \beta^{1/d})$, where the direct sum runs through all $\vartheta \in \oplus_{j \in J} \kappa^{(b)}_{K^{\mathrm{alg}}} \frac{du_j}{u_j} \oplus \kappa^{(b)}_{K^{\mathrm{alg}}} \frac{dt}{t}$, *such that $M_\vartheta \otimes F_\eta$ has pure refined intrinsic radii $t^a\vartheta$ for all $\eta \in I$.*

We may obtain the decomposition (1.2.6.8) *over K if we group Galois conjugates of ϑ's.*

Proof. Without loss of generality, we assume that M is indecomposable. By Proposition 1.2.5.11, we have the stated dichotomy there. We apply Theorem 1.2.6.7 to the ∂_j for which case (a) of Proposition 1.2.5.11 holds for M. The decompositions for different ∂_j's are compatible. This gives the desired decomposition. □

Notation 1.2.6.9. Keep the notation as in Theorem 1.2.6.7. We use $\mathcal{I}\Theta(M)$ to denote the set of ϑ in (1.2.6.8) with multiplicity rank M_ϑ.

1.2.7 Variation of extrinsic subsidiary radii

In this subsection, we will consider the variation and decomposition by extrinsic subsidiary radii for multi-derivations (under some assumptions). This will be used to study differential Artin conductors in the next subsection. The proofs in this subsection are similar to the ones in Subsections 1.3.3 and 1.2.5, but much simpler.

We keep Hypothesis 1.2.1.1 as usual. We also assume the following.

Hypothesis 1.2.7.1. Assume $|u_j| = 1$ for $j \in J$.

Notation 1.2.7.2. Let M be a ∂_{J^+}-differential module of rank d over $A^1_K(\alpha, \beta)$, with $0 \leq \alpha < \beta \leq 1$. For $r \in (-\log\beta, -\log\alpha)$ and $i \in \{1, \dots, d\}$, denote

$$\hat{f}_i(M, r) = -\log R(M \otimes F_{e^{-r}}; i), \qquad \hat{F}_i(M, r) = \hat{f}_1(M, r) + \cdots + \hat{f}_i(M, r).$$

Notation 1.2.7.3. Denote $\widetilde{K}$ to be the completion of $K(x_J)$ with respect to the $(1, \dots, 1)$-Gauss norm; view $\widetilde{K}$ as a differential field of order m with derivations ∂_J. For $0 \leq \alpha < \beta \leq 1$, Taylor series (as in Lemma 1.1.2.16) gives rise to an injective homomorphism $\tilde{f}^* : K\{\{\alpha/t, t/\beta\}\} \to \widetilde{K}\{\{\alpha/t, t/\beta\}\}$ such that $\tilde{f}^*(u_j) = u_j + x_j t$.

For $\eta \in (\alpha, \beta)$, we use $\widetilde{F}_\eta$ to denote the completion of $\widetilde{K}(t)$ with respect to the η-Gauss norm. Then $\tilde{f}^*$ extends to an injective isometric homomorphism $\tilde{f}^* : F_\eta \hookrightarrow \widetilde{F}_\eta$.

Lemma 1.2.7.4. *For any $0 \leq \alpha < \beta \leq 1$ and any ∂_{J^+}-differential module M on $A^1_K[\alpha,\beta)$, $\tilde{f}^*M$ gives a ∂_0-differential module on $A^1_{\widetilde{K}}[\alpha,\beta)$. Moreover, for $\eta \in (\alpha,\beta)$,*

$$R_{\partial_0}(M \otimes \widetilde{F}_\eta) = \min_{j \in J^+} \{R_{\partial_j}(M \otimes F_\eta)\} = R(M \otimes F_\eta).$$

Proof. This follows from the fact that $\partial_0|_{\tilde{f}^*M} = \partial_0|_M + \sum_{j\in J} x_j \partial_j|_M$. □

Theorem 1.2.7.5. *Let M be a ∂_{J^+}-differential module of rank d on $A^1_K(\alpha,\beta)$ with $0 \leq \alpha < \beta \leq 1$.*

(a) *(Linearity) For $i = 1,\dots,d$, the functions $\hat{f}_i(M,r)$ and $\hat{F}_i(M,r)$ are continuous and piecewise affine.*

(b) *(Integrality) If $i = d$ or $\hat{f}_i(M,r_0) > \hat{f}_{i+1}(M,r_0)$, then the slopes of $\hat{F}_i(M,r)$ in some neighborhood of r_0 belong to $\mathbb{Z}$. Consequently, the slopes of each $\hat{f}_i(M,r)$ and $\hat{F}_i(M,r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \dots \cup \frac{1}{d}\mathbb{Z}$.*

(c) *(Monotonicity) Suppose that M is defined over $A^1_K[0,\beta)$. Then the slopes of $\hat{F}_i(M,r)$ are nonpositive, and each $\hat{F}_i(M,r)$ is constant for r sufficiently large.*

(d) *(Convexity) For $i = 1,\dots,d$, the function $\hat{F}_i(M,r)$ is convex.*

(e) *Suppose for some $i \in \{1,\dots,d-1\}$, the function $\hat{F}_i(M,r)$ is affine and $\hat{f}_i(M,r) > \hat{f}_{i+1}(M,r)$ for $r \in (-\log\beta, -\log\alpha)$. Then M admits a unique direct sum decomposition separating the first i subsidiary extrinsic radii of $M \otimes F_\eta$ for any $\eta \in (\alpha,\beta)$.*

Proof. Consider $\tilde{f}^*M$ as a ∂_0-differential module over $\widetilde{K}\{\{\alpha/t, t/\beta\}\}$. (For (c), we view $\tilde{f}^*M$ as a module over $\widetilde{K}\{\{t/\beta\}\}$.) By Lemma 1.2.7.4, we have $f_i^{(0)}(\tilde{f}^*M,r) = \hat{f}_i(M,r)$ for $r \in (-\log\beta, -\log\alpha)$. (For (c), we have the equality for $r \in (-\log\beta, +\infty)$.) The theorem follows from Theorems 1.2.2.5 and 1.2.3.5. □

1.2.8 Differential conductors

As promised earlier (Remark 1.1.6.9), we can use the results of this subsection to extend the results of [Ked07a] by relaxing [Ked07a, Hypothesis 2.1.3] to the hypothesis that K is of rational type. As this is straightforward to do, we merely summarize the outcome by stating and deducing a result which includes [Ked07a, Theorems 2.7.2 and 2.8.2].

Definition 1.2.8.1. Let M be a ∂_{J^+}-differential module of rank d on $A^1_K(\eta_0, 1)$ for some $\eta_0 \in (0,1)$. We say that M is *solvable* if $IR(M \otimes F_\eta) \to 1$ as $\eta \to 1^-$.

Theorem 1.2.8.2. *Let M be a solvable ∂_{J^+}-differential module of rank d over $A^1_K(\eta_0, 1)$ for some $\eta_0 \in (0,1)$. Then by making η_0 closer to 1, there exist a decomposition $M = M_1 \oplus \cdots \oplus M_r$ over $A^1_K(\eta_0, 1)$ and nonnegative distinct rational numbers $b_1, \dots, b_r$ with $b_i \cdot \mathrm{rank}\,(M_i) \in \mathbb{Z}$, such that*

$$IR(M_i \otimes F_\eta; j) = \eta^{b_i} \qquad (i = 1, \dots, r;\ j = 1, \dots, \mathrm{rank}\,(M_i);\ \eta \in (\eta_0, 1)).$$

Under the same hypothesis and by making η_0 closer to 1, there exist a decomposition $M = \hat{M}_1 \oplus \cdots \oplus \hat{M}_{r'}$ over $A^1_K(\eta_0, 1)$ and nonnegative rational numbers $\hat{b}_1, \dots, \hat{b}_r$ with $\hat{b}_i \cdot \mathrm{rank}\,(\hat{M}_i) \in \mathbb{Z}$, such that

$$R(\hat{M}_i \otimes F_\eta; j) = \eta^{\hat{b}_i} \qquad (i = 1, \dots, r';\ j = 1, \dots, \mathrm{rank}\,(\hat{M}_i);\ \eta \in (\eta_0, 1)).$$

Proof. The two statements can be proved using the same argument as follows. By Theorems 1.2.4.4 and 1.2.7.5(a)(b)(d), for $l = 1, \dots, d$, the functions $d!F_l(M, r)$ and $d!\hat{F}_l(M, r)$ on $(0, -\log\eta_0)$ are continuous, convex, and piecewise affine with integer slopes. By hypothesis, $d!F_i(M, r) \to 0$ and hence $d!\hat{F}_i(M, r) \to 0$ as $r \to 0^+$; because of this and the fact that $d!F_i(M, r) \geq 0$ and $d!\hat{F}_i(M, r) \geq 0$ for all r, the slopes of $F_i(M, r)$ and $\hat{F}_i(M, r)$ are forced to be nonnegative. Hence there is a least such slope, that is, $d!F_i(M, r)$ and $d!\hat{F}_i(M, r)$ are linear in a right neighborhood of $r = 0$.

We can thus choose $\eta_0 \to 1^-$ so that $d!F_i(M, r)$ and $d!\hat{F}_i(M, r)$ are linear on $(0, -\log\eta_0)$ for $i = 1, \dots, d$. We obtain the desired decomposition by Theorems 1.2.5.4 and Theorems 1.2.7.5(e), respectively; the integrality of $b_i \cdot \mathrm{rank}\,(M_i)$ and $\hat{b}_i \cdot \mathrm{rank}\,(\hat{M}_i)$ follows from the fact that $F_{\dim M_i}(M_i, r)$ and $\hat{F}_{\dim \hat{M}_i}(\hat{M}_i, r)$ have integral slopes, again by Theorems 1.2.4.4 and 1.2.7.5(b). $\square$

Definition 1.2.8.3. Let M be a solvable ∂_{J^+}-differential module of rank d over $A_K^1(\eta_0, 1)$ for some $\eta_0 \in (0,1)$. Define the *differential log-breaks* of M to be the multiset consisting of b_i from Theorem 1.2.8.2 above with multiplicity rank (M_i). We define the *differential Swan conductor* of M to be the sum of the differential log-breaks, that is $\mathrm{Swan}(M) = \sum_{i=1}^r b_i \cdot \mathrm{rank}\,(M_i)$; it is an integer by Theorem 1.2.8.2 above. Similarly, we define the *differential (non-log)-breaks* to be the multiset consisting of $\hat{b}_i$ from Theorem 1.2.8.2 above with multiplicity rank $(\hat{M}_i)$. We define the *differential Artin conductor* of M to be the sum of the differential non-log-breaks; it is also an integer by Theorem 1.2.8.2 above.

1.2.9 Subharmonicity for residual characteristic zero

When $m = 0$, the functions $F_i(M, r)$ obey a certain subharmonicity property [Ked**, Theorem 11.3.2]. When the residual characteristic p is equal to 0, one can obtain a similar result even when K carries derivations. (See Remark 1.2.2.8 for discussion of the case $p > 0$.)

We continue to assume Hypothesis 1.2.1.1 for this subsection. Moreover, we assume the following.

Hypothesis 1.2.9.1. Throughout this subsection, we assume $p = 0$.

Definition 1.2.9.2. For $\overline{\mu} \in (\kappa_K^{\mathrm{alg}})^\times$, let μ be a lift of $\overline{\mu}$ in some finite extension L of K. Let E be a finite unramified extension of the completion of $\mathcal{O}_K[t]_{(t)} \otimes_{\mathcal{O}_K} L$ for the 1-Gauss norm. For $\alpha \leq 1 \leq \beta$, define the substitution

$$T_\mu : K\langle \alpha/t, t/\beta \rangle \to E, \qquad t \mapsto t + \mu.$$

Definition 1.2.9.3. Fix $j \in J^+$. Let M be a ∂_j-differential module of rank d on $A_K^1[\alpha, \beta]$ for some $\alpha \leq 1 \leq \beta$. For $i = 1, \dots, n$, let $s_{\infty,i}^{(j)}(M)$ and $s_{0,i}^{(j)}(M)$ be the left (if $\beta \neq 1$) and right (if $\alpha \neq 1$) slopes of $F_i^{(j)}(M, r)$ at $r = 0$. For $\overline{\mu} \in (\kappa_K^{\mathrm{alg}})^\times$, pick any $\mu \in \mathcal{O}_L$ lifting $\overline{\mu}$ in a finite unramified extension L of K, and let $s_{\overline{\mu},i}^{(j)}(M)$ be the right slope of $F_i^{(j)}(T_\mu^*(M), r)$ at $r = 0$. Note that $T_\mu^*(M)$ is still a ∂_j-differential module by Lemma 1.1.4.6.

If M is a ∂_{J^+}-differential module of rank d on $A^1_K[\alpha,\beta]$ for some $\alpha \leq 1 \leq \beta$, for $i = 1,\dots,n$ and $\overline{\mu} \in \kappa_K^{\mathrm{alg}}$, we similarly define $s_{\infty,i}(M)$ and $s_{\overline{\mu},i}(M)$ as the slopes of the corresponding functions $F_i(M,r)$ or $F_i(T^*_\mu(M),r)$.

Theorem 1.2.9.4. *Fix $j \in J^+$. Let M be a ∂_j-differential module of rank d on $A^1_K[\alpha,\beta]$ for some $\alpha < 1 < \beta$. Choose $i \in \{1,\dots,d\}$ such that $f_i^{(j)}(M,0) > 0$.*

(a) *The quantity $s^{(j)}_{\overline{\mu},i}(M)$ does not depend on the lift μ and the unramified extension L/K.*

(b) *We have $s^{(j)}_{\overline{\mu},i}(M) \leq 0$ for all $\overline{\mu} \neq 0$, with equality for all but finitely many $\overline{\mu}$.*

(c) *We have*

$$s^{(j)}_{\infty,i}(M) \leq \sum_{\overline{\mu}\in\kappa_K^{\mathrm{alg}}} s^{(j)}_{\overline{\mu},i}(M),$$

with equality if either $i = n$ and $f_n^{(j)}(M,0) > 0$, or $i < n$ and $f_i^{(j)}(M,0) > f_{i+1}^{(j)}(M,0)$.

Proof. When $j = 0$, this is [Ked**, Theorem 11.3.2(d)]. When $j \in J$, the proof of Theorem 1.2.2.6 reduces the problem to [Ked**, Theorem 11.2.1(c)]. Note that we do not have to use the Frobenius pushforward. □

Remark 1.2.9.5. Let L be a complete extension of K such that ∂_j extends to L with the same operator norm. Then $M\otimes L$ becomes a ∂_j-differential module over $A^1_L[\alpha,\beta]$. For $\overline{\mu} \notin \kappa_K^{\mathrm{alg}}$, we always have $s^{(j)}_{\overline{\mu},i}(M) = 0$; this can be seen either by inspecting the proof of Theorem 1.2.9.4, or by deducing the claim directly from (b). Namely, (b) implies that the equality $s^{(j)}_{\overline{\mu},i}(M) = 0$ holds with only finitely many exceptions; on the other hand, if $\overline{\mu}$ were an exception not in κ_K^{alg}, then so would be each of its infinitely many conjugates in κ_L^{alg}.

Theorem 1.2.9.6. *Let M be a ∂_{J^+}-differential module of rank d on $A^1_K[\alpha,\beta]$ for some $\alpha < 1 < \beta$. Choose $i \in \{1,\dots,d\}$ such that $f_i(M,0) > 0$.*

(a) *The quantity $s_{\overline{\mu},i}(M)$ does not depend on the lift μ and the unramified extension L/K.*

(b) *We have* $s_{\overline{\mu},i}(M) \leq 0$ *for all* $\overline{\mu} \neq 0$, *with equality for all but finitely many* $\overline{\mu}$.

(c) *We have*

$$s_{\infty,i}(M) \leq \sum_{\overline{\mu} \in \kappa_K^{\mathrm{alg}}} s_{\overline{\mu},i}(M).$$

Proof. Suppose first that ∂_0 is dominant for each irreducible component of $M \otimes F_1$ which contributes to $F_i(M, 0)$. Then $s_{\infty,i}(M)$ is less than or equal to the left slope of $F_i^{(0)}(M, r)$ at $r = 0$, whereas $s_{\overline{\mu},i}(M)$ is greater than or equal to the right slope of $F_i^{(0)}(T_\mu^*(M), r)$ at $r = 0$. We may thus reduce to the case $m = 0$, which is [Ked**, Theorem 11.3.2(c)].

It suffices to reduce to the case where ∂_0 is dominant for each irreducible component of $M \otimes F_1$ which contributes to $F_i(M, 0)$. This proceeds as in Step 2 of the proof of Theorem 1.2.4.4, except that we may end up working over an enlargement of K. This causes no harm in (a) or (b), but in (c) the sum may end up running over a larger field. However, the argument of Remark 1.2.9.5 shows that the extra terms do not contribute: that is, we may use (b) to show that $s_{\overline{\mu},i}(M) = 0$ if $\overline{\mu} \notin \kappa_K^{\mathrm{alg}}$, so (c) holds as written. □

Remark 1.2.9.7. The proof given above does not achieve the equality in (c) for $m > 0$, because the reduction in the last paragraph does not maintain equality.

As in [Ked**, Subsection 12.2], we can study decomposition theorems over closed annuli or discs using subharmonicity.

Definition 1.2.9.8. Fix $j \in J^+$. Let M be a ∂_j-differential module over $K\langle \alpha/t, t/\beta \rangle$ with $\alpha \leq 1 \leq \beta$. Define the *i-th ∂_j-discrepancy* of M at $r = 0$ as

$$\mathrm{disc}_i^{(j)}(M, 0) = -\sum_{\overline{\mu} \in (\kappa_K^{\mathrm{alg}})^\times} s_{\overline{\mu},i}^{(j)}(M);$$

it is nonnegative by Theorem 1.2.9.4. By Remark 1.2.9.5, this definition is invariant under enlarging K. We may extend the definition to general $r \in [-\log\beta, -\log\alpha]$ by pulling back M along

$$K\langle \alpha/t, t/\beta\rangle \to K(c)^\wedge\langle \alpha e^r/t, t/\beta e^r\rangle, \quad t \mapsto ct,$$

where c is transcendental over K and $K(c)^\wedge$ is the completion with respect to the e^{-r}-Gauss norm.

If M is a ∂_{J^+}-differential module over $K\langle \alpha/t, t/\beta\rangle$ with $\alpha \leq 1 \leq \beta$, we similarly define the *i-th discrepancy* $\mathrm{disc}_i(M, 0)$ of M at $r = 0$ as the sum of $-s_{\overline{\mu},i}(M)$ over $\overline{\mu} \in (k^{\mathrm{alg}})^\times$. This quantity is again nonnegative, and is again invariant under enlarging K (this time by the final remark in the proof of Theorem 1.2.9.6). This definition can similarly be extended to $r \in [-\log\beta, -\log\alpha]$.

Remark 1.2.9.9. If $r \notin \mathbb{Q} \cdot \log|K^\times|$, then Remark 1.2.4.7 implies that $F_i(M, r)$ is affine in a neighborhood of r. By Theorem 1.2.9.6, it follows that $\mathrm{disc}_i(M, r) = 0$.

Theorem 1.2.9.10. *Fix $j \in J^+$. Let M be a ∂_j-differential module over $K\langle \alpha/t, t/\beta\rangle$ of rank d. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

(a) *We have $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ for $r \in [-\log\beta, -\log\alpha]$.*

(b) *The function $F_i^{(j)}(M, r)$ is affine for $r \in [-\log\beta, -\log\alpha]$.*

(c) *We have $\mathrm{disc}_i^{(j)}(M, -\log\alpha) = \mathrm{disc}_i^{(j)}(M, -\log\beta) = 0$.*

Then there is a direct sum decomposition of M inducing, for each $\eta \in [\alpha, \beta]$, the decomposition of $M \otimes F_\eta$ separating the first i subsidiary ∂_j-radii from the others.

Proof. Similar to Theorem 1.2.3.5 but invoking [Ked**, Lemma 12.1.3] instead. □

Theorem 1.2.9.11. *Fix $j \in J^+$. Let M be a ∂_j-differential module over $K\langle t/\beta\rangle$ of rank d. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

(a) *We have $f_i^{(j)}(M, -\log\beta) > f_{i+1}^{(j)}(M, -\log\beta)$.*

(b) *The function $F_i^{(j)}(M, r)$ is constant for r in a neighborhood of $-\log\beta$.*

(c) *We have $\mathrm{disc}_i^{(j)}(M, -\log\beta) = 0$.*

Then there is a direct sum decomposition of M inducing, for each $\eta \in (0, \beta]$, the decomposition of $M \otimes F_\eta$ separating the first i subsidiary ∂_j-radii from the others.

Proof. One can prove this similarly to Theorem 1.2.3.5 by invoking [Ked**, Lemma 12.1.2] instead. It is also an immediate corollary of Theorems 1.2.9.10 and 1.2.3.10; note that Theorem 1.2.9.4 verifies the condition (c) in Theorem 1.2.9.10. □

Theorem 1.2.9.12. *Let M be a ∂_{J^+}-differential module over $K\langle \alpha/t, t/\beta \rangle$ of rank d. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

(a) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $r \in [-\log\beta, -\log\alpha]$.*

(b) *The function $F_i(M, r)$ is affine for $r \in [-\log\beta, -\log\alpha]$.*

(c) *We have $\mathrm{disc}_i(M, -\log\alpha) = \mathrm{disc}_i(M, -\log\beta) = 0$.*

Then there is a direct sum decomposition of M inducing, for each $\eta \in [\alpha, \beta]$, the decomposition of $M \otimes F_\eta$ separating the first i subsidiary radii from the others.

Proof. Similar to Theorem 1.2.5.1 but invoking Theorem 1.2.9.10 instead on the boundary. □

Theorem 1.2.9.13. *Let M be a ∂_{J^+}-differential module over $K\langle t/\beta \rangle$ of rank d. Suppose that the following conditions hold for some $i \in \{1, \dots, d-1\}$.*

(a) *We have $f_i(M, -\log\beta) > f_{i+1}(M, -\log\beta)$.*

(b) *The function $F_i(M, r)$ is constant for r in a neighborhood of $-\log\beta$.*

(c) *We have $\mathrm{disc}_i(M, -\log\beta) = 0$.*

Then there is a direct sum decomposition of M inducing, for each $\eta \in (0, \beta]$, the decomposition of $M \otimes F_\eta$ separating the first i subsidiary radii from the others.

Proof. It follows from Theorems 1.2.9.12 and 1.2.3.10; note also that Theorem 1.2.9.6 verifies the condition (c) in Theorem 1.2.9.12. □

1.3 Differential modules on higher-dimensional spaces

We now study the variation of subsidiary radii of differential modules on some simple higher-dimensional spaces. Rather than derive these directly, we deduce these from the corresponding results on 1-dimensional spaces from the previous section, using some properties of convex functions.

Throughout this section, we retain Hypothesis 1.2.1.1.

1.3.1 Convex functions

In this subsection, we set some terminology for convex functions, as in [Ked08+a, Section 2].

Definition 1.3.1.1. For a subset $C \subseteq \mathbb{R}^n$, we denote its interior by $\mathrm{int}(C)$. We say it is *convex* if for all $x, y \in C$ and all $t \in [0,1]$, $tx + (1-t)y \in C$. For $C \subseteq \mathbb{R}^n$ convex, a function $f : C \to \mathbb{R}$ is *convex* if for all $x, y \in C$ and all $t \in [0,1]$,

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y). \tag{1.3.1.2}$$

Such a function is continuous on $\mathrm{int}(C)$.

Definition 1.3.1.3. An *affine functional* on $\mathbb{R}^n$ is a map $\lambda : \mathbb{R}^n \to \mathbb{R}$ of the form $\lambda(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n + b$ for some $a_1, \dots, a_n, b \in \mathbb{R}$. If $a_1, \dots, a_n \in \mathbb{Z}$, we say λ is *transintegral* (short for "integral after translation"); if also $b \in \mathbb{Z}$, we say λ is *integral.* For $\lambda : \mathbb{R}^n \to \mathbb{R}$ an affine functional, define the *slope* of λ as the linear functional $\tilde{\lambda}(x) = \lambda(x) - \lambda(0)$.

Definition 1.3.1.4. For $f : C \to \mathbb{R}^n$ convex, a *domain of affinity* of f is a subset U of C with nonempty interior (in $\mathbb{R}^n$) on which f agrees with an affine functional λ. The nonempty interior condition ensures that λ is uniquely determined; we call it the *ambient functional* on U.

Lemma 1.3.1.5. *Let $f : C \to \mathbb{R}^n$ be a convex function, and let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be an affine functional which agrees with f on a subset of C with nonempty interior in $\mathbb{R}^n$.*

(a) *We have* $f(x) \geq \lambda(x)$ *for all* $x \in C$.

(b) *The set of* $x \in C$ *for which* $f(x) = \lambda(x)$ *is a convex subset of* C.

(c) *If* λ' *is another affine functional with the same slope as* λ, *and* λ' *occurs as the ambient functional of some domain of affinity of* f, *then* $\lambda = \lambda'$.

Proof. For (a), choose y in the interior of a domain of affinity U of f with ambient functional λ. For $\epsilon > 0$ sufficiently small, the quantity z defined by $\epsilon x + (1-\epsilon)z = y$ will also belong to U. By convexity of f, $\epsilon f(x) + (1-\epsilon)\lambda(z) \geq \lambda(y)$, so

$$f(x) \geq \frac{\lambda(y) - (1-\epsilon)\lambda(z)}{\epsilon} = \lambda(x).$$

We may deduce (b) and (c) immediately from (a). □

Definition 1.3.1.6. A subset $C \subseteq \mathbb{R}^n$ is *polyhedral* if there exist finitely many affine functionals $\lambda_1, \dots, \lambda_r$ such that

$$C = \{x \in \mathbb{R}^n : \lambda_i(x) \geq 0 \qquad (i = 1, \dots, r)\}. \tag{1.3.1.7}$$

(We do not require C to be bounded.) If the λ_i can all be taken to be (trans)integral, we say that C is *(trans)rational polyhedral.* (We use *RP* and *TRP* as shorthand for *rational polyhedral* and *transrational polyhedral.*) For $C \subseteq \mathbb{R}^n$ a convex subset of $\mathbb{R}^n$, a continuous convex function $f : C \to \mathbb{R}^n$ is *polyhedral* if there exist finitely many affine functionals $\lambda'_1, \dots, \lambda'_s$ such that

$$f(x) = \max\{\lambda'_1(x), \dots, \lambda'_s(x)\} \qquad (x \in C). \tag{1.3.1.8}$$

(In particular, such a function extends continuously to a convex function on the closure of C, or even to all of $\mathbb{R}^n$.) Similarly, if C is (trans)rational polyhedral, we say f is *(trans)integral polyhedral* if (1.3.1.8) holds for some (trans)integral affine functionals $\lambda'_1, \dots, \lambda'_s$.

Remark 1.3.1.9. If C is a convex subset of $\mathbb{R}^n$, then a continuous convex function $f : C \to \mathbb{R}^n$ is polyhedral if and only if C is covered by finitely many domains of affinity for f, by [Ked08+a, Lemma 2.2.6]. Moreover, if C is compact, then it suffices to check that every point in C has a neighborhood covered by finitely many domains of affinity for f, as then compactness will imply the existence of finitely many domains of affinity which cover C.

1.3.2 Detecting polyhedral functions

In this subsection, we establish a theorem that can be used to detect polyhedrality of certain convex functions based on integrality properties of certain values of the functions. We start with a weaker result in the same spirit, from [Ked08+a, Section 2].

Notation 1.3.2.1. In this subsection, for a point $x \in \mathbb{Q}^n$, we write $x_1, \dots, x_n$ for the coordinates of x.

Theorem 1.3.2.2. *Let C be a bounded RP subset of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}$ be a continuous convex function. Then f is integral polyhedral if and only if*

$$f(x) \in \mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n \qquad (x \in C \cap \mathbb{Q}^n). \tag{1.3.2.3}$$

Proof. See [Ked08+a, Theorem 2.4.2]. □

One cannot hope to similarly detect transintegral polyhedral functions by sampling them at individual points, i.e., on zero-dimensional TRP subsets of $\mathbb{R}^n$. The best one can do is detect them by sampling on 1-dimensional TRP subsets of $\mathbb{R}^n$, as follows.

Definition 1.3.2.4. Let C be a convex subset of $\mathbb{R}^n$. We say a function $f : C \to \mathbb{R}$ is *convex transintegral polyhedral in dimension* 1 if its restriction to the intersection of C with any 1-dimensional TRP subset of $\mathbb{R}^n$ is continuous, convex, and transintegral polyhedral. In other words, for any $x \in C, a \in \mathbb{Q}^n$, if we put $I_{x,a} = \{t \in \mathbb{R} : x + ta \in C\}$, then the function $g : I_{x,a} \to \mathbb{R}$ defined by $g(t) = f(x + ta)$ is continuous, convex, piecewise affine with slopes in $a_1\mathbb{Z} + \cdots + a_n\mathbb{Z}$, and has only finitely many slopes. (The latter is automatic if $I_{x,a}$ is closed and bounded, which always occurs if C is compact.)

Theorem 1.3.2.5. *Let C be a TRP subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a function which is convex transintegral polyhedral in dimension* 1*. Then f itself is convex and transintegral polyhedral (hence continuous).*

The proof is somewhat complicated, and will occupy the rest of this subsection. We first tackle the case where C is compact, for which we assemble several lemmas.

Definition 1.3.2.6. Let C be a TRP subset of $\mathbb{R}^n$. For $x \in C$, define the *angle* of C at x, denoted $\angle_x C$, to be the set of $z \in \mathbb{R}^n$ such that for some $t_0 > 0$, $x + tz \in C$ for $t \in [0, t_0]$. It is clear that $\angle_x C$ is an RP subset of $\mathbb{R}^n$ stable under multiplication by $\mathbb{R}_{>0}$.

Lemma 1.3.2.7. *Let C be a TRP subset of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}$ be a function which is convex transintegral polyhedral in dimension* 1*. Then f is convex.*

Proof. We may assume $\dim(C) = n$, by replacing $\mathbb{R}^n$ by a plane of the appropriate dimension. It suffices to verify (1.3.1.2) for any $x, y \in C$ and any $t \in [0, 1]$. By applying a change of basis in $\mathrm{GL}_n(\mathbb{Z})$, we may reduce to the case where the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ belong to $\angle_x C$.

We now choose $x'_1, \dots, x'_n > 0$ in turn so that for $i = 1, \dots, n$, $x_i + x'_i - y_i \in \mathbb{Q}$, $x + x'_1\mathbf{e}_1 + \cdots + x'_i\mathbf{e}_i \in \mathrm{int}(C)$, and

$$|f(x + x'_1\mathbf{e}_1 + \cdots + x'_i\mathbf{e}_i) - f(x + x'_1\mathbf{e}_1 + \cdots + x'_{i-1}\mathbf{e}_{i-1})| < \epsilon/n$$
$$|f(t(x + x'_1\mathbf{e}_1 + \cdots + x'_i\mathbf{e}_i) + (1-t)y) - f(t(x + x'_1\mathbf{e}_1 + \cdots + x'_{i-1}\mathbf{e}_{i-1}) + (1-t)y)| < \epsilon/n.$$

Namely, given $x'_1, \dots, x'_{i-1}$, the eligible choices of x'_i form a dense subset of an open interval with left endpoint 0. (Here we are using the continuity of the restriction of f to TRP sets of dimension 1.)

Put $x' = x + x'_1\mathbf{e}_1 + \cdots + x'_n\mathbf{e}_n$. Since $x' - y \in \mathbb{Q}^n$, the segment from x' to y is TRP. Hence

$$tf(x') + (1-t)f(y) \geq f(tx' + (1-t)y),$$

and so

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y) - 2\epsilon.$$

Since ϵ was arbitrary, this implies (1.3.1.2), yielding convexity of f. □

Definition 1.3.2.8. Let C be a TRP subset of $\mathbb{R}^n$. For $f : C \to \mathbb{R}$ a convex function, $x \in C$, and $z \in \angle_x C$, define $f'(x,z)$ to be the directional derivative of f at x in the direction of z, i.e.,

$$f'(x,z) = \lim_{t \to 0^+} \frac{f(x+tz) - f(x)}{t}.$$

Note that this is a limit taken over a decreasing sequence; for it to exist in all cases, we must allow it to take the value $-\infty$.

Lemma 1.3.2.9. *Let C be a TRP subset of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}$ be a convex function. For any fixed $x \in C$, the function $z \mapsto f'(x,z)$ is convex as a function from $\angle_x C$ to $\mathbb{R} \cup \{-\infty\}$ (in the sense of satisfying* (1.3.1.2)*).*

Proof. Take any $z_1, z_2 \in \angle_x C$. We assume first that $f'(x,z_1), f'(x,z_2) > -\infty$. Pick $u \in [0,1]$ and put $z_3 = uz_1 + (1-u)z_2$. Given $\epsilon > 0$, choose $t > 0$ for which

$$x + tz_i \in C \quad (i = 1,2,3), \qquad f'(x,z_i) \geq \frac{f(x+tz_i) - f(x)}{t} - \epsilon \quad (i = 1,2).$$

Then

$$\begin{aligned} uf'(x,z_1) + (1-u)f'(x,z_2) &\geq u\frac{f(x+tz_1) - f(x)}{t} + (1-u)\frac{f(x+tz_2) - f(x)}{t} - \epsilon \\ &\geq \frac{f\big(u(x+tz_1) + (1-u)(x+tz_2)\big) - f(x)}{t} - \epsilon \\ &\geq f'(x,z_3) - \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this proves the claim when both $f'(x,z_1)$ and $f'(x,z_2)$ are not $-\infty$. If one of them is $-\infty$, the same argument would imply that $f'(x,z_3) = -\infty$;

this completes the proof. $\square$

Lemma 1.3.2.10. *Assume that Theorem 1.3.2.5 holds for compact C with n replaced by $n-1$. Let C be a compact TRP subset of $\mathbb{R}^n$, and let $f: C \to \mathbb{R}$ be a function which is convex transintegral polyhedral in dimension 1. Then for any $x \in C$, the function $z \mapsto f'(x,z)$ on $\angle_x C$ is itself convex transintegral polyhedral in dimension 1.*

Proof. By Lemma 1.3.2.7, f is convex. By Lemma 1.3.2.9, $f'(x,z)$ is convex on $\angle_x C$, hence continuous on $\operatorname{int}(\angle_x C)$. By hypothesis, for $z \in \angle_x C \cap \mathbb{Q}^n$, $f'(x,z) \in \mathbb{Z}z_1 + \cdots + \mathbb{Z}z_n$. By Theorem 1.3.2.2, $f'(x,z)$ is integral polyhedral on any bounded RP subset of $\operatorname{int}(\angle_x C)$.

By subdividing C by hyperplanes, we may reduce to the case where $\angle_x C$ admits a bounded cross-section by a rational hyperplane. Pick any $z \in \angle_x C$ and $a \in \mathbb{Q}^n$ such that the set $I_{z,a} = \{u \in \mathbb{R} : z + ua \in \angle_x C\}$ is bounded. We must show that the function $g(u) = f'(x, z+ua)$ is continuous, convex, and transintegral polyhedral on $I_{z,a}$. (This suffices because we can recover all values of $f'(x,z)$ from the values on a bounded cross-section by a rational hyperplane.) By what we know about f', we already know all of these on $\operatorname{int}(I_{z,a})$. Consequently, it suffices to check that g is affine in a neighborhood of an endpoint of $I_{z,a}$.

For this, we may assume that the endpoint in question is a left endpoint at $u = 0$. Then z lies on the boundary of $\angle_x C$, so we can choose a codimension 1 facet D of C containing x, such that the ray from x in the direction of z has nontrivial intersection with D. By the hypothesis that Theorem 1.3.2.5 holds on compact TRP subsets of dimension $n-1$, the restriction of f to D must be transintegral polyhedral. In particular, we can rescale z so that for $t \in [0, 1+\epsilon]$ for some $\epsilon > 0$, $x + tz \in C$ and $f(x+tz) = f(x) + tf'(x,z)$.

Consider the function $h(t) = f'(x+tz, a)$ for $t \in [0, 1+\epsilon]$. Since the difference quotient $(f(x+tz+ua) - f(x+tz))/u$ is convex in t (the term $f(x+tz+ua)$ is convex, the term $-f(x+tz)$ is affine, and dividing by u has no effect), so is $h(t)$. However, $h(t) \in \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n$ for all t. This means that for $t \in (0, 1+\epsilon)$, $h(t)$ is continuous but takes values in a discrete subset of $\mathbb{R}$; this can only happen if $h(t)$ is equal to a constant value c on $(0, 1+\epsilon)$.

Rescale a if necessary so that $x+z+a \in C$ and $f(x+z+ua) = f(x)+f'(x,z)+uc$ for $u \in [0,1]$. We now claim that $f(x+tz+ua) = f(x)+tf'(x,z)+uc$ for $t \in [0,1], u \in [0,t]$. Since equality holds at $(t,u) = (0,0),(1,0),(1,1)$, we have by convexity of f that $f(x+tz+ua) \leq f(x)+tf'(x,z)+uc$ in the entire region. On the other hand, for any $t \in [0,1]$, the function $f(x+tz+ua)$ in u is convex, and equals $f(x)+tf'(x,z)+uc$ for u in a right neighborhood of 0. Consequently, $f(x+tz+ua) \geq f(x)+tf'(x,z)+uc$ for $u \in [0,t]$, yielding the desired equality.

We may rewrite the last claim as $f(x+tz+tua) = f(x)+tf'(x,z)+tuc$ for $t \in [0,1], u \in [0,1]$. From this, we may deduce that $g(u) = f'(x, z+ua) = f'(x,z)+uc$ for $u \in [0,1]$. This proves affinity of g near an endpoint, completing the argument. □

We now establish the compact case of Theorem 1.3.2.5.

Lemma 1.3.2.11. *The conclusion of Theorem 1.3.2.5 holds if C is compact.*

Proof. We may assume that C has nonempty interior, by replacing $\mathbb{R}^n$ by a plane containing C of the appropriate dimension. With this extra hypothesis, we proceed by induction on n, with trivial base case $n = 1$.

We have convexity of f by Lemma 1.3.2.7. It thus suffices to prove that f is transintegral polyhedral (and hence continuous) in a neighborhood of any $x \in C$. By Lemma 1.3.2.10, the restriction of $f'(x,z)$ to any compact TRP subset of $\angle_x C$ is convex transintegral polyhedral in dimension 1. By applying the induction hypothesis to the intersection of $\angle_x C$ with a rational hyperplane, we may deduce that $f'(x,z)$ is continuous, convex, and transintegral polyhedral. By Theorem 1.3.2.2, $f'(x,z)$ is in fact integral polyhedral.

To prove that f is transintegral polyhedral in a neighborhood of x, it suffices to do so after cutting C into finitely many pieces. We may thus reduce to the case where $f'(x,z)$ is affine on $\angle_x C$. Since $\angle_x C$ is a rational polyhedral cone, we may pick $z_1, \ldots, z_l \in \angle_x C \cap \mathbb{Q}^n$ such that $\angle_x C$ is the convex hull of the rays from 0 through $z_1, \ldots, z_l$. We may then rescale $z_1, \ldots, z_l$ so that $f(x+tz_i) = f(x)+tf'(x,z_i)$ for $i = 1, \ldots, l$ and $t \in [0,1]$.

For any z in the convex hull of $z_1, \dots, z_l$, we now deduce (using the affinity of $f'(x,z)$) that $f(x+z) \leq f(x) + f'(x,z)$. Since $f(x+tz)$ is convex in t, this is only possible if $f(x+tz) = f(x) + tf'(x,z)$ for $t \in [0,1]$. We conclude that f agrees with an integral affine functional on the convex hull of $x, x+z_1, \dots, x+z_l$. As noted above, this completes the proof. $\square$

We now allow C which are no longer necessarily bounded.

Definition 1.3.2.12. Let C be a TRP subset of $\mathbb{R}^n$. Define the *small cone* of C at x, denoted $\angle'_x C$, to be the set of $z \in \mathbb{R}^n$ such that $x + tz \in C$ for *all* $t > 0$; this is again a convex rational polyhedral cone in $\mathbb{R}^n$. Moreover, it does not depend on x by the following reasoning. Write $C = \{x \in \mathbb{R}^n : \lambda_1(x), \dots, \lambda_m(x) \geq 0\}$ for some transintegral affine functionals $\lambda_1, \dots, \lambda_m$. Write $\lambda_i(x) = \lambda_{i,0}(x) + c_i$ with $\lambda_{i,0}$ linear. Then $z \in \angle'_x C$ if and only if $x \in C$ and $\lambda_{i,0}(z) \geq 0$ for $i = 1, \dots, m$. In particular, $\angle'_x C$ does not depend on the choice of $x \in C$; we thus notate it also by $\angle' C$.

Lemma 1.3.2.13. *The conclusion of Theorem 1.3.2.5 holds.*

Proof. We may again assume that C has nonempty interior in $\mathbb{R}^n$; by slicing C with hyperplanes, we may further assume that the small cone $\angle' C$ is strictly convex (i.e., $\angle' C \cap (-\angle' C) = \{0\}$). We now induct on n, where we may assume $n \geq 2$ because the case $n = 1$ is trivial. By the induction hypothesis, the restriction of f to each boundary facet of C is convex transintegral polyhedral.

As in the proof of Lemma 1.3.2.10, for each boundary facet D of C, each $i \in \{1, \dots, n\}$, and each $a \in \mathbb{Q}^n$, the function $x \mapsto f'(x,a)$ is constant on the interior of each domain of affinity of the restriction of f to D. In particular, for $x \in D$ outside of a set of measure zero, $f'(x,a)$ takes only finitely many values.

By Lemma 1.3.2.11, f is polyhedral on any compact TRP subset of C. In particular, C is covered by domains of affinity of f; to prove that f is polyhedral on all of C, it suffices to show that C can be covered by finitely many domains of affinity of f (see Remark 1.3.1.9). By Lemma 1.3.1.5, it suffices to check that the ambient functionals on domains of affinity of f can have only finitely many slopes.

Let U be a domain of affinity of f with ambient functional λ. Choose a basis $a_1, \dots, a_n$ of $\mathbb{Q}^n$ none of whose elements is contained in $\angle' C \cup (-\angle' C)$ (this is possible because $\angle' C$ is strictly convex and $n \geq 2$). For $x \in U$ and $i \in \{1, \dots, n\}$, the function $f(x + ta_i)$ on I_{x,a_i} is convex transintegral polyhedral, so has a limiting slope at each endpoint of I_{x,a_i}. (Note that our hypothesis that $a_i \notin \angle' C \cup (-\angle' C)$ ensures that I_{x,a_i} is compact.) By the previous paragraph, for x away from a set of measure zero, these limiting slopes are themselves confined to a finite set. Since f is convex, the slope of $f(x + ta_i)$ at $t = 0$ is now also constrained to a finite set. This conclusion for $i = 1, \dots, n$ constrains the slope of λ to a finite set, proving the claim. □

1.3.3 Variation of subsidiary radii

In this subsection, we will extend Theorem 1.2.4.4 into a higher-dimensional generalization (Theorem 1.3.3.9). We keep Hypothesis 1.2.1.1 and Notation 1.2.1.2. We begin by introducing the setup of [Ked08+a, Section 4.1].

Notation 1.3.3.1. Throughout this subsection, we put $I = \{1, \dots, n\}$ for notational simplicity.

Notation 1.3.3.2. For X an n-tuple:

- for A an $n \times n$ matrix, write X^A for the n-tuple whose j-th entry is $\prod_{i=1}^n x_i^{A_{ij}}$;
- for c a number, put $X^c = (x_1^c, \dots, x_n^c)$.

Definition 1.3.3.3. For a subset $C \subset \mathbb{R}^n$, let e^{-C} denote the subset $\{e^{-r_I} : r_I \in C\} \subset (0, +\infty)^n$. A subset S of $[0, +\infty)^n$ is *log-(T)RP* if S is the closure of $\mathring{S} = e^{-C}$ for some (T)RP subset C of $\mathbb{R}^n$. We say S is *ind-log-(T)RP* if it is a union of an increasing sequence of log-(T)RP sets S_α; we denote $\mathring{S} = \cup_\alpha \mathring{S}_\alpha$. For instance, any open subset of $[0, +\infty)^n$ is covered by ind-log-RP subsets.

Caution 1.3.3.4. The subset $(0, 1]$ is an ind-log-TRP subset but not a log-TRP subset. By contrast, $[0, 1]$ is a log-TRP subset.

Definition 1.3.3.5. Let $C \subset \mathbb{R}^n$ be a TRP subset defined by (1.3.1.7), where $\lambda_s(x_I) = a_{s,1}x_1 + \cdots + a_{s,n}x_n + b_s$ for $a_{s,i} \in \mathbb{Z}$ and $s = 1, \ldots, r$. Denote the closure of e^{-C} in $[0, +\infty)^n$ by S. Define $A_K(S)$ to be the subspace of the (Berkovich) analytic n-space with coordinates $t_1, \ldots, t_n$ satisfying the condition $(|t_1|, \ldots, |t_n|) \in S$. Precisely,

$$\begin{aligned}\Gamma(A_K(S), \mathcal{O}) &= K[t_1, \ldots, t_n]\langle t_I^{a_{1,I}}/e^{-b_1}, \ldots, t_I^{a_{r,I}}/e^{-b_r}\rangle \\ &= \Big\{ \sum_{e_I \in \mathbb{Z}^n} \alpha_{e_I} t_I^{e_I} \Big| \lim_{e_I} |\alpha_{\alpha_I}| \eta_I^{e_I} = 0, \text{ for all } \eta_I \in S \Big\}.\end{aligned}$$

For an ind-log-TRP subset $S = \cup_\alpha S_\alpha$, we define $A_K(S) = \cap_\alpha A_K(S_\alpha)$.

Definition 1.3.3.6. Let S be an ind-log-TRP subset of $[0, +\infty)^n$. A ($\partial_{I\cup J}$-) *differential module* M over $X = A_K(S)$ is a locally free coherent sheaf together with an integrable connection

$$\nabla : M \to M \otimes \left(\bigoplus_{j=1}^{m} \mathcal{O}_X \cdot du_j \oplus \bigoplus_{i=1}^{n} \mathcal{O}_X \cdot dt_i \right).$$

We label the derivations $\partial_1, \ldots, \partial_m$ as usual, and put $\partial_{m+1} = \partial_{t_1}, \ldots, \partial_{m+n} = \partial_{t_n}$.

Notation 1.3.3.7. For $\eta_I = (\eta_1, \ldots, \eta_n) \in \mathring{S}$, let F_{η_I} be the completion of $K(t_I)$ with respect to the η_I-Gauss norm. Write $f_l(M, r_I) = -\log IR(M \otimes F_{e^{-r_I}}; l)$ and $F_l(M, r_I) = f_1(M, r_I) + \cdots + f_l(M, r_I)$ for $l = 1, \ldots, \operatorname{rank} M$.

Lemma 1.3.3.8. *Given $\eta_I \in (0, +\infty)^n$ and $A \in \mathrm{GL}_n(\mathbb{Z})$, let M be a differential module over $F_{\eta_I^A}$, and let $h_A^* : F_{\eta_I^A} \to F_{\eta_I}$ be given by $t_I \mapsto t_I^A$. Then $IR(M) = IR(h_A^* M)$.*

Proof. This follows from [Ked08+a, Proposition 4.2.7] (which is itself an immediate consequence of [Ked08+a, Lemma 4.1.5]) applied to A and A^{-1}. □

Theorem 1.3.3.9. *Let S be an ind-log-TRP subset of $[0, +\infty)^n$, and let M a differential module of rank d over $A_K(S)$.*

(a) *(Continuity) For $l = 1, \ldots, d$, the functions $f_l(M, r_I)$ and $F_l(M, r_I)$ are continuous.*

(b) *(Convexity) For $l = 1, \dots, d$, the function $F_l(M, r_I)$ is convex.*

(c) *(Polyhedrality) For $r_I \in -\log \overset{\circ}{S}$, if $l = d$ or $f_l(M, r_I) > f_{l+1}(M, r_I)$, then $F_l(M, r_I)$ is transintegral polyhedral in some neighborhood of r_I. Moreover, on any TRP subset of $-\log \overset{\circ}{S}$, $d!F_l(M, r_I)$ and $F_d(M, r_I)$ are transintegral polyhedral functions.*

(d) *(Monotonicity) Assume that S is log-TRP. Then for any $r_I, r'_I \in -\log \overset{\circ}{S}$, if $r_i \leq r'_i$ for $i \in I$ and $(1-t)r_I + tr'_I \in -\log \overset{\circ}{S}$ for any $t \in [0, +\infty)$, then $F_l(M, r_I) \geq F_l(M, r'_I)$ for $l = 1, \dots, d$.*

Proof. We first prove (a)-(c). We need only verify that, for $l = 1, \dots, d$, $d!F_l(M, r_I)$ and $F_d(M, r_I)$ satisfy the conditions of Theorem 1.3.2.5. Moreover, by translating and enlarging K if necessary, it suffices to check the hypothesis of Theorem 1.3.2.5 for $I_{x,a}$ in the case $x = 0$.

It suffices to consider $a = a_I \in \mathbb{Z}^n$ with $\gcd(a_I) = 1$. Let us describe $f_l(M, a_I t)$ and $F_l(M, a_I t)$ for $l = 1, \dots, d$ and $t \in I_{0,a_I}$. Pick an $n \times n$ invertible integral matrix A with (a_I) as the first row. Equip $A_K(S^{A^{-1}})$ with the coordinates (s_I), and define the toroidal transform $\phi : A_K(S^{A^{-1}}) \to A_K(S)$ by $\phi^*(t_I) = s_I^A$, where $S^{A^{-1}} = \{X^{A^{-1}} | X \in S\}$. By Lemma 1.3.3.8, $f_l(M, a_I t) = f_l(\phi^* M, (a_I A^{-1})t)$. The theorem follows from Theorem 1.2.4.4.

To prove (d), by continuity, we may assume that $r_I - r'_I$ are all rational numbers. By an argument as in the previous paragraph, we may reduce to the 1-dimensional case. In this case, we get a differential module over a disc, so the desired statement follows from Theorem 1.2.4.4(c). □

1.3.4 Decomposition by subsidiary radii

To conclude, we extend the theorems of §1.2.5 to higher-dimensional spaces.

Lemma 1.3.4.1. *Suppose $r \in \{0, \dots, n\}$. Put $C = \{(x_I) | x_I \geq 0, x_1 + \cdots + x_r \leq 1\} \subset \mathbb{R}^n$, and let C_ϵ be any TRP subset of $\mathbb{R}^n$ containing C in its interior. Let S (resp. S_ϵ) denote the closure of e^{-C} (resp. e^{-C_ϵ}) in $[0, +\infty)^n$, which is a log-TRP subset.*

Let M be a differential module of rank d over $A_K(S_\epsilon)$. Suppose that the following conditions hold for some $l \in \{1, \dots, d-1\}$.

(a) *The function $F_l(M, r_I)$ is affine for $(r_I) \in C_\epsilon$.*

(b) *We have $f_l(M, r_I) > f_{l+1}(M, r_I)$ for $(r_I) \in C_\epsilon$.*

Then M admits a unique direct sum decomposition over $A_K(S)$ separating the first l subsidiary radii of $M \otimes F_{e^{-r_I}}$ for any $(r_I) \in C$.

Proof. Note that $\Gamma(A_K(S), \mathcal{O}) = K\langle t_I, e^{-1}/t_1 \cdots t_r \rangle$ may be embedded into the completion $F_{1,\dots,1}$ of $K(t_1, \dots, t_n)$ for the $(1, \dots, 1)$-Gauss norm. For $i = 1, \dots, n$, let $F^{(i)}_{1,\dots,1}$ be the completion of $K(t_1, \dots, \widehat{t_i}, \dots, t_n)$ for the $(1, \dots, 1)$-Gauss norm; then the image of $\Gamma(A_K(S), \mathcal{O})$ also belongs to each of the subrings

$$F^{(i)}_{1,\dots,1}\langle e^{-1}/t_i, t_i \rangle \quad (i = 1, \dots, r); \qquad F^{(i)}_{1,\dots,1}\langle t_i \rangle \quad (i = r+1, \dots, n),$$

In fact, it is equal to the intersection of these subrings; this is true because C is the convex hull of the union of the segments

$$\{(x_1, \dots, x_n) : 0 \leq x_i \leq 1; \quad x_j = 0 \quad (j \neq i)\} \qquad (i = 1, \dots, r)$$
$$\{(x_1, \dots, x_n) : 0 \leq x_i; \quad x_j = 0 \quad (j \neq i)\} \qquad (i = r+1, \dots, n).$$

Consequently, by Lemma 1.2.3.2, it suffices to prove the decomposition over the rings $F^{(i)}_{1,\dots,1}\langle e^{-1}/t_i, t_i \rangle$ for $i = 1, \dots, r$ and $F^{(i)}_{1,\dots,1}\langle t_i \rangle$ for $i = r+1, \dots, n$. The former case follows by applying Theorem 1.2.5.1 to $M \otimes F_{1,\dots,1}\langle e^{-1-\epsilon}/t_i, t_i/e^{\epsilon} \rangle$ for $i = 1, \dots, r$ for some $\epsilon > 0$; the latter case follows by applying Theorem 1.2.5.4 to $F_{1,\dots,1}\langle t_i/e^{\epsilon} \rangle$ for $i = r+1, \dots, n$ for some $\epsilon > 0$. □

Theorem 1.3.4.2. *Let S be a ind-log-TRP subset of $[0, +\infty)^n$, and let M a differential module of rank d over $A_K(\mathrm{int}(S))$. Suppose that the following conditions hold for some $l \in \{1, \dots, d-1\}$.*

(a) *The function $F_l(M, r_I)$ is affine for $(r_I) \in \mathrm{int}(-\log \mathring{S})$.*

(b) *We have* $f_l(M, r_I) > f_{l+1}(M, r_I)$ *for* $(r_I) \in \mathrm{int}(-\log \mathring{S})$.

Then M *admits a unique direct sum decomposition over* $A_K(\mathrm{int}(S))$ *separating the first* l *subsidiary radii of* $M \otimes F_{e^{-r_I}}$ *for any* $(r_I) \in \mathrm{int}(-\log \mathring{S})$.

Proof. We can cover $\mathrm{int}(S)$ by log-TRP subsets $S_\alpha \subset \mathrm{int}(S)$ such that for each point of $x \in \mathrm{int}(S)$, there exists a neighborhood of x contained in some S_α. Moreover, we can choose those S_α to be simplicial, i.e., under a toroidal transform and rescaling, each S_α can be transformed into the form desired for Lemma 1.3.4.1. Since S_α lies in the interior of S, the decomposition follows from Lemma 1.3.4.1 by gluing the decompositions obtained on each of the S_α. □

Lemma 1.3.4.3. *Suppose* $r \in \{0, \dots, n\}$. *Put* $C = \{(x_I) | x_I \geq 0, x_1 + \cdots + x_r < 1\} \subset \mathbb{R}^n$, *and let* C_ϵ *be any TRP subset of* $\mathbb{R}^n$ *containing* C *in its interior. Let* S_ϵ *denote the closure of* e^{-C_ϵ} *in* $[0, +\infty)^n$, *which is a log-TRP subset. Let* S *be the set of points* $(s_I) \in S_\epsilon$ *such that* $s_I \leq 1$ *and* $s_1 \cdots s_r > e^{-1}$. *Let* R *be the subring of* $\Gamma(A_K(S_\epsilon), \mathcal{O})$ *consisting of those* f *for which* $|f|_{s_I}$ *is bounded over* $(s_I) \in S$. *Let* M *be a differential module of rank* d *over* $A_K(S_\epsilon)$. *Suppose that the following conditions hold for some* $l \in \{1, \dots, d-1\}$.

(a) *The function* $F_l(M, r_I)$ *is affine for* $(r_I) \in C_\epsilon$.

(b) *We have* $f_l(M, r_I) > f_{l+1}(M, r_I)$ *for* $(r_I) \in C_\epsilon$.

Then $M \otimes R$ *admits a unique direct sum decomposition separating the first* l *subsidiary radii of* $M \otimes F_{e^{-r_I}}$ *for any* $(r_I) \in C$.

Proof. Let F be the completion of $\mathrm{Frac} R$ for the $(1, \dots, 1)$-Gauss norm. Define $F^{(i)}_{1,\dots,1}$ as in the proof of Lemma 1.3.4.1. Then inside F, R is the intersection of the rings

$$F^{(i)}_{1,\dots,1}\langle 1/t_i^{-1}, t_i^{-1}/e]\!]_0 \qquad (i = 1, \dots, r); \qquad F^{(i)}_{1,\dots,1}\langle t_i \rangle \quad (i = r+1, \dots, n).$$

We may thus argue as in Lemma 1.3.4.1, but using Theorem 1.2.5.8 instead of Theorems 1.2.5.1 and 1.2.5.4. □

Theorem 1.3.4.4. *Let S be a log-TRP subset of $[0,+\infty)^n$. Let R be the subring of $\Gamma(A_K(\mathrm{int}(S)),\mathcal{O})$ consisting of those f for which $|f|_{s_I}$ is bounded over $s_I \in \mathrm{int}(S)$. Let M be a differential module of rank d over $A_K(S)$. Suppose that the following conditions hold for some $l \in \{1,\dots,d-1\}$.*

(a) *The function $F_l(M, r_I)$ is affine for $(r_I) \in -\log \mathring{S}$.*

(b) *We have $f_l(M, r_I) > f_{l+1}(M, r_I)$ for $(r_I) \in -\log \mathring{S}$.*

Then $M \otimes R$ admits a unique direct sum decomposition separating the first l subsidiary radii of $M \otimes F_{e^{-r_I}}$ for any $(r_I) \in \mathrm{int}(-\log \mathring{S})$.

Proof. Analogous to Theorem 1.3.4.2, except using Lemma 1.3.4.3 instead of Lemma 1.3.4.1. □

Remark 1.3.4.5. It may be helpful to illustrate the argument needed to reduce Theorem 1.3.4.4 to Lemma 1.3.4.3 with an explicit example. Take $S = [0,1]^2$, so that $R = \mathcal{O}_K[\![x,y]\!] \otimes_{\mathcal{O}_K} K$. We must partition $\mathrm{int}(-\log \mathring{S}) = (0,+\infty)^2$ into regions to which Lemma 1.3.4.3 may be applied. One such partition consists of

$$\{(x,y) \in \mathbb{R}^2 : 0 < x, 0 < y \leq \min\{x,1\}\},$$
$$\{(x,y) \in \mathbb{R}^2 : 0 < y, 0 < x \leq \min\{y,1\}\},$$
$$\{(x,y) \in \mathbb{R}^2 : 1 \leq x, 1 \leq y\}.$$

Since the parts all contain $(1,1)$, we can glue the three resulting decompositions together by matching them on $M \otimes F_{e^{-1},e^{-1}}$.

Remark 1.3.4.6. Note that Lemma 1.3.4.3 is not a special case of Theorem 1.3.4.4. We will not discuss the formulation and proof of a common generalization because it is just a somewhat awkward exercise.

Remark 1.3.4.7. By Remark 1.2.5.10, in Theorem 1.3.4.4, if $\log|K^\times| \subseteq \mathbb{Q}$ and $-\log \mathring{S}$ is RP, we may also take M to be defined over R. For example, if K carries the trivial valuation (forcing $p = 0$) and

$$S = \{(x,y) \in (0,1]^2 : xy = e^{-1}\},$$

then $R = K[\![x,y]\!][x^{-1},y^{-1}]$. This example can be used in the study of good formal structures for flat holomorphic connections; however, one needs to refine Theorem 1.3.4.4 slightly in case $p = 0$, to remove the need for strict inequality on the boundary of $-\log \mathring{S}$. For this, please consult to [Ked08+b].

Chapter 2

Ramification Theory for Local Fields: Overview

2.1 Introduction

Let K be a complete discretely valued field and let G_K be the Galois group of a separable closure K^{sep} of K. When the residue field κ_K is perfect, one has a classical ramification theory as well as Artin conductors and Swan conductors, which measure the ramification of a representation of G_K of finite local monodromy (i.e., the image of the inertia group being finite). Also, we have the Hasse-Arf theorem in this case, which states that these conductors are integers. It is one of the fundamental and amazing theorem in the classical ramification theory.

A goal of this thesis is to generalize the ramification theory to the case when the residue field κ_K is not perfect. The key result in this case is the analogue of Hasse-Arf theorem proved in Theorem 2.2.2.19.

2.1.1 Why imperfect residue case?

Let X be a connected proper smooth curve over an algebraically closed field k of characteristic $p > 0$, with geometric generic point η. Let D be a finite set of closed points. Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}_l}$-sheaf of rank d over $U = X\backslash D$, where l is a prime number different from char k. In other words, $\mathcal{F}$ is given by a representation $\pi_1(U,\eta) \to GL_d(\overline{\mathbb{Q}}_l)$. From this, we can read off Swan conductors $\mathrm{Swan}_x(\mathcal{F})$ associated to the representation $G_{(\mathrm{Frac}\mathcal{O}_X)^{\wedge,x}} \to \pi_1(U,\eta) \to GL_d(\overline{\mathbb{Q}}_l)$, where $(\mathrm{Frac}\mathcal{O}_X)^{\wedge,x}$ is the completion of the function field of X with respect to the norm at x. If we use $\chi_c(\mathcal{F}) = \sum_{i=0}^{2}(-1)^i \dim H_c^i(U,\mathcal{F})$ to denote the Euler characteristic of $\mathcal{F}$, the Grothendieck-Ogg-Shafarevich formula states that

$$\chi_c(\mathcal{F}) = d \cdot \chi_c(\overline{\mathbb{Q}}_l) - \sum_{x\in D} \mathrm{Swan}_x(\mathcal{F}), \tag{2.1.1.1}$$

where $\overline{\mathbb{Q}}_l$ is the trivial sheaf on U. In other words, this formula says that the Euler characteristic can be obtained from the global geometric information and the local ramification information. Since $\chi_c(\mathcal{F})$ is an (alternating) sum of integers, it suggests that each single conductor $\mathrm{Swan}_x(\mathcal{F})$ should be an integer.

In order to generalize this formula to higher dimensional cases, we need to measure the ramification along a divisor D with simple normal crossings on a smooth variety X. It is natural to pass to (the completion of) the local ring at the generic point η_i of one irreducible component D_i of D, which is a (complete) discrete valuation ring $\mathcal{O}_K$. The residue field of $\mathcal{O}_K$ is exactly the function field of D_i, which is typically *imperfect* if $\dim X > 1$. We still want to understand the ramification for the Galois group of its fraction field $\mathrm{Frac}(\mathcal{O}_K)$, a complete discretely valued field. However, some notable technical difficulties arise then, for example, the poorly behaved nature of Herbrand functions ϕ and ψ; non-monogeneration of ring of integers; and fierce ramification.

2.1.2 Some historical review

Motivated by the questions above, we need to understand the situation when the residue field κ_K is not perfect. In [Kat89a], Kato made a pioneer attempt. He defined arithmetic Swan conductors for one-dimensional representations. Even more, he also introduced the refined Swan conductors, which give secondary information of the ramification. Then, there have been fifteen years with no essential breakthrough until Abbes and Saito [AS02, AS03] gave a general definition of arithmetic non-logarithmic ramification filtration and arithmetic logarithmic ramification filtration on G_K by counting geometric connected components on certain rigid spaces $AS^a_{L/K}$ over K. One can define arithmetic Artin conductors and Swan conductors using the two filtrations. Later, Saito [Sai07+] gave a general definition of refined Swan conductors in the equal characteristic case.

Abbes and Saito showed [AS06+] that their definition of Swan conductors coincide with Kato's definition for characters in the equal characteristic case. Moreover, they proved some important properties of their filtrations. For example, the subquotients of both log and non-log filtrations are p-abelian groups [AS03]. (See also [Sai07+], where Saito proved that the subquotients of the log filtrations are annihilated by p.) However, they were not able to establish a Hasse-Arf theorem of the filtrations. This has become one of the central problem in ramification theory and arithmetic geometry.

Another approach to the problem using p-adic differential modules emerged in the mid 1990s. If the field K is of equal characteristic $p > 0$, the work of Christol, Matsuda, Mebkhout, Kedlaya, and Tsuzuki [Mat95, Ked05a] gave an alternative way to understand the classical Swan conductors. They first associated a p-adic differential module over the Robba ring to any p-adic Galois representation with finite local monodromy (that is to say the image of the inertia group is finite). Then they gave an interpretation of the Swan conductors by measuring the spectral norms of differential operators.

Partly inspired by Matsuda [Mat04], Kedlaya realized that this framework can be generalized to the case when the residue field κ_K is not perfect. In [Ked07a], he adopted the same construction and took into account of the effects of the differential operators corresponding to a p-basis of κ_K. Vaguely speaking, he defined the differential Artin / Swan conductor to be the maximal number computed by the differential operators under certain normalization. Kedlaya showed that the definition of Artin and Swan conductors turned out to give filtrations on the Galois group [Ked07a, Definition 3.5.12] (Theorem 3.2.3.5(4)). Also, he was able to prove a Hasse-Arf theorem for differential conductors (Theorem 3.2.3.5(1)(4)).

In [Ked07a], Kedlaya asked, as Matsuda suggested, whether the differential conductors are the same as the arithmetic ones, in which case the Hasse-Arf theorem for the arithmetic filtrations in the equal characteristic case would follow from that for the differential ones. Chiarellotto and Pulita [CP07+] gave an affirmative answer to this question when the representations are one-dimensional, using the setting of Kato's conductor [Kat89a]. We will give an affirmative answer to the general case in Chapter 3. This proof is close to Matsuda's approach [Mat04], however, we do not know how to explicitly link two methods. We will also show in Chapter 4 that a slightly modified proof may be applied to prove the Hasse-Arf theorem for K of mixed characteristic!

Another method of understanding general ramification theory is due to Borger, who introduced the notion of generic perfection [Bor04] of a complete discretely valued field, which is universal for all perfections of the residue field. He defines the Artin conductors to be the ones obtained by base changing to (the completion of) the fraction field of the generic perfection. The Hasse-Arf theorem of these conductors follows immediately from that of the classical ones. Kedlaya in [Ked07a, P.297] asked if Borger's definition also coincides with the two definitions above. We will show in Proposition 3.2.4.8 that, in the equal characteristic case, the differential Artin conductors are invariant under the operation of "adding a generic p^∞-th root" (see Definition 2.3.2.7), and hence it is the same as Borger's definition. However, in the mixed characteristic case, we can only deduce that the Artin conductors are invariant under the operation of "adding a generic p-th root" (see Definition 2.3.2.7), and we do not know if Borger's Artin conductors agree with the arithmetic Artin conductors.

2.1.3 Structure of this chapter

In Section 2.2, we give a brief review of the classical ramification theory, to motivate Abbes and Saito's definition. Then, we give Abbes and Saito's definition of arithmetic ramification filtrations, following [AS02].

In Section 2.3, we outline a fake proof of the Hasse-Arf theorem, leaving out some fake-assumptions that we cannot meet. We hope that this would help the readers better understand the big picture first. In particular, we would like to point out that the concept of generic rotation is actually used in previous chapters already, but with more number theoretic picture involved here.

In Section 2.4, we first review Borger's definition of generic perfection and Artin conductors. Then we prove the comparison theorem between Borger's Artin conductors with differential Artin conductors, provided that we know the invariance of the arithmetic Artin conductors under the operation of "adding a generic p^∞-th root". We briefly discuss why the same argument fails for mixed characteristic case.

In Section 2.5, we first discuss an application of ramification filtration to finite flat group schemes, given by Abbes and Mokrane [AM04]. Then we prove that the Hasse-Arf theorem for non-logarithmic ramification filtrations implies a Hasse-Arf type theorem for finite flat group schemes.

2.2 Ramification Filtrations

2.2.1 Classical ramification theory

We now discuss the classical ramification theory for a complete discretely valued field, following [Ser79, Chapter IV].

Hypothesis 2.2.1.1. In this subsection, let K be a complete discretely valued field whose residue field κ_K is perfect. Let L be a finite Galois extension of K with ramification degree $e_{L/K}$.

Definition 2.2.1.2. The *lower numbering filtration* of $G_{L/K}$ is defined as follows: for $i \geq -1$ an integer,

$$G_{L/K,i} \stackrel{\text{def}}{=} \mathrm{Ker}\,(G_{L/K} \to \mathrm{Aut}(\mathcal{O}_L/\mathfrak{m}_L^{i+1})).$$

In particular,

$$G_{L/K,-1} = G_{L/K};$$
$$G_{L/K,0} = I_{L/K}, \text{ the inertia group;}$$
$$G_{L/K,1} = P_{L/K}, \text{ the wild inertia group.}$$

For $i \geq -1$ real, we define $G_{L/K,i} = G_{L/K,\lceil i \rceil}$. The lower numbering filtration behaves nicely with respect to subgroups of $G_{L/K}$ but not quotients; it thus cannot be defined on the absolute Galois group G_K.

Definition 2.2.1.3. For $i \geq -1$, the *upper numbering filtration* of $G_{L/K}$ is defined by the relation $G_{L/K}^{\phi_{L/K}(i)} \stackrel{\text{def}}{=} G_{L/K,i}$, where

$$\phi_{L/K}(i) = \int_0^i \frac{1}{[G_{L/K,0} : G_{L/K,t}]}\,dt.$$

Note that the indices where the filtration jumps are now rational numbers, but not necessarily integers. In any case, Proposition 2.2.1.4 below implies that there is a unique filtration G_K^i on G_K which induces the upper numbering filtration on each $G_{L/K}$ (that is, $G_{L/K}^i$ is the image of G_K^i under the surjection $G_K \to G_{L/K}$).

Proposition 2.2.1.4 (Herbrand). *Let L' be a Galois subextension of L/K, and put $H = \mathrm{Gal}(L/L')$, so that H is normal in $G_{L/K}$ and $G_{L/K}/H = G_{L'/K}$. Then $G_{L'/K}^i = (G_{L/K}^i H)/H$; that is, the upper numbering filtration is compatible with forming quotients of $G_{L/K}$.*

Proof. See *[Ser79, § IV.3 Proposition 14].* □

Definition 2.2.1.5. We call $b \in \mathbb{R}$ a *(ramification) break* of L/K if $G_{L/K}^{b-1} \supsetneq \cup_{a>b-1} G_{L/K}^a$. From Definition 2.2.1.3, all breaks of L/K are rational numbers. Among all breaks of L/K, the biggest one is called the *highest (ramification) break* of L/K, denoted by $b(L/K)$. (If $L = K$, we set $b(L/K) = 0$ by convention.) This quantity has a more direct interpretation in terms of generators, stated in Proposition 2.2.1.6 below.

We also define the *highest log-break* to be $b_{\log}(L/K) = 0$ if $b(L/K) \leq 1$ and $b_{\log}(L/K) = b(L/K) - 1$ otherwise; it measures the wild ramification of G_K.

Proposition 2.2.1.6. *There exists $x \in \mathcal{O}_L$ that generates $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra. For such an x,*

$$b(L/K) = \frac{1}{e_{L/K}}\Big(\sum_{1\neq g\in G_{L/K}} v_L(gx-x) + \max_{1\neq g\in G_{L/K}} v_L(gx-x)\Big).$$

In particular, if L/K is totally ramified, we may take x to be any uniformizer π_L of L.

Proof. The existence of x is proved in [Ser79, § III.6 Proposition 12]. The rest is just plain calculation, which may be found in, for instance, [Col03+, Proposition 1.2]. □

A key observation, which Saito attributes to Kato, is the following.

Proposition 2.2.1.7. *Let P be the minimal polynomial of x in the proposition above. Then, the rigid analytic space $X^a = \{u \mid |u| \leq 1, |P(u)| < |\pi_K|^a\}$ has $[L:K]$ geometric connected components ([BGR84, 9.1.4/8]) if and only if $a \geq b(L/K)$.*

Proof. A rigorous proof may be found in [Col03+, Lemme 2.4] or [AS02, Lemma 6.6]. We will, instead, give a rough idea of why this is true.

The picture here is that if a is very large, we confine u in very small neighborhoods of the roots of $P(u) = 0$, or equivalently, the conjugates of x. The rigid space X^a should be geometrically disjoint union of very small discs centered at each conjugate of x. When a becomes smaller, the discs grow larger and, at some point, some of them crash into one disc, which decreases the number of geometric connected components; the number $b(L/K)$ records this moment.

The cut-off condition is obviously $|u - x| < |gx - x|$ for any $g \in G_{L/K}\backslash\{1\}$; in other words, u is closer to x than any other conjugates of x. Note that $P(u) = \prod_{g\in G_{L/K}}(u - gx)$. Hence, one has $|u - gx| = |gx - x|$. Thus,

$$|P(u)| = \prod_{g\in G_{L/K}} |u - gx| = |u - x| \prod_{1\neq g\in G_{L/K}} |x - gx| < |\pi_K|^{b(L/K)}.$$

In fact, this gives the essential ingredient of a rigorous proof. □

2.2.2 Review of Abbes-Saito's definition

We will sketch the definition of arithmetic ramification filtrations on the Galois group of a complete discretely valued field K. For more details, one can consult [AS02, AS03].

Hypothesis 2.2.2.1. In this subsection, let K be a complete discretely valued field with possibly imperfect residue field κ_K. Assume $\operatorname{char}\kappa_K = p > 0$. Let L be a finite Galois extension of K with (naïve) ramification degree $e = e_{L/K}$ (see Notation 1.1.1.7).

Notation 2.2.2.2. Denote $\theta = |\pi_K|$. When K is of mixed characteristic, we denote $\beta_K = v_K(p)$, the ***absolute ramification degree***. We say K is ***absolutely unramified*** if $\beta_K = 1$, equivalently, p is a uniformizer of K.

Definition 2.2.2.3. Take $Z = (z_j)_{j\in J} \subset \mathcal{O}_L$ to be a finite set of elements generating $\mathcal{O}_L$ over $\mathcal{O}_K$, i.e., $\mathcal{O}_K[(u_j)_{j\in J}]/\mathcal{I} \simeq \mathcal{O}_L$ mapping u_j to z_j for $j \in J = \{1, \dots, m\}$ and for some appropriate ideal $\mathcal{I}$. Let $(f_i)_{i=1,\dots,n}$ be a finite set of generators of $\mathcal{I}$. For $a > 0$, define the *Abbes-Saito space* to be

$$AS^a_{L/K,Z} = \left\{(u_J) \in A^m_K[0,1] \;\middle|\; |f_i(u_J)| \leq \theta^a,\ 1 \leq i \leq n\right\}. \tag{2.2.2.4}$$

We denote the *geometric* connected components of $AS^a_{L/K,Z}$ by $\pi_0^{\mathrm{geom}}(AS^a_{L/K,Z})$. The *highest ramification break* $b(L/K)$ of the extension L/K is defined to be the minimal b such that $\forall a > b$, $\#\pi_0^{\mathrm{geom}}(AS^a_{L/K,Z}) = [L : K]$.

Caution 2.2.2.5. Recall from Notation 1.1.1.22 that $\mathcal{O}_K\langle u_J\rangle$ denotes the Tate algebra in m variables over $\mathcal{O}_K$. S. Zerbes pointed out that a set of polynomial generators of $\operatorname{Ker}(\mathcal{O}_K\langle u_J\rangle \to \mathcal{O}_L)$ may not generate $\mathcal{I}$ over $\mathcal{O}_K[u_J]$. For example, $K = \mathbb{F}_p(x,y)((\pi_K))$ with $p > 2$ and L is generated by z, w with relation $z^p = z^{p-1}w^{p-1}\pi_K + x$, $w^p = z^{p-1}w^{p-1}\pi_K + y$; these two equations generate the kernel of $\mathcal{O}_K\langle u_J\rangle \to \mathcal{O}_L$ but not the kernel of $\mathcal{O}_K[u_J] \to \mathcal{O}_L$ because $\mathcal{O}_K[u_J]$ is not π_K-adically complete.

Remark 2.2.2.6. When the residue field κ_K is perfect, we may choose the generator in Definition 2.2.2.3 to be the one in Proposition 2.2.1.6. Then the rigid analytic space X^a in Proposition 2.2.1.7 is the union $\cup_{a'>a} AS^{a'}_{L/K,x}$. Hence, Abbes and Saito's definition is a natural generalization of the classical ramification breaks.

Similarly to the proof of Proposition 2.2.1.7, we give an intuitive way of understanding the above definitions following [AS02].

First, if $a \to 0^+$, the conditions on $f_1, \dots, f_n$ in 2.2.2.4 are almost vacuous. So, $AS^a_{L/K,Z}$ is almost the whole polydisc. In particular, it is geometrically connected. In contrast, if $a \to \infty$, the conditions on $f_1, \dots, f_n$ in 2.2.2.4 basically restrain the possible u_J to be very close to z_J or other solutions to the equations $f_1 = 0, \dots, f_n = 0$, which are exactly Galois conjugates of z_J. Thus, $AS^a_{L/K,Z}$ has exactly $[L : K]$ geometric connected components. From these two extreme cases, we know that, when we increase a, the Abbes-Saito space shrinks from a whole polydisc to smaller polydiscs and, at some a it breaks apart into different polydiscs. The highest ramification break captures the last break.

Remark 2.2.2.7. It might be more natural to view $AS^a_{L/K,Z}$ together with a morphism $\pi : AS^a_{L/K,Z} \to A^n_K[0, \theta^a]$ mapping (u_J) to $\big(f_1(u_J), \dots, f_n(u_J)\big)$. The similar view should also be taken for $AS^a_{L/K,\log,Z,P}$ below. We will come back to this in Subsection 2.3.3.

Definition 2.2.2.8. Keep the notation from Definition 2.2.2.3. Moreover, take a subset $P \subset Z$ and assume that P and hence Z contain π_L. Let $e_j = v_L(z_j)$. Take $g_j \in \mathcal{O}_K[(u_j)_{j\in J}]$ as a lift of $z_j^e/\pi_K^{e_j}$, $\forall j \in P$, and take $h_{i,j} \in \mathcal{O}_K[(u_j)_{j\in J}]$ as a lift of $z_j^{e_i}/z_i^{e_j}$, $\forall i, j \in P^2$. For $a > 0$, define the *logarithmic Abbes-Saito space* to be

$$AS^a_{L/K,\log,Z,P} = \left\{ (u_J) \in A^m_K[0,1] \;\middle|\; \begin{array}{cc} |f_i(u_j)| \leq \theta^a, & 1 \leq i \leq n \\ |u_j^e - \pi_K^{e_j} g_j| \leq \theta^{a+e_j} & \forall j \in P \\ |u_j^{e_i} - u_i^{e_j} h_{i,j}| \leq \theta^{a+e_ie_j/e_{L/K}} & \forall (i,j) \in P^2 \end{array} \right\}. \tag{2.2.2.9}$$

Similarly, the *highest logarithmic ramification break* $b_{\log}(L/K)$ of the extension L/K is defined to be the minimal b such that $\forall a > b$, $\#\pi_0^{\mathrm{geom}}(AS^a_{L/K,\log,Z,P}) = [L : K]$.

Remark 2.2.2.10. The additional structure on the special subset P is to give a log-variant of Definition 2.2.2.3, where we equip $\mathcal{O}_K$ and $\mathcal{O}_L$ the natural log-structures given by $\pi_K^{\mathbb{Z}} \hookrightarrow \mathcal{O}_K$ and $\pi_L^{\mathbb{Z}} \hookrightarrow \mathcal{O}_L$, respectively.

We reproduce several statements from [AS02, AS03].

Proposition 2.2.2.11. *The Abbes-Saito spaces have the following properties.*

(1) *For $a > 0$, the Abbes-Saito spaces $AS^a_{L/K,Z}$ and $AS^a_{L/K,\log,Z,P}$ do not depend on the choices of the generators $(f_i)_{i=1,\dots,n}$ of $\mathcal{I}$ and the lifts g_j and $h_{i,j}$ for $i,j \in P$* [AS02, Section 3]. *(This justifies the omission of $f_i, g_j, h_{i,j}$ from the notation.)*

(1') *In the definition of both Abbes-Saito spaces, if we choose polynomials $(f_i)_{i=1,\dots,n}$ as generators of* $\mathrm{Ker}\,(\mathcal{O}_K\langle u_J\rangle \to \mathcal{O}_L)$ *instead of* $\mathcal{I} = \mathrm{Ker}\,(\mathcal{O}_K[u_J] \to \mathcal{O}_L)$*, the spaces do not change.*

(2) *If we use another pair of generating sets Z and P satisfying the same properties, then we have a canonical bijection on the sets of the geometric connected components $\pi_0^{\mathrm{geom}}(AS^a_{L/K,Z})$ and $\pi_0^{\mathrm{geom}}(AS^a_{L/K,\log,Z,P})$ for different generating sets, where $a > 0$. In particular, both highest ramification breaks are well-defined* [AS02, Section 3].

(3) *The highest ramification break (resp. highest logarithmic ramification break) gives rise to a filtration on the Galois group G_K consisting of normal subgroups $\mathrm{Fil}^a G_K$ for (resp., $\mathrm{Fil}^a_{\log} G_K$) for $a \geq 0$ such that $b(L/K) = \inf\{a|\mathrm{Fil}^a G_K \subseteq G_L\}$ (resp. $b_{\log}(L/K) = \inf\{a|\mathrm{Fil}^a_{\log} G_K \subseteq G_L\}$)* [AS02, Theorems 3.3, 3.11]. *Moreover, for L/K a finite Galois extension, both highest ramification breaks are rational numbers* [AS02, Theorems 3.8, 3.16].

(4) *Let K'/K be a (not necessarily finite) extension of complete discretely valued fields. If K'/K is unramified, then $\mathrm{Fil}^a G_{K'} = \mathrm{Fil}^a G_K$ for $a > 0$* [AS02, Proposition 3.7]. *If K'/K is tamely ramified with ramification index $e < \infty$, then $\mathrm{Fil}^{ea}_{\log} G_{K'} = \mathrm{Fil}^a_{\log} G_K$ for $a > 0$* [AS02, Proposition 3.15].

(4') *Let K'/K be a complete extension of discretely valued fields with the* same *value group and linearly independent of a given finite extension L/K. Denote $L' = K'K$. If $\mathcal{O}_{L'} = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$, then $b(L/K) = b(L'/K')$* [AM04, Lemme 2.1.5].

(5) *For* $a \geq 0$, *define* $\mathrm{Fil}^{a+}G_K = \overline{\cup_{b>a}\mathrm{Fil}^b G_K}$ *and* $\mathrm{Fil}^{a+}_{\log}G_K = \overline{\cup_{b>a}\mathrm{Fil}^b_{\log}G_K}$. *Then, the subquotients* $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+}G_K$ *are abelian p-groups if* $a \in \mathbb{Q}_{>1}$ *and are* 0 *if* $a \notin \mathbb{Q}$, *except possibly when K is of mixed characteristic and absolutely unramified* ([AS02, Theorem 3.8] and [AS03, Theorem 1]). *The subquotients* $\mathrm{Fil}^a_{\log}G_K/\mathrm{Fil}^{a+}_{\log}G_K$ *are abelian p-groups if* $a \in \mathbb{Q}_{>0}$ *and are* 0 *if* $a \notin \mathbb{Q}$ ([AS02, Theorem 3.16], [AS03, Theorem 1]).

Moreover, if K is of equal characteristic $p > 0$, *the subquotients* $\mathrm{Fil}^a_{\log}G_K/\mathrm{Fil}^{a+}_{\log}G_K$ *are abelian p-groups killed by p if* $a \in \mathbb{Q}_{>0}$. [Sai07+, Corollary 1.3.6]

(6) *For* $a > 0$, $\mathrm{Fil}^{a+1}G_K \subseteq \mathrm{Fil}^a_{\log}G_K \subseteq \mathrm{Fil}^a G_K$ [AS02, Theorem 3.15(1)].

(7) *The inertia subgroup is* $\mathrm{Fil}^a G_K$ *for* $a \in (0, 1]$ *and the wild inertia subgroup is* $\mathrm{Fil}^{1+}G_K = \mathrm{Fil}^{0+}_{\log}G_K$ [AS02, Theorems 3.7 and 3.15].

(8) *When the residue field* κ_K *is perfect, the arithmetic ramification filtrations agree with the classical upper numbered filtration* [Ser79] *in the following way:* $\mathrm{Fil}^a G_K = \mathrm{Fil}^{a-1}_{\log}G_K = G_K^{a-1}$ *for* $a \geq 1$, *where* G_K^a *is the classical upper numbered filtration on* G_K [AS02, Section 6.1].

Proof. For the convenience of readers, we point out some ingredients of the proof. For details, one can consult original papers.

(1) is straightforward by matching up points.

(1') is not in any literature. However, it can be proved verbatim as (1).

(2) One can show that if we add a new (dummy) generator in Z or P, the new Abbes-Saito space admits a fibration over the original Abbes-Saito space whose fibers are closed discs of radius θ^a.

(3) The first statement is just abstract nonsense. The second one is essentially because Abbes-Saito spaces are defined over K and the geometric connected components can be detected over the algebraic closure K^{alg}, which has value group $|K^\times|^{\mathbb{Q}}$. However, realizing this principle needs formal models of rigid spaces. As we will reprove this result in the main theorem, we refer to the original paper for the formal model proof.

(4) and (4') When $\mathcal{O}_{LK'} \simeq \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$, one can match up the non-logarithmic Abbes-Saito space for LK'/K' and the extension of scalar of the non-logarithmic

Abbes-Saito space for L/K in a natural way. In the tamely ramified and the logarithmic case, one can also identify two logarithmic Abbes-Saito spaces [AS02, Proposition 9.8]; it is slightly more complicated.

(5) The proof uses the formal models of the Abbes-Saito spaces and their stable reductions, which is in an orthogonal direction of the present thesis. One may consult [AS03] and [Sai07+] for a complete treatment.

(6) and (7) are easy facts.

(8) follows from an explicit calculation in monogenic case, carried out in Propositions 2.2.1.6 and 2.2.1.7 and Remark 2.2.2.6. □

Remark 2.2.2.12. To avoid confusion, we point out that our approach to the Hasse-Arf Conjecture 2.2.2.17 below does not use (5) and the second statement of (3) on the rationality of the breaks in the proposition above. In fact, we will derive these properties from the properties by reducing to the classical case, or using the comparison with differential conductors.

Definition 2.2.2.13. By a representation of G_K, we mean a continuous homomorphism $\rho : G_K \to GL(V_\rho)$, where V_ρ is a finite dimensional vector space over a field F of characteristic zero. We allow F to have a nonarchimdedean topology; hence the image of G_K may not be finite. For $\sigma : H \to G_K$ a continuous homomorphism, we write $\rho|_H$ for $\rho \circ \sigma$.

We say that a representation ρ of G_K has *finite local monodromy* if the image of the inertia subgroup of G_K is finite.

Definition 2.2.2.14. Let $\rho : G_K \to GL(V_\rho)$ be a representation of finite local monodromy. Define the *arithmetic Artin and Swan conductors* as

$$\mathrm{Art}(\rho) \stackrel{\mathrm{def}}{=} \sum_{a \in \mathbb{Q}_{\geq 0}} a \cdot \dim \left(V_\rho^{\mathrm{Fil}^{a+} G_K} / V_\rho^{\mathrm{Fil}^a G_K} \right), \tag{2.2.2.15}$$

$$\mathrm{Swan}(\rho) \stackrel{\mathrm{def}}{=} \sum_{a \in \mathbb{Q}_{\geq 0}} a \cdot \dim \left(V_\rho^{\mathrm{Fil}^{a+}_{\log} G_K} / V_\rho^{\mathrm{Fil}^a_{\log} G_K} \right). \tag{2.2.2.16}$$

They are actually finite sums.

Conjecture 2.2.2.17. *(Hasse-Arf Theorem) Let K be a complete discretely valued field. For any representation ρ of G_K of finite local monodromy, the arithmetic conductors are nonnegative integers, namely,* $\mathrm{Art}_{ar}(\rho) \in \mathbb{Z}_{\geq 0}$ *and* $\mathrm{Swan}_{ar}(\rho) \in \mathbb{Z}_{\geq 0}$.

Moreover, the subquotients of the filtration $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ *for $a \in \mathbb{Q}_{>1}$ and* $\mathrm{Fil}^a_{\log} G_K/\mathrm{Fil}^{a+}_{\log} G_K$ *for $a \in \mathbb{Q}_{>0}$ are abelian groups killed by p.*

Proposition 2.2.2.18. *If the residue field κ_K is perfect, Conjecture 2.2.2.17 holds.*

Proof. By Proposition 2.2.2.11(8), we are reduced to the classical Hasse-Arf theorem [Ser79, §VI.2 Theorem 1' and §IV.2 Corollary 3]. Note that in this case, $\mathrm{Swan}(\rho) = \mathrm{Art}(\rho) - \dim V_\rho / V_\rho^{I_K}$. □

We will prove Conjecture 2.2.2.17 in Chapters 3 and 4, except for some special cases. The precise statement is as follows; it summaries the results from Corollary 3.4.3.3 and Theorems 4.2.3.5, 4.3.1.14, and 4.3.3.3.

Theorem 2.2.2.19. *Let K be a complete discretely valued field and let G_K be its absolute Galois group.*

(1) *(Hasse-Arf Theorem) Let $\rho: G_K \to GL(V_\rho)$ be a continuous representation of finite local monodromy. Then the Artin conductor* $\mathrm{Art}(\rho)$ *is a nonnegative integer except possibly when K is of mixed characteristic and is absolutely unramified; the Swan conductor* $\mathrm{Swan}(\rho)$ *is a nonnegative integer except possibly when K is of mixed characteristic and $p = 2$, in which case, we have* $\mathrm{Swan}(\rho) \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

(2) *The subquotients* $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ *for $a > 1$ and* $\mathrm{Fil}^a_{\log} G_K/\mathrm{Fil}^{a+}_{\log} G_K$ *for $a > 0$ of the ramification filtrations are trivial if $a \notin \mathbb{Q}$ and are abelian groups killed by p if $a \in \mathbb{Q}$, except possibly in the mixed characteristic, absolutely unramified and non-logarithmic case.*

Remark 2.2.2.20. The restriction on not being absolutely unramified also occurs in [AS03]. It reflects the failure of deforming the uniformizer p (not even "slightly"). Explicitly, we have a dichotomy (assuming that K has a finite set of lifted p-basis

$b_1, \ldots, b_m$)

$$\Omega^1_{\mathcal{O}_K/\mathbb{Z}_p} \otimes \kappa_K = \begin{cases} \bigoplus_{j=1}^m \kappa_K \cdot db_j & K \text{ is absolutely unramified,} \\ \kappa_K \cdot d\pi_K \oplus \bigoplus_{j=1}^m \kappa_K \cdot db_j & \text{otherwise.} \end{cases}$$

Compare Remarks 2.3.4.14 and 4.1.1.9 for another interpretation of the restriction from a more technical point of view. This does not affect the logarithmic ramification filtration because it well-behavior under tame base change helps avoid the restriction on β_K.

The restriction on $p \neq 2$ is purely technical. Please see the proof of Theorem 4.3.1.14 as well as Remark 4.3.1.15 for more detailed description.

We do not know any counterexample of Conjecture 2.2.2.17 in the case when either of the two conditions fails.

For the rest of this chapter, we first give some idea of the proof of the above theorem, and deduce some applications of it.

Before doing so, we point out a relevant conjecture on the arithmetic ramification filtration, which we will prove in Corollary 3.4.3.5 for the equal characteristic case.

Conjecture 2.2.2.21. *Let K be a complete discretely valued field. For $a \in \mathbb{Q}_{\geq 0}$, let $\mathrm{Fil}^a_{\log} G_K$ be the arithmetic logarithmic ramification filtration. Then for $a \in \mathbb{Q}_{>0}$, the conjugation action of $\mathrm{Fil}^{0+}_{\log} G_K / \mathrm{Fil}^a_{\log} G_K$ on $\mathrm{Fil}^a_{\log} G_K / \mathrm{Fil}^{a+}_{\log} G_K$ is trivial.*

Remark 2.2.2.22. By [Ser79, §IV.2 Proposition 10], the above Conjecture 2.2.2.21 is true if the residue field κ_K is perfect. We will prove it for the equal characteristic case in Corollary 3.4.3.5.

2.3 A fake proof of Hasse-Arf conjecture

In this section, we give a fake proof of the Hasse-Arf Conjecture 2.2.2.17, which will be the prototype of the real proofs in Chapters 3 and 4.

2.3.1 A brief sketch of the proof

This subsection is designed to give a vague overview of the proof. A slightly more detailed argument is presented in later subsections of this section.

Some preliminary reductions may reduce the problem to studying the ramification break of a finite totally ramified and wildly ramified Galois extension L/K. The problem is that the residue field extension κ_L/κ_K may not be separable. The idea of the proof is that if we could add the p-th roots of some elements of K without changing the ramification break $b(L/K)$, then we would practically "perfectify" the residue field of K and hence reduce to the separable residue field extension case, which can easily reduce further to the perfect residue field case.

The reality is not as ideal as we hoped, but one expects that, if adding the p-th roots of b_j, an element in a p-basis of K, changes the ramification break, then adding the p-th roots of $b_j + a\pi_K$ would not change the ramification break for all $a \in \mathcal{O}_K^\times$. In order to give a systematic approach, we introduce the operations of adding generic p-th or p^∞-th roots of elements in a lifted p-basis, that is to replace K by $\widetilde{K} = K(x)^{\mathrm{unr},\wedge}\big((b+x\pi_K)^{1/p}\big)$ or $\widetilde{K} = K(x)^{\mathrm{unr},\wedge}(b+x\pi_K)^{1/p^n})$, namely, we first add a dummy variable x and then adjoin a p-th or p^∞-th root of an element involving x. Fortunately, knowing that the ramification breaks are invariant under adding generic p-th or p^∞-th roots is enough for proving Conjecture 2.2.2.17, because we can still reduce to the separable residue field extension case (see Proposition 2.3.2.10).

Now, it suffices to show the invariance of ramification breaks under the operations of adding generic p-th or p^∞-th roots. For this, we need to link the ramification breaks with the generic radii of convergence of some differential modules and apply the tool of differential modules from Chapter 1. Assume for a moment that pulling back an Abbes-Saito space $AS^a_{L/K,Z} \to A^n_K[0,\theta^a]$ along some map $A^n_{\widetilde{K}}[0,\theta^a] \to A^n_K[0,\theta^a]$ gives an Abbes-Saito space for the extension $\widetilde{K}L/\widetilde{K}$, i.e., we have the following Cartesian diagram.

$$\begin{array}{ccc} AS^a_{L/K,Z} & \longleftarrow & AS^a_{\widetilde{K}L/\widetilde{K},\widetilde{Z}} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ A^n_K[0,\theta^a] & \overset{f}{\longleftarrow} & A^n_{\widetilde{K}}[0,\theta^a] \end{array} \tag{2.3.1.1}$$

The key here is that the morphism π is finite étale of degree $[L:K]$ (for some standard Abbes-Saito spaces which will discuss below). Thus, we can push forward the structure sheaf along π to get a differential module $\mathcal{E}$ over $A^n_K[0,\theta^a]$. Using simple Taylor expansion, we know that $\pi_0^{\mathrm{geom}}(AS^a_{L/K,Z}) = [L:K]$ if and only if $\mathcal{E}$ is trivial over $A^n_K[0,\theta^b]$ for $b<a$, which can be seen from the generic radii of $\mathcal{E}$. (This is not false when $\mathrm{char}\, K = p > 0$; see Subsection 2.3.6 for the explanation.) When adding a generic p-th or p^∞-th root, it is equivalent to considering an Abbes-Saito space for $\widetilde{K}L/\widetilde{K}$ and considering the corresponding differential module. By the base change property (2.3.1.1), we need to match up the generic radii of $\mathcal{E}$ with those of $f^*\mathcal{E}$. This is exactly where the tool of differential module from Chapter 1 comes into play.

Now, we discuss the base change diagram (2.3.1.1) above. In practice, we work with a special type of Abbes-Saito spaces (Definition 2.3.3.7). They basically arise from a good set of generators of $\mathcal{O}_L/\mathcal{O}_K$, by choosing a lifted p-bases of κ_L and a uniformizer π_L. A more serious problem we encounter is that the definition of Abbes-Saito space is not "functorial" with respect to the base field K and hence the expected base change diagram (2.3.1.1) does not hold. The key point here is to slightly change the morphism $\pi : AS^a_{L/K,Z} \to A^n_K[0,\theta^a]$ to $\Pi : TS^a_{L/K} \to A^n_K[0,\theta^a]$, where we introduce a space $TS^a_{L/K}$ isomorphic to $AS^a_{L/K,Z}$, which carries a "functorial" morphism down to $A^n_K[0,\theta^a]$. We call this $AS = TS$ theorem (Fake-theorem 2.3.5.2). We will see in Subsection 2.3.5 that this needs to be more carefully studied in the mixed characteristic case.

2.3.2 Generic p-th roots and generic p^∞-th roots

The notion of generic p^∞-th roots was first (implicitly) introduced by Borger in [Bor04]. Kedlaya [Ked07a] realized that in the equal characteristic case, adding generic p-th roots into the field extension will not change the (differential) non-logarithmic ramification filtration; hence, one can prove the non-logarithmic Hasse-Arf theorem by reducing to the perfect residue field case.

Hypothesis 2.3.2.1. In this subsection unless otherwise specified, let K be a discretely valued field with *separably closed* and *imperfect* residue field. Let L be a finite Galois extension of K. Assume that K admits a *finite* lifted p-basis (see Definition 1.1.1.11).

Remark 2.3.2.2. This is a mild hypothesis because the conductors behave well under unramified base changes, and the tamely ramified case is well-studied (Proposition 2.2.2.11(7)). Also, one can easily reduce to the finite p-basis case (see Proposition 2.3.2.13).

Notation 2.3.2.3. Let $J = \{1, \dots, m\}$ for notational convenience. For rest of the thesis, we reserve the notation j and m for indexing p-basis. We also use J^+ to denote $J \cup \{0\}$, where 0 refers to the uniformizer π_K.

Notation 2.3.2.4. Let x be transcendental over K. Define $K(x)^\wedge$ to be the completion of $K(x)$ with respect to the 1-Gauss norm and define K' to be the completion of the maximal unramified extension of $K(x)^\wedge$. Set $L' = K'L$.

Lemma 2.3.2.5. *Let $L(x)^\wedge$ be the completion of $L(x)$ with respect to the 1-Gauss norm. Then, L' is the completion of the maximal unramified extension of $L(x)^\wedge$. In particular, the residue field of L' is $\kappa_{L'} = \kappa_K(x)^{\mathrm{sep}} \cdot \kappa_L$, which is separably closed.*

Proof. First, $L(x)^\wedge = LK(x)^\wedge$ because the latter is complete and is dense in the former. So, it suffices to prove that L' is complete and has separable residue field. Since L'/K' is finite, L' is complete. Moreover, the residue field $\kappa_{L'}$ of L' is separably closed because it is a finite extension of a separably closed field $\kappa_K(\bar{x})^{\mathrm{sep}}$. □

Proposition 2.3.2.6. *The highest ramification breaks do not change if we make a base change from K to K'. In other words, $b(L/K) = b(L'/K')$ and $b_{\log}(L/K) = b_{\log}(L'/K')$.*

Proof. Since π_L is a uniformizer of L' and $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ surjects onto $\kappa_{L'}$ by previous lemma, we have $\mathcal{O}_{L'} = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. The result follows from Proposition 2.2.2.11(4'). $\square$

Definition 2.3.2.7. Let b_{j_0} be an element in a lifted p-basis of K. We will often need to make a base change $K \hookrightarrow \widetilde{K} = K'((b_{j_0} + x\pi_K)^{1/p^n}; n \in \mathbb{N})^\wedge$(resp. $K \hookrightarrow \widetilde{K} = K'((b_{j_0} + x\pi_K)^{1/p})$), a process which we shall refer to as *adding a generic p^∞-th (resp. p-th) root (of b_{j_0}).* If we start with a finite field extension L/K, adding a generic p^∞-th (resp. p-th) root will mean considering the extension $\widetilde{L} = L\widetilde{K}/\widetilde{K}$. We have $G_{\widetilde{L}/\widetilde{K}} = G_{L/K}$ as $\widetilde{K}$ is linearly independent from L over K. By convention, we take $\pi_{\widetilde{K}} = \pi_K$ as $\widetilde{K}/K$ since unramified. If we add a generic p^∞-th root, we provide $\widetilde{K}$ with a lifted p-basis $\{b_{J\setminus\{j_0\}}, x\}$, which has the same number of elements as the original lifted p-basis. If we add a generic p-th root, we provide $\widetilde{K}$ with a lifted p-basis $\{b_{J\setminus\{j_0\}}, (b_{j_0} + x\pi_K)^{1/p}, x\}$, which has one more element than the original lifted p-basis.

Remark 2.3.2.8. The reason to introduce the distinction between the generic p-th roots and the p^∞-th roots is that, on one hand, to prove the comparison to Borger's conductor, we need the invariance under adding generic p^∞-th root operations; and on the other hand, we can only prove the invariance under adding generic p-th root operations in the mixed characteristic case, which is fortunately enough to deduce the Hasse-Arf theorem. see also Remark 2.4.2.2.

Remark 2.3.2.9. If K is of mixed characteristic, adding a generic p^∞-th or p-th root does not change the absolute ramification degree of K (Notation 2.2.2.2).

The purpose of adding generic p-th or p^∞-th roots is to reduce to the monogenic case. The proof of the following proposition is essentially the same as [Ked07a, Lemma 3.5.4]. It is also implicitly contained in Borger's construction of Artin conductors (Section 2.4).

Proposition 2.3.2.10. *Let L/K be a finite Galois extension of complete discretely valued fields satisfying Hypothesis 2.3.2.1. Then after finitely many operations of* *adding generic p^∞-th or p-th roots, the field extension we start with has separable residue field extension.*

Moreover, if K is of mixed characteristic, then in this process, the absolute ramification degree β_K does not change.

Proof. First, the tamely ramified part is always preserved under these operations. So, we can assume that L/K is totally wildly ramified and hence the Galois group $G_{L/K}$ is a p-group. We can filter the extension L/K as $K = K_0 \subset \cdots \subset K_n = L$, where K_i/K_{i-1} is a (wildly ramified) $\mathbb{Z}/p\mathbb{Z}$-Galois extension and K_i/K is Galois for each $i = 1, \dots, n$. Each of these subextensions

(a) either has inseparable residue field extension (and hence has naïve ramification degree 1),

(b) or has separable residue field extension (and hence has naïve ramification degree p).

Moreover, $\mathcal{O}_{K_i}/\mathcal{O}_{K_{i-1}}$ is generated by one element, a lift of the a generator of the residue field in case (a), or a uniformizer of $\mathcal{O}_{K_i}$ in case (b).

Let i_0 be the maximal number such that K_i/K_{i-1} has separable residual extension for $i = 1, \dots, i_0$. Obviously adding a generic p^∞-th or p-th root does not decrease i_0 because after adding a generic p-th root, the naïve ramification degree of $\widetilde{K}_{i_0}/\widetilde{K}$ still equals to the degree p^{i_0}. Now, it suffices to show that after finitely many operations of adding generic p^∞-th or p-th roots, K_{i_0+1}/K_{i_0} has separable residue field extension (if $i_0 < n$). Suppose the contrary.

Let $g \in G_{K_{i_0+1}/K_{i_0}} \simeq \mathbb{Z}/p\mathbb{Z}$ be a generator. We claim that

$$\gamma = \min_{x \in \mathcal{O}_{K_{i_0+1}}} \left(v_{K_{i_0+1}}(g(x) - x) \right)$$

decreases by at least 1 after adding generic p-th roots of each of the elements in the p-basis. This would be enough to conclude the proposition because γ is always a nonnegative integer.

Let z be a generator of $\mathcal{O}_{K_{i_0+1}}$ over $\mathcal{O}_{K_{i_0}}$. It satisfies an equation

$$z^p + a_1 z^{p-1} + \cdots + a_p = 0, \tag{2.3.2.11}$$

where $a_1, \ldots, a_{p-1} \in \mathfrak{m}_{K_{i_0}}$ and $a_p \in \mathcal{O}_{K_{i_0}}^\times$ with $\bar{a}_p \in \kappa_{K_{i_0}}^\times \backslash (\kappa_{K_{i_0}}^\times)^p = \kappa_K^\times \backslash (\kappa_K^\times)^p$. It is easy to see that $\gamma = v_{K_{i_0+1}}(g(z) - z)$.

Adding generic p-th or p^∞-th roots of each of the element in a lifted p-basis gives us a field $\widehat{K}$. Now, the field extension $\widehat{K}K_{i_0+1}/\widehat{K}K_{i_0}$ is also generated by z as above. But we can write $a_p = \alpha^p + \beta$ for $\alpha \in \mathcal{O}_{\widehat{K}K_{i_0}}$ and $\beta \in \mathfrak{m}_{\widehat{K}K_{i_0}}$. Hence if we substitute $z' = z + \alpha$ into (2.3.2.11), we get $z'^p + a'_1 z'^{p-1} + \cdots + a'_p = 0$, with $a'_1, \ldots, a'_p \in \mathfrak{m}_{\widehat{K}K_{i_0}}$. Hence, $v_{\widehat{K}K_{i_0+1}}(z') > 0$. By assumption that the extension $\widehat{K}K_{i_0+1}/\widehat{K}K_{i_0}$ has naïve ramification degree 1, $\pi_{K_{i_0}}$ is a uniformizer for $\widehat{K}K_{i_0+1}$ and hence $z'/\pi_{K_{i_0}}$ lies in $\mathcal{O}_{\widehat{K}K_{i_0+1}}$. Thus,

$$\begin{aligned} \gamma' &= \min_{x \in \mathcal{O}_{\widehat{K}K_{i_0+1}}} \left(v_{\widehat{K}K_{i_0+1}}(g(x) - x)\right) \leq v_{\widehat{K}K_{i_0+1}}\big(g(z'/\pi_{K_{i_0}}) - z'/\pi_{K_{i_0}}\big) \\ &= v_{K_{i_0+1}}\big(g(z) - z\big) - 1 = \gamma - 1. \end{aligned}$$

This proves the claim and hence the proposition. ☐

Remark 2.3.2.12. It is worthwhile to point out that, if we only add generic p-th roots, then after these operations, the number of elements in the lifted p-basis of the resulting field will be more than that of the original field.

For the following proposition, we drop Hypothesis 2.3.2.1.

Proposition 2.3.2.13. *Assume that the highest non-logarithmic ramification breaks $b(L/K)$ are invariant under the operation of adding a generic p^∞-th or p-th root if*

(a) *either K is of equal characteristic and L/K verifies Hypothesis 2.3.2.1,*

(b) *or K is of mixed characteristic with a fixed absolute ramification degree β_K and L/K verifies Hypothesis 2.3.2.1.*

Then, we have for all such K,

(1) $\mathrm{Art}(\rho)$ *is a nonnegative integer for any representation* $\rho : G_K \to GL(V_\rho)$ *with finite local monodromy;*

(2) *the subquotients* $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ *are trivial if* $a \notin \mathbb{Q}$ *and are abelian groups killed by* p *if* $a \in \mathbb{Q}_{>1}$.

Proof. (1) Since the conductor is additive and is invariant when base changing to the completion of the maximal unramified extension of K (Proposition 2.2.2.11(4)), we may assume that ρ is irreducible and exactly factors through the Galois group of a totally ramified Galois extension L/K. We may also assume that the residue field κ_K is imperfect and the extension is wildly ramified since the classical case is well-known (Propositions 2.2.2.11(7) and 2.2.2.18). We need only to show that $\mathrm{Art}(\rho) = b(L/K) \cdot \dim \rho \in \mathbb{Z}$.

Now we reduce to the finite p-basis case. Choose a finite subset $J_0 \subset J$ such that $\kappa_K(\bar{b}_j^{1/p})$ is linearly independent from κ_L for any $j \in J \backslash J_0$. Pick lifts $b_j \in \mathcal{O}_K$ of $\bar{b}_j$ for each $j \in J \backslash J_0$. Define $K_1 = K\left(b_j^{1/p^n}; j \in J\backslash J_0, n \in \mathbb{N}\right)^\wedge$ and $L_1 = K_1 L$. It is easy to see that $[L_1 : K_1] = [L : K]$, $e_{L_1/K_1} \geq e_{L/K}$, and $[\kappa_{L_1} : \kappa_{K_1}] \geq [\kappa_L : \kappa_K]$, where κ_{K_1} and κ_{L_1} are the residue fields of K_1 and L_1, respectively. Thus, all the inequalities are forced to be equalities. This implies $G_{L_1/K_1} = G_{L/K}$ and $\mathcal{O}_{L_1} = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_1}$. By Proposition 2.2.2.11(4'), $b(L_1/K_1) = b(L/K)$. Therefore, we may reduce to the case when Hypothesis 2.3.2.1 holds.

Now, we can apply Proposition 2.3.2.10 and the assumption we have to reduce to the case when L/K has separable residue field extension. In this case, Proposition 2.2.2.11(4') implies that replacing K by $K\left(b_j^{1/p^n}; j \in J, n \in \mathbb{N}\right)^\wedge$ does not change the conductor, where b_J is a lifted p-basis of K. Hence, we reduce to the classical case; the statement follows from Proposition 2.2.2.18.

Now we prove (2), following the idea of [Ked07a, Theorem 3.5.13]. Let L be a finite Galois extension of K with Galois group $G_{L/K}$; then we obtain an induced filtration on $G_{L/K}$. It suffices to check that $\mathrm{Fil}^a G_{L/K}/\mathrm{Fil}^{a+} G_{L/K}$ is abelian and killed by p; moreover, we may quotient further to reduce to the case where $\mathrm{Fil}^{a+} G_{L/K}$ is the trivial group but $\mathrm{Fil}^a G_{L/K}$ is not. As above, we may reduce to the classical case because the ramification break of any intermediate extension between L and K is also preserved under the operations above. The statement follows from Proposition 2.2.2.18. □

2.3.3 Standard Abbes-Saito spaces

In this subsection, we introduce the standard Abbes-Saito spaces by choosing a distinguished set of generators of $\mathcal{O}_L/\mathcal{O}_K$. We continue to use Notation 2.3.2.3 and assume the following.

Hypothesis 2.3.3.1. Let K be a discretely valued field with *imperfect* residue field and let L be a finite Galois extension of K. Assume that K admits a *finite* lifted p-basis b_J.

Notation 2.3.3.2. We define a norm on the polynomial ring $\mathcal{O}_K[u_{J^+}]$: for $h = \sum_{e_{J^+}} \alpha_{e_{J^+}} u_{J^+}^{e_{J^+}}$, where $\alpha_{e_{J^+}} \in \mathcal{O}_K$, we set $|h| = \max_{e_{J^+}}\{|\alpha_{e_{J^+}}| \cdot \theta^{e_0/e}\}$. For $a \in \frac{1}{e}\mathbb{Z}_{\geq 0}$, denote N^a to be the set of elements with norm $\leq \theta^a$; it is in fact an ideal.

Construction 2.3.3.3. Choose lifted p-bases $b_J \subset \mathcal{O}_K$ and $c_J \subset \mathcal{O}_L$ of K and L, respectively. Let $\mathbf{k}_0 = \kappa_K$ with p-basis $(\bar{b}_j)_{j\in J}$. By possibly rearranging the indexing in b_J, we can filter the extension κ_L/κ_K by subextensions $\mathbf{k}_j = \kappa_K(\bar{c}_1, \dots, \bar{c}_j)$ with p-bases $\{\bar{c}_1, \dots, \bar{c}_j, \bar{b}_{j+1}, \dots, \bar{b}_m\}$ for $j \in J$. Moreover, if $[\mathbf{k}_j : \mathbf{k}_{j-1}] = p^{r_j}$, then $\bar{c}_j^{p^{r_j}} \in \mathbf{k}_{j-1}$. We also choose uniformizers π_K and π_L of K and L.

Write $\Delta : \mathcal{O}_K\langle u_{J^+}\rangle/\mathcal{I}_{L/K} \xrightarrow{\sim} \mathcal{O}_L$ mapping u_j to c_j for $j \in J$ and u_0 to π_L, where $\mathcal{I}_{L/K}$ is some proper ideal. Let $\overline{\Delta}$ be the composite of Δ with the reduction $\mathcal{O}_L \twoheadrightarrow l$. Hence,

$$\left\{u_{J^+}^{e_{J^+}} | e_j \in \{0, \dots, p^{r_j} - 1\} \text{ for all } j \in J, \text{ and } e_0 \in \{0, \dots, e-1\}\right\} \qquad (2.3.3.4)$$

form a basis of $\mathcal{O}_K\langle u_{J^+}\rangle/\mathcal{I}_{L/K}$ as a free $\mathcal{O}_K$-module. We choose a set of generators p_{J^+} of $\mathcal{I}_{L/K}$ by writing each $u_j^{p^{r_j}}$ (for $j \in J$) or u_0^e (for $j = 0$) in terms of the basis (2.3.3.4). We say that p_j *corresponds* to c_j. Obviously, p_{J^+} generates $\mathcal{I}_{L/K}$. Moreover,

$$\begin{aligned} p_j &\in u_j^{p^{r_j}} - \mathfrak{b}_j(u_1, \dots, u_{j-1}) + N^{1/e} \cdot \mathcal{O}_K[u_{J^+}], \quad j \in J, \\ p_0 &\in u_0^e - \mathfrak{d}(u_1, \dots, u_m)\pi_K + \pi_K N^{1/e} \cdot \mathcal{O}_K[u_{J^+}], \end{aligned}$$

where $\mathfrak{b}_j(u_1, \ldots, u_{j-1}) \in \mathcal{O}_K[u_1, \ldots, u_{j-1}]$ with powers on u_i smaller than p^{r_i} for all $i = 1, \ldots, j-1$, and $\mathfrak{d}(u_1, \ldots, u_m) \in \mathcal{O}_K[u_1, \ldots, u_m]$ such that $\Delta(\mathfrak{d}) \in \mathcal{O}_L^\times$.

Remark 2.3.3.5. It is attractive to hope that one can find p-basis $(\bar{b}_j)_{j\in J}$ of κ_K so that $\kappa_L = \kappa_K(\bar{b}_j^{p^{-r_j}})$ for some $r_j \in \mathbb{Z}_{\geq 0}$. This however is false in general, as pointed out to the author by S. Ohkubo. In fact, Sweedler [Swe68] studied this phenomenon and called the above case modular. He also gave the following non-modular example [Swe68, Example 1.1].

Let κ_0 be a perfect field of characteristic p and let X, Y, Z be indeterminants. Let $\kappa = \kappa_0(X^p, Y^p, Z^{p^2})$ and $\ell = \kappa(Z, XY + Z)$. Then $[\ell : \ell \cap \kappa^{p^{-1}}] = p^2$ and $[\ell \cap \kappa^{p^{-1}} : \kappa] = p$. Hence, ℓ/κ cannot be modular.

Remark 2.3.3.6. It is also not true in general that one can take uniformizers π_L and π_K of L and K so that $\pi_L^{\mathfrak{e}}/\pi_K \equiv 1 \bmod \mathfrak{m}_L$, as notified to the author by Shun Ohkubo. He gave the following counterexample.

Let K be a complete discretely valued field with imperfect residue field κ_k. Let $b \in \mathcal{O}_K$ be such that $\bar{b} \in \kappa_K \backslash \kappa_K^p$. Choose $\alpha, \beta \in \overline{K}$ as follows: let α be a root of the polynomial $X^p + \pi_K X + b \in K[X]$ and β a root of the polynomial $Y^p + \pi_K Y + \pi_K \alpha \in K(\alpha)[Y]$. Denote $L = K(\alpha, \beta)$. Then L/K is a separable extension of degree p^2 with naïve ramification degree p. The ring of integers of $K(\alpha)$ and $K(\alpha, \beta)$ are $\mathcal{O}_K[\alpha]$ and $\mathcal{O}_K[\alpha, \beta]$, respectively. We claim that we cannot choose uniformizers π_L and π_K so that $\pi_L^p/\pi_K \equiv 1 \bmod \mathfrak{m}_L$.

It is clear that β is a uniformizer of L. For any uniformizer π_L of L,

$$\frac{\pi_L^p}{\pi_K} = \frac{\beta^p}{\pi_K}\left(\frac{\pi_L}{\beta}\right)^p \in (-\alpha - \beta)(\mathcal{O}_L^\times)^p \xrightarrow{(\mathrm{mod}\ \mathfrak{m}_L)} (-\alpha)\kappa_L^p \subset \kappa_L.$$

In particular, π_L^p/π_K is not congruent to 1 modulo $\mathfrak{m}_L$.

We expect (though we do not know at the moment) a counterexample for which L/K is Galois.

Definition 2.3.3.7. The *(standard) Abbes-Saito spaces* $AS^a_{L/K}$ and $AS^a_{L/K,\log}$ for $a > 0$ are defined by taking generators to be $\{c_J, \pi_L\}$ and relations to be p_{J+} (see

Proposition 2.2.2.11(1')). In particular, their rings of functions are

$$\mathcal{O}^a_{AS,L/K} = K\langle u_{J^+}, \pi_K^{-a} V_{J^+}\rangle / \big(p_0(u_{J^+}) - V_0, \dots, p_m(u_{J^+}) - V_m\big), \text{ and}$$
$$\mathcal{O}^a_{AS,L/K,\log} = K\langle u_{J^+}, \pi_K^{-a-1} V_0, \pi_K^{-a} V_J\rangle / \big(p_0(u_{J^+}) - V_0, \dots, p_m(u_{J^+}) - V_m\big).$$

Remark 2.3.3.8. The additional π_K^{-1} added on V_0 reflects the log-structure. (See Remark 2.2.2.10.

2.3.4 Cohen rings and ψ_K-functions

We insert here a discussion of Cohen rings and their functoriality with respect to the p-bases. For more detailed study, one may consult [Whi02]. We also give an interpretation of the functoriality by differential module using Taylor series. Then, we introduce the fake function $\psi_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$, which is a deformation of the uniformizer π_K and the lifted p-basis b_J.

Definition 2.3.4.1. Let κ be a field of characteristic $p > 0$. A *Cohen ring* C_κ is a *absolutely unramified* complete discrete valuation ring with residue field κ. If κ is perfect, C_κ is exactly the ring of Witt vectors.

A *based field* of characteristic $p > 0$ is a field κ equipped with a distinguished p-basis $\bar{b}_J$, where J is an index set. We view based fields as forming a category whose morphisms from $(\kappa, \bar{b}_J)$ to $(\kappa', \bar{b}'_{J'})$ are morphisms $\kappa \to \kappa'$ of fields carrying $\bar{b}_J$ into $\bar{b}'_{J'}$ as a set.

For $(\kappa, \bar{b}_J)$ a based field, a *based Cohen ring* for $(\kappa, \bar{b}_J)$ is a pair (C_κ, B_J), where C_κ is a Cohen ring for κ and B_J is a subset of C_κ which lifts $\bar{b}_J$.

Proposition 2.3.4.2. *There is a functor from based fields to based Cohen rings which is a quasi-inverse of the residue field functor.*

Proof. This is implicit in Cohen's original paper [Coh46]; an explicit proof is given by [Whi02, Theorem 2.1]. An alternative proof may be found in [Ked07a, Proposition 3.1.4]. □

The following proposition, proved in [Whi02, Theorem 2.1], is stronger than the proposition above. For convenience of the reader, we include the proof.

Proposition 2.3.4.3. *Keep the notation as above and let R be a complete noetherian local ring with the maximal ideal $\mathfrak{m}_R$ containing p. Assume that we have a homomorphism $\bar{\psi} : \kappa \hookrightarrow R/\mathfrak{m}_R$. Then, for any $B'_J \subseteq R$ lifting $\bar{\psi}(\bar{b}_J)$, there exists a unique continuous homomorphism $\psi : C_\kappa \to R$ lifting $\bar{\psi}$ and sending B_j to B'_j for all $j \in J$.*

Proof. For any $n \in \mathbb{N}$, a *level n expression* of an element $g \in C_\kappa$ is a (non-canonical) way of writing g as

$$g = \sum_{i,i' \geq 0} \sum_{e_J=0}^{p^n-1} p^i A_{i,i',e_J}^{p^n} B_J^{e_J} \tag{2.3.4.4}$$

for some $A_{i,i',e_J} \in C_\kappa$ and for a fixed i, $A_{i,i',e_J} = 0$ when $i' \gg 0$ for all e_J. Then we set

$$\psi_n(g) = \sum_{i,i' \geq 0} \sum_{e_J=0}^{p^n-1} p^i \tilde{A}_{i,i',e_J}^{p^n} B_J'^{e_J} \tag{2.3.4.5}$$

where $\tilde{A}_{i,i',e_J}$ is some lift of $\bar{\psi}(a_{i,i',e_J})$ in R with a_{i,i',e_J} being the reduction of A_{i,i',e_J} in κ. Different choices of lifts $\tilde{A}_{i,i',e_J}$ may change the definition of $\psi_n(g)$ by an element in $\mathfrak{m}^n$; a different level n expression as in (2.3.4.4) may also vary $\psi_n(g)$ by some element in $\mathfrak{m}_R^n$. For a level n expression of g as in (2.3.4.4) with $n \geq 1$, we can rewrite it as

$$g = \sum_{i,i' \geq 0} \sum_{e'_J=0}^{p-1} \sum_{e_J=0}^{p^{n-1}-1} p^i (A_{i,i',e_J}^{p} B_J^{e'_J})^{p^{n-1}} B_J^{e_J},$$

which is a level $n-1$ expression for g. From this, we conclude that $\psi_n(g) \equiv \psi_{n-1}(g) \bmod \mathfrak{m}_R^{n-1}$. Taking $n \to \infty$, we get our map $\psi(g) = \lim_{n\to\infty} \psi_n(g)$. It is not hard to check that ψ is actually a homomorphism; this is because for $g, h \in C_\kappa$, the formal sum and product of level n expressions of g and h are level n expressions of $g + h$ and gh, respectively.

To prove the uniqueness, take another continuous homomorphism $\psi' : C_\kappa \to R$

satisfying all the conditions. Then for a level n expression of g as in (2.3.4.4),

$$\psi'\Big(\sum_{i,i'\geq 0}\sum_{e_J=0}^{p^n-1} p^i A_{i,i',e_J}^{p^n} B_J^{e_J}\Big) = \sum_{i,i'\geq 0}\sum_{e_J=0}^{p^n-1} p^i \psi'(A_{i,i',e_J})^{p^n} B_J'^{e_J}$$

is exactly one possible definition for ψ_n. As we pointed out above, $\psi'(g) \equiv \psi_n(g) \equiv \psi(g) \bmod \mathfrak{m}_R^n$. Let $n \to \infty$ and we have $\psi = \psi'$. □

Hypothesis 2.3.4.6. From now on, we assume that $J = \{1, \ldots, m\}$ is a finite set.

Corollary 2.3.4.7. *Keep the notation as above. There exists a unique continuous homomorphism $\psi : C_\kappa \to C_\kappa[\![\delta_1, \ldots, \delta_m]\!]$ such that for all $j \in J$, $\psi(B_j) = B_j + \delta_j$ and for any $g \in C_\kappa$, $\psi(g) - g$ lies in the ideal generated by $\delta_1, \ldots, \delta_m$. Moreover, if $\kappa_0 = \cap_n \kappa^{p^n}$ and C_{κ_0} is the ring of Witt vectors of the perfect field κ_0, the homomorphism ψ above is a C_{κ_0}-homomorphism.*

Proof. The first statement follows from the proof of previous proposition. By the functoriality of Witt vectors, ψ has to be identity when restricted to C_{κ_0} because κ_0 is perfect. Hence, ψ is a C_{κ_0}-homomorphism. □

Corollary 2.3.4.8. *Let $K = \kappa_K((\pi_K))$ be a complete discretely valued field of characteristic $p > 0$ with lifted p-basis b_J. Fix $j \in J$ and let $b_j' \in \mathcal{O}_K$ be an element such that $b_j' \equiv b_j \pmod{\mathfrak{m}_K}$. Then there exists a (unique) continuous automorphism $g^* : K \to K$ such that $g^*(\pi_K) = \pi_K$, $g^*(b_j) = b_j'$, and $g^*(b_{J\setminus j}) = b_{J\setminus j}$.*

Proof. Applying Proposition 2.3.4.3 to $\kappa = \kappa_K$, $R = \kappa_K[\![\pi_K]\!]$, and $\mathfrak{m}_R = (\pi_K)$ gives us a homomorphism $g^* : C_{\kappa_K}/(p) = \kappa_K \to \kappa_K[\![\pi_K]\!]$ such that $g^*(b_j) = b_j'$, and $g^*(b_{J\setminus j}) = b_{J\setminus j}$. One can extend this to an automorphism $g^* : K \to K$ by setting $g^*(\pi_K) = \pi_K$. □

Proposition 2.3.4.9. *The homomorphism ψ in Corollary 2.3.4.7 can be also constructed via Taylor series as follows. For $x \in C_\kappa$,*

$$\psi(x) = \sum_{e_J} \frac{\partial_J^{e_J}}{(e_J)!}(x) \cdot \delta_J^{e_J},$$

where $\partial_j = \partial_{u_j}$ are differential operators as introduced in Situation 1.1.6.7.

Proof. By Lemma 1.1.6.8, Taylor series gives the desired homomorphism; Definition 1.1.4.1 (or in fact Corollary 1.1.2.17) verifies that we can divide the factorials. By the uniqueness in Proposition 2.3.4.3, this gives the homomorphism we are looking for. □

Corollary 2.3.4.10. *Modulo p, the homomorphism ψ in Corollary 2.3.4.7 gives a continuous homomorphism $\bar{\psi} : \kappa \to \kappa[\![\delta_J]\!]$. Moreover, if for $\bar{g} \in \kappa$, we can write $d\bar{g} = \bar{g}_1 d\bar{b}_1 + \cdots + \bar{g}_m d\bar{b}_m$ in $\Omega^1_{\kappa/\mathbb{F}_p}$, then $\psi(\bar{g}) \equiv \bar{g} + \bar{g}_1\delta_1 + \cdots + \bar{g}_m\delta_m$ modulo $(\delta_J)^2 \cdot \kappa[\![\delta_J]\!]$.*

Proof. This follows from Proposition 2.3.4.9 above immediately. □

From now on, we assume Hypothesis 2.3.3.1 for the rest of the subsection; we fix a finite p-basis (b_J) and a uniformizer π_K of K.

Fake-assumption 2.3.4.11. Pretend that we have a *continuous* homomorphism $\psi_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$ such that $\psi_K(\pi_K) = \pi_K + \delta_0$ and $\psi_K(b_j) = b_j + \pi_K$ for all $j \in J$.

Remark 2.3.4.12. The reason to use δ_0/π_K instead of δ_0 is for convenience of notation.

Caution 2.3.4.13. Such a homomorphism exists when K is of equal characteristic $p > 0$ (Corollary 2.3.4.7). However, when K is of mixed characteristic, there is never such a continuous homomorphism. In particular, we cannot make $\psi_K(p) = p$ and $\psi_K(\pi_K) = \pi_K + \delta_0$ happen at the same time. This is because one cannot "deform" the uniformizer in the mixed characteristic case.

Moreover, since K will not be absolutely unramified in applications, (lifted) p-basis may not deform freely either. In other words, we may not be able to find a continuous morphism $\psi'_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_J]\!]$ such that $\psi'(b_j) = b_j + \delta_j$ for all $j \in J$. However, this is true if K is absolutely unramified (Notation 2.2.2.2), as proved in Corollary 2.3.4.7. More generally, this is true if K is the composite of an absolutely unramified field and a field which is a totally ramified extension of a complete discretely valued field with perfect residue field. (See Situation 1.1.6.7 and the following discussions.)

Remark 2.3.4.14. This fake assumption will be fixed in Chapter 4, by taking $\psi_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$ to be just a function, without requiring to be a homomorphism (Construction 4.1.1.2). It turns out that we may take ψ_K to be an approximate homomorphism modulo $p(\delta_0/\pi_K, \delta_J)$ (Proposition 4.1.1.8).

This also explains the reason that Theorem 2.2.2.19 is only stated for the case $\beta_K > 1$; indeed when $\beta_K = 1$, we have no control over the approximate homomorphism ψ_K.

It is very interesting to compare this more technical interpretation of the restriction $\beta_K > 1$ with the explanation in [AS03] recalled in Remark 2.2.2.20. Since the differentials often measure the ability of deforming an object, the lack of the direct summand $d\pi_K$ in the absolutely unramified case is reflected here as the failure of "deforming" the uniformizer, even infinitesimally.

Although ψ_K is just a fake function, but $\bar{\psi}_K : \kappa_K \to \kappa_K[\![\delta_J]\!]$ still satisfies the conclusion of Corollary 2.3.4.10, by Lemma 4.1.1.11. Before moving on, we deduce a technical lemma that is useful in the proof of $AS = TS$ theorem.

Lemma 2.3.4.15. *Assume that $\bar{\psi}_K$ satisfies the conclusion of Corollary 2.3.4.10, the determinant*

$$\det\Big(\frac{\partial(\bar{\psi}_K^*(\bar{p}_i) - \bar{p}_i)}{\partial \delta_j}\Big)_{i,j\in J}\Big|_{\delta_{J^+}=0} \in \big(\kappa_K[u_{J^+}]/(\bar{p}_J, u_0)\big)^\times = \kappa_L^\times,$$

where $\bar{p}_J$ are reductions of p_J in $\kappa_K[u_{J^+}]/(u_0)$.

Proof. By the expression in Cosntruction 2.3.3.3.

$$\Big(\frac{\partial(\bar{\psi}(\bar{p}_i) - \bar{p}_i)}{\partial \delta_j}\Big)_{i,j\in J} \bmod (\delta_J) = \Big(\frac{\partial(\bar{\psi}(\bar{\mathfrak{b}}_i) - \bar{\mathfrak{b}}_i)}{\partial \delta_j}\Big)_{i,j\in J} \bmod (\delta_J) \qquad (2.3.4.16)$$

Let $\bar{\alpha}_{ij} \in \kappa_L$ denote the entries in the matrix on the right hand side of (2.3.4.16), where we identify $\mathcal{O}_K\langle u_{J^+}\rangle/(p_{J^+}, u_0) \xrightarrow{\sim} \kappa_L$. Under this identification, $\bar{\mathfrak{b}}_i$ will become $\bar{c}_i^{p^{r_i}}$ for all $i \in J$. It suffices to show that the i-th row is κ_L-linearly independent from the first $i-1$ rows for all i. If we write

$$\bar{c}_i^{p^{r_i}} = \sum_{e_1=0}^{p^{r_0}-1} \cdots \sum_{e_{i-1}=0}^{p^{r_{i-1}}-1} \bar{\lambda}_{e_1,\dots,e_{i-1}} \bar{c}_1^{e_1} \cdots \bar{c}_{i-1}^{e_{i-1}},$$

where $\bar{\lambda}_{e_1,\dots,e_{i-1}} \in \kappa_K$ for which $d\bar{\lambda}_{e_1,\dots,e_{i-1}} = \bar{\mu}_{e_1,\dots,e_{i-1},1} d\bar{b}_1 + \cdots + \bar{\mu}_{e_1,\dots,e_{i-1},m} d\bar{b}_m$, then by Corollary 2.3.4.10,

$$\begin{aligned}\bar{\alpha}_{i1} d\bar{b}_1 + \cdots + \bar{\alpha}_{im} d\bar{b}_m &= \sum_{e_1=0}^{p^{r_0}-1} \cdots \sum_{e_{i-1}=0}^{p^{r_{i-1}}-1} u_1^{e_1} \cdots u_{i-1}^{e_{i-1}} \big(\bar{\mu}_{e_1,\dots,e_{i-1},1} d\bar{b}_1 + \cdots + \bar{\mu}_{e_1,\dots,e_{i-1},m} d\bar{b}_m\big) \\ &\equiv d(\bar{c}_i^{p^{r_i}}) \bmod \big(d\bar{c}_1, \dots, d\bar{c}_{i-1}\big)\end{aligned}$$

in $\Omega^1_{\mathbf{k}_{i-1}/\mathbb{F}_p}$; it is in fact nontrivial because $d\bar{c}_1, \dots, d\bar{c}_m$ form a basis of $\Omega^1_{\kappa_L/\mathbb{F}_p}$ and hence there should not be any auxiliary relation among $d\bar{c}_1, \dots, d\bar{c}_j$ in $\Omega^1_{\mathbf{k}_j/\mathbb{F}_p}$. But we know that the sums $\bar{\alpha}_{i'1} d\bar{b}_1 + \cdots + \bar{\alpha}_{i'm} db_m$ for $i' < i$ all lie in the submodule of $\Omega^1_{\mathbf{k}_{i-1}/\mathbb{F}_p}$ generated by $d\bar{c}_1, \dots, d\bar{c}_{i-1}$. Hence the i-th row of the matrix in (2.3.4.16) is ($\mathbf{k}_{i-1}$-)linearly independent from the first $i-1$ rows. The lemma follows. □

Remark 2.3.4.17. When κ_L/κ_K is modular in the sense of [Swe68], we can choose the p-basis of κ_K so that $\bar{c}_j^{p^{r_j}} = \bar{b}_j$ for all $j \in J$; in that case, the above lemma is much easier to prove because the matrix in (2.3.4.16) is the identity matrix. However, this may not be the case in general; see also Remark 2.3.3.5.

2.3.5 Thickening spaces and $AS = TS$ theorem

In this subsection, we introduce the thickening spaces for the extension L/K and state the $AS = TS$ theorem. Since we have Fake-assumption 2.3.4.11, we will not be able to give meaningful definition of the thickening spaces. For precise definition in the equal and mixed characteristic case, please consult Section 3.3 and Definition 4.1.1.13, respectively. (The definition in Section 3.3 actually looks different from the construction below. This is because we need to solve another problem first, which is explained in Remark 2.3.6.4.)

We continue to assume Hypothesis 2.3.2.1.

Fake-definition 2.3.5.1. Let $a > 1$. We define the *(non-logarithmic) thickening space (of level a)*, denoted by $TS^a_{L/K}$, to be the rigid space associated to the algebra

$$\mathcal{O}^a_{TS,L/K} = K\langle \pi_K^{-a}\delta_{J^+}\rangle \otimes_{\psi_K,K} L.$$

Similar, for $a > 0$, we defined the *logarithmic thickening space (of level a)*, denoted by $TS^a_{L/K,\log}$ to be the rigid space associated to the algebra

$$\mathcal{O}^a_{TS,L/K,\log} = K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle \otimes_{\psi_K,K} L.$$

The thickening spaces are equipped with compatible projections Π to the polydiscs, which give rise to the following Cartesian diagram for $a > 0$.

$$\begin{array}{ccccc}
TS^{a+1}_{L/K} & \hookrightarrow & TS^a_{L/K,\log} & \hookrightarrow & TS_{L/K} \\
\downarrow \Pi & & \downarrow \Pi & & \downarrow \Pi \\
A^{m+1}_K[0,\theta^{a+1}] & \hookrightarrow & A^1_K[0,\theta^{a+1}] \times A^m_K[0,\theta^a] & \hookrightarrow & A^1_K[0,\theta) \times A^m_K[0,1)
\end{array}$$

where $TS_{L/K} = \cup_{a>0} TS^a_{L/K,\log}$.

Fake-theorem 2.3.5.2. We have an isomorphism of K-algebras:

$$\mathcal{O}^a_{AS,L/K} \simeq \mathcal{O}^a_{TS,L/K} \quad \text{if} \quad a > 1,$$
$$\mathcal{O}^a_{AS,L/K,\log} \simeq \mathcal{O}^a_{TS,L/K,\log} \quad \text{if} \quad a > 0.$$

The correct version of this theorem is proved in Theorems 3.4.2.2 and 4.1.2.2, respectively.

Remark 2.3.5.3. As discussed in Subsection 2.3.1, the real purpose of constructing the thickening spaces isomorphic to standard Abbes-Saito spaces is to replace the morphism $\pi : AS^a_{L/K} \to A^m_K[0,\theta^a]$ by the morphism $\Pi : TS^a_{L/K} = A^m_K[0,\theta^a] \times_{\psi^*_K,K} L \to A^m_K[0,\theta^a]$. The thickening has the obvious advantage that it is functorial on K, as can be easily seen from its Fake-Definition 2.3.5.1.

The idea of proving Theorem 2.3.5.2 is simply approximation. We may get some sense from the following.

Example 2.3.5.4. It is also good to see the difference between Abbes-Saito spaces and thickening spaces in an example. Consider the extension of $\mathbb{F}_p((x))$ given by $y^p - x^{p-1}y = x$. A standard Abbes-Saito space is given by

$$\{(u,\delta)||u| \leq 1, |\delta| < \theta^a, u^p - x^{p-1}u = x + \delta\},$$

whereas a thickening space is given by

$$\{(u,\delta)||u| \leq 1, |\delta| < \theta^a, u^p - (x+\delta)^{p-1}u = x + \delta\}.$$

In other words, an Abbes-Saito space $AS^a_{L/K}$ consists of the points which are close to the solutions to those equations; in contrast, a thickening space $TS^a_{L/K}$ consists of points which are solutions to some equations whose coefficients are close to the original equations. So it should not be surprising that they consist of the same set of points. This can be also justified by the non-vanishing of the Jacobian matrix in Lemma 2.3.4.15.

2.3.6 Interpretation by differential modules

In this subsection, we interpret the ramification breaks using differential modules. This naïve approach actually fails in the equal characteristic case, we defer this failure to later discussion. Also we give the ideal picture of realizing the ramification break as the maximum of the breaks with respect to the uniformizer and each element in the lifted p-basis.

As a reminder, we continue to assume Hypothesis 2.3.2.1.

Fake-construction 2.3.6.1. Consider the projection $\Pi : TS^a_{L/K} \to A^{m+1}_K[0, \theta^a]$; it is finite and étale (under Fake-assumption 2.3.4.11). Then $\mathcal{E} = \Pi_* \mathcal{O}_{TS,L/K}$ become a differential module over $A^{m+1}_K[0, \theta^a]$ with respect to $\partial/\partial\delta_j$ for $j \in J^+$; we call $\mathcal{E}$ the *differential module associated to L/K.*

Remark 2.3.6.2. In the equal characteristic case, the étaleness of thickening space over the polydisc is stated in a slightly different way, as in Subsection 3.3.4. In the mixed characteristic case, the étaleness of thickening space is in fact a nontrivial result, proved in Subsection 4.1.3.

Fake-assumption 2.3.6.3. Assume that the theory of differential modules from Chapter 1 applies to differential modules over $A_K^{m+1}[0, \theta^a]$ (even if K is of characteristic $p > 0$).

Remark 2.3.6.4. This is indeed a very problematic assumption. For example, $\frac{d^p}{dx^p}$ is zero on $K[x]$ if K is of characteristic $p > 0$. So the theory of differential modules does not work at all.

Our remedy carried out in the next chapter is to, roughly speaking, lift the whole picture from over K to over an annulus $A_E^1[\eta_0, 1)$, where $E = \mathrm{Frac} C_{\kappa_K}$ is the fraction field of a Cohen ring of κ_K, and η_0 is some real number in $(0, 1)$ close enough to 1. It turns out that taking the limit $\eta_0 \to 1$ as in Subsection 1.2.8 will give the ramification information for K.

Fake-theorem 2.3.6.5. Let L/K be a finite Galois extension. For $a > 1$, the following statements are equivalent.

(a) The ramification break $b(L/K) \leq a$.

(b) The number of geometric connected components $\#\pi_0^{\mathrm{geom}}(TS^{a'}_{L/K}) = [L : K]$ for any $a' > a$.

(c) The differential module $\mathcal{E}$ is trivial on $A_K^m[0, \theta^{a'}]$ for any $a' > a$.

(d) The generic radii of convergence of $\mathcal{E}$ at $A_K^m[\theta^{a'}, \theta^{a'}]$ is $\theta^{a'}$ for any $a' > a$.

(e) The generic radii of convergence of $\mathcal{E}$ at $A_K^m[\theta^a, \theta^a]$ is θ^a.

Proof. We will just give the idea of the proof since the notations here are not quite well-defined. In the equal characteristic case, this fake theorem is realized by Theorem 3.3.4.6; in the mixed characteristic case, we use a slight variant Theorem 4.1.4.4.

Now, we prove the theorem under Fake-assumption 2.3.6.3.

(a) $\Longleftrightarrow$ (b) follows from Fake-theorem 2.3.5.2.

(b) $\Longrightarrow$ (c) is true because the generic radii of convergence is not sensitive to base extension; we are just pushing forward geometrically $[L:K]$ copies of $A_K^1[0,\theta^{a'}]$.

(c) $\Longrightarrow$ (b) is less trivial. Note that for any $a' > a$, the Taylor series induces a *ring* isomorphism

$$\mathcal{E}|_{\delta_{J^+}=0} \otimes_K L \to H^0_\nabla(A_K^m[0,\theta^{a'}],\mathcal{E}) \otimes_K L \subset \mathcal{O}_{TS,L/K,a'} \otimes_K L. \tag{2.3.6.6}$$

But we know the left hand of (2.3.6.6) is isomorphic to $L \otimes L \xrightarrow{\sim} \prod_{g\in G} L$. So the idempotent elements on the left hand side of (2.3.6.6) map to idempotent elements on the right hand side; this forces $\#\pi_0(TS^{a'}_{L/K} \times_K L) = [L:K]$.

(d) $\Longrightarrow$ (c) can be obtained by constructing horizontal sections using Taylor series. Conversely, (c) $\Longrightarrow$ (d) is trivial.

(d) $\Longleftrightarrow$ (e) follows from Theorem 1.2.4.4(a) and (c). □

Remark 2.3.6.7. The differential module $\mathcal{E}$ is defined over $A_K^1[0,\theta) \times A_K^m[0,1)$. We know that the geometric connected components of $TS^a_{L/K} = \Pi^{-1}(A_K^{m+1}[0,\theta^a])$ determine the ramification break. The key observation is that it may happen that when $a_{j_0} < b(L/K)$ for some j_0 and $a_j > b(L/K)$ for $j \in J^+\backslash\{j_0\}$, $\Pi^{-1}\big(A_K^1[0,\theta^{a_0}] \times \cdots \times A_K^1[0,\theta^{a_m}]\big)$ still have $[L:K]$ geometric connected components. This is because the generic radii of $\frac{\partial}{\partial\delta_{j_0}}$ for a single j_0 can be bigger than $\theta^{b(L/K)}$.

Fake-assumption 2.3.6.8. Ideally, for each $j \in J^+$, there should be a ramification break $b_j(L/K)$ associated to b_j ($j \in J$) or π_K ($j = 0$), which is given by the log of generic radii of $\frac{\partial}{\partial\delta_j}$ with θ as base. By Theorem 2.3.6.5, the ramification break $b(L/K) = \max_{j\in J^+}\{b_j(L/K)\}$.

Remark 2.3.6.9. This fake assumption is again wrong in two different ways for equal or mixed characteristic cases.

In the mixed characteristic case, aside from the problem in Fake-assumption 2.3.4.11, Fake-assumption 2.3.6.8 is not valid because we cannot use a single number to measure the generic radii over a polydisc; when the radii of the polydisc vary, the generic radii will change. The same phenomenon, however, does not show up in the equal characteristic case, because as we will show in Sections 3.2 and 3.3 that the differential module $\mathcal{E}$ is a pull-back of a differential module over an annulus and the generic radii of $\frac{\partial}{\partial \delta_j}$ does not change as the radii vary.

In the equal characteristic case, this is equivalent to fixing Fake-assumption 2.3.6.3. See Subsection 3.2.4 for more details.

Definition 2.3.6.10. we say b_j for $j \in J$ (resp. π_K) is *dominant* if $b(L/K)$ is the same as $b_j(L/K)$ (resp. $b_0(L/K)$).

Remark 2.3.6.11. In the classical case, there is no p-basis and hence π_K is always dominant. The inseparable residue field extension causes a possibility that π_K might no longer be dominant but some b_j dominates the ramification break of L/K.

2.3.7 Adding generic roots

We study the behavior of thickening spaces, differential modules, and ramification breaks under the operation of adding generic p-th or p^∞-th roots (Definition 2.3.2.7).

Fake-assumption 2.3.7.1. Pretend that the continuous homomorphism ψ_K in Fake-assumption 2.3.4.11 is functorial in the following sense: If $\widetilde{K}$ is obtained by adding a generic p^∞-th (resp. p-th) root of b_{j_0} for an element in the lifted p-basis of K, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_K & \xrightarrow{\psi_K} & \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!] \\ \big\downarrow & & \big\downarrow f \\ \mathcal{O}_{\widetilde{K}} & \xrightarrow{\psi_{\widetilde{K}}} & \mathcal{O}_{\widetilde{K}}[\![\eta_0/\pi_K, \eta_{J\backslash\{j_0\}}, \eta_{m+1}]\!] \end{array} \qquad \left(\text{resp. } \begin{array}{ccc} \mathcal{O}_K & \xrightarrow{\psi_K} & \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!] \\ \big\downarrow & & \big\downarrow f \\ \mathcal{O}_{\widetilde{K}} & \xrightarrow{\psi_{\widetilde{K}}} & \mathcal{O}_{\widetilde{K}}[\![\delta_0/\pi_K, \eta_J, \eta_{m+1}]\!] \end{array} \right)$$

where $\psi_{\widetilde{K}}$ is constructed by Fake-assumption 2.3.4.11 for $\widetilde{K}$, with respect to the uniformizer π_K and lifted p-basis $\{b_{J\backslash\{j_0\}}, x\}$ (resp. $\{b_{J\backslash\{j_0\}}, (b_{j_0} + x\pi_K)^{1/p}, x\}$), and where

$$\begin{aligned} f^*(\delta_j) &= \eta_j, \quad \text{for } j \in J^+ \backslash \{j_0\}, \\ f^*(\delta_{j_0}) &= (x + \eta_{m+1})(\pi_K + \eta_0) - x\pi_K \\ (\text{resp. } f^*(\delta_{j_0}) &= \left((b_{j_0} + x\pi_K)^{1/p} + \eta_{j_0}\right)^p - (x + \eta_{m+1})(\pi_K + \eta_0) - b_{j_0}) \quad). \end{aligned}$$

The homomorphism f^* induces a K-morphism of rigid spaces $A_{\widetilde{K}}^{m(+1)}[0, \theta^a] \to A_K^m[0, \theta^a]$ if we add a generic p^∞-th root (resp. p-th root).

Remark 2.3.7.2. When K is of mixed characteristic, we need to invent the notion of approximate commutative diagram (Defintion 4.1.1.7) to save this Fake-assumption. See Lemma 4.2.1.4.

Fake-proposition 2.3.7.3. The differential module associated to $\widetilde{L}/\widetilde{K}$ is exactly $f^*\mathcal{E}$ with f defined as above.

Proof. This follows immediately from the following Cartesian diagram (under the Fake-assumptions 2.3.4.11 and 2.3.7.1) for adding a generic p^∞-th (resp. p-th) root.

$$\begin{array}{ccccccc} TS^a_{L/K} & = & A_K^m[0,\theta^a] \times_{\psi_K, K} L & \xleftarrow{f \times 1} & A_{\widetilde{K}}^{m(+1)}[0,\theta^a] \times_{\psi_{\widetilde{K}}, \widetilde{K}} \widetilde{L} & = & TS^a_{\widetilde{L}/\widetilde{K}} \\ & & \downarrow \Pi & & \Pi \downarrow & & \\ & & A_K^m[0,\theta^a] & \xleftarrow{\quad f \quad} & A_{\widetilde{K}}^{m(+1)}[0,\theta^a] & & \end{array}$$

□

Remark 2.3.7.4. In the mixed characteristic case, the above fake-proposition is the core of the proof of the Hasse-Arf theorem. See Theorem 4.2.1.7.

Fake-proposition 2.3.7.5. The action of differential operations on $f^*\mathcal{E}$ are related to the action of differential operators on $\mathcal{E}$ as follows.

$$\frac{\partial}{\partial \eta_j} = \frac{\partial}{\partial \delta_j} \quad \text{for all } j \in J\backslash\{j_0\},$$
$$\frac{\partial}{\partial \eta_0} = \frac{\partial}{\partial \delta_0} + (x + \eta_{m+1})\frac{\partial}{\partial \delta_{j_0}},$$
$$\frac{\partial}{\partial \eta_{m+1}} = (\pi_K + \eta_0) \cdot \frac{\partial}{\partial \delta_{j_0}},$$
$$\frac{\partial}{\partial \eta_{j_0}} = p\big((b_{j_0} + x\pi_K)^{1/p} + \eta_{j_0}\big)^{p-1}\frac{\partial}{\partial \delta_{j_0}} \quad \text{(if adding a generic } p\text{-th root)}.$$

Proof. It follows immediately from the expression of f^* in Fake-assumption 2.3.7.1. □

Fake-proposition 2.3.7.6. Under the Fake-assumption 2.3.6.8, the ramification breaks $b_j(L/K)$ for each j vary as follows under the operation of adding a generic p^∞-th or p-th root of $b_{j_0}(L/K)$.

$$b_j(\widetilde{L}/\widetilde{K}) = b_j(L/K) \quad \text{for all } j \in J\backslash\{j_0\},$$
$$b_0(\widetilde{L}/\widetilde{K}) = \max\{b_0(L/K), b_{j_0}(L/K)\},$$
$$b_{m+1}(\widetilde{L}/\widetilde{K}) = b_{j_0}(L/K) - 1,$$
$$b_{j_0}(\widetilde{L}/\widetilde{K}) < b_{j_0}(L/K) \quad \text{(if adding a generic } p\text{-th root)}.$$

In particular,

$$b(\widetilde{L}/\widetilde{K}) = \max_{j\in J^+\cup\{m+1\}}\{b_j(\widetilde{L}/\widetilde{K})\} = \max_{j\in J^+}\{b_j(L/K)\} = b(L/K). \tag{2.3.7.7}$$

Proof. It follows from previous proposition. □

Remark 2.3.7.8. If b_{j_0} is dominant, as a result of adding generic p^∞-th or p-th root, π_K becomes dominant. In the language of Remark 2.3.6.11, we return to a situation closer to the the classical case.

Fake-theorem 2.3.7.9. Let K be a complete discretely valued field and let G_K be its absolute Galois group. Assume that K is not absolutely unramified if K is of mixed characteristic. Let $\rho : G_K \to GL(V_\rho)$ be a continuous representation of finite local monodromy. Then the Artin conductor $\mathrm{Art}(\rho) \in \mathbb{Z}_{\geq 0}$. Moreover, the graded piece $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ of the ramification filtration are trivial if $a \notin \mathbb{Q}$ and are abelian groups killed by p if $a \in \mathbb{Q}_{>1}$.

Proof. It follows from Proposition 2.3.2.13 and Fake-proposition 2.3.7.6, which are subject to numerous fake-assumptions. The restriction of K not being absolutely unramified and of mixed characteristic reflects the complete failure of rescuing Fake-assumption 2.3.4.11. □

2.3.8 Integrality of Swan conductors

In this subsection, we introduce a dichotomy for the relation between non-log and log ramification breaks. Then from this ideal situation, we deduce the fake proof of the integrality of Swan conductors.

Fake-assumption 2.3.8.1. Continuing with Fake-assumption 2.3.6.8, we assume that the logarithmic ramification break is computed by $b_{\log}(L/K) = \max\{b_0(L/K) - 1, b_j(L/K); j \in J\}$. We say b_j for $j \in J$ (resp. π_K) is *log-dominant* if $b_{\log}(L/K)$ is the same as $b_j(L/K)$ (resp. $b_0(L/K) - 1$).

Moreover, we assume the following behavior of the logarithmic breaks under tame base change. Fix $n \in \mathbb{N}$ prime to p and let $K_n = K(\pi_K^{1/n})$ and $L_n = LK_n$. Let $\pi_{K_n} = \pi_K^{1/n}$ be the uniformizer of K_n and we continue to take b_J as the set of lifted p-basis of K_n. Then, we assume that $b_0(L_n/K_n) = nb_0(L/K) - (n-1)$ and $b_j(L_n/K_n) = nb_j(L/K)$ for $j \in J$.

Remark 2.3.8.2. The motivation of the fake assumption is that it is true (under modification) in the equal characteristic case. See subsection 3.2.4. In the mixed characteristic case, we actually use a slightly different strategy because we do not have well-defined $b_j(L/K)$ as explained in Remark 2.3.6.9. See also Subsection 4.3.1 for the actual proof, which is a variant of the one proposed here.

Fake-theorem 2.3.8.3 (Dichotomy of Swan conductors). Let L/K be a finite Galois extension of complete discretely valued field, satisfying Hypothesis 2.3.2.1. There is a dichotomy for the relation between the non-log and log ramification breaks as follows.

(1) Either π_K is log-dominant, in which case $b(L/K) = b_{\log}(L/K) + 1$;

(2) or π_K is not log-dominant, in which case there exists $N \in \mathbb{N}$ such that for any integer $n > N$ prime to p, $b(L_n/K_n) = b_{\log}(L_n/K_n) = nb_{\log}(L/K)$.

Proof. We deduce this theorem by combining Fake-assumptions 2.3.6.8 and 2.3.8.1. Indeed, if π_K is log-dominant, $b_0(L/K) > b_j(L/K)$ for all $j \in J$. Hence, $b(L/K) = b_0(L/K) = b_{\log}(L/K) + 1$. If π_K is not log-dominant, b_j is dominant for some $j \in J$. In other words, there exists some $N \in \mathbb{N}$ such that $b_j(L/K) > b_0(L/K) - 1 + 1/N$. Then, for any integer $n > N$ prime to p,

$$b_0(L_n/K_n) = nb_0(L/K) - (n-1) = n\big(b_0(L/K) - (n-1)/n\big) < nb_j(L/K) = b_j(L_n/K_n).$$

Hence, $b(L_n/K_n) = b_j(L_n/K_n) = b_{\log}(L_n/K_n) = nb_{\log}(L/K)$. □

Fake-theorem 2.3.8.4. Let K be a complete discretely valued field and let G_K be its absolute Galois group. Let $\rho : G_K \to GL(V_\rho)$ be a continuous representation of finite local monodromy. Then the Swan conductor $\mathrm{Swan}(\rho) \in \mathbb{Z}_{\geq 0}$.

Proof. Similarly to the reduction steps in Proposition 2.3.2.13, we may reduce to the case when ρ is induced by a faithful irreducible representation of $\mathrm{Gal}(L/K)$ of a finite Galois extension L/K satisfying Hypothesis 2.3.2.1. We first observe that to prove $\mathrm{Swan}(\rho) \in \mathbb{Z}$, it suffices to show that for two integers $n_1, n_2 \in \mathbb{Z}$ coprime and both prime to p, $\mathrm{Swan}(\rho|_{G_{K_{n_1}}}), \mathrm{Swan}(\rho|_{G_{K_{n_2}}})$ are both integers, where $K_{n_i} = K(\pi_K^{1/n_i})$ is a tamely ramified extension of K. Indeed, knowing $\mathrm{Swan}(\rho|_{G_{K_{n_1}}}) = n_1\mathrm{Swan}(\rho) \in \mathbb{Z}$ and $\mathrm{Swan}(\rho|_{G_{K_{n_2}}}) = n_2\mathrm{Swan}(\rho) \in \mathbb{Z}$ would imply that $\mathrm{Swan}(\rho) \in \mathbb{Z}$.

Therefore, we may assume that K is not absolutely unramified in the mixed characteristic case. Now, we use the dichotomy Fake-theorem 2.3.8.3. If π_K is log-dominant, $b_{\log}(L/K) = b(L/K) - 1$ and $\mathrm{Swan}(\rho) = \mathrm{Art}(\rho) - \dim(\rho)$. The theorem follows from Fake-theorem 2.3.7.9. If π_K is not log-dominant, we take $n_1, n_2 \in \mathbb{Z}_{>N}$

such that $p \nmid n_1 n_2$. By Fake-theorem 2.3.8.3,

$$n_1 \mathrm{Swan}(\rho) = \mathrm{Swan}(\rho|_{K_{n_1}}) = \mathrm{Art}(\rho|_{K_{n_1}}) \in \mathbb{Z},$$
$$n_2 \mathrm{Swan}(\rho) = \mathrm{Swan}(\rho|_{K_{n_2}}) = \mathrm{Art}(\rho|_{K_{n_2}}) \in \mathbb{Z}.$$

This also implies that $\mathrm{Swan}(\rho) \in \mathbb{Z}$. □

Remark 2.3.8.5. As we stated in Theorem 2.2.2.19 that when $p = 2$ and K is of mixed characteristic, we have a little trouble in proving strong integrality for Swan conductors. One can compare the above fake proof with the actual proof of Theorem 4.3.1.14.

2.4 Borger's conductors

In this section, we first review Borger's definition of Artin conductors by generic perfection, following [Bor04]. Then, we prove the comparison theorem linking this to arithmetic and differential conductors.

2.4.1 Borger's definition

In this subsection, we review the definition of Borger's Artin conductors following [Bor04].

Let K be a complete discretely valued field with residue field κ_K. Assume κ_K is of characteristic $p > 0$. We do not impose Hypothesis 2.3.2.1 on K.

Definition 2.4.1.1. An $\mathbb{F}_p$-algebra R is called *perfect* if $F : x \mapsto x^p$ is an isomorphism. For a $\mathbb{F}_p$-algebra R, we use $R^{\mathrm{pf}} = \cup_{n\in\mathbb{N}} R^{1/p^n}$ to denote its *perfection.* Let $\mathtt{CRP}_{\mathcal{O}_K}$ be the subcategory of the category of $\mathcal{O}_K$-algebras consisting of flat $\mathcal{O}_K$-algebras A, complete with respect to $\mathfrak{m}_K$-adic topology and $A/\mathfrak{m}_K A$ is perfect.

Proposition 2.4.1.2. [Bor04, Theorem 1.4] *The category has an initial object $\mathcal{O}_K^u$, the* universal residual perfection *of $\mathcal{O}_K$. We have an equivalence of categories*

$$\mathtt{CRP}_{\mathcal{O}_K} \xrightarrow{\sim} \mathtt{PerfAlg}_{\overline{\mathcal{O}_K^u}}, \quad A \mapsto A/\mathfrak{m}_K A,$$

where $\mathtt{PerfAlg}_{\overline{\mathcal{O}_K^u}}$ *is the category of perfect $\mathcal{O}_K^u/\mathfrak{m}_K\mathcal{O}_K^u$-algebras.*

Definition 2.4.1.3. Let $\mathcal{O}_K^g$ be the inverse image of $\mathrm{Frac}(\mathcal{O}_K^u/\mathfrak{m}_K\mathcal{O}_K^u)$, called the *generic residual perfection* of $\mathcal{O}_K$. Denote $K^g = \mathrm{Frac}(\mathcal{O}_K^g)$. By Proposition 2.4.1.2, $\mathcal{O}_K^g$ is a complete discrete valuation ring with perfect residue field.

We have a homomorphism of Galois groups $G_{K^g} \to G_K$. Thus, given a representation ρ of G_K, we define the *Borger's Artin conductor* $\mathrm{Art}_B(\rho)$ to be $\mathrm{Art}(\rho|_{G_{K^g}})$, where the latter term is the classical definition [Ser79]. (See also Definition 2.2.2.14.)

Obviously, Borger's Artin conductors have a Hasse-Arf property naturally inherited from the one for K^g, a complete discrete valuation field with perfect residue field.

Proposition 2.4.1.4. [Bor04, Theorem A] *Keep the notation as above,* $\mathrm{Art}_B(\rho)$ *is a nonnegative integer and it coincides with classical definition when the residue field κ_K is perfect.*

[Bor04, Proposition 2.3] *Furthermore,* $\mathrm{Art}_B(\rho)$ *is unchanged after a (not necessarily finite) unramified complete extension of K.*

Moreover, Borger proved that his definition coincides with a variant of arithmetic Artin conductor Art_K for characters using the language of [Kat89a]. The author is not sure if this variant fits into Abbes and Saito's definition.

Proposition 2.4.1.5. [Bor04, Theorem B] *If χ is a class in $H^1(G_K, \mathbb{Q}/\mathbb{Z})$ and χ' is its image in $H^1(G_{K^g}, \mathbb{Q}/\mathbb{Z})$, then* $\mathrm{Art}_K(\chi) = \mathrm{Art}_K(\chi')$. *In particular, for a rank one representation ρ of G_K,* $\mathrm{Art}_K(\rho) = \mathrm{Art}_B(\rho)$.

Borger gives the following explicit descriptions of K^u and K^g.

Proposition 2.4.1.6. *Let b_J be a set of lifted p-basis of K.*

If K is of equal characteristic $p > 0$, $K^u = \kappa_K[v_{i,j}; j \in J, i \in \mathbb{Z}_{>0}]^{\mathrm{pf}}((\pi_{K^u}))$. The homomorphism $K \to K^u$ is determined by $\pi_K \mapsto \pi_{K^u}$ and $b_j \mapsto b_j + \sum_{i>0} v_{i,j}\pi_{K^u}^i$. $K^g = \mathrm{Frac}(\kappa_K[v_{i,j}; j \in J, i \in \mathbb{Z}_{>0}]^{\mathrm{pf}})((\pi_{K^g}))$ and the homomorphism $K \to K^g$ is given by composing $K \hookrightarrow K^u$ with the natural morphism $K^u \hookrightarrow K^g$, sending π_{K^u} to π_{K^g}.

If K is of mixed characteristic, We have a Cohen ring C_{κ_K} with respect to the lifted p-basis b_J. Let W be the Witt vectors of $\kappa_K[v_{i,j}; j \in J, i \in \mathbb{Z}_{>0}]^{\mathrm{pf}}$ and W^g the Witt vectors of $\mathrm{Frac}(\kappa_K[v_{i,j}; j \in J, i \in \mathbb{Z}_{>0}]^{\mathrm{pf}})$. We have $K^u = K \otimes_{C_{\kappa_K}} W$ and $K^g = K \otimes_{C_{\kappa_K}} W^g$. The homomorphisms $K \to K^u \hookrightarrow K^g$ is determined by $b_j \to [\bar{b}_j]_W + \sum_{i>0}[v_{i,j}]_W\pi_{K^u}^i$, where $[\cdot]_W$ denotes the Teichmüller lift.

2.4.2 Comparison theorem

A key ingredient to prove the comparison between Borger's Artin conductors and arithmetic non-logarithmic conductors is to study how arithmetic non-logarithmic conductors behave under the operations of adding generic p^∞-th roots.

In this subsection, we do not impose any Hypothesis on K.

Proposition 2.4.2.1. *Assume that the highest non-logarithmic ramification breaks $b(L/K)$ are invariant under the operation of adding a generic p^∞-th root if*

(a) *either K is of equal characteristic and L/K verifies Hypothesis 2.3.2.1,*

(b) *or K is of mixed characteristic with a fixed absolute ramification degree β_K and L/K verifies Hypothesis 2.3.2.1.*

Then, we have for all such K and all representation ρ of G_K of finite local monodromy, $\mathrm{Art}_B(\rho) = \mathrm{Art}(\rho)$.

Proof. Let ρ be a representation of G_K of finite local monodromy. Without lost of generality, we may assume that κ_K is separably closed because both conductors stay the same under unramified complete extension (Propositions 2.2.2.11(4) and 2.4.1.4). Thus, we may assume that ρ exactly factors through the Galois group $G_{L/K}$ of a finite totally ramified Galois extension L/K. Moreover, we may assume that ρ is irreducible. As K^g has perfect residue field, $\mathrm{Art}_B(\rho|_{G_{K^g}}) = \mathrm{Art}(\rho|_{G_{K^g}})$ are all the same as the classical definition (Propositions 2.2.2.11(8) and 2.4.1.4). We need only to show that $\mathrm{Art}(\rho) = \mathrm{Art}(\rho|_{G_{K^g}})$.

Similar to the proof of Theorem 2.3.2.13, one may add a p^∞-root of some elements of the lifted p-basis into K without changing the Abbes-Saito conductors and hence reduce to the case of finite p-basis. In other words, there exists $K \hookrightarrow K_1 = K(b_j^{p^{-n}} | j \in J \backslash J_0, n \in \mathbb{N})^\wedge$ for some $J_0 \subset J$ and $\#J_0 < +\infty$, such that $\mathrm{Art}(\rho) = \mathrm{Art}(\rho|_{G_{K_1}})$. We easily see that there exists $K_1 \to K^g$ extending $K \to K^g$.

By Proposition 2.3.2.10, we can do finitely many operations of adding generic p^∞-th roots and make sure that the result field extension K_2L/K_2 has separable residue field extension but $\mathrm{Art}(\rho|_{G_{K_1}}) = \mathrm{Art}(\rho|_{G_{K_2}})$. We also have to show that we have a homomorphism $K_2 \hookrightarrow K^g$ extending $K_1 \hookrightarrow K^g$, for which we go back to the proof of Proposition 2.3.2.10 and construct the homomorphism step by step.

The r-th $(1 \leq r \leq r_0)$ step of adding generic p^∞-th roots is to construct

$$K_1^{(r)} = \Big(K_1^{(r-1)}(u_{r,J_0})\big((u_{r-1,j} + u_{r,j}\pi_K)^{1/p^n};\ j \in J_0, n \in \mathbb{N}\big)\Big)^\wedge.$$

where $u_{0,j} = b_j$, $\forall j \in J_0$ and $K_1^{(0)} = K_1$. In equal characteristic case, we map

$$u_{r,j} \mapsto \sum_{r' \geq r} v_{r',j}\pi_{K^g}^{r'-r}, \quad \forall j \in J_0, r = 1, \ldots, r_0;$$

in mixed characteristic case, we map

$$u_{r,j} \mapsto \sum_{r' \geq r} [v_{r',j}]_W \pi_{K^g}^{r'-r}, \quad \forall j \in J_0, r = 1, \ldots, r_0.$$

One checks easily that this map extends to a homomorphism $K_2 \hookrightarrow K^g$.

Now, K_2L/K_2 has naïve ramification the same as the degree, so $\mathcal{O}_{K^gL} = \mathcal{O}_{K^g} \otimes_{\mathcal{O}_{K_2}} \mathcal{O}_{K_2L}$. Hence, $\mathrm{Art}(\rho|_{G_{K_2}}) = \mathrm{Art}(\rho|_{G_{K^g}})$ via Proposition 2.2.2.11(4'). □

Remark 2.4.2.2. Be careful that the above proposition may not hold if we change generic p^∞-th roots to generic p-th roots. This is because, in a sense, Borger's Artin conductors are not quite generic as they do not allow rotations like $\big((b_j + u\pi_K)^{1/p} + v\pi_K\big)^{1/p}$ for a dummy variable v at the next step; it only allows the specialization to $v = 0$. This is not a problem if Fake-assumption 2.3.6.8 were true, because, after the first rotation, the direction corresponds to $(b_j + u\pi_K)^{1/p}$ is not dominant so we can forget about this direction and obtain a comparison theorem. However, we can only save Fake-assumption 2.3.4.11 in the equal characteristic case (see Remark 2.3.6.9). In the mixed characteristic case, we do not know if we can obtain a comparison theorem with Borger's conductors. Nevertheless, we do not expect that specializing to $v = 0$ would mess up the conductors.

2.5 Hasse-Arf theorem for finite flat group schemes

2.5.1 Ramification filtration for finite flat group schemes

We first recall some definitions and basic properties from [AM04] and [Hat08]. Then, we use a theorem by Raynaud [BBM82, Théorèm 3.1.1] to reduce the integrality result to the case of finite Galois extension of complete discretely valued fields.

Keep the notation as in previous sections. We do not assume any hypothesis on K (and there will be no L in this subsection).

Convention 2.5.1.1. All finite flat groups schemes are commutative.

The construction of the canonical filtration on a generically étale finite flat group scheme is similar to that of the arithmetic ramification filtration.

Definition 2.5.1.2. Let A be a finite flat $\mathcal{O}_K$-algebra. Write $A = \mathcal{O}_K[x_1, \ldots, x_n]/\mathcal{I}$ with $\mathcal{I}$ an ideal generated by $f_1, \ldots, f_r$. For $a \in \mathbb{Q}_{>0}$, define the rigid analytic space

$$X^a = \big\{(x_1, \ldots, x_n) \in A^n_K[0,1] \,\big|\, |f_\alpha(x_1, \ldots, x_n)| \leq \theta^a, \alpha = 1, \ldots, r\big\},$$

where $\theta = |\pi_K|$ as in Notation 2.2.2.2. The *highest break* $b(A/\mathcal{O}_K)$ is the smallest number such that for all $a > b(A/\mathcal{O}_K)$, $\#\pi_0^{\mathrm{geom}}(X^a) = \mathrm{rank}_{\mathcal{O}_K} A$. This is the same as Definition 2.2.2.3 if $A = \mathcal{O}_L$, except here we use the ring of integers instead of the fields in the notation.

Lemma 2.5.1.3. [AM04, Lemme 2.1.5] *Let K'/K be a (not necessarily finite) extension of complete discrete valuation fields of naïve ramification degree e. Let A be a finite flat $\mathcal{O}_K$-algebra which is a complete intersection relative to $\mathcal{O}_K$. Put $A' = A \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$; then $b(A'/\mathcal{O}_{K'}) = e \cdot b(A/\mathcal{O}_K)$.*

Notation 2.5.1.4. For a finite flat group scheme $G = \operatorname{Spec} A$, it is *generically étale* if $G \times_{\mathcal{O}_K} K$ is étale over K; it is *generically trivial* if $G \times_{\mathcal{O}_K} K$ is a disjoint union of copies of $\operatorname{Spec} K$.

Definition 2.5.1.5. For a geometrically étale finite flat group scheme $G = \operatorname{Spec} A$, we have a natural map of points $G(K^{\mathrm{alg}}) \hookrightarrow X^a(K^{\mathrm{alg}})$; further composing with the map for geometric connected components, we obtain a map

$$\sigma^a : G(K^{\mathrm{alg}}) \hookrightarrow X^a(K^{\mathrm{alg}}) \to \pi_0^{\mathrm{geom}}(X^a).$$

Define G^a to be the closure of $\operatorname{Ker} \sigma^a$. We use $b(G/\mathcal{O}_K)$ to denote the highest break $b(A/\mathcal{O}_K)$; then for $a > b(G/\mathcal{O}_K)$, $G^a = \operatorname{Spec} \mathcal{O}_K$.

Proposition 2.5.1.6. [AM04, Lemme 2.3.2] *Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of finite flat group schemes. Then for $a > 0$, $0 \to G'^a \to G^a \to G''^a \to 0$ is exact.*

2.5.2 Hasse-Arf theorem for finite flat group schemes

The following question is first raised in [Hat08]; and its proof is essentially due to Hattori. The author would like to thank him for clarifying this and the permission to include the proof here.

Theorem 2.5.2.1. *Assume that Theorem 2.2.2.19 holds. Let $\mathcal{O}_K$ be a complete discrete valuation ring. For any generically trivial finite flat groups scheme G over $\mathcal{O}_K$, $b(G/\mathcal{O}_K)$ is a nonnegative integer.*

Proof. Since Theorem 2.2.2.19 requires that $\beta_K > 1$ when K is of mixed characteristic, we need to get around this restriction first. Let $n_1, n_2 \in \mathbb{N}$ be two coprime numbers such that $p \nmid n_1 n_2$. Let K_{n_1} and K_{n_2} be two tamely ramified extensions of K with ramification degree n_1 and n_2, respectively. By Lemma 2.5.1.3, it suffices to prove the theorem for $G \times_{\mathcal{O}_K} \mathcal{O}_{K_{n_1}}/\mathcal{O}_{K_{n_1}}$ and $G \times_{\mathcal{O}_K} \mathcal{O}_{K_{n_2}}/\mathcal{O}_{K_{n_2}}$, respectively. Thus, we may assume that $\beta_K \geq 2$ when K is of mixed characteristic.

We may assume that G is connected by taking the connected component of the identity. By a theorem of Raynaud [BBM82, Théorèm 3.1.1], we may realize G as the kernel of an isogeny $f : \mathfrak{B} \to \mathfrak{A}$ of two abelian schemes over $\operatorname{Spec} \mathcal{O}_K$. Let α and β be generic points of the special fibers of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then by [AM04, Lemme 2.1.6], $b(\mathcal{O}^{\wedge}_{\mathfrak{B},\beta}/\mathcal{O}^{\wedge}_{\mathfrak{A},\alpha}) = b(G/\mathcal{O}_K)$.

Since the generic fiber of G is a disjoint union of copies of $\operatorname{Spec} K$, we know that $\mathcal{O}^{\wedge}_{\mathfrak{B},\beta}/\mathcal{O}^{\wedge}_{\mathfrak{A},\alpha}$ is a generically étale finite Galois extension of complete discrete valuation rings, with Galois group $G(K)$; in particular, all irreducible representations of this Galois group over an algebraically closed field are one dimensional. By Hasse-Arf Theorem 2.2.2.19, $b(\mathcal{O}^{\wedge}_{\mathfrak{B},\beta}/\mathcal{O}^{\wedge}_{\mathfrak{A},\alpha}) = b(G/\mathcal{O}_K)$ is an integer. $\square$

Chapter 3

Ramification Theory for Local Fields: Equal Characteristic Case

Plan of this chapter

The aim of this chapter is to prove the Hasse-Arf Conjecture 2.2.2.17 in the equal characteristic. The idea is that we first define the differential conductors which do have the desired properties and then we should that they are in fact the same.

In Section 3.1, we make a construction, which lifts a rigid space over K to a rigid space over an annulus over K. In particular, we prove that the connected components of the original rigid space are in one-to-one correspondence with the connected components of the lifting space, when the annulus is "thin" enough. This construction suggests strong connection to Berthelot's construction of rigid cohomology.

In Section 3.2, we discuss how to associate a differential module $\mathcal{E}_\rho$ to a representation ρ of G_K of finite local monodromy. Then we introduce differential Artin and Swan conductors following [Ked07a] and discuss their properties in Subsection 3.2.3. In Subsection 3.2.4, we introduce a calculation of breaks by p-basis. In Subsection 3.2.5, we discuss refined Swan conductors.

In Section 3.3, we introduce a thickening construction. In Subsection 3.3.1, as an example, we first construct the thickening space when K can be realized geometrically. Then in Subsection 3.3.2, we define the thickening spaces for general K and discuss spectral properties of the differential module obtained by pulling back $\mathcal{E}_\rho$ to the thickening spaces. In Subsections 3.3.3 and 3.3.4, we link the (highest) differential breaks and spectral norms with the connected components of certain base change of the thickening spaces.

In Section 3.4, we first define the lifts $\widetilde{AS}^a_{L/K}$ of standard Abbes-Saito spaces $AS^a_{L/K}$. Then we prove in Subsection 3.4.2 that the lifted Abbes-Saito spaces and (the base change of) the thickening spaces are isomorphic (Theorem 3.4.2.2). From this, in Subsection 3.4.3, we deduce our main Theorem 3.4.3.1: the differential conductors coincide with the arithmetic conductors; the Hasse-Arf theorem for arithmetic conductors follows.

3.1 Lifting rigid spaces

In this section, we introduce a construction, which lifts a rigid space over a field of characteristic $p > 0$ to a rigid space over an annulus over a field of characteristic zero.

Most of the content in this section should be credited to Kedlaya. The author would like to thank him for allowing to include the proofs.

3.1.1 A Gröbner basis argument

In this subsection, we introduce a division algorithm using Gröbner basis, which enables us to find a representative in the quotient ring achieving the quotient norm.

Hypothesis 3.1.1.1. Let F be a complete discretely valued field of mixed characteristic $(0,p)$, with ring of integers $\mathcal{O}_F$ and residue field κ. We fix a uniformizer π_F of F.

Notation 3.1.1.2. Fix a positive integer n, and put

$$R^{\mathrm{int}} = \mathcal{O}_F\langle u_1, \dots, u_n\rangle((S)),$$

$$R = R^{\mathrm{int}} \otimes_{\mathcal{O}_F} F,$$

$$R_\kappa = R^{\mathrm{int}} \otimes_{\mathcal{O}_F} \kappa \cong \kappa[u_1, \dots, u_n]((S)) = \kappa((S))\langle u_1, \dots, u_n\rangle.$$

For $\eta \in (0,1]$, let $|\cdot|_\eta$ (for short) denote the $(1,\dots,1,\eta)$-Gauss norm on R.

Definition 3.1.1.3. We equip R_κ with the lexicographic term ordering induced by the correspondence $u_1^{i_1}\cdots u_n^{i_n}S^j \mapsto (-j,i_1,\dots,i_n)$, i.e., we write $\bar{\alpha}u_1^{i_1}\cdots u_n^{i_n}S^j \succeq \bar{\beta}u_1^{i'_1}\cdots u_n^{i'_n}S^{j'}$ if $(-j,i_1,\dots,i_n)\succeq(-j',i'_1,\dots,i'_n)$ under the lexicographic order (Notation 1.1.1.2), where $\bar{\alpha},\bar{\beta}\in\kappa^\times$.

Using this ordering, we define the *leading term* $\mathrm{lead}(\bar{f})$ of a nonzero element $\bar{f}\in R_\kappa$ to be its largest term under the ordering. In particular, for $\bar{f},\bar{g}\in R_\kappa\backslash\{0\}$, $\mathrm{lead}(\bar{f}\bar{g})=\mathrm{lead}(\bar{f})\mathrm{lead}(\bar{g})$.

For an ideal I_κ of R_κ, a *Gröbner basis* of I_κ is a finite subset $\{\bar{r}_1,\dots,\bar{r}_m\}\subset I_\kappa$ such that $\mathrm{lead}(\bar{r}_i)$ do not have exponents on S and the ideal consisting of the leading terms of all elements of I_κ is generated by $\mathrm{lead}(\bar{r}_1),\dots,\mathrm{lead}(\bar{r}_m)$. Such a basis exists because R_κ is noetherian. By [Eis95, Lemma 15.5], $\bar{r}_1,\dots,\bar{r}_m$ also generate I_κ.

Proposition 3.1.1.4. *For any $\bar{f}\in R_\kappa$, there exists $\bar{g}_1,\dots,\bar{g}_m,\bar{f}'\in R_\kappa$ such that*

$$\bar{f}=\bar{g}_1\bar{r}_1+\cdots\bar{g}_m\bar{r}_m+\bar{f}', \tag{3.1.1.5}$$

where any term of $\bar{f}'$ is not divisible by any $\mathrm{lead}(\bar{r}_h)$*, and* $\mathrm{lead}(\bar{f})\succeq\mathrm{lead}(\bar{g}_h\bar{r}_h)$ *for all h.*

Proof. Let j be the exponent of S in $\mathrm{lead}(\bar{f})$ and let $S^j\bar{f}_{(j)}$ be the sum of terms in $\bar{f}$ for which the exponents of S are j. Applying [Eis95, Proposition-Definition 15.6] to $\bar{f}_{(j)}$, we can write

$$\bar{f}_{(j)}\equiv\bar{g}_{1,(j)}\bar{r}_1+\cdots+\bar{g}_{m,(j)}\bar{r}_m+\bar{f}'_{(j)}\pmod{S\cdot\kappa[u_1,\dots,u_m][\![S]\!]},$$

where $\bar{g}_{h,(j)}\in\kappa[u_1,\dots,u_m]$ and $\mathrm{lead}(\bar{g}_{h,(j)}\bar{r}_h)\preceq\mathrm{lead}(\bar{f}_{(j)})$ for $h=1,\dots,m$ and any term in $\bar{f}'_{(j)}\in\kappa[u_1,\dots,u_m]$ is not divisible by any $\mathrm{lead}(\bar{r}_h)$.

If we repeat the above argument for $\bar{f}_{(j)}-S^j\big(\bar{g}_{1,(j)}\bar{r}_1+\cdots+\bar{g}_{m,(j)}\bar{r}_m+\bar{f}'_{(j)}\big)\in S^{j+1}\cdot\kappa[u_1,\dots,u_m][\![S]\!]$ in place of $\bar{f}$, we will obtain $\bar{f}'_{(j')}$ and $\bar{g}_{h,(j')}$ for $h=1,\dots,m$ and for some $j'\geq j+1$. We can then iterate this process.

For $h = 1, \dots, m$, put $\bar{g}_h = S^j \bar{g}_{h,(j)} + S^{j+1} \bar{g}_{h,(j+1)} + \cdots$ and $\bar{f}' = S^j \bar{f}'_{(j)} + S^{j+1} \bar{f}'_{(j+1)} + \cdots$; the power series converge to elements in R_κ we are looking for. □

Definition 3.1.1.6. For $f \in R$, write

$$f = \sum_{i_1,\dots,i_n,j} f_{i_1,\dots,i_n,j} u_1^{i_1} \cdots u_n^{i_n} S^j. \tag{3.1.1.7}$$

Of the monomials for which $|f_{i_1,\dots,i_n,j}| = |f|_1$, there must be one which is lexicographically largest; we call the corresponding term $f_{i_1,\dots,i_n,j} u_1^{i_1} \cdots u_n^{i_n} S^j$ the 1-*leading term* of f, denoted by $\mathrm{Lead}(f)$.

Hypothesis 3.1.1.8. Let I^{int} be an ideal of R^{int} such that $R^{\mathrm{int}}/I^{\mathrm{int}}$ is *flat* over $\mathcal{O}_F$.

Notation 3.1.1.9. Denote $I = I^{\mathrm{int}} \otimes_{\mathcal{O}_F} F$ and $I_\kappa = I^{\mathrm{int}} \otimes_{\mathcal{O}_F} \kappa$; the latter is an ideal in R_κ by the flatness hypothesis above. Choose $r_1, \dots, r_m \in I^{\mathrm{int}}$ which project to elements of a Gröbner basis $\bar{r}_1, \dots, \bar{r}_m$ of I_κ.

For $f \in R$, let $\mathbf{j}_f$ denote the minimal exponent of S in the expression (3.1.1.7) of f. Denote $\mathbf{j}_I = \min\{\mathbf{j}_{r_h}; h = 1, \dots, m\}$; it is a nonpositive integer.

Notation 3.1.1.10. In this subsection, fix $\eta_0 \in (|\pi_F|^{-1/\mathbf{j}_I}, 1)$. In particular, $|\pi_F|\eta_0^{\mathbf{j}_I} < 1$.

Notation 3.1.1.11. Let $\mathcal{R}_{\eta_0}$ be the Fréchet completion of R for $|\cdot|_\eta$ for $\eta \in [\eta_0, 1)$. Let $R_{\eta_0}^{\mathrm{int}}$ denote $\{f \in \mathcal{R}_{\eta_0} | |f|_1 \leq 1\}$ and put $R_{\eta_0} = R_{\eta_0}^{\mathrm{int}} \otimes_{\mathcal{O}_F} F$ and $I_{\eta_0} = I \otimes_R R_{\eta_0}$.

Notation 3.1.1.12. For an element $f \in \mathcal{R}_{\eta_0}$ written as in (3.1.1.7) and $l \in \mathbb{Z}$, let $\pi_F^l f_{(l)}$ be the sum of all terms $f_{i_1,\dots,i_n,j} u_1^{i_1} \cdots u_n^{i_n} S^j$ for which $v_F(f_{i_1,\dots,i_n,j}) = l$. Thus, $f_{(l)} \in R_{\eta_0}^{\mathrm{int}}$; we use $\bar{f}_{(l)}$ denote its reduction in R_κ.

Lemma 3.1.1.13. *For $h = 1, \dots, m$ and $\eta \in [\eta_0, 1]$,*

$$|r_h|_\eta = 1, \qquad |r_{h,(l)}|_\eta \leq \eta^{\mathbf{j}_I}, \text{ for } l \in \mathbb{Z}_{\geq 0}.$$

Proof. The former inequality follows from the choice of η_0 in Notation 3.1.1.10. The latter follows from the definition of $\mathbf{j}_I$ in Notation 3.1.1.9. □

Construction 3.1.1.14. For $f \in R_{\eta_0}$ with $|f|_1 = |\pi_F|^{l_0}$, the *division algorithm* is the following procedure. Put $f_{l_0} = f$. Given f_l for $l \geq l_0$, we apply Proposition 3.1.1.4 to write

$$\bar{f}_{l,(l)} = \bar{g}_{l,1}\bar{r}_1 + \cdots + \bar{g}_{l,m}\bar{r}_m + \bar{f}'_{l,(l)},$$

where $\bar{g}_{l,h} \in R_\kappa$ and $\mathrm{lead}(\bar{g}_{l,h}\bar{r}_h) \preceq \mathrm{lead}(\bar{f}_{l,(l)})$ for $h = 1, \ldots, m$ and any term of $\bar{f}'_{l,(l)} \in R_\kappa$ is not divisible by any $\mathrm{lead}(\bar{r}_h)$. For each h, pick lifts $g_{l,h}$ of $\bar{g}_{l,h}$ in R^{int} so that $g_{l,h} = g_{l,h,(0)}$, namely, we only lift nonzero terms. Put

$$f_{l+1} = f_l - \pi_K^l\big(g_{l,1}r_1 + \cdots + g_{l,m}r_m\big).$$

Remark 3.1.1.15. Division algorithm depends on many choices but we will prove in Proposition 3.1.1.19 that the outcome $\lim_{l\to+\infty} f_l$ is uniquely determined by f.

Lemma 3.1.1.16. *At each step of the division algorithm, for $\eta \in [\eta_0, 1]$ and $h = 1, \ldots, m$,*

$$|g_{l,h}|_\eta \leq |f_{l,(l)}|_\eta, \qquad |f_{l+1,(l')} - f_{l,(l')}|_\eta \begin{cases} \leq \eta^{\mathrm{j}_I}|f_{l,(l)}|_\eta & l' > l \\ \leq |f_{l,(l)}|_\eta & l' = l \\ = 0 & l' < l \end{cases}. \tag{3.1.1.17}$$

Proof. The former inequality holds because $\mathrm{lead}(\bar{g}_{l,h}\bar{r}_h) \preceq \mathrm{lead}(\bar{f}_{l,(l)})$. The latter relation follows from the former one, using Lemma 3.1.1.13. □

Corollary 3.1.1.18. *For $h = 1, \ldots, m$, the series $g_h = \pi_F^{l_0}g_{l_0,h} + \pi_F^{l_0+1}g_{l_0+1,h} + \cdots$ converges under $|\cdot|_\eta$ for $\eta \in [\eta_0, 1)$. Consequently, $g_h \in R_{\eta_0}$ for $h = 1, \ldots, m$.*

Proof. By Lemma 3.1.1.16,

$$\begin{aligned}
|\pi_F^l g_{l,h}|_\eta \leq |\pi_F^l f_{l,(l)}|_\eta &\leq |\pi_F|^l \max\left\{\eta^{\mathrm{j}_I}|f_{l-1,(l-1)}|_\eta, |f_{l-1,(l)}|_\eta\right\} \\
&\leq |\pi_F|^l \max\left\{\eta^{2\mathrm{j}_I}|f_{l-2,(l-2)}|_\eta, \eta^{\mathrm{j}_I}|f_{l-2,(l-1)}|_\eta, \eta^{\mathrm{j}_I}|f_{l-2,(l-2)}|_\eta, |f_{l-2,(l)}|_\eta\right\} \leq \cdots \\
&\leq |\pi_F|^l \max_{l'<l}\left\{\eta^{(l-l')\mathrm{j}_I}|f_{(l')}|_\eta\right\} \leq \max_{l'<l}\left\{\left(|\pi_F|\eta_0^{\mathrm{j}_I}\right)^{l-l'}\left|\pi_F^{l'}f_{(l')}\right|_\eta\right\};
\end{aligned}$$

this goes to zero as $l \to +\infty$. □

Proposition 3.1.1.19. *Keep the notation as above. The quantity $f - g_1r_1 - \cdots - g_mr_m$ is the unique element of $f + I_{\eta_0}$ for which none of its term is divisible by any* $\mathrm{Lead}(r_h)$.

Proof. It follows from the definition of $g_1, \ldots, g_m$ that none of the term of $f - g_1r_1 - \cdots - g_mr_m$ is divisible by any $\mathrm{Lead}(r_h)$.

Assume that $f \in R_{\eta_0}$ does not contain any term divisible by any of $\mathrm{Lead}(r_h)$, then we need to show that for any nonzero $g \in I_{\eta_0}$, there is a term in $f + g$ divisible by some of $\mathrm{Lead}(r_h)$. Assume the contrary. Let $n = \log_{|\pi_F|}|g|_1$. Then $\bar{g}_{(n)} \in I_\kappa$ does not contain any term which divides any of $\mathrm{lead}(\bar{r}_h)$. This forces $\bar{g}_{(n)} = 0$ because the leading term of any nonzero element in I_κ is divisible by some $\mathrm{lead}(\bar{r}_h)$. Contradiction. The lemma follows. □

Lemma 3.1.1.20. *For $\eta \in [\eta_0, 1]$, $|f - g_1r_1 - \cdots - g_mr_m|_\eta$ equals the minimum η-norm of any element of $f + I_{\eta_0}$. Moreover, this continues to hold if we pass from R_{η_0} to its completion $R_{\eta_0}^{\wedge,\eta}$ under $|\cdot|_\eta$.*

Proof. For $\eta \in [\eta_0, 1]$, by Lemma 3.1.1.16, $|f_{l+1}|_\eta \leq |f_l|_\eta$ and hence $|f - g_1r_1 - \cdots - g_mr_m|_\eta \leq |f|_\eta$. By Proposition 3.1.1.19, starting with any element in $f + I_{\eta_0}$, the division algorithm will eventually lead to a unique element $f - g_1r_1 - \cdots - g_mr_m$; hence the first statement follows.

The second statement follows from the fact that any element in $f + I_{\eta_0}R_{\eta_0}^{\wedge,\eta}$ is a limit of elements in $f + I_{\eta_0}$. □

Proposition 3.1.1.21. *Let f be a rigid analytic function on the space*

$$\mathbf{X}_{\eta_0} = \big\{(u_1, \ldots, u_n, S) \in A_F^{n+1}[0,1] \big| \eta_0 \leq |S| < 1; r_1, \ldots, r_m = 0\big\}.$$

Then the following are equivalent.

(a) f is induced by an element of $R_{\eta_0}^{\mathrm{int}}$.

(b) There exists a function $r : [\eta_0, 1) \to \mathbb{R}$ with $\lim_{\eta \to 1^-} r(\eta) \leq 1$, such that for each $\eta \in [\eta_0, 1)$, f lifts to an element of the $|\cdot|_\eta$-completion of R_{η_0} having η-norm less than or equal to $r(\eta)$.

Proof. It is clear that (a) implies (b), so assume (b). We can write f as a Fréchet limit of the projections of some sequence of elements $f_1, f_2, \dots$ of R, under the quotient norms associated to the $|\cdot|_\eta$ for $\eta \in [\eta_0, 1)$. Use the division algorithm to write $f_l = g_{l,1}r_1 + \dots + g_{l,m}r_m + h_l$ with $g_{l,1}, \dots, g_{l,m}, h_l \in R_{\eta_0}$. Moreover, as $f_l - f_{l+1}$ tends to zero under the Fréchet topology, so is $h_l - h_{l+1}$ since it can be obtained from the division algorithm of $f_l - f_{l+1}$ and Lemma 3.1.1.16 ensures that $|f_l - f_{l+1}|_\eta \geq |h_l - h_{l+1}|_\eta$. Hence, the h_l form a Féchet convergent sequence; denote the limit by h, which is a lift of f. Note that for a fixed η, $|h_l|_\eta$ equals the η-quotient norm of f_l, which in turn equals the η-quotient norm of f when l is large enough. Thus, $|h|_\eta \leq r(\eta)$ for all $\eta \in [\eta_0, 1)$. Hence it lies in $R_{\eta_0}^{\text{int}}$. □

Notation 3.1.1.22. Define

$$A^{\text{int}} = R^{\text{int}}/I^{\text{int}} \qquad A = R/I$$

$$A_{\eta_0} = R_{\eta_0}/I_{\eta_0} \qquad A_\kappa = A^{\text{int}} \otimes_{\mathcal{O}_F} \kappa \cong R_\kappa/I_\kappa;$$

we may view A_κ as an affinoid algebra over $\kappa((S))$, whose corresponding rigid analytic space is denoted by X.

3.1.2 Quotient norms versus spectral norms

In this subsection, we compare spectral norms with the quotient norms discussed in previous section. As an application, we deduce that the connected components of $\mathbf{X}_{\eta_0}$ when $\eta_0 \to 1^-$ as a rigid space over F are the same as the connected components of X as a rigid space over $\kappa((S))$.

Keep the notation as above and assume the following.

Hypothesis 3.1.2.1. In this subsection, we assume that A_κ is reduced.

Notation 3.1.2.2. Let $|\cdot|_{\kappa,\text{qt}}$ denote the quotient norm on A_κ induced by the Gauss norm on R_κ. Let $|\cdot|_{\kappa,\text{sp}} = \lim_{n\to+\infty} |\cdot^n|_{\kappa,\text{qt}}^{1/n}$ be the spectral norm; it is a norm because A_κ is reduced. By [BGR84, Theorem 6.2.4/1], there exists $c > 0$ such that $|\cdot|_{\kappa,\text{sp}} \leq |\cdot|_{\kappa,\text{qt}} \leq |S|_\kappa^{-c}| \cdot |_{\kappa,\text{sp}}$, where $|S|_\kappa$ is the norm of S in $\kappa((S))$.

Notation 3.1.2.3. In this subsection, fix $\eta_0 \in \left(|\pi_F|^{1/(-\mathrm{j}_I+pc)}, 1\right)$; in particular, $|\pi_F|\eta_0^{\mathrm{j}_I} < \eta_0^{pc}$ and $\eta_0 > p^{-1/pc}$.

Notation 3.1.2.4. For $\eta \in [\eta_0, 1]$, let $|\cdot|_{\eta,\mathrm{qt}}$ denote the quotient norm on A_{η_0} or A induced by the η-Gauss norm on R_{η_0} or R. Similarly, we have the η-spectral (semi)norm $|\cdot|_{\eta,\mathrm{sp}} = \lim_{n\to+\infty} |\cdot^n|_{\eta,\mathrm{qt}}^{1/n}$; we will see in Lemma 3.1.2.6 that it is a norm.

Proposition 3.1.2.5. *The quotient norm $|\cdot|_{1,\mathrm{qt}}$ on A is the same as the spectral (semi)norm $|\cdot|_{1,\mathrm{sp}}$. As a consequence, the map $A^{\mathrm{int}} \to A_\kappa$ induces an isomorphism $A^\circ/A^{\circ\circ} \cong A_\kappa$, where $A^\circ = \{f \in A \| f|_{1,\mathrm{sp}} \leq 1\}$ and $A^{\circ\circ} = \{f \in A \| f|_{1,\mathrm{sp}} < 1\}$.*

Proof. Since $A^{\mathrm{int}}/\mathfrak{m}_F A^{\mathrm{int}} = A_\kappa$ is reduced, by [BGR84, 6.2.1/4(iii)], the quotient norm on A is equal to the spectral seminorm, $A^\circ = A^{\mathrm{int}}$, and $A^{\circ\circ} = \mathfrak{m}_F A^{\mathrm{int}}$. This proves the claim. □

Lemma 3.1.2.6. *For $\eta \in [\eta_0, 1)$, we have $|\cdot|_{\eta,\mathrm{sp}} \leq |\cdot|_{\eta,\mathrm{qt}} \leq \eta^{-pc/(p-1)}|\cdot|_{\eta,\mathrm{sp}}$ on A_{η_0}. The same is true when extending both norm to the completion of A_{η_0} with respect to $|\cdot|_{\eta,\mathrm{qt}}$ (which is the same as the completion with respect to the spectral norm). In particular, this shows that $|\cdot|_{\eta,\mathrm{sp}}$ is a norm on A_{η_0}.*

Proof. It suffices to show that for any $f \in A_{\eta_0}$, $|f^p|_{\eta,\mathrm{qt}} \geq \eta^{pc}|f|_{\eta,\mathrm{qt}}^p$; then it would follow that $|f^{p^n}|_{\eta,\mathrm{qt}} \geq \eta^{(p^n-1)pc/(p-1)}|f|_{\eta,\mathrm{qt}}^{p^n}$ for all $n \in \mathbb{N}$ by iteration, and hence the statement follows by taking limit.

Pick a representative $\tilde{f}$ of f in R_{η_0} containing no terms divisible by any $\mathrm{Lead}(r_h)$ (hence by Proposition 3.1.1.19, $|\tilde{f}|_\eta = |f|_{\eta,\mathrm{qt}}$). Fix $\eta \in [\eta_0, 1)$, we will show that

$$|\tilde{f}^p|_{\eta,\mathrm{qt}} = \Big|\sum_l (\pi_F^l \tilde{f}_{(l)})^p\Big|_{\eta,\mathrm{qt}} \geq \eta^{pc}|\tilde{f}|_\eta^p = \eta^{pc}|f|_{\eta,\mathrm{qt}}^p. \tag{3.1.2.7}$$

First, we remark that, given the middle inequality, the former equality follows; this is because $\tilde{f}^p - \sum_l(\pi_F^l\tilde{f}_{(l)})^p$ consists of products of $\pi_F^l\tilde{f}_{(l)}$ with an extra multiple p from the multinomial coefficients and then $|\tilde{f}^p - \sum_l(\pi_F^l\tilde{f}_{(l)})^p|_{\eta,\mathrm{qt}} \leq |\tilde{f}^p - \sum_l(\pi_F^l\tilde{f}_{(l)})^p|_\eta \leq p^{-1}|\tilde{f}|_\eta^p < \eta^{pc}|\tilde{f}|_\eta^p$, for $\eta \in [\eta_0, 1)$. So it suffices to prove the middle inequality in

(3.1.2.7). For any l, we have

$$\left|(\bar{\bar{f}}_{(l)})^p\right|_{\kappa,\mathrm{qt}} \geq \left|(\bar{\bar{f}}_{(l)})^p\right|_{\kappa,\mathrm{sp}} = \left|\bar{\bar{f}}_{(l)}\right|_{\kappa,\mathrm{sp}}^p \geq |S|_\kappa^{pc} \cdot \left|\bar{\bar{f}}_{(l)}\right|_{\kappa,\mathrm{qt}}^p.$$

Let $(\tilde{f}_{(l)})^p = g_{l,1}r_1 + \cdots + g_{l,m}r_m + h_l$ be the result of the first step of applying the division algorithm to $(\tilde{f}_{(l)})^p$. Then $\log_\eta |h_{l,(0)}|_\eta = \log_{|S|_\kappa} \left|(\bar{\bar{f}}_{(l)})^p\right|_{\kappa,\mathrm{qt}}$ and hence $|h_{l,(0)}|_\eta \geq \eta^{pc}|\tilde{f}_{(l)}|_\eta^p$. Moreover, by Lemma 3.1.1.16, $|h_l - h_{l,(0)}|_\eta \leq \eta^{\mathbf{j}_I}|\pi_F||\tilde{f}_{(l)}|_\eta^p < \eta^{pc}|\pi_F|^{-pl}|\tilde{f}|_\eta^p$; this implies that $|h_{l,(0)}|_{\eta,\mathrm{qt}} = |h_{l,(0)}|_\eta$.

Now, we can write

$$\sum_l (\pi_F^l \tilde{f}_{(l)})^p = \sum_l \pi_F^{pl} h_{l,(0)} + \sum_l \pi_F^{pl}(h_l - h_{l,(0)}) \qquad (3.1.2.8)$$

in the quotient ring. The former term on the right hand side of (3.1.2.8) has (quotient) norm at least $\eta^{pc}|\tilde{f}|_\eta^p$ because none of them is divisible by any $\mathrm{Lead}(r_h)$. In contrast, the latter term on the right hand side of (3.1.2.8) has norm strictly less than $\eta^{pc}|\tilde{f}|_\eta^p$. Thus, the inequality in (3.1.2.7) holds. ☐

Remark 3.1.2.9. It is attractive to think that $|\cdot|_{\eta,\mathrm{sp}} \leq |\cdot|_{\eta,\mathrm{qt}} \leq \eta^{-c}|\cdot|_{\eta,\mathrm{sp}}$ when $\eta \to 1^-$. However, the best we know is that for any $c' > c$, we have an ϵ depending on c', for which $|\cdot|_{\eta,\mathrm{sp}} \leq |\cdot|_{\eta,\mathrm{qt}} \leq \eta^{-c'}|\cdot|_{\eta,\mathrm{sp}}$ for all $\eta \in [\epsilon, 1)$.

Corollary 3.1.2.10. *For a rigid analytic function f on $\mathbf{X}_{\eta_0}$, the following are equivalent.*

(a) *f is an element in $A_{\eta_0}^{\mathrm{int}}$.*

(b) *There exists a function $r : [\eta_0, 1) \to \mathbb{R}$ with $\lim_{\eta\to 1} r(\eta) \leq 1$, such that for each $\eta \in [\eta_0, 1)$, $|f|_{\eta,\mathrm{sp}} \leq r(\eta)$.*

Proof. It follows from combining Lemma 3.1.2.6 with Proposition 3.1.1.21. ☐

Proposition 3.1.2.11. *There are one-to-one correspondences among the following four sets.*

(a) *the idempotent elements of A_κ;*

(b) *the idempotent elements of* $A_{\eta_0}^{\mathrm{int}}$*;*

(c) *the idempotent elements of* A_{η_0}*;*

(d) *the idempotent elements on* $\mathbf{X}_{\eta_0}$.

Proof. By Corollary 3.1.2.10, the sets (b), (c), and (d) are the same because idempotent elements have spectral norms 1. It suffices to match up (a) and (b). We have a map from the set of idempotent elements of $A_{\eta_0}^{\mathrm{int}}$ to the set of idempotent elements of A_κ by reducing modulo π_F. We first show the injectivity. Let $f, g \in R_{\eta_0}^{\mathrm{int}}$ be idempotents whose reductions modulo π_F are the same, i.e., $\bar{f} = \bar{g} \in A_\kappa$. This implies that $\bar{f}^{p-1} + \bar{f}^{p-2}\bar{g} + \cdots + \bar{g}^{p-1} = 0$ in A_κ. Since $f - g = f^p - g^p = (f-g)(f^{p-1} + f^{p-2}g + \cdots + g^{p-1})$, we have

$$\begin{aligned} |f-g|_{1,\mathrm{qt}} &= \big|(f-g)(f^{p-1} + f^{p-2}g + \cdots + g^{p-1})\big|_{1,\mathrm{qt}} \\ &\leq |f-g|_{1,\mathrm{qt}} \cdot |f^{p-1} + f^{p-2}g + \cdots + g^{p-1}|_{1,\mathrm{qt}} \leq |f-g|_{1,\mathrm{qt}} \cdot |\pi_F|. \end{aligned}$$

This forces $|f-g|_{1,\mathrm{qt}} = 0$ and hence $f = g$.

To see the surjectivity, we start with an idempotent $\bar{f} \in A_\kappa$, viewed as an element in R_κ with none of its terms divisible by any of $\mathrm{Lead}(\bar{r}_h)$; pick a lift $\tilde{f}_0 \in R^{\mathrm{int}}$ of $\bar{f}$ which only contains the terms that $\bar{f}$ has and let $f_0 \in A^{\mathrm{int}}$ denote its image in A^{int}. If we set $\tilde{h}_0$ be the result of applying the division algorithm to $\tilde{f}_0^2 - \tilde{f}_0$ and $h_0 = f_0^2 - f_0$, then $|h_0|_{1,\mathrm{qt}} = |\tilde{h}_0|_{1,\mathrm{qt}} \leq |\pi_F|$ and $|h_0|_{\eta,\mathrm{qt}} = |\tilde{h}_0|_{\eta,\mathrm{qt}} \leq p^{-1}\eta^{-2c} < 1$ for all $\eta \in [\eta_0, 1)$, where the latter inequality holds because all terms in $\tilde{f}_0$ come from the terms in $\bar{f}$ which have norms $\leq |\bar{f}|_{\kappa,\mathrm{qt}} \leq |S|_\kappa^{-c}|\bar{f}|_{\kappa,\mathrm{sp}} = |S|_\kappa^{-c}$. We apply a Hensel lemma type iteration to f_0 as follows. For $\alpha \geq 0$, we set $f_{\alpha+1} = f_\alpha + h_\alpha - 2h_\alpha f_\alpha$ and

$$h_{\alpha+1} := f_{\alpha+1}^2 - f_{\alpha+1} = (f_\alpha + h_\alpha - 2h_\alpha f_\alpha)^2 - (f_\alpha + h_\alpha - 2h_\alpha f_\alpha) = 4h_\alpha^2(h_\alpha - 1).$$

Hence, $|h_{\alpha+1}|_{\eta,\mathrm{qt}} \leq |h_\alpha|_{\eta,\mathrm{qt}}^2$ for all $\eta \in [\eta_0, 1]$. Thus $|h_\alpha|_{\eta,\mathrm{qt}} \to 0$ as $\alpha \to +\infty$; hence f_α converges to an element $f \in A_{\eta_0}^{\mathrm{int}}$ which is idempotent. It is clear from the construction that the reduction of f modulo π_F is the same as $\bar{f}$. This proves the surjectivity. □

Corollary 3.1.2.12. *When $\eta_0 \in p^{\mathbb{Q}}$, there is a one-to-one correspondence between the connected components of X and the connected components of $\mathbf{X}_{\eta_0}$.*

Remark 3.1.2.13. This is the first place where we need the rationality of $\log_p \eta_0$ to ensure that we are in the classical rigid analytic space setting to talk about connected components.

3.1.3 Lifting construction

In order to apply the results from previous two subsections later in the paper, we, reversing the picture, start with a rigid analytic space X and try to construct $\mathbf{X}_{\eta_0}$ out from it.

Let κ and F be as before.

Definition 3.1.3.1. Let X be a *reduced* affinoid rigid space over $\kappa((S))$ with ring of analytic functions $A_\kappa = R_\kappa / I_\kappa$ where $R_\kappa = \kappa((S))\langle u_1, \dots, u_n \rangle$ and I_κ is some ideal. The *lifting construction* refers to the following.

(1) find an ideal I^{int} in $R^{\mathrm{int}} = F\langle u_1, \dots, u_n \rangle((S))$ so that $R^{\mathrm{int}}/I^{\mathrm{int}}$ is *flat* over $\mathcal{O}_F$ and $I^{\mathrm{int}} \otimes_{\mathcal{O}_F} \kappa = I_\kappa$.

(2) Choose a Gröbner basis of I_κ and lift them to $r_1, \dots, r_m \in I^{\mathrm{int}}$ as in Definition 3.1.1.6 and define η_0 as in Notation 3.1.2.3.

(3) We call the rigid analytic space

$$\mathbf{X}_{\eta_0} = \left\{ (u_1, \dots, u_n, S) \in A_F^{n+1}[0,1] \middle| \, \eta_0 \leq |S| < 1; r_1, \dots, r_m = 0 \right\}$$

the *lifting space* of X; it depends only on the choices of I^{int} and η_0 as well as the lifts of $r_1, \dots, r_m$ in I^{int}.

Remark 3.1.3.2. We do not know if such a lifting space always exists in general. The only obstruction is to find an ideal I^{int} lifting I_κ such that $R^{\mathrm{int}}/I^{\mathrm{int}}$ is flat over $\mathcal{O}_F$.

Question 3.1.3.3. It would be interesting to know if the above lifting construction can be globalized for arbitrary rigid spaces over $\kappa((S))$. In particular, given a morphism between two rigid spaces over $\kappa((S))$, can we lift the morphism (non-canonically) to a morphism between (some strict neighborhood of) their lifting spaces? Can we "glue" the lifting spaces up to homotopy? This situation is very similar to Berthelot's construction of rigid cohomology [Brt96+].

For an affinoid subdomain of a polydisc, we explicate this lifting process.

Example 3.1.3.4. Let $p_1, \ldots, p_m \in \kappa[\![S]\!][u_1, \ldots, u_n]$ be polynomials and $a_1, \ldots, a_m \in \mathbb{N}$. We consider the following affinoid subdomain of the unit polydisc

$$X = \left\{ (u_1, \ldots, u_n) \in A^n_{\kappa((S))}[0,1] \middle| \, |p_1| \leq |S|^{a_1}, \ldots, |p_m| \leq |S|^{a_m} \right\}.$$

The ring of analytic functions on X is

$$\kappa((S))\langle u_1, \ldots, u_n, v_1, \ldots, v_m \rangle / (v_1 S^{a_1} - p_1, \ldots, v_m S^{a_m} - p_m).$$

For each i, let P_i be a lift of p_i in $\mathcal{O}_F[\![S]\!][u_1, \ldots, u_n]$ (here we allow P_i to have new terms other than the terms of p_i). We claim that the ring

$$\mathcal{O}_F\langle u_1, \ldots, u_n, v_1, \ldots, v_m \rangle((S)) / (v_1 S^{a_1} - P_1, \ldots, v_m S^{a_m} - P_m) \tag{3.1.3.5}$$

is flat over $\mathcal{O}_F$. This is because the ring

$$\mathcal{O}_F((S))[u_1, \ldots, u_n, v_1, \ldots, v_m] / (v_1 S^{a_1} - P_1, \ldots, v_m S^{a_m} - P_m) \simeq \mathcal{O}_F((S))[u_1, \ldots, u_n]$$

is flat and hence torsion free over $\mathcal{O}_F$, and so is its completion (3.1.3.5) with respect to the topology induced by $(p, S)^r \mathcal{O}_F[\![S]\!][u_1, \ldots, u_n, v_1, \ldots, v_m]$ for $r \in \mathbb{N}$.

Therefore, by Definition 3.1.3.1,

$$\mathbf{X}_{\eta_0} = \left\{ (u_1, \ldots, u_n, S) \in A^{n+1}_F[0,1] \middle| \, \eta_0 \leq |S| < 1; |P_1| \leq |S|^{a_1}, \ldots, |P_m| \leq |S|^{a_m} \right\}$$

is a lifting space for X, for some $\eta_0 \in (0, 1)$.

3.2 Differential conductors

In this section, we give the definition of differential Artin and Swan conductors following [Ked07a].

3.2.1 Construction of differential modules

In this subsection we review Tsuzuki's construction [Tsu98a] of differential modules over the Robba ring associated to p-adic Galois representations. For a systematic treatment, one may consult, for example, [Ked07a, Section 3].

Notation 3.2.1.1. For the rest of this chapter, let K be a complete discretely valued field of equal characteristic $p > 0$. Fix a uniformizer π_K and a non-canonically isomorphism

$$\kappa_K((\pi_K)) \simeq K. \tag{3.2.1.2}$$

Let $\bar{b}_J \subset \kappa_K$ be a p-basis of κ_K, where J is an index set. Then the image b_J of $\bar{b}_J$ in K under the isomorphism (3.2.1.2) form a lifted p-basis of K. Hence, $(db_j)_{j\in J}$ and $d\pi_K$ form a basis of $\Omega^1_{\mathcal{O}_K/\mathbb{F}_p}$. We set $\kappa_0 = \cap_{n>0} K^{p^n} \cong \cap_{n>0} \kappa_K^{p^n}$; it is a perfect field.

Notation 3.2.1.3. Let $\mathcal{O}_F$ denote the Cohen ring of κ_K with respect to $\bar{b}_J$ and let $B_J \subset \mathcal{O}_F$ be the canonical lifts of the p-basis. (For more about Cohen rings, see [Ked07a, Section 3.1] or [Whi02].) Denote $F = \mathrm{Frac}\mathcal{O}_F$. We use $\mathcal{O}_{F_0}$ to denote the ring of Witt vectors $W(\kappa_0)$, as a subring of $\mathcal{O}_F$. Denote $F_0 = \mathrm{Frac}\mathcal{O}_{F_0}$.

Definition 3.2.1.4. Let $\mathcal{O}$ denote the ring of integers in a finite extension of $\mathbb{Q}_p$ and let $\mathbb{F}_q$ be its residue field, where $q = p^\lambda$ for $\lambda \in \mathbb{N}$. Write $\mathbb{Q}_q$ for the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and write $\mathbb{Z}_q$ for its ring of integer.

By an *$\mathcal{O}$-representation* of G_K, we mean a continuous homomorphism $\rho : G_K \to \mathrm{GL}(\Lambda_\rho)$ with Λ_ρ a finite free $\mathcal{O}$-module. We say that ρ has *finite local monodromy* if the image of the inertia subgroup of G_K is finite.

By a *p-adic representation*, we mean a continuous representation $\rho : G_K \to \mathrm{GL}(V_\rho)$ with V_ρ a finite vector space over $\mathrm{Frac}(\mathcal{O})$. Since G_K is compact, we can always find a (not necessarily unique) $\mathcal{O}$-lattice $\Lambda_\rho \subset V_\rho$, stable under the action of G_K. Hence, we can obtain an $\mathcal{O}$-representation $\rho : G_K \to \mathrm{GL}(\Lambda_\rho)$.

Hypothesis 3.2.1.5. We always assume that $\mathbb{F}_q \subseteq \kappa_0$ (see the Remark 3.2.3.7).

Notation 3.2.1.6. Denote $\mathcal{O}_{F'} = \mathcal{O}_F \otimes_{\mathbb{Z}_q} \mathcal{O}$. Since $\mathrm{Frac}(\mathcal{O})/\mathbb{Q}_q$ is totally ramified, $\mathcal{O}_{F'}$ is a complete discrete valuation ring; we denote its fraction field to be F'.

Notation 3.2.1.7. Let C_K be the Cohen ring of K with respect to the p-basis $\{(b_j)_{j\in J}, \pi_K\}$. By functoriality of Cohen ring (Proposition 2.3.4.3) of the isomorphism (3.2.1.2), $\mathcal{O}_F$ sits naturally inside C_K. Moreover, B_J are the canonical lift of b_J in C_K. We denote the canonical lift of π_K in C_K by S.

Put $\Gamma = C_K \otimes_{\mathbb{Z}_q} \mathcal{O}$; it is a complete discrete valuation ring since $\mathcal{O}$ is totally ramified over $\mathbb{Z}_q$.

Definition 3.2.1.8. A *(qth-power) Frobenius lift* on Γ is a homomorphism $\phi : \Gamma \to \Gamma$ which acts trivially on $\mathcal{O}$ and induces the qth-power Frobenius on K. The *standard Frobenius lift (with respect to B_J)* is the Frobenius lift which sends B_j to B_j^q for $j \in J$ and S to S^q; it is unique by Proposition 2.3.4.3.

Remark 3.2.1.9. We will see below that the construction of differential module is valid for any Frobenius lifts, but only the standard Frobenius lift gives rise to the Frobenius antecedent for the differential modules.

Definition 3.2.1.10. Recall the definition of differential modules in Definition 1.1.1.24. Let R be a ring equipped with an endomorphism $\phi : R \to R$. A (ϕ, ∇)*-module* over R is a ∇-module over R equipped with an isomorphism $\phi^* M \to M$ of ∇-modules.

Definition 3.2.1.11. For every $\mathcal{O}$-representation $\rho : G_K \to \mathrm{GL}(\Lambda_\rho)$, define its associated (ϕ, ∇)-module over Γ by

$$D(\rho) = \left(\Lambda_\rho \otimes_{\mathcal{O}} \widehat{\Gamma^{\mathrm{unr}}}\right)^{G_K},$$

where $\widehat{\Gamma^{\mathrm{unr}}}$ is the p-adic completion of the maximal unramified extension of Γ.

Proposition 3.2.1.12. *For any Frobenius lift ϕ on Γ, the functor D from $\mathcal{O}$-representations of G_K to (ϕ, ∇)-modules over Γ is an equivalence of categories.*

Proof. For convenience of the reader, we briefly describe the functor here; for more details, one may consult [Ked07a, Propositions 3.2.7 and 3.2.8]. It is well-known that D establishes an equivalence between the category of representations and the category of ϕ-modules over Γ (finite free Γ-modules with semi-linear ϕ-actions), with $V(M) = \big(M \otimes_\Gamma \widehat{\Gamma^{\mathrm{unr}}}\big)^{\phi=1}$ as the inverse. The non-trivial part is that every ϕ-module over Γ admits a unique structure of (ϕ, ∇)-module; this involves a standard approximation argument. □

For an $\mathcal{O}$-representation ρ of finite monodromy, one can refine the (ϕ, ∇)-module associated to ρ as follows.

Construction 3.2.1.13. Since C_K has an $\mathcal{O}_F$-algebra structure, any element $x \in \Gamma$ can be uniquely written in the form of $\sum_{i\in\mathbb{Z}} x_i S^i$ for $x_i \in \mathcal{O}_F \otimes_{\mathbb{Z}_q} \mathcal{O} = \mathcal{O}_{F'}$ such that the indices i for which $v(x_i) \leq n$ are bounded below.

For $r > 0$, put $\Gamma^r = \{x \in \Gamma | \lim\limits_{n\to-\infty} v(x_n) + rn = \infty\}$ and $\Gamma^\dagger = \cup_{r>0}\Gamma^r$; the latter is commonly known as the *integral Robba ring* over F'. It is not hard to show that any Frobenius lift ϕ preserves $\Gamma^\dagger$ and that $\Omega^1_{\Gamma^\dagger/\mathcal{O}} = \bigoplus_{j\in J} \Gamma^\dagger dB_j \oplus \Gamma^\dagger dS$.

Since $\mathcal{O}_{F'} \hookrightarrow \Gamma^\dagger$, we can identify $\mathcal{O}_{F'}^{\mathrm{unr}} \hookrightarrow (\Gamma^\dagger)^{\mathrm{unr}}$, where the superscript unr means taking the maximal unramified extensions of discrete valuation rings. Put $\widetilde{\Gamma}^\dagger = \widehat{\mathcal{O}_F^{\mathrm{unr}}} \otimes_{\mathcal{O}_F^{\mathrm{unr}}} (\Gamma^\dagger)^{\mathrm{unr}} \subset \widehat{\Gamma^{\mathrm{unr}}}$, where we take the p-adic completion. For a p-adic representation ρ with finite local monodromy, define

$$D^\dagger(\rho) = D(\rho) \cap \big(V_\rho \otimes_{\mathcal{O}} \widetilde{\Gamma}^\dagger\big) = \big(V_\rho \otimes_{\mathcal{O}} \widetilde{\Gamma}^\dagger\big)^{G_K}. \tag{3.2.1.14}$$

Theorem 3.2.1.15. [Ked07a, Theorem 3.3.6] *Let ϕ be a Frobenius lift on Γ acting on $\Gamma^\dagger$. Then $D^\dagger$ induces an equivalence between the category of $\mathcal{O}$-representations with finite local monodromy and the category of (ϕ, ∇)-modules over $\Gamma^\dagger$.*

Lemma 3.2.1.16. [Ked05a, Proposition 3.20] *The integral Robba ring $\Gamma^\dagger$ is an henselian ring.*

Notation 3.2.1.17. For $\eta_0 \in (0,1)$, we use $Z_K^{\geq \eta_0}$ for short to denote $A_F^1[\eta_0, 1)$. Denote the ring of analytic functions on it by $\mathcal{R}_F^{\eta_0}$. We define the *Robba ring over* F to be $\mathcal{R}_F = \cup_{\eta \in [\eta_0, 1)} \mathcal{R}_F^{\eta}$. Also denote $\mathcal{R}_{F'}^{\eta_0} = \mathcal{R}_F^{\eta_0} \otimes_{\mathbb{Z}_q} \mathcal{O}$ and $\mathcal{R}_{F'} = \mathcal{R}_F \otimes_{\mathbb{Z}_q} \mathcal{O}$. We will be only interested in the behavior when η_0 is close to 1.

Remark 3.2.1.18. We use K in the subscript of $Z_K^{\geq \eta_0}$ because the space is functorially in K but not in F, as we made a non-canonical choice in (3.2.1.2).

Now, we restrict the (ϕ, ∇)-module $D^\dagger(\rho)$ to the Robba ring over F' as follows.

Construction 3.2.1.19. Consider the natural injection $\Gamma^\dagger \hookrightarrow \mathcal{R}_{F'}$. Note that the Frobenius ϕ extends by continuity to $\mathcal{R}_{F'}$. Thus, from an $\mathcal{O}$-representation ρ with finite local monodromy, we obtain a differential module $\mathcal{E}_\rho = D^\dagger(\rho) \otimes_{\Gamma^\dagger} \mathcal{R}_{F'}$ over $\mathcal{R}_{F'}$.

Moreover, if we start with a p-adic representation $\rho : G_K \to \mathrm{GL}(V_\rho)$ of finite local monodromy, we can choose an $\mathcal{O}$-lattice Λ_ρ of V_ρ stable under the action of G_K as in Defintion 3.2.1.4. Then we associate a differential module $\mathcal{E}_\rho$ to the $\mathcal{O}$-representation given by Λ_ρ. It is clear that $\mathcal{E}_\rho$ does not depend on the choice of the lattice Λ_ρ. We call $\mathcal{E}_\rho$ the *differential module associated to* ρ.

Proposition 3.2.1.20. [Ked07a, Proposition 3.5.1] *The* (ϕ, ∇)*-module* $\mathcal{E}_\rho$ *over* $\mathcal{R}_{F'}$ *is independent of the choice of the p-basis and the lifts to* K *(up to a canonical isomorphism).*

Proposition 3.2.1.21. *The differential module* $\mathcal{E}_\rho$ *descends to a differential module over* $\mathcal{R}_{F'}^{\eta_0}$ *for some* $\eta_0 \in (0,1)$.

Proof. Indeed, defining a differential module needs only finite data. So, we can realize it on a certain annulus. See for instance [Ked07a, Remark 3.4.1]. □

Remark 3.2.1.22. The current construction of associating differential module to a p-adic representation (Constructions 3.2.1.13 and 3.2.1.19) is *not* functorial with respect to the base field $\mathrm{Frac}(\mathcal{O})$ of the representation. If $\mathcal{O}'$ is a finite extension of $\mathcal{O}$, for a p-adic representation ρ over $\mathrm{Frac}(\mathcal{O})$ of finite local monodromy, one can naturally obtain $\rho' = \rho \otimes_{\mathcal{O}} \mathcal{O}'$ as a p-adic representation over $\mathrm{Frac}(\mathcal{O}')$. Assume that

κ_K contains the residue field $\mathbb{F}_{q'}$ of $\mathcal{O}'$. Then the differential modules associated to ρ and ρ' are the same if $\mathcal{O}'/\mathcal{O}$ is unramified, and $\mathcal{E}_\rho \otimes_{\mathcal{O}} \mathcal{O}' = \mathcal{E}_{\rho'}$ if $\mathcal{O}'/\mathcal{O}$ is totally ramified.

There are two reasons of keeping this non-functoriality flaw. For one, the differential conductors we define later will be the same if we change ρ to $\rho \otimes_{\mathcal{O}} \mathcal{O}'$. For the other, if we define Γ in Notation 3.2.1.7 using the tensor over $\mathbb{Z}_p$ instead of $\mathbb{Z}_q$, in which case we do have the functoriality, we will get the direct sum of $[\mathbb{F}_q : \mathbb{F}_p] = \lambda$ copies of $\mathcal{E}_\rho$ as differential modules. When proving the integrality of Swan conductors, we have to come back to study $\mathcal{E}_\rho$ because $K \otimes_{\mathbb{Z}_p} \mathcal{O} \simeq F'^{\oplus\lambda}$ is not a field if $q > p$.

3.2.2 Differential modules with Frobenius structure

In this subsection, we study further structures of the differential module $\mathcal{E}_\rho$. In particular, we deduce that $\mathcal{E}_\rho$ is solvable in the sense of Definition 1.2.8.1. This would enable us to invoke Theorem 1.2.8.2 to define and deduce properties of differential conductors.

In this subsection, we assume the following.

Hypothesis 3.2.2.1. Assume that K as a finite lifted p-basis $b_J \subset \mathcal{O}_K$, where $J = \{1, \dots, m\}$. We also retrieve Notation 2.3.2.3.

Notation 3.2.2.2. Let $\partial_0 = \partial/\partial S, \partial_1 = \partial/\partial B_1, \dots, \partial_m = \partial/\partial B_m$ denote a dual basis of $\Omega^1_{\mathcal{O}_F[\![S]\!]/\mathcal{O}_{F_0}}$ with respect to $dS, dB_1, \dots, dB_m$; they give rise to a set of derivations on $\mathcal{R}^{\eta_0}_{F'}$ of rational type for all $\eta_0 \in (0,1)$. For a (ϕ, ∇)-module $\mathcal{E}$ over $\mathcal{R}^{\eta_0}_{F'}$, these differential operators act on $\mathcal{E}$, commuting with each other and commuting with the Frobenius action; this gives $\mathcal{E}$ a structure of ∂_{J^+}-differential modules in the sense of Definition 1.1.6.2.

We also use F'_η to denote the completion of $F'(t)$ with respect to the η-Gauss norm.

Proposition 3.2.2.3. *Let ϕ be the standard qth-power Frobenius lift on Γ. Recall the notation of Frobenius antecedent from Subsection 1.1.4. Then the pull back by Frobenius $\phi : F'_{\eta^p} \to F'_\eta$ is the same as $\varphi^{(\partial_0, \lambda)} \circ \cdots \circ \varphi^{(\partial_m, \lambda)}$, where $q = p^\lambda$.*

Proof. We may assume that F' contains ζ_q a q-th root of unity. It suffices to show that the image $\phi(F'_{\eta^p})$ is stable under the action of $(\mathbb{Z}/q\mathbb{Z})^{m+1}$ in the sense of Construction 1.1.4.9 (one $\mathbb{Z}/q\mathbb{Z}$ from each ∂_j-Frobenius for $j \in J^+$) and that the degree of F'_η over $\phi(F'_{\eta^p})$ is q^{m+1}.

For $\underline{i} = (i_0, \ldots, i_m) \in (\mathbb{Z}/q\mathbb{Z})^{m+1}$, we have $S^{(\underline{i})} = \zeta_q^{i_0}$ and $(B_j)^{(\underline{i})} = \zeta_q^{i_j} B_j$ for $j \in J$. Hence, both the standard Frobenius lift ϕ and $(\cdot)^{(\underline{i})} \circ \phi$ are continuous homomorphism from $\mathcal{O}_F[\![S]\!]$ to itself sending B_j to B_j^q and S to S^q. By Proposition 2.3.4.3, they must be the same. Hence the image of ϕ is stable under the $(\mathbb{Z}/q\mathbb{Z})^{m+1}$-action.

To see that F'_η has degree q^{m+1} over $\phi(F'_{\eta^p})$, it suffices to show that the degree of F has degree q^m over $\phi(F)$, because the S part is obvious. Note that $\phi : \mathcal{O}_F \to \mathcal{O}_F$ is a flat homomorphism because (the latter) $\mathcal{O}_F$ is torsion free. Hence the degree of $\phi : F \to F$ is the same as the degree of $\bar{\phi} : \kappa_K \to \kappa_K$, which is q^m. □

Proposition 3.2.2.4. *Let ϕ be the standard qth-power Frobenius lift on Γ. Let $\mathcal{E}$ be a (ϕ, ∇)-module over $A^1_{F'}[\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. Then $\mathcal{E}$ is solvable.*

Proof. By Corollary 1.1.4.26, we have

$$f_i(\phi^* M, r) = \max\left\{p^{-\lambda} f_i(M, qr), p^{1-\lambda}(f_i(M, qr) - \log p), \ldots, f_i(M, qr) - \lambda \log p\right\},$$

where $\lambda = \log_p q$. Since $\phi^* M \xrightarrow{\sim} M$, we actually get an equality comparing subsidiary radii of M at different radii. Taking the supremum limit $r \to 0^+$ or equivalently $\eta \to 1^-$, gives a number $g_i(M)$ satisfying

$$g_i(M) = \max\left\{p^{-\lambda} g_i(M), p^{1-\lambda}(g_i(M) - \log p), \ldots, g_i(M) - \lambda \log p\right\}.$$

This forces $g_i(M)$ to be zero. By continuity of $f_i(M, r)$, $\lim_{r \to 0^+} f_i(M, r) = 0$. Hence, $\mathcal{E}$ is solvable. □

Proposition 3.2.2.5. *Let ϕ be the standard qth-power Frobenius lift and ϕ' be another Frobenius lift on Γ. Assume that $\mathcal{E}$ is a (ϕ, ∇)-module over $A^1_{F'}[\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. Then $\mathcal{E}$ is naturally equipped with a (ϕ', ∇)-module structure.*

Proof. Define the Frobenius structure for ϕ' by Taylor series as follows.

$$\phi'(v) = \sum_{e_{J+}=0}^{\infty} \frac{(\phi'(S) - \phi(S))^{e_0} \prod_{j\in J}(\phi'(B_j) - \phi(B_j))^{e_j}}{e_{J+}!} \phi\Big(\frac{\partial^{e_0}}{\partial S^{e_0}} \frac{\partial^{e_1}}{\partial B_1^{e_1}} \cdots \frac{\partial^{e_m}}{\partial B_m^{e_m}}(v)\Big).$$

Since $|\phi'(S) - \phi(S)|_1 < 1$ and $|\phi'(B_j) - \phi(B_j)|_1 < 1$ for all $j \in J$. This series converges under $|\cdot|_\eta$ for $\eta \in (0,1)$ sufficiently close to 1 and also for $|\cdot|_1$. □

Remark 3.2.2.6. One may also approach the results of this subsection without using the standard Frobenius first but using a generalized version of Corollary 1.1.4.26. This point of view is taken in [Ked**, Chap. 17].

3.2.3 Differential conductors

Combining the result of Subsections 1.2.8 and 3.2.2, we can now define the differential conductors associated to a p-adic representation ρ of finite local monodromy. This is a slight improvement over Kedlaya's original construction in [Ked07a, Section 3.5], as the tool of differential modules in Section 1 was not available at the time.

We continue to assume Hypothesis 3.2.2.1 in this subsection.

Theorem 3.2.3.1. *Let ρ be a p-adic representation of G_K of finite local monodromy. Let $\mathcal{E}_\rho$ be the (ϕ, ∇) module over $A^1_{F'}[\eta_0, 1)$ associated to ρ in Construction 3.2.1.19. Then after making η_0 sufficiently close to 1^-, there exists a* unique *decomposition of (ϕ, ∇)-modules $\mathcal{E} = \bigoplus_{b\in\mathbb{Q}_{\geq 1}} \mathcal{E}_b$ (resp. $\mathcal{E} = \bigoplus_{b\in\mathbb{Q}_{\geq 0}} \mathcal{E}_{b,\log}$) over $A^1_{F'}[\eta_0, 1)$, where each of $\mathcal{E}_b$ (resp. $\mathcal{E}_{b,\log}$) has uniform break (resp. log-break) b. Moreover, each of $\mathcal{E}_b$ (resp. $\mathcal{E}_{b,\log}$) corresponds to a p-adic subrepresentation ρ_b (resp. $\rho_{b,\log}$) of G_K and $\rho = \oplus_{b\in\mathbb{Q}_{\geq 1}} \rho_b$ (resp. $\rho = \oplus_{b\in\mathbb{Q}_{\geq 0}} \rho_{b,\log}$).*

Proof. By Proposition 3.2.2.4, $\mathcal{E}_\rho$ is solvable. Now, we can invoke Theorem 1.2.8.2 to get the desired decomposition as ∇-modules. Since this decomposition is canonical and the action of ∇ commutes with the action of the Frobenius, we actually obtain a decomposition of (ϕ, ∇)-modules.

Moreover, by the slope filtration [Ked07a, Theorem 3.4.6], the Frobenius action on each direct summand of $\mathcal{E}$ is of unit-root; the decomposition of the representation follows by the equivalence of category in Theorem 3.2.1.15. □

Definition 3.2.3.2. We define the *differential ramification breaks* of ρ to be the differential non-log-breaks of $\mathcal{E}_{\rho/\rho^{I_K}}$ (See Definition 1.2.8.3), where ρ^{I_K} denote the subrepresentation of ρ fixed by the inertia group; the biggest break is denoted by $b(\rho)$. We also define the *differential Artin conductor* of ρ to be that of $\mathcal{E}_{\rho/\rho^{I_K}}$.

Similarly, we define the *differential ramification log-breaks* of ρ to be the differential log-breaks of $\mathcal{E}_\rho$ (See Definition 1.2.8.3); the biggest break is denoted by $b_{\log}(\rho)$. We also define the *differential Swan conductor* of ρ to be that of $\mathcal{E}_\rho$.

Remark 3.2.3.3. In the above definition of differential Artin conductors, we split off the unramified part because the multiset of convergence radii can not distinguish the unramified part from the tame part which contributes differently to the Artin conductor. This does not matter for Swan conductors.

Remark 3.2.3.4. By [Ked07a, Proposition 2.6.6], the definition of the differential conductors does not depend on the choice of the uniformizer S and the lifted p-basis B_J. Moreover, we may also lift the Hypothesis 3.2.2.1 and define the differential conductors for arbitrary complete discretely valued fields of equal characteristic p [Ked07a, Corollary 3.5.7].

Theorem 3.2.3.5. *Differential conductors satisfy the following properties:*

(0) *When the residue field κ_K is perfect, the differential Artin and Swan conductors are the same as classical ones in* [Ser79].

(1) *For any representation ρ of finite local monodromy,* $\mathrm{Swan}_{\mathrm{dif}}(\rho) \in \mathbb{Z}_{\geq 0}$ *and* $\mathrm{Art}_{\mathrm{dif}}(\rho) \in \mathbb{Z}_{\geq 0}$.

(2) *Let K'/K be a tamely ramified extension of ramification degree e'. Let ρ be a representation of G_K of finite local monodromy and let ρ' denote the restriction of ρ to $G_{K'}$. Then* $\mathrm{Swan}_{\mathrm{dif}}(\rho') = e' \cdot \mathrm{Swan}_{\mathrm{dif}}(\rho)$. *If $e' = 1$, i.e., K'/K is unramified,* $\mathrm{Art}_{\mathrm{dif}}(\rho') = \mathrm{Art}_{\mathrm{dif}}(\rho)$.

(3) *Let ρ be a faithful p-adic representation of the Galois group of a Galois extension L/K. If L/K is tamely ramified and not unramified, $b(\rho) = 1$ and $b_{\log}(\rho) = 0$. If L/K is unramified, $b(\rho) = b_{\log}(\rho) = 0$.*

(4) *Put* $\mathrm{Fil}^0_{\mathrm{dif}}G_K = G_K$ *and* $\mathrm{Fil}^a_{\mathrm{dif}}G_K = I_K$ *for* $a \in (0,1]$. *For* $a > 1$, *let* R_a *be the set of finite image representations* ρ *with differential ramification break less than* a. *Define* $\mathrm{Fil}^a_{\mathrm{dif}}G_K = \bigcap_{\rho \in R_a} (I_K \cap \mathrm{Ker}(\rho))$ *and write* $\mathrm{Fil}^{a+}_{\mathrm{dif}}G_K$ *for the closure of* $\cup_{b>a}\mathrm{Fil}^b_{\mathrm{dif}}G_K$. *This defines a differential filtration on* G_K *such that for any finite image representation* ρ, $\rho(\mathrm{Fil}^a_{\mathrm{dif}}G_K)$ *is trivial if and only if* $\rho \in R_a$.

Similarly, put $\mathrm{Fil}^0_{\mathrm{dif,log}}G_K = G_K$. *For* $a > 0$, *let* $R_{a,\log}$ *be the set of finite image representations* ρ *with logarithmic differential ramification break less than* a. *Define* $\mathrm{Fil}^a_{\mathrm{dif,log}}G_K = \bigcap_{\rho \in R_{a,\log}} (I_K \cap \mathrm{Ker}(\rho))$ *and write* $\mathrm{Fil}^{a+}_{\mathrm{dif,log}}G_K$ *for the closure of* $\cup_{b>a}\mathrm{Fil}^b_{\mathrm{dif,log}}G_K$. *This defines a differential logarithmic filtration on* G_K *such that for any finite image representation* ρ, $\rho(Fil^a_{\mathrm{dif,log}}G_K)$ *is trivial if and only if* $\rho \in R_{a,\log}$.

Moreover,

$$\text{for } a > 0,\ \mathrm{Fil}^a_{\mathrm{dif}}G_K/\mathrm{Fil}^{a+}_{\mathrm{dif}}G_K = \begin{cases} 0 & a \notin \mathbb{Q} \\ \text{an abelian group killed by } p & a \in \mathbb{Q} \end{cases}$$

$$\text{for } a > 1,\ \mathrm{Fil}^a_{\mathrm{dif,log}}G_K/\mathrm{Fil}^{a+}_{\mathrm{dif,log}}G_K = \begin{cases} 0 & a \notin \mathbb{Q} \\ \text{an abelian group killed by } p & a \in \mathbb{Q} \end{cases}$$

Proof. For (0), see [Ked05a, Theorem 5.23].

To prove the rest of the statement, as in [Ked07a, Section 3.5], we may first reduce to the case when Hypothesis 3.2.2.1 holds.

(1) This is Theorem 1.2.8.2.

(2) If L/K is unramified, we can use the same π_K as the uniformizer of L. The corresponding differential module $\mathcal{E}_{\rho'}$ of ρ' is just a simple extension of scalar. Since the calculation of spectral norms does not depend on the base field (Remark 1.1.2.7), we compute the same result on spectral norms and hence have the same Artin conductor. If L/K is tamely ramified with ramification degree e', we may first make a further unramified extension and assume that the $L = K(\pi_K^{1/e'})$. Hence, the corresponding differential module $\mathcal{E}_{\rho'}$ of ρ' is just the tame pullback and the statement follows from Proposition 1.2.1.7.

(3) is an immediate consequence of (2). But be caution that the differential

ramification breaks can not distinguish unramified extensions from tamely ramified extensions (see also Remark 3.2.3.3).

(4) The proof for Swan conductors is in [Ked07a, Theorem 3.5.13], which we will not repeat. The proof for the non-logarithmic differential filtration is much simpler than the logarithmic case because of the different normalization in the Subsection 1.2.8. Indeed, it suffices to show that we can do some rotation so that ∂_0 becomes dominant; this is exactly the content of Subsection 1.2.7. □

Remark 3.2.3.6. The converse of (3) is a well-known fact for experts. However, we are unable to find good references to support a proof. As it will become an easy consequence of the comparison Theorem 3.4.3.1 and properties of arithmetic ramification conductors (Proposition 2.2.2.11(6)), we do not state it here.

Remark 3.2.3.7. Note that the invariance of the differential conductors under unramified base changes enables us to assume that κ_0 is algebraically closed. This justifies the assumption we made in Hypothesis 3.2.1.5.

3.2.4 Breaks by p-basis

In this subsection, we try to fix Fake-assumption 2.3.6.8. Please consult Remark 2.3.6.7 for motivation.

We keep Hypothesis 3.2.2.1 for this subsection and we keep the notation from previous subsections.

Proposition 3.2.4.1. *For each $j \in J^+$, there is a ramification break $b_j(L/K)$ associated to b_j ($j \in J$) or π_K ($j = 0$), such that $R_{\partial_j}(\mathcal{E}_\rho \otimes F'_\eta) = \eta^{b_j}$ for all $\eta \to 1^-$. Hence,*

$$b(\rho) = \max_{j \in J^+}\{b_j(\rho)\}, \quad b_{\log}(\rho) = \max\{b_0(\rho) - 1; b_j(\rho) \textit{ for } j \in J\}.$$

Proof. By applying the same argument as in the proof of Proposition 3.2.2.4, we know $IR_{\partial_j}(\mathcal{E}_\rho \otimes F'_{\eta^q}) = IR_{\partial_j}(\mathcal{E}_\rho \otimes F'_\eta)^q$ for $\eta \to 1^-$. Therefore, by the convexity given by Theorem 1.2.2.6(d), $f_1^{(j)}(\mathcal{E}_\rho, r)$ is affine as $r \to 0^+$. The proposition follows. □

Definition 3.2.4.2. We call $b_{J^+}(\rho)$ the *breaks by p-basis* of ρ with respect to the lifted p-basis b_J and the uniformizer π_K.

When doing operation on K, we want to understand the corresponding effect on the $b_j(\rho)$.

Lemma 3.2.4.3. *Fix $j_0 \in J$. Let $b'_{J^+}(\rho)$ be the breaks by p-basis of ρ with respect to the lifted p-basis $\{b_{J\setminus\{j_0\}}, b_{j_0} + \pi_K\}$ and the uniformizer π_K. Then $b'_j(\rho) = b_j(\rho)$ for $j \in J$ and*

$$b'_0(\rho) \begin{cases} = \max\{b_0(\rho), b_{j_0}(\rho)\} & \text{if } b_0(\rho) \neq b_{j_0}(\rho), \\ \leq b_0(\rho) & \text{if } b_0(\rho) = b_{j_0}(\rho). \end{cases}$$

Proof. Let ∂'_{J^+} denote the derivation dual to the basis $dB_{J\setminus\{j_0\}}, dS, d(B_{j_0} + S)$ of $\Omega^1_{\mathcal{O}_F[\![S]\!]/\mathcal{O}_{F_0}}$, as in Notation 3.2.2.2. Then $\partial'_J = \partial_J$ and $\partial'_0 = \partial_0 + \partial_{j_0}$. The lemma follows immediately. □

Remark 3.2.4.4. This lemma is in fact much stronger than it looks. Applying the same argument to $b_{j_0} + \alpha\pi_K$ for $\alpha \in \kappa_0$, we find out that for all but possibly one $\alpha \in \kappa_0$, $b_0(\rho) \geq b_j(\rho)$. So, the direction for uniformizer is "generically dominant". This motivates the following lemma.

Lemma 3.2.4.5. *Fix $j_0 \in J$. Let K' be the completion of $K(x)$ with respect to the 1-Gauss norm, equipped with lifted p-basis $\{b_{J\setminus\{j_0\}}, b_{j_0} + x\pi_K, x\}$ and the uniformizer π_K. Let ρ' be the representation $G_{K'} \to G_K \xrightarrow{\rho} GL(V_\rho)$. Let $b'_{J^+\cup\{m+1\}}(\rho')$ denote the breaks by p-basis with respect to the lifted p-basis and the uniformizer above, where $b'_{J\setminus\{j_0\}}(\rho')$ corresponds to $b_{J\setminus\{j_0\}}$, $b'_{j_0}(\rho')$ corresponds to $b_{j_0} + x\pi_K$, $b'_0(\rho')$ corresponds to π_K, and $b'_{m+1}(\rho')$ corresponds to x. Then we have $b'_j(\rho') = b_j(\rho)$ for $j \in J$, $b'_{m+1}(\rho') = b_{j_0}(\rho) - 1$, $b'_0(\rho') = \max\{b_0(\rho), b_{j_0}(\rho)\}$.*

Proof. Let $\widetilde{F}'$ denote the completion of $F'(X)$ with respect to the 1-Gauss norm, where X is a lift of the x. Let $f : A^1_{\widetilde{F}'}[\eta_0, 1) \to A^1_{F'}[\eta_0, 1)$ be the natural morphism and then $f^*\mathcal{E}_\rho$ is the differential module associated to ρ'. Let $\partial'_{J^+\cup\{m+1\}}$ be the differential operators corresponding to the lifted basis as in the lemma. Then under the identification by f^*, we have

$$\partial'_J = \partial_J, \quad \partial'_{m+1} = S\partial_{j_0}, \quad \partial'_0 = \partial_0 + X\partial_{j_0}.$$

The lemma follows because X is transcendental over F'. □

Lemma 3.2.4.6. *Fix $j_0 \in J$. Denote $K' = K(b_{j_0}^{1/p})$ equipped with lifted p-basis $\{b_{J\backslash\{j_0\}}, b_{j_0}^{1/p}\}$. Let $b'_{J^+}(\rho|_{G_{K'}})$ be the breaks by p-basis of $\rho|_{G_{K'}}$ with respect to the p-basis above. Then $b'_j(\rho) = b_j(\rho)$ for $j \in J^+\backslash\{j_0\}$ and $b'_{j_0}(\rho|_{G_{K'}}) = \frac{1}{p}b_{j_0}(\rho)$.*

Proof. Replacing K by K' is equivalent to use $\varphi^{(\partial_j)}$ to pullback the differential module $\mathcal{E}_\rho$. The lemma follows from Corollary 1.1.4.26 applying to $\mathcal{E} \otimes F'_\eta$ when $\eta \to 1^-$. □

Lemma 3.2.4.7. *Fix $j_0 \in J$. Denote K' the completion of $K(b_{j_0}^{1/p^n}; n \in \mathbb{N})$ equipped with lifted p-basis $b_{J\backslash\{j_0\}}$. Let $b'_{J^+}(\rho|_{G_{K'}})$ be the breaks by p-basis of $\rho|_{G_{K'}}$ with respect to the p-basis above. Then $b'_j(\rho) = b_j(\rho)$ for $j \in J^+\backslash\{j_0\}$.*

Proof. Obvious. □

As promised earlier (Subsection 2.1.2 and Remark 2.4.2.2), we can also show that the differential Artin conductor is invariant under the operations of adding generic p^∞th roots. This would establish the comparison with Borger's conductor by Proposition 2.4.2.1 (and by invoking the comparison Theorem 3.4.3.1).

Proposition 3.2.4.8. *Fix $j_0 \in J$. Let K' be the field obtained after adding a generic p^∞-th root of b_{j_0} and let ρ' be the representation $G_{K'} \to G_K \xrightarrow{\rho} GL(V_\rho)$. Then $b(\rho) = b(\rho')$.*

Proof. Let K_1 be the completion of $K(x)$ with respect to the 1-Gauss norm and let K_2 be the completion of the maximal unramified extension of K_1. Let ρ_2 denote the representation $G_{K_2} \to G_K \xrightarrow{\rho} GL(V_\rho)$. By Lemma 3.2.4.5, $b(\rho_2) = b(\rho)$ and if $b'_{J^+\cup\{m+1\}}(\rho_2)$ denotes the breaks by p-basis with respect to the lifted p-basis as in Lemma 3.2.4.5, then $b'_{j_0}(\rho_2) \leq b'_0(\rho_2)$. The proposition follows by applying Lemma 3.2.4.7 to K_2 with respect to the $b_{j_0} + x\pi_K$. □

3.2.5 Refined Swan conductors

In this subsection, we define the refined Swan conductors. This provides secondary information of the graded pieces of the differential ramification filtration. We keep the notation as in previous subsections and continue to assume Hypothesis 3.2.2.1.

Notation 3.2.5.1. Fix a Dwork pi $\boldsymbol{\pi} = (-p)^{1/(p-1)}$ for this subsection.

Construction 3.2.5.2. Let ρ be a p-adic representation of G_K of pure log-break $b = b_{\log}(\rho)$. Let $\mathcal{E}_\rho$ denote the (ϕ, ∇)-module associated to ρ. By Theorem 3.2.3.1, there exists $\eta_0 \in (0,1)$ such that $IR(\mathcal{E}_\rho \otimes F'_\eta) = \eta^b$ for $\eta \in [\eta_0, 1)$. By Theorem 1.2.6.7, after making a finite *unramified* extension of F', $\mathcal{E}_\rho$ admits a decomposition

$$\mathcal{E}_\rho = \bigoplus_{\vartheta \in \mathcal{I}\Theta(\mathcal{E}_\rho)} \mathcal{E}_\vartheta \tag{3.2.5.3}$$

over $A^1_{F'(\boldsymbol{\pi})}(\eta_0^{1/d}, 1)$ with d the prime-to-p part of the denominator of b, where $\mathcal{E}_\vartheta$ has pure refined intrinsic radii ϑ. (Here we only need to make an unramified extension of F' because $\omega^{-1}IR(\mathcal{E}_\rho \otimes F'_\eta)/\eta^b \in |F'(\boldsymbol{\pi})^\times|$ and an unramified extension of F' is enough to have (1.2.6.8) with each of M_ϑ with pure intrinsic radii.)

We define the set of *refined Swan conductors* of ρ to be

$$\mathrm{rsw}(\rho) = \big\{\frac{1}{\boldsymbol{\pi}}\vartheta\pi_K^{-b} \mid \vartheta \in \mathcal{I}\Theta(\mathcal{E}_\rho)\big\} \subset \Omega^1_K(\log) \otimes \pi_K^{-b}\kappa_{K^{\mathrm{alg}}}.$$

Lemma 3.2.5.4. *Construction 3.2.5.2 of refined Swan conductor does not depend on the choice of lifted p-basis of K or the uniformizer π_K.*

Proof. For another choice of lifted p-basis and uniformizer, we will end up considering another set of differential operators $\partial'_j = d/dB'_j$ for $j \in J$ and $\partial'_0 = d/dS'$. We assume that

$$\frac{dB'_{j'}}{B'_{j'}} = \sum_{j\in J} \alpha_{j',j}\frac{dB_j}{B_j} + \alpha_{j',0}\frac{dS}{S} \text{ for } j' \in J,$$
$$\frac{dS'}{S'} = \sum_{j\in J} \alpha_{0,j}\frac{dB_j}{B_j} + \alpha_{0,0}\frac{dS}{S} \text{ for } j' \in J,$$

where $\alpha_{j',j} \in \mathcal{O}_F[\![S]\!]$ for $j, j' \in J^+$.

Moreover, we may assume that $\mathcal{E}_\rho$ has pure differential log-break and has pure refined Swan conductors with respect to the lifted p-basis b_J and uniformizer π_K. Furthermore, we may assume that $\mathcal{E}_\rho$ (by gluing Frobenius antecedent) extends to $[\eta_0, 1)$ such that $\mathcal{E}_\rho \otimes F'_\eta$ has visible intrinsic radii for some $\eta \in (\eta_0, 1)$, and hence has pure refined intrinsic radii $\theta_0 \frac{dS}{S} + \theta_1 \frac{dB_1}{B_1} + \cdots + \theta_m \frac{dB_m}{B_m}$.

By applying Proposition 1.1.6.16, we have, for any $j \in J^+$,

$$\mathcal{I}\Theta_{\partial'_j}(\mathcal{E}_\rho \otimes F'_\eta) = \left\{\alpha_{j,0}\theta_0 + \cdots + \alpha_{j,m}\theta_m \ (\mathrm{rank}\,(\mathcal{E}_\rho)\ \text{times})\right\} \subset \kappa^{(s)}_{K^{\mathrm{alg}}}$$

for some s, if $IR_{\partial'_j}(\mathcal{E}_\rho \otimes F'_\eta) = IR(\mathcal{E}_\rho \otimes F'_\eta)$; we use J_0 denote those j for which this is the case. By Proposition 1.1.6.16, we also have $\alpha_{j,0}\theta_0 + \cdots + \alpha_{j,m}\theta_m = 0$ in $\kappa^{(s)}_{K^{\mathrm{alg}}}$ for $j \in J \backslash J_0$. Note that, we have

$$\begin{aligned}&\big(\alpha_{0,0}\theta_0 + \cdots + \alpha_{0,m}\theta_m\big)\frac{dS'}{S} + \sum_{j \in J}\big(\alpha_{j,0}\theta_0 + \cdots + \alpha_{j,m}\theta_m\big)\frac{dB'_j}{B'_j} \\ &= \theta_0\frac{dS}{S} + \theta_1\frac{dB_1}{B_1} + \cdots + \theta_m\frac{dB_m}{B_m}.\end{aligned}$$

Hence, they compute the same refined intrinsic radii and the lemma is proved. □

Theorem 3.2.5.5. *Let K be a complete discretely valued field of equal characteristic $p > 0$.*

(a) *Let ρ be a p-adic representation of G_K with finite local monodromy which has pure (differential) log-break $b = b_{\log}(\rho)$. Then there exists a finite tamely ramified extension K'/K such that, we have a canonical decomposition of representations of $G_{K'}$ over a finite extension of* Frac$(\mathcal{O})$:

$$\rho|_{G_{K'}} = \bigoplus_{\vartheta \in \mathrm{rsw}(\rho)} \rho_\vartheta,$$

where ρ_ϑ has pure refined Swan conductors $\vartheta \in \Omega^1_K(\log) \otimes \pi_K^{-b}\kappa_{K^{\mathrm{alg}}}$. By Galois descent, we have a decomposition of $\rho = \oplus_{\{\vartheta\} \subset \mathrm{rsw}(\rho)} \rho_{\{\vartheta\}}$, where the direct sum *runs through all Galois conjugacy classes in* $\mathrm{rsw}(\rho)$ *and* $\mathrm{rsw}(\rho_{\{\vartheta\}})$ *consists of only the Galois conjugacy class* $\{\vartheta\}$ *(with same multiplicity on each element).*

(b) *The refined Swan conductor gives rise to an* injective *homomorphism for* $b \in \mathbb{Q}_{\geq 0}$,

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{Fil}^{b}_{\mathrm{dif,log}} G_K / \mathrm{Fil}^{b+}_{\mathrm{dif,log}} G_K, \mathbb{F}_p) \to \Omega^1_K(\log) \otimes \pi_K^{-b} \kappa_{K^{\mathrm{alg}}}. \qquad (3.2.5.6)$$

Proof. (a) We first replace K and $\mathrm{Frac}(\mathcal{O})$ by an unramified extension of K and a finite extension of $\mathrm{Frac}(\mathcal{O})$, respectively, corresponding to the unramified extension of F' made for the decomposition (3.2.5.3) (so that Hypothesis 3.2.1.5 holds). Since the decomposition is canonical, it is a decomposition for (ϕ, ∇)-modules. By the slope filtration [Ked07a, Theorem 3.4.6], the Frobenius action on each direct summand of $\mathcal{E}$ is of unit-root; the decomposition of the representation follows by the equivalence of categories in Theorem 3.2.1.15.

(b) We need to show the following.

(i) for any p-adic representations ρ of pure break b and any p-adic representations ρ' of break smaller than b, $\mathrm{rsw}(\rho \otimes \rho')$ is just $\dim \rho'$ copies of $\mathrm{rsw}(\rho)$;

(ii) for any p-adic representations ρ and ρ' of pure break b and of pure refined Swan conductor ϑ, then $\rho \otimes \rho'^{\vee}$ has smaller break.

They follows from Lemmas 1.1.5.12 and 1.1.5.12, respectively. □

Remark 3.2.5.7. It would be interesting to know if the homomorphism rsw defined here coincides with the one defined in [Sai07+, Corollary 1.3.4]. The choice of the Dwork pi here is expected to correspond to the choice of the Artin-Scheier sheaf in [Sai07+].

We temporarily drop Hypothesis 3.2.2.1 for the following proposition. It is an analogue of Conjecture 2.2.2.21 for differential ramification. We will see in Corollary 3.4.3.5 that Proposition 3.2.5.8 implies Conjecture 2.2.2.21 for the equal characteristic case.

Proposition 3.2.5.8. *Let K be a complete discretely valued field of equal characteristic $p > 0$. Then for $a \in \mathbb{Q}_{>0}$, the conjugation action of $\mathrm{Fil}^{0+}_{\mathrm{dif,log}}/\mathrm{Fil}^{a}_{\mathrm{dif,log}}G_K$ on $\mathrm{Fil}^{a}_{\mathrm{dif,log}}G_K/\mathrm{Fil}^{a+}_{\mathrm{dif,log}}G_K$ is trivial.*

Proof. It suffices to show that for a p-adic representation ρ of G_K with finite local monodromy which has pure (differential) log-break b, if it is absolutely irreducible under any tamely ramified extension, then $\rho|_{\mathrm{Fil}^b_{\mathrm{dif,log}}G_K/\mathrm{Fil}^{b+}_{\log}G_K}$ is a direct sum of a *single* character $\chi : \mathrm{Fil}^b_{\log}G_K/\mathrm{Fil}^{b+}_{\log}G_K \to \mathcal{O}^\times$. This is equivalent to showing that the action of $\mathrm{Fil}^b_{\mathrm{dif,log}}$ on $\rho \otimes \rho^\vee$ is trivial, and to showing that $\rho \otimes \rho^\vee$ has smaller log-break.

Since only finitely many elements in a lifted p-basis can be dominant in $\mathcal{E}_\rho$, we may assume Hypothesis 3.2.2.1. By Theorem 3.2.5.5(a), such condition implies that ρ must have pure refined Swan conductor and hence $\rho \otimes \rho^\vee$ must have smaller log-break. □

3.3 Thickening technique

In this section, we introduce a thickening technique. Vaguely speaking, it is to construct a reasonable object which can be thought of as a tubular neighborhood of the "diagonal embedding of $A^1_F[\eta_0, 1)$ into $A^1_F[\eta_0, 1) \times_{F_0} A^1_F[\eta_0, 1)$". Be caution that the latter rigid space is not really well-defined.

We first start with a geometric interpretation of this construction and then move on to the abstract definition of the thickening space.

We keep the Hypothesis 3.2.2.1 throughout this section.

3.3.1 Geometric thickening

In this subsection, we describe the thickening technique when the residue field κ_K can be realized as the field of rational functions on a smooth κ_0-variety. We hope this can provide some geometric intuition of the thickening construction in the next subsection; the content in this subsection will not be used in the rest of the thesis.

Hypothesis 3.3.1.1. *Only* in this subsection, we assume that the field κ_K is a finite separable extension of $\kappa_0(\bar{b}_1, \ldots, \bar{b}_m)$.

Construction 3.3.1.2. Let $\overline{X}$ be a smooth variety over κ_0 whose field of rational functions is κ_K; such an $\overline{X}$ exists because we may realize it as an open subscheme of an étale cover of $\mathrm{Spec}\,\kappa_0[\bar{b}_1, \dots, \bar{b}_m]$ which induces the extension $\kappa_K/\kappa_0(\bar{b}_1, \dots, \bar{b}_m)$. We may further shrink $\overline{X}$ so that it is the special fiber of a smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_{F_0}$ of topological finite type, i.e. $\mathfrak{X} \times_{\mathrm{Spf}\mathcal{O}_{F_0}} \mathrm{Spec}\,\kappa_0 = \overline{X}$. We may further shrink $\mathfrak{X}$ and $\overline{X}$ so that we have lifts $B_1, \dots, B_m$ of $\bar{b}_1, \dots, \bar{b}_m$ on $\mathfrak{X}$ and $dB_1, \dots, dB_m$ form a basis of the sheaf of differentials $\Omega^1_{\mathfrak{X}/\mathcal{O}_{F_0}}$. We use $\mathbf{X}$ to denote the "generic fiber" of $\mathfrak{X}$ as a rigid space over $\mathrm{Spm}(F_0)$, in the sense of Raynaud.

Consider the following commutative diagram

$$\begin{array}{ccccc}
\overline{X} & \longrightarrow & \mathfrak{X} & \longleftarrow & \mathbf{X}\\
\downarrow && \downarrow && \downarrow\\
\overline{P}=\overline{X}\times_{\kappa_0}\mathbb{A}^1_{\kappa_0} & \longrightarrow & \mathcal{P}=\mathfrak{X}\times_{\mathrm{Spf}\mathcal{O}_{F_0}}\mathbb{A}^1_{\mathcal{O}_{F_0}} & \longleftarrow & \mathbf{P}=\mathbf{X}\times_{F_0}A^1_{F_0}[0,1]\\
\downarrow && \downarrow && \downarrow\\
\mathrm{Spec}\,\kappa_0 & \longrightarrow & \mathrm{Spf}\mathcal{O}_{F_0} & \longleftarrow & \mathrm{Spm}(F_0)
\end{array}$$

where the vertical arrows from the first row to the second row are all embedding of zero sections and the coordinates of $\mathbb{A}^1_{\kappa_0}$ and $\mathbb{A}^1_{\mathcal{O}_{F_0}}$ are π_K and S, respectively.

The tube of $\overline{X}$ in $\mathbf{P}$, denoted by $]\overline{X}[_{\mathcal{P}}$, is isomorphic to $\mathbf{X} \times A^1_{F_0}[0, 1)$. Let $\mathcal{O}_{\mathbf{X}}$ be the ring of rigid analytic functions on $\mathbf{X}$; then F is exactly the p-adic completion of $\mathrm{Frac}\mathcal{O}_{\mathbf{X}}$. If we base change the tube $]\overline{X}[_{\mathcal{P}}$ from $\mathbf{X}$ over to F, we get $A^1_F[0, 1)$. We are interested in the annulus $A^1_F[\eta_0, 1)$ for some $\eta_0 \in (0, 1)$, which can be obtained from base changing $\mathbf{X} \times A^1_{F_0}[\eta_0, 1)$ from $\mathbf{X}$ to F.

Now, we consider the thickening space of this annulus $A^1_F[\eta_0, 1)$.

Construction 3.3.1.3. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
\overline{X} & \hookrightarrow & \overline{P} & \xrightarrow{\Delta_{\overline{P}}} & \overline{P}\times_{\kappa_0}\overline{P} & \hookrightarrow & \mathcal{P}\times_{\mathcal{O}_{F_0}}\mathcal{P} & \longleftarrow & \mathbf{P}\times_{F_0}\mathbf{P}\\
 & \searrow & \downarrow & \swarrow & & & \downarrow & & \downarrow\\
 & & \mathrm{Spec}\,\kappa_0 & & \longrightarrow & & \mathrm{Spf}\mathcal{O}_{F_0} & \longleftarrow & \mathrm{Spm}(F_0)
\end{array}$$

where we denote $\mathrm{pr}_i : \mathcal{P} \times_{\mathcal{O}_{F_0}} \mathcal{P} \to \mathcal{P}$ the projection to the i-th factor, $i = 1, 2$. Then

$\mathcal{P} \times_{\mathcal{O}_{F_0}} \mathcal{P}$ has a set of local parameter given by $B_1 = \mathrm{pr}_1^*(B_1), \dots, B_m = \mathrm{pr}_1^*(B_m), S = \mathrm{pr}_1^*(S), B_1' = \mathrm{pr}_2^*(B_1), \dots, B_m' = \mathrm{pr}_2^*(B_m), S' = \mathrm{pr}_2^*(S)$. By Berthelot's Fibration Theorem [Brt96+, THÉORÈME 1.3.2], we have an isomorphism

$$]\overline{X}[_{\mathcal{P} \times_{\mathcal{O}_{F_0}} \mathcal{P}} \simeq]\overline{X}[_{\mathcal{P}} \times_{F_0} A_{F_0}^{m+1}[0,1),$$

where the factor $]\overline{X}[_{\mathcal{P}}$ respects the projection pr_1 and the coordinates for the open polydisc on the right hand side are given by $\delta_0 = S - S', \delta_1 = B_1 - B_1', \dots, \delta_m = B_m - B_m'$. The geometric thickening space is the subspace of $]\overline{X}[_{\mathcal{P} \times_{\mathcal{O}_{F_0}} \mathcal{P}}$ where $|\delta_0| = |S - S'| < |S|$, or more precisely,

$$\mathbf{X} \times_{F_0} \left\{ (S, \delta_0) \in A_{F_0}^2[0,1) \big| |\delta_0| < |S| \right\} \times_{F_0} A_{F_0}^m[0,1).$$

Thus, *the thickening space*, denoted by TS_K^a, of $A_F^1[\eta_0, 1)$ is the space obtained by base changing

$$\mathbf{X} \times_{F_0} \left\{ (S, \delta_0) \in A_{F_0}^2[0,1) \big| |S| \geq \eta_0, |\delta_0| < |S| \right\} \times_{F_0} A_{F_0}^m[0,1).$$

from $\mathbf{X}$ to F.

The projection $\mathrm{pr}_1 : \mathbf{P} \times_{F_0} \mathbf{P} \to \mathbf{P}$ gives a F-morphism of rigid spaces $\pi : TS_K^a \to A_F^1[\eta_0, 1)$; the projection $\mathrm{pr}_2 : \mathbf{P} \times_{F_0} \mathbf{P} \to \mathbf{P}$ gives a F_0-morphism of rigid spaces $\tilde{\pi} : TS_K^a \to A_F^1[\eta_0, 1)$. The morphism $\tilde{\pi}$ does not respect the F-rigid space structure.

3.3.2 General thickening construction

In this subsection, we introduce the thickening spaces and study basic properties of differential modules over them.

We keep Hypothesis 2.3.2.1 in this subsection. Note that Hypothesis 3.3.1.1 is no longer in force from now on.

Definition 3.3.2.1. For $\eta \in (0,1)$, we write $Z_K^\eta = A_F^1[\eta, \eta]$. For $a \in \mathbb{Q}_{>1}$ and

$\eta_0 \in (0,1)$, we define the *thickening space (of $A^1_K[\eta_0, 1)$ and level a)* to be

$$TS_K^{a,\geq\eta_0} = \big\{(S, \delta_0, \dots, \delta_m) \in A_F^{m+2}[0,1)\big||S| \geq \eta_0; |\delta_j| \leq |S|^a \text{ for } j \in J^+\big\}. \quad (3.3.2.2)$$

For $\eta \in [\eta_0, 1)$, we put

$$TS_K^{a,\eta} = A_F^1[\eta, \eta] \times_F A_F^{m+1}[0, \eta^a].$$

Similarly, for $a \in \mathbb{Q}_{>0}$ and $\eta_0 \in (0,1)$, we define the *log-thickening space (of $A^1_K[\eta_0, 1)$ and level a)* to be

$$TS_{K,\log}^{a,\geq\eta_0} = \big\{(S, \delta_0, \dots, \delta_m) \in A_F^{m+2}[0,1)\big||S| \geq \eta_0; |\delta_0| \leq |S|^{a+1}; |\delta_j| \leq |S|^a \text{ for } j \in J\big\}. \quad (3.3.2.3)$$

For $\eta \in [\eta_0, 1)$, denote

$$TS_{K,\log}^{a,\eta} = A_F^1[\eta, \eta] \times_F A_F^1[0, \eta^{a+1}] \times_F A_F^m[0, \eta^a].$$

As in Notation 3.2.1.17, we may drop the superscript $\geq \eta_0$ for simplicity.

Caution 3.3.2.4. One may want to write $TS_K^{a,\geq\eta_0} = \bigcup_{\eta\in[\eta_0,1)} A_F^1[\eta, 1) \times_F A_F^{m+1}[0, \eta^a]$ for simplicity as in the introduction. However, this will not define the same space as in (3.3.2.2), because the union does *not* give an admissible cover of $TS_K^{a,\geq\eta_0}$. Similar expression for log-thickening space is not valid either. Nevertheless, it might be helpful to think the space and picture the geometry this way.

Remark 3.3.2.5. We need $a \in \mathbb{Q}$ in Definition 3.3.2.1 to make sure that (3.3.2.2) and (3.3.2.3) actually define (Berkovich) rigid analytic space. For individual $TS_K^{a,\eta}$ and $TS_{K,\log}^{a,\eta}$, one may just take $a \in \mathbb{R}$.

Notation 3.3.2.6. Let $|\cdot|_{Z_K^\eta}$ denote the η-Gauss norm on Z_K^η. For $a \in \mathbb{Q}_{>1}$, let $|\cdot|_{TS_K^{a,\eta}}$ denote the Gauss norm on $TS_K^{a,\eta}$; for $a > 0$, let $|\cdot|_{TS_{K,\log}^{a,\eta}}$ denote the Gauss norm on $TS_{K,\log}^{a,\eta}$.

Notation 3.3.2.7. For $a \in \mathbb{Q}_{>1}$ (resp. $a \in \mathbb{Q}_{>0}$) and $\eta_0 \in (0,1)$, denote the natural embedding of $Z_K^{\geq \eta_0}$ into the locus where $\delta_j = 0$ for $j \in J^+$ by $\Delta : Z_K^{\geq \eta_0} \hookrightarrow TS_K^{a,\geq \eta_0}$ (resp. $\Delta : Z_K^{\geq \eta_0} \hookrightarrow TS_{K,\log}^{a,\geq \eta_0}$). Also, we have the naïve projection $\pi : TS_K^{a,\geq \eta_0} \to Z_K^{\geq \eta_0}$ (resp. $\pi : TS_{K,\log}^{a,\geq \eta_0} \to Z_K^{\geq \eta_0}$) by projecting to the first factor. These morphisms are compatible when changing a and η_0, or replacing $\geq \eta_0$ by η for some $\eta \in [\eta_0, 1)$.

To simplify notation, for a and η_0 as above, we identify $\mathcal{O}_{Z_K^{\geq \eta_0}}$ as a subring of $\mathcal{O}_{TS_K^{a,\geq \eta_0}}$ and $\mathcal{O}_{TS_{K,\log}^{a,\geq \eta_0}}$ via π^*; same for η instead of $\geq \eta_0$. It is worthwhile to point out that π^* is an isometry; hence the identification will not change any calculation on norms.

Proposition 3.3.2.8. *We have a unique continuous $\mathcal{O}_{F_0}$-homomorphism $\tilde{\pi}^* : \mathcal{O}_F[\![S]\!] \to \mathcal{O}_F[\![S, \delta_{J^+}]\!]$ such that $\tilde{\pi}^*(S) = S + \delta_0$ and $\tilde{\pi}^*(B_j) = B_j + \delta_j$ for all $j \in J$. Moreover, for $g \in \mathcal{O}_F$, $\tilde{\pi}^*(g) - g \in (\delta_1, \dots, \delta_m)(g)\mathcal{O}_F[\![\delta_1, \dots, \delta_m]\!]$.*

Proof. It follows from Corollary 2.3.4.7 immediately. □

Theorem 3.3.2.9. *For $a \in \mathbb{Q}_{>1}$ (resp. $a \in \mathbb{Q}_{>0}$) and $\eta_0 \in (0,1)$, the homomorphism $\tilde{\pi}^*$ induces a F_0-morphism $\tilde{\pi} : TS_K^{a,\geq \eta_0} \to Z_K^{\geq \eta_0}$ (resp. $\tilde{\pi} : TS_{K,\log}^{a,\geq \eta_0} \to Z_K^{\geq \eta_0}$) such that $\tilde{\pi} \circ \Delta = \mathrm{id}$; same if replacing $\geq \eta_0$ by η for some $\eta \in [\eta_0, 1)$.*

For any $g \in \mathcal{O}_{Z_K^\eta}$ and for $a > 1$ (resp. $a > 0$),

$$|\tilde{\pi}^*(g) - g|_{TS_K^{a,\eta}} \leq \eta^{a-1} \cdot |g|_{Z_K^\eta} \quad (\textit{resp. } |\tilde{\pi}^*(g) - g|_{TS_{K,\log}^{a,\eta}} \leq \eta^a \cdot |g|_{Z_K^\eta}). \tag{3.3.2.10}$$

In particular, $|\tilde{\pi}^*(g)|_{TS_K^{a,\eta}} = |\tilde{\pi}^*(g)|_{TS_{K,\log}^{a,\eta}} = |g|_{Z_K^\eta}$. Moreover, we have the following bound for $TS_K^{a,\eta}$: if $g \in \mathcal{O}_{Z_K^\eta} \cap \mathcal{O}_F[\![S]\!]$, then

$$|\tilde{\pi}^*(g) - g|_{TS_K^{a,\eta}} \leq \eta^a. \tag{3.3.2.11}$$

Proof. We need only to establish the bound on the norms. Let $g = \sum_{i \in \mathbb{Z}} a_i S^i \in F\langle \eta/S, S/\eta \rangle$, we have

$$\tilde{\pi}^*(g) - g = \sum_{i \in \mathbb{Z}} \left(\tilde{\pi}^*(a_i)(S+\delta_0)^i - a_i S^i\right) = \sum_{i \in \mathbb{Z}} \left((\tilde{\pi}^*(a_i) - a_i)(S+\delta_0)^i + a_i((S+\delta_0)^i - S^i)\right). \tag{3.3.2.12}$$

Since $\tilde{\pi}^*(a_i) - a_i \in (\delta_1, \dots, \delta_m)(a_i)\mathcal{O}_F[\![\delta_1, \dots, \delta_m]\!]$, we have

$$|\tilde{\pi}(a_i) - a_i|_{TS_K^{a,\eta}} \leq |a_i|\eta^a, \quad |\tilde{\pi}(a_i) - a_i|_{TS_{K,\log}^{a,\eta}} \leq |a_i|\eta^a. \tag{3.3.2.13}$$

Moreover, we have can bound $(S+\delta_0)^i - S^i$ by

$$\left|(S+\delta_0)^i - S^i\right|_{TS_K^{a,\eta}} \leq \eta^{a+i-1}, \quad \left|(S+\delta_0)^i - S^i\right|_{TS_{K,\log}^{a,\eta}} \leq \eta^{a+i}. \tag{3.3.2.14}$$

Plugging the estimates (3.3.2.13) and (3.3.2.14) into (3.3.2.12), we obtain (3.3.2.10). When $g \in \mathcal{O}_F[\![S]\!]$, (3.3.2.14) always gives $\left|(S+\delta_0)^i - S^i\right|_{TS_K^{a,\eta}} \leq \eta^a$ for $i \geq 0$ (when $i = 0$, we have zero); the equation (3.3.2.11) follows. $\square$

Remark 3.3.2.15. For $a > 0$, one can factor the morphism $\tilde{\pi}$ for non-log thickening space as $TS_K^{a+1,\geq\eta_0} \to TS_{K,\log}^{a,\geq\eta_0} \xrightarrow{\tilde{\pi}} Z_K^{\geq\eta_0}$, where the second morphism is the $\tilde{\pi}$ for the log-thickening space.

Notation 3.3.2.16. For a ∇-module over $Z_K^{\geq\eta_0}$, we call $\tilde{\pi}^*\mathcal{E}$ the *thickened differential module* of $\mathcal{E}$, denoted by $\mathcal{F}$. We view $\mathcal{F}$ as a differential module over $TS_K^{a,\geq\eta_0}$ or $TS_{K,\log}^{a,\geq\eta_0}$ with respect to the differential operators $\partial/\partial\delta_0, \dots, \partial/\partial\delta_m$, i.e. we view $\mathcal{F}$ as a differential module on $TS_K^{a,\geq\eta_0}$ or $TS_{K,\log}^{a,\geq\eta_0}$ relative to $Z_K^{\geq\eta_0}$.

Proposition 3.3.2.17. *Let $\eta \in [\eta_0, 1)$. The radii of convergence of ∂_{J^+} on $\mathcal{E}_\eta$ over Z_K^η and the radii of convergence of $\partial/\partial\delta_{J^+}$ on $\mathcal{F}_{a,\eta} = \mathcal{F} \otimes \mathrm{Frac}(\mathcal{O}_{TS_K^{a,\eta}})^\wedge$ and $\mathcal{F}_{a,\eta,\log} = \mathcal{F} \otimes \mathrm{Frac}(\mathcal{O}_{TS_{K,\log}^{a,\eta}})^\wedge$ are related as follows.*

$$\begin{aligned} R_{\partial/\partial\delta_j}(\mathcal{F}_{a,\eta}) &= \min\{R_{\partial_j}(\mathcal{E}_\eta), \eta^a\}, \ j \in J^+; \\ IR_{\partial/\partial\delta_j}(\mathcal{F}_{a,\eta,\log}) &= \min\left\{IR_{\partial_j}(\mathcal{E}_\eta)/\eta^a, 1\right\}, \ j \in J^+. \end{aligned}$$

Proof. Note that $\tilde{\pi}^*(dB_j) = dB_j + d\delta_j$ for $j \in J$ and $\tilde{\pi}^*(dS) = dS + d\delta_0$. The actions of $\partial/\partial\delta_j$, $j \in J$ (resp. $j = 0$), on $\mathcal{F}_{a,\eta}$ and $\mathcal{F}_{a,\eta,\log}$ are the same as the action of ∂_j (resp. ∂_0) on $\mathcal{E}_\eta$. The statement follows from the facts that δ_J are transcendental over $\mathcal{O}_{Z_K^\eta}$ and the homomorphism $\tilde{\pi}^*$ is isometric (via Theorem 3.3.2.9). $\square$

3.3.3 Thickened differential modules

When the p-adic representation factors through the Galois group of a *finite* totally ramified and wildly ramified Galois extension L/K, the thickened differential module $\mathcal{E}$ can be reconstructed from a finite Galois cover of the thickening space TS_K.

We keep Hypothesis 2.3.2.1 for this subsection. Moreover, we impose the following.

Hypothesis 3.3.3.1. For the rest of this subsection, we assume that L/K is a finite totally ramified and wildly ramified Galois extension.

Notation 3.3.3.2. Let L be as above. Given a uniformizer π_L of L, we fix a non-canonical isomorphism $\kappa_L((\pi_L)) \simeq L$. For a p-basis $\bar{c}_J$ of κ_L, we use c_J to denote the image of $\bar{c}_J$ under this isomorphism; we may use the same index set J because κ_L/κ_K is a finite extension.

Let $\mathcal{O}_E$ be the Cohen ring of κ_L with respect to $\bar{c}_J$ and let C_J be the canonical lifts of $\bar{c}_J$. Set $E = \mathrm{Frac}\mathcal{O}_E$.

Caution 3.3.3.3. The residue field extension κ_L/κ_K is typically not separable and hence *cannot* be embedded into the extension $L = \kappa_L((\pi_L))/K = \kappa_K((\pi_K))$.

Construction 3.3.3.4. We retrieve the notation from Construction 2.3.3.3. For each $j \in J$, fix an element in $\mathcal{O}_E[\![T]\!]$ lifting $b_j \in \mathcal{O}_K \subset \kappa_L[\![t]\!]$ for $j \in J$ and fix an element in $T^e + T^{e+1}\mathcal{O}_E[\![T]\!]$ lifting $\pi_K \in \mathcal{O}_K \subset \kappa_L[\![\pi_L]\!]$. By Proposition 2.3.4.3, there exists a continuous homomorphism $f^* : C_K \hookrightarrow C_L$ sending B_J and S to the elements chosen above; it naturally restricts to $f^* : \mathcal{O}_F[\![S]\!] \hookrightarrow \mathcal{O}_E[\![T]\!]$.

Lemma 3.3.3.5. *Keep the notation as above.*

(1) *The homomorphism f^* is finite and $C_1, \dots, C_m$ and T generate $\mathcal{O}_E[\![T]\!]$ over $\mathcal{O}_F[\![S]\!]$. Hence, f^* induces a surjective map $\mathcal{O}_F[\![S]\!]\langle U_0, \dots, U_m\rangle \twoheadrightarrow \mathcal{O}_E[\![T]\!]$ sending U_0 to T and U_j to C_j for $j \in J$. Moreover, one can choose generators $P_0, \dots, P_m$ of the kernel so that, modulo p, they are exactly p_{J^+} in Construction 2.3.3.3. In particular,*

$$\begin{aligned} P_0 &\in U_0^e - \mathfrak{D}(U_1, \dots, U_m)S + (p, U_0S, S^2) \cdot \mathcal{O}_F[\![S]\!]\langle U_0, \dots, U_m\rangle, \\ P_j &\in U_j^{p^{r_j}} - \mathfrak{B}_j + (p, U_0, S) \cdot \mathcal{O}_F[\![S]\!]\langle U_0, \dots, U_m\rangle, \end{aligned}$$

where $\mathfrak{B}_j$ is a polynomial in $U_0, \dots, U_{j-1}$ with coefficients in $\mathcal{O}_F$ and with degree on U_0 smaller than e and degree on $U_{j'}$ smaller than $p^{r_{j'}}$ for $j' = 1, \dots, j-1$, and $\mathfrak{D}(U_1, \dots, U_m) \in \mathcal{O}_F[U_1, \dots, U_m]$. Moreover, $\{U_{J^+}^{e_{J^+}} \,|\, 0 \le e_0 < e; 0 \le e_j < p^{r_j}, j \in J\}$ form a basis of $\mathcal{O}_F[\![S]\!]\langle U_0, \dots, U_m\rangle/(P_{J^+})$ over $\mathcal{O}_F[\![S]\!]$.

(2) The map f^* extends to a map $f^*_\eta : F\langle \eta/S, S/\eta\rangle \to E\langle \eta^{1/e}/T, T/\eta^{1/e}\rangle$ for $\eta \in [0,1)$. Thus f^* extends by continuity to a homomorphism $f^* : \mathcal{R}_F^{\eta_0} \to \mathcal{R}_E^{\eta_0^{1/e}}$ and a map $f : A^1_F[\eta_0, 1) \to A^1_E[\eta_0^{1/e}, 1)$ for $\eta_0 \in (0,1)$.

(3) Let $\Gamma_F^\dagger$ and $\Gamma_E^\dagger$ be the integral Robba rings over F and E, respectively, similarly constructed as in Construction 3.2.1.13 but without tensoring with F. Let $\mathcal{R}_E$ be the Robba ring over L as in Notation 3.2.1.17. Then $\Gamma_E^\dagger$ is a finite étale extension of $\Gamma_F^\dagger$ with Galois group $G_{L/K}$. Moreover, $\mathcal{R}_E \simeq \Gamma_E^\dagger \otimes_{\Gamma_F^\dagger} \mathcal{R}_F$.

(4) For some $\eta_0 \in (0,1)$, $A^1_E[\eta_0^{1/e}, 1)$ is Galois étale over $\eta \in [\eta_0, 1)$ via f^* with Galois group $G_{L/K}$. Hence, $\mathcal{R}_E^{\eta_0^{1/e}}$ becomes a regular $G_{L/K}$-representation over $\mathcal{R}_F^{\eta_0}$ via f^*.

Proof. (1) is equivalent to its mod p version, which is exactly Construction 2.3.3.3.

(2) It suffices to prove that f^* is continuous respecting the norms $|\cdot|_{Z_K^\eta}$ on C_K and $|\cdot|_{Z_L^{\eta^{1/e}}}$ on C_L, for all $\eta \in [\eta_0, 1)$. Since $f^*(\mathcal{O}_F) \in \mathcal{O}_E[\![T]\!]$ and $f^*(S) \in T^e + T^{e+1}\mathcal{O}_E[\![T]\!]$, we have $|g|_{Z_K^\eta} = |f^*(g)|_{Z_L^{\eta^{1/e}}}$ for any $g \in C_K$. Hence the map f^* extends continuously to $f^*_\eta : K\langle \eta/S, S/\eta\rangle \to L\langle \eta^{1/e}/T, T/\eta^{1/e}\rangle$.

(3) The first statement follows from Lemma 3.2.1.16. The second statement is true because $\Gamma_E^\dagger \otimes_{\Gamma^\dagger} \mathcal{R}_F$ is complete and dense in $\mathcal{R}_E$.

(4) It follows from (2) and (3) since $\mathcal{R}_F$ (resp. $\mathcal{R}_E$) is a limit of $\mathcal{R}_F^{\eta_0}$ (resp. $\mathcal{R}_E^{\eta_0^{1/e}}$). □

Remark 3.3.3.6. The morphism f does not respect the naïve K-space structure on $A^1_E[\eta_0^{1/e}, 1)$; this is precisely because of Caution 3.3.3.3. But it respects the K-space structure on $A^1_E[\eta_0^{1/e}, 1)$ induced by $\mathcal{O}_F \hookrightarrow \mathcal{O}_F[\![S]\!] \xrightarrow{f^*} \mathcal{O}_E[\![T]\!]$.

Construction 3.3.3.7. Keep the notation as in Construction 3.2.1.19. Let $\rho : G_{L/K} \to GL(V_\rho)$ be a p-adic representation, where V_ρ is a finite dimensional vector space over $\mathrm{Frac}(\mathcal{O})$. We have

$$\mathcal{E}_\rho = D^\dagger(\rho)\otimes_{\Gamma_F^\dagger}\mathcal{R}_F = \left(V_\rho\otimes_{\mathcal{O}}\tilde{\Gamma}^\dagger\right)^{G_K}\otimes_{\Gamma_F^\dagger}\mathcal{R}_F = \left(V_\rho\otimes_{\mathbb{Z}_q}\Gamma_E^\dagger\right)^{G_{L/K}}\otimes_{\Gamma_F^\dagger}\mathcal{R}_F = \left(V_\rho\otimes_{\mathbb{Z}_q}\mathcal{R}_E\right)^{G_{L/K}}.$$

Hence, for some $\eta_0 \in (0,1)$, the differential module $\mathcal{E}_\rho$ descents to

$$\mathcal{E}_\rho = \left(V_\rho \otimes_{\mathbb{Z}_q} f_*\mathcal{O}_{Z_L^{\geq\eta_0^{1/e}}}\right)^{G_{L/K}};$$

this is a differential module over $\mathcal{R}_F^{\eta_0} \otimes_{\mathbb{Z}_q} \mathcal{O} = \mathcal{R}_{F'}^{\eta_0}$. This construction respect tensor products, i.e., given another p-adic representation ρ' of $G_{L/K}$ over $\mathrm{Frac}(\mathcal{O})$, we have

$$\mathcal{E}_{\rho\otimes\rho'} = \mathcal{E}_\rho \otimes_{\mathcal{R}_{F'}^{\eta_0}} \mathcal{E}_{\rho'}$$

Hypothesis 3.3.3.8. From now on, we always assume that $\eta_0 \in (0,1)$ is close enough to 1 so that all statements in Lemma 3.3.3.5 hold and $\mathcal{E}_\rho$ descents to $\mathcal{R}_{F'}^{\eta_0}$.

3.3.4 Spectral norms and connected components of thickening spaces

In this subsection, we relate the spectral norms of differential operators on $\mathcal{E}$ to the connected components of certain rigid spaces.

We keep Hypotheses 3.2.2.1, 3.3.3.1, and 3.3.3.8 in this subsection.

Definition 3.3.4.1. Let $a \in \mathbb{Q}_{>1}$. We define the spaces $TS_{L/K}^{a,\geq\eta_0}$, $TS_{K\backslash L}^{a,\geq\eta_0}$, and $TS_{L/K\backslash L}^{a,\geq\eta_0}$ by the following Cartesian diagrams.

$$\begin{array}{ccccc}
Z_L^{\geq\eta_0^{1/e}} & \longleftarrow & TS_{L/K}^{a,\geq\eta_0} & \xleftarrow{\tilde{f}} & TS_{L/K\backslash L}^{a,\geq\eta_0} \\
\downarrow f & & \downarrow f\times 1 & & \downarrow \\
Z_K^{\geq\eta_0} & \xleftarrow{\pi} & TS_K^{a,\geq\eta_0} & \longleftarrow & TS_{K\backslash L}^{a,\geq\eta_0} \\
 & & \downarrow \tilde{\pi} & & \downarrow \\
 & & Z_K^{\geq\eta_0} & \xleftarrow{f} & Z_L^{\geq\eta_0^{1/e}}
\end{array} \qquad (3.3.4.2)$$

We make similar constructions for the logarithmic version of all spaces if $a \in \mathbb{Q}_{>0}$.

Remark 3.3.4.3. The naïve base change $f \times 1$ helps to realize geometric connected components as connected components (see Theorem 3.3.4.6). The base change $\tilde{f}$ encodes the ramification information, which is what we are interested in.

Remark 3.3.4.4. It is natural to relate $TS^{a,\geq\eta_0}_{L/K\backslash L}$ to the thickening space of $Z_L^{\geq\eta_0^{1/e}}$. However, it is not clear how to compare the levels or radii of the two spaces. We do not need this result in our paper.

Corollary 3.3.4.5. *The space $TS^{a,\geq\eta_0}_{L/K\backslash L}$ admits an action of $G_{L/K}$ by pulling back the action on $Z_L^{\geq\eta_0^{1/e}}$ over $Z_K^{\geq\eta_0}$ via $\tilde{\pi}\circ(f\times 1)$. Under this action, $\tilde{f}_*\mathcal{O}_{TS^{a,\geq\eta_0}_{L/K\backslash L}}$ is a regular representation of $G_{L/K}$ over $\mathcal{O}_{TS^{a,\geq\eta_0}_{L/K}}$. For a p-adic representation ρ of $G_{L/K}$ over* Frac($\mathcal{O}$)*, define*

$$\widetilde{\mathcal{F}}_\rho = \left(V_\rho \otimes_{\mathbb{Q}_q} \tilde{f}_*\mathcal{O}_{TS^{a,\geq\eta_0}_{L/K\backslash L}}\right)^{G_{L/K}};$$

this is a differential module over $TS^{a,\geq\eta_0}_{L/K} \times_{\mathbb{Q}_q} \mathrm{Frac}(\mathcal{O})$. Then $\widetilde{\mathcal{F}}_\rho \simeq (f\times 1)^*\tilde{\pi}^*\mathcal{E}_\rho$.

The same statement also holds for log-space.

Proof. This is an easy consequence of flat base change for the two Cartesian squares on the right in Diagram (3.3.4.2). □

Theorem 3.3.4.6. *Let $\rho : G_{L/K} \to GL(V_\rho)$ be a* faithful *p-adic representation over F with L/K satisfying the Hypotheses 3.2.2.1 and 3.3.3.1. Then, for $b > 1$, the following conditions are equivalent:*

(1) *ρ has differential ramification break $\leq b$.*

(2) *For any rational number $c > b$, when $\eta_0 \to 1^-$, $\widetilde{\mathcal{F}}$ is a trivial ∇-module over $TS^{c,\geq\eta_0}_{L/K} \times_{\mathbb{Q}_q} \mathrm{Frac}(\mathcal{O})$.*

(3) *For any rational number $c > b$, when $\eta_0 \to 1^-$, $TS^{c,\geq\eta_0}_{L/K\backslash L}$ has exactly $[L : K]$ connected components.*

(4) *For any rational number $c > b$, when $\eta_0 \to 1^-$, $Z_{L'}^{\geq\eta_0^{1/e'}} \times_{Z_L^{\geq\eta_0^{1/e}},\pi} TS^{c,\geq\eta_0}_{L/K\backslash L}$ has exactly $[L : K]$ connected components for some finite extension L'/L, where e' is the naïve ramification degree of L'/K.*

Also, the similar conditions for logarithmic spaces are equivalent provided $b > 0$.

Proof. The idea of the proof is essentially presented in Theorem 2.3.6.5.

We will only prove the statement for the non-logarithmic spaces and the statement for the logarithmic spaces can be proved completely in the same way (by using intrinsic radii instead of extrinsic radii).

We make a general remark that the Proposition 3.3.2.17 is unchanged if we replace $\mathcal{F}$ by $\widetilde{\mathcal{F}}$ as the spectral norms are invariant under scalar extensions.

We first prove the equivalence between (1) and (2). If ρ has differential ramification break $\leq b$, Proposition 3.3.2.17 and Remark 1.1.2.7 imply that for $\eta \in [\eta_0, 1)$ as before, $R_{\partial/\partial\delta_j}(\mathcal{F}_{b,\eta}) \geq \eta^b$ and hence $\widetilde{\mathcal{F}}$ has a basis of horizontal sections over $TS^{c,\geq\eta_0}_{L/K} \times_{\mathbb{Q}_q} \mathrm{Frac}(\mathcal{O})$ for any rational number $c > b$ via Taylor series. This proves (2). Now, assume (2), i.e., $\widetilde{\mathcal{F}}$ is trivial over $TS^{c,\geq\eta_0}_{L/K} \times_{\mathbb{Q}_q} \mathrm{Frac}(\mathcal{O})$ for any rational number $c > b$ and some $\eta_0 \in (0, 1)$. It follows that $R_{\partial/\partial\delta_j}(\mathcal{F}_{c,\eta}) = \eta^c$. By Proposition 3.3.2.17, $R_{\partial_j}(\mathcal{E}_\eta) \geq \eta^{-c}$, for any $j \in J^+$, $\eta \in [\eta_0, 1)$, and $c \in \mathbb{Q}_{>b}$. This implies that the differential ramification break $\leq b$, as rational numbers are dense in the real numbers.

Obviously, (3)$\Rightarrow$(2). To see the converse, fix some rational number $c > b$. If $\widetilde{\mathcal{F}}$ corresponds to a trivial ∇-module on $TS^{c,\geq\eta_0}_{L/K} \times_{\mathbb{Q}_q} \mathrm{Frac}(\mathcal{O})$, then for any $n \in \mathbb{N}$, $\widetilde{\mathcal{F}}^{\otimes n}$ is also a trivial ∇-module, which corresponds to $V_\rho^{\otimes n}$ by functoriality (Construction 3.3.3.7). By Lemma 3.3.4.9 below from the theory of representations of finite groups, the differential module

$$\left(\mathrm{Frac}(\mathcal{O})[G] \otimes_{\mathbb{Q}_q} f_*\mathcal{O}_{TS^{c,\geq\eta_0}_{L/K\backslash L}}\right)^{G_{L/K}}$$

is a direct summand of a direct sum of some $\widetilde{\mathcal{F}}^{\otimes n}$ and hence is a trivial ∇-module. We claim that $f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{L/K\backslash L}}$ is a trivial ∇-module for all rational numbers $c' > c$.

To see this, we look at the following *isomorphism* of differential modules

$$\begin{array}{ccc} \left(\mathrm{Frac}(\mathcal{O})[G] \otimes_{\mathbb{Q}_q} f_*\mathcal{O}_{TS^{c,\geq\eta_0}_{L/K\backslash L}}\right)^{G_{L/K}} & \xrightarrow{\simeq} & \mathrm{Frac}(\mathcal{O}) \otimes_{\mathbb{Q}_q} f_*\mathcal{O}_{TS^{c,\geq\eta_0}_{L/K\backslash L}}; \\ \sum_{g\in G_{L/K}} \mathbf{f}g \otimes g\mathbf{v} & \longmapsto & \mathbf{f} \cdot \mathbf{v}, \end{array} \tag{3.3.4.7}$$

where $\mathbf{f} \in \mathrm{Frac}(\mathcal{O})$ and $\mathbf{v} \in f_*\mathcal{O}_{TS^{c,\geq\eta_0}_{L/K\backslash L}}$. This map does *not* respect the $\mathrm{Frac}(\mathcal{O})[G]$-modules. From this, we know that the right hand side of (3.3.4.7), denoted by $\mathcal{G}$, is a trivial differential module; in particular, for all $j \in J^+$, the radii of convergence of $\partial/\partial\delta_j$ on $\mathcal{G}$ are $R_{\partial/\partial\delta_j}(\mathcal{G}_\eta) = \eta^c$, which equal the radii of convergence of $\partial/\partial\delta_j$ on $f_*\mathcal{O}_{TS^{c,\geq\eta_0}_{L/K\backslash L}}$ at radius η. Using Taylor series, we can construct horizontal sections on $TS^{c',\geq\eta_0}_{L/K}$ for any rational number $c' > c$. This proves the claim.

As a consequence, any section of $f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{L/K\backslash L}}$ on $V(\delta_{J^+} = 0) = Z_L^{\geq\eta_0^{1/e}} \stackrel{\Delta}{\hookrightarrow} TS^{c',\geq\eta_0}_{L/K}$ can be *uniquely* extended to a horizontal section over $TS^{c',\geq\eta_0}_{L/K}$ relative to $Z_L^{\geq\eta_0^{1/e}}$ via π. In other words, we have an homomorphism

$$\sigma : \Gamma\Big(V(\delta_{J^+} = 0), f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{L/K\backslash L}}\Big) \cong H^0_\nabla\Big(TS^{c',\geq\eta_0}_{L/K}, f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{L/K\backslash L}}\Big) \hookrightarrow \Gamma\Big(TS^{c',\geq\eta_0}_{L/K\backslash L}, \mathcal{O}_{TS^{c',\geq\eta_0}_{L/K\backslash L}}\Big), \tag{3.3.4.8}$$

where H^0_∇ denotes the sections killed by ∂/∂_{J^+}. Since the first map of (3.3.4.8) is given by Taylor series, σ is in fact a *ring* homomorphism. The ring on the left hand side is isomorphic to the functions on $Z_L^{\geq\eta_0^{1/e}} \times_{Z_K^{\geq\eta_0}} Z_L^{\geq\eta_0^{1/e}}$ because the restrictions of $\tilde{\pi}$ and π to $V(\delta_{J^+} = 0)$ are both the same as f. Moreover, since $Z_L^{\geq\eta_0^{1/e}}$ is finite étale Galois over $Z_K^{\geq\eta_0}$ (Lemma 3.3.3.5), $Z_L^{\geq\eta_0^{1/e}} \times_{Z_K^{\geq\eta_0}} Z_L^{\geq\eta_0^{1/e}} = \coprod_{g\in G_{L/K}} Z_L^{\geq\eta_0^{1/e}}$. Via the homomorphism σ in (3.3.4.8), we can lift the idempotent elements on $Z_L^{\geq\eta_0^{1/e}} \times_{Z_K^{\geq\eta_0}} Z_L^{\geq\eta_0^{1/e}}$ to idempotent elements in $\mathcal{O}_{TS^{c',\geq\eta_0}_{L/K\backslash L}}$. Therefore, (3) holds.

The equivalence between (2) and (4) can be proved similarly. We need base change at least to $Z_L^{\geq\eta_0^{1/e}}$ in (3) so that we can split the fiber over $V(\delta_{J^+} = 0)$. $\square$

Lemma 3.3.4.9. *Let G be a finite group and $F_\mathcal{O}$ be a field of characteristic* 0. *Let $\rho : G \longrightarrow \mathrm{GL}(V_\rho)$ be a faithful representation over $F_\mathcal{O}$. Then the regular representation $F_\mathcal{O}[G]$ is a direct summand of a direct sum of some self-tensor products of V_ρ.*

This is an easy exercise of finite group representations but we do not know a good reference. The author would like to thank Xuhua He for providing the following proof.

Proof. Let χ be the character of V_ρ and let d be the dimension of V_ρ. Since the representation is injective, $\chi(1) = d$ and $\chi(g) \neq d$ for all $g \in G$ nontrivial. (This is because all the eigenvalues of $\rho(g)$ are roots of unity and cannot all be 1.)

Therefore, for each $g \neq 1$ there exists a polynomial P_g in χ with integer coefficients such that $P_g(\chi(g)) = 0$ but $P_g(d) \neq 0$. Let $P = \prod_{1\neq g\in G} P_g$ and then $P(d) \neq 0$ but

$P(\chi(g)) = 0$ for all $g \neq 1$. Moreover, by multiplying a constant, we may assume that $\#G$ divides $P(d)$.

We know that the character of a tensor product is the product of the characters. Therefore, if we take the terms in P which have positive coefficients, those terms together will correspond to the desired direct sum of $V_\rho^{\otimes n}$, containing the regular representation. □

3.4 Comparison theorem

3.4.1 Lifts of standard Abbes-Saito spaces

Recall the Standard Abbes-Saito spaces defined in Subsection 2.3.3. In this subsection, we define their lifts to a rigid analytic space over the annulus $A_F^1[\eta_0, 1)$ for some η_0, following Section 3.1.

In this subsection, we continue to assume Hypotheses 3.2.2.1 and 3.3.3.1.

Construction 3.4.1.1. Let P_{J^+} be the lifts of p_{J^+} as in Lemma 3.3.3.5. For $a \in \mathbb{Q}_{>0}$ and $\eta_0 \in (0, 1)$, we define the *lifted Abbes-Saito spaces* to be

$$\begin{aligned} \widetilde{AS}_{L/K}^{a,\geq\eta_0} &= \left\{ (U_{J^+}, S) \in A_F^{m+2}[0,1] \middle| \begin{array}{c} \eta_0 \leq |S| < 1 \\ |P_0(U_{J^+}, S)| \leq |S|^a, \dots, |P_m(U_{J^+}, S)| \leq |S|^a \end{array} \right\} \\ \widetilde{AS}_{L/K,\log}^{a,\geq\eta_0} &= \left\{ (U_{J^+}, S) \in A_F^{m+2}[0,1] \middle| \begin{array}{c} \eta_0 \leq |S| < 1,\ |P_0(U_{J^+}, S)| \leq |S|^{a+1}, \\ |P_1(U_{J^+}, S)| \leq |S|^a, \dots, |P_m(U_{J^+}, S)| \leq |S|^a \end{array} \right\}; \end{aligned}$$

they are viewed as rigid spaces over $Z_K^{\geq\eta_0}$.

Lemma 3.4.1.2. *Let K'/K be a finite Galois extension of naïve ramification degree e'. If we identify C_K as a subring of $C_{K'}$ as in Construction 3.3.3.4, we may view P_{J^+} as polynomials in U_{J^+} with coefficients in $\mathcal{O}_{F'}[\![S']\!]$, where F' is the fraction field of the Cohen ring of $\kappa_{K'}$ and S' is a lift of the uniformizer $\pi_{K'}$ in K'. Then, for*

$\eta_0 \in (0,1)$ *and* $a \in \mathbb{Q}_{>0}$*, we have*

$$Z_{K'}^{\geq \eta_0^{1/e'}} \times_{Z_K^{\geq \eta_0}} \widetilde{AS}_{L/K}^{a,\geq \eta_0} \cong \left\{ (U_{J^+}, S') \in A_{F'}^{m+2}[0,1] \,\middle|\, \begin{array}{c} \eta_0^{1/e'} \leq |S'| < 1 \\ |P_0| \leq |S'|^{e'a}, \dots, |P_m| \leq |S'|^{e'a} \end{array} \right\}$$

$$Z_{K'}^{\geq \eta_0^{1/e'}} \times_{Z_K^{\geq \eta_0}} \widetilde{AS}_{L/K,\log}^{a,\geq \eta_0} \cong \left\{ (U_{J^+}, S') \in A_{F'}^{m+2}[0,1] \,\middle|\, \begin{array}{c} \eta_0^{1/e'} \leq |S'| < 1,\ |P_0| \leq |S'|^{e'(a+1)}, \\ |P_1| \leq |S'|^{e'a}, \dots, |P_m| \leq |S'|^{e'a} \end{array} \right\};$$

Proof. The only thing not obvious is that we replace $|P_j| \leq |S|^{a(+1)}$ by $|P_j| \leq |S'|^{e'a(+e')}$; this is because $|S| = |S'|^{e'}$ as proved in Lemma 3.3.3.5(2). □

Theorem 3.4.1.3. *For* $a \in \mathbb{Q}_{>0}$*, there is a one-to-one correspondence between the geometric connected components of* $AS^a_{L/K,(\log)}$ *and the following limit of connected components:*

$$\varprojlim_{K'/K} \lim_{\eta_0 \to 1^-} \pi_0^{\mathrm{geom}} \Big(Z_{K'}^{\geq \eta_0^{1/e'}} \times_{Z_K^{\geq \eta_0}} \widetilde{AS}_{L/K(,\log)}^{a,\geq \eta_0} \Big),$$

where e' *is the naïve ramification degree of* K'/K *and the second limit only takes* $\eta_0 \in p^{\mathbb{Q}} \cap (0,1)$.

Proof. By Lemma 3.4.1.2 and Example 3.1.3.4, when $e'a \in \mathbb{Z}$, $Z_{K'}^{\geq \eta_0^{1/e'}} \times_{Z_K^{\geq \eta_0}} \widetilde{AS}_{L/K(,\log)}^{a,\geq \eta_0}$ is a lifting space of $AS^a_{L/K(,\log)}$. The theorem then follows from Corollary 3.1.2.12. □

Remark 3.4.1.4. Here, we need $\eta_0 \in p^{\mathbb{Q}} \cap (0,1)$ to invoke Corollary 3.1.2.12.

Remark 3.4.1.5. Introducing this ramified extension K'/K to make $e'a \in \mathbb{Z}$ may not be essential, but it eases the proof.

3.4.2 Comparison of rigid spaces

In this subsection, we will prove that the lifted Abbes-Saito spaces are isomorphic to some thickening spaces we constructed in Subsection 3.3.4. In this subsection, we continue to assume Hypotheses 3.2.2.1 and 3.3.3.1.

Lemma 3.4.2.1. *Keep the notation as in Section 3.3.3. We have*

$$\det \left(\frac{\partial(\tilde{\pi}^*(P_i) - P_i)}{\partial \delta_j} \right)_{i,j \in J^+} \Bigg|_{\delta_{J^+}=0} \in \big(\mathcal{O}_F[\![S]\!]\langle U_{J^+}\rangle/(P_{J^+})\big)^\times = (\mathcal{O}_E[\![T]\!])^\times.$$

In particular, the corresponding matrix is invertible.

Proof. It is enough to prove that the matrix is of full rank modulo (p, T). First, modulo (p, T), the first row will be all zero except the first element which is $\mathfrak{d}(c_1, \ldots, c_m) \in \kappa_L$. Hence, we need only to look at

$$\left(\frac{\partial(\tilde{\pi}^*(P_i) - P_i)}{\partial \delta_j}\right)_{i,j \in J} \bmod (p, T, \delta_{J^+}) = \det\left(\frac{\partial(\bar{\psi}^*(\bar{p}_i) - \bar{p}_i)}{\partial \delta_j}\right)_{i,j \in J}\Bigg|_{\delta_{J^+}=0}.$$

This is invertible by Lemma 2.3.4.15. □

Theorem 3.4.2.2. *There exists $\eta_0' \in (0,1)$ such that for any $a \in \mathbb{Q}_{>1}$ and any $\eta_0 \in (\max\{p^{-1/a}, \eta_0'\}, 1)$, there exists an isomorphism of rigid spaces over $Z_K^{\geq \eta_0}$:*

$$\widetilde{AS}_{L/K}^{a, \geq \eta_0} \simeq TS_{K \backslash L}^{a, \geq \eta_0}. \tag{3.4.2.3}$$

Similarly, There exists $\eta_0' \in (0,1)$ such that for any $a \in \mathbb{Q}_{>0}$ and any $\eta_0 \in (\max\{p^{-1/a}, \eta_0'\}, 1)$, there exists an isomorphism of rigid spaces over $Z_K^{\geq \eta_0}$:

$$\widetilde{AS}_{L/K,\log}^{a, \geq \eta_0} \simeq TS_{K \backslash L,\log}^{a, \geq \eta_0}. \tag{3.4.2.4}$$

Proof. We will give the proof for the log-spaces and make changes for the non-log spaces when necessary. The proofs in two cases will be almost the same except that when constructing the morphism χ_2, we have slightly different approximations. We will match up the ring of functions on the two rigid spaces in (3.4.2.4) (resp. (3.4.2.3)). Fix an $\eta_0 \in (p^{-1/a}, 1)$ satisfying Hypothesis 3.3.3.8. (η_0' is given by the conditions in Hypothesis 3.3.3.8.)

Recall that $\mathcal{O}_{TS_{K,\log}^{a,\geq\eta_0}} = \mathcal{R}_F^{\eta_0}\langle S^{-a-1}\delta_0, S^{-a}\delta_J\rangle$ (resp. $\mathcal{O}_{TS_K^{a,\geq\eta_0}} = \mathcal{R}_F^{\eta_0}\langle S^{-a}\delta_{J^+}\rangle$). For each $j \in J^+$, $\tilde{\pi}^*(P_j)$ is the polynomial P_j with coefficients replaced by their pullbacks to $\mathcal{O}_{TS_{K,\log}^{a,\geq\eta_0}}$ (resp. $\mathcal{O}_{TS_K^{a,\geq\eta_0}}$) via $\tilde{\pi}^*$. So the ring of functions on $TS_{K\backslash L,\log}^{a,\geq\eta_0}$ (resp. $TS_{K\backslash L}^{a,\geq\eta_0}$) is

$$\begin{aligned} \mathcal{R}_{1,\log} &= \mathcal{R}_F^{\eta_0}\langle S^{-a-1}\delta_0, S^{-a}\delta_J\rangle\langle U_{J^+}\rangle / \big(\tilde{\pi}^*(P_{J^+})\big) \\ (\text{resp. } \mathcal{R}_1 &= \mathcal{R}_F^{\eta_0}\langle S^{-a}\delta_{J^+}\rangle\langle U_{J^+}\rangle / \big(\tilde{\pi}^*(P_{J^+})\big)\ ). \end{aligned} \tag{3.4.2.5}$$

By Lemma 3.3.3.5(1),

$$
\begin{aligned}
\tilde{\pi}^*(P_j) &\in U_j^{p^{r_j}} - \tilde{\pi}^*(\mathfrak{B}_j) + (p, U_0, S, \delta_0) \cdot \mathcal{O}_F[\![\delta_{J^+}, S]\!][U_{J^+}], \\
\tilde{\pi}^*(P_0) &\in U_0^e - \tilde{\pi}^*(\mathfrak{D})S - \delta_0 + (p, U_0S, S^2, U_0\delta_0, S\delta_0, \delta_0^2) \cdot \mathcal{O}_F[\![\delta_{J^+}, S]\!][U_{J^+}],
\end{aligned}
$$

Thus, we can view $\mathcal{R}_{1,\log}$ (resp. $\mathcal{R}_1$) as a finite free module over $\mathcal{O}_{TS^{a,\geq\eta_0}_{K,\log}}$ (resp. $\mathcal{O}_{TS^{a,\geq\eta_0}_K}$) with basis $\{U_{J^+}^{e_{J^+}} \,|\, 0 \leq e_0 < e;\, 0 \leq e_j < p^{r_j}, j \in J\}$. For each $\eta \in [\eta_0, 1)$, we provide $\mathcal{R}_{1,\log}$ (resp. $\mathcal{R}_1$) the following norm: for $g = \sum \lambda_{e_{J^+}} U_{J^+}^{e_{J^+}}$ with $\lambda_{e_{J^+}} \in \mathcal{O}_{TS^{a,\geq\eta_0}_{K,\log}}$ (resp. $\lambda_{e_{J^+}} \in \mathcal{O}_{TS^{a,\geq\eta_0}_K}$), summed over $e_0 = 0, \ldots, e-1$ and $e_j = 0, \ldots, p^{r_j}-1$ for $j \in J$, we define

$$
|g|_{\mathcal{R}_{1,\log},\eta} = \max_{e_{J^+}}\{|\lambda_{e_{J^+}}|_{TS^{a,\eta}_{K,\log}} \cdot \eta^{e_0/e}\} \quad (\text{resp. } |g|_{\mathcal{R}_1,\eta} = \max_{e_{J^+}}\{|\lambda_{e_{J^+}}|_{TS^{a,\eta}_K} \cdot \eta^{e_0/e}\}\ ).
$$

It is clear that $\mathcal{R}_{1,\log}$ (resp. $\mathcal{R}_1$) is the Fréchet completion for the norms $|\cdot|_{\mathcal{R}_{1,\log},\eta}$ (resp. $|\cdot|_{\mathcal{R}_1,\eta}$) for all $\eta \in [\eta_0, 1)$.

On the other hand, the ring of functions of $AS^{a,\geq\eta_0}_{L/K,\log}$ (resp. $AS^{a,\geq\eta_0}_{L/K}$) is

$$
\begin{aligned}
\mathcal{R}_{2,\log} &= \mathcal{R}_F^{\eta_0}\langle S^{-a-1}V_0, S^{-a}V_J\rangle\langle U_{J^+}\rangle/(P_{J^+} - V_{J^+}) \\
&(\text{resp. } \mathcal{R}_2 = \mathcal{R}_F^{\eta_0}\langle S^{-a}V_{J^+}\rangle\langle U_{J^+}\rangle/(P_{J^+} - V_{J^+})\ ),
\end{aligned}
$$

which is clearly a finite free module over

$$
W_{\log} = \mathcal{R}_F^{\eta_0}\langle V_0/\eta^{a+1}, V_J/\eta^a\rangle \quad (\text{resp. } W = \mathcal{R}_F^{\eta_0}\langle V_{J^+}/\eta^a\rangle\ )
$$

with basis $\{U_{J^+}^{e_{J^+}} \,|\, 0 \leq e_0 < e;\ 0 \leq e_j < p^{r_j}, j \in J\}$. Similarly, for $\eta \in [\eta_0, 1)$, we provide $\mathcal{R}_{2,\log}$ (resp. $\mathcal{R}_2$) with the following norm: for $g = \sum \lambda_{e_{J^+}} U_{J^+}^{e_{J^+}}$ with $\lambda_{e_{J^+}} \in W_{\log}$ (resp. $\lambda_{e_{J^+}} \in W$) summed over $e_0 = 0, \ldots, e-1$ and $e_j = 0, \ldots, p^{r_j} - 1$ for $j \in J$, we define

$$
|g|_{\mathcal{R}_{2,\log},\eta} = \max_{e_{J^+}}\{|\lambda_{e_{J^+}}|_{W_{\log}} \cdot \eta^{e_0/e}\} \quad (\text{resp. } |g|_{\mathcal{R}_2,\eta} = \max_{e_{J^+}}\{|\lambda_{e_{J^+}}|_W \cdot \eta^{e_0/e}\}\ ).
$$

It is clear that $\mathcal{R}_{2,\log}$ (resp. $\mathcal{R}_2$) is the Fréchet completion for the norms $|\cdot|_{\mathcal{R}_{2,\log},\eta}$ (resp. $|\cdot|_{\mathcal{R}_2,\eta}$) for all $\eta \in [\eta_0, 1)$.

We will identify U_{J^+} in different rings but the V_{J^+} will not be same as δ_{J^+}. Be caution that the two norms will not be the same under the identification, but they will give the same topology.

Now, we define a continuous K-homomorphism $\chi_1 : \mathcal{R}_{2,\log} \to \mathcal{R}_{1,\log}$ (resp. $\chi_1 : \mathcal{R}_2 \to \mathcal{R}_1$) so that $\chi_1(S) = S$, $\chi_1(U_j) = U_j$, $\chi(V_j) = P_j(U_{J^+})$ for all $j \in J^+$. We need only to check that for any $\eta \in [\eta_0, 1)$,

$$|\chi_1(V_j)|_{\mathcal{R}_{1,\log},\eta} \leq \begin{cases} \eta^{a+1} & j = 0 \\ \eta^a & j \in J \end{cases} \quad (\text{resp. } |\chi_1(V_j)|_{\mathcal{R}_1,\eta} \leq \eta^a,\ \forall j \in J^+). \qquad (3.4.2.6)$$

Here we need separate arguments for logarithmic case and non-logarithmic case. In the logarithmic case, Inequality (3.3.2.10) tells us $|P_j - \tilde{\pi}^*(P_j)|_{\mathcal{R}_{1,\log},\eta} \leq \eta^a |P_j|_{\mathcal{R}_{2,\log},\eta}$ for $j \in J^+$, which exactly gives the bound in (3.4.2.6) because $|P_0|_{\mathcal{R}_{2,\log},\eta} \leq \eta$ and $|P_j|_{\mathcal{R}_{2,\log},\eta} \leq 1$ for $j \in J$ by Lemma 3.3.3.5(1). In the non-logarithmic case, combining Lemma 3.3.3.5(1) and inequality (3.3.2.11), one has $|P_j - \tilde{\pi}^*(P_j)|_{\mathcal{R}_1,\eta} \leq \eta^a$ for $j \in J^+$; Inequality (3.4.2.6) follows.

Conversely, we will define a continuous K-homomorphism $\chi_2 : \mathcal{R}_{1,\log} \to \mathcal{R}_{2,\log}$ (resp. $\chi_2 : \mathcal{R}_1 \to \mathcal{R}_2$) as the inverse to χ_1. Obviously, we need $\chi_2(S) = S$, $\chi_2(U_j) = U_j$ for all $j \in J^+$. The only thing not clear is $\chi_2(\delta_j)$ for all $j \in J^+$.

By Lemma 3.4.2.1, let

$$A := \Big(\frac{\partial(\tilde{\pi}^*(P_i) - P_i)}{\partial \delta_j}\Big)_{i,j\in J^+}\Big|_{\delta_{J^+}=0} \in \mathrm{GL}_{m+1}(\mathcal{O}_E[\![T]\!]) \cong \mathrm{GL}_{m+1}\big(\mathcal{O}_F[\![S]\!]\langle U_{J^+}\rangle/(P_{J^+})\big).$$

Let A^{-1} denote the inverse matrix in $\mathrm{GL}_{m+1}\big(\mathcal{O}_F[\![S]\!]\langle U_{J^+}\rangle/(P_{J^+})\big)$, whose entries are written as polynomials in U_{J^+} (using the basis in Lemma 3.3.3.5(1)). Thus,

$$A^{-1} \cdot A - I \in \mathrm{Mat}_{m+1}\big((\delta_{J^+}) \cdot \mathcal{O}_F[\![S]\!]\langle U_{J^+}\rangle\big), \qquad (3.4.2.7)$$

where I is the $(m+1)\times(m+1)$ identity matrix. Now, we write

$$\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} = (I - A^{-1}A)\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} - A^{-1}\left(\begin{pmatrix}\tilde{\pi}^*(P_0)-P_0\\ \vdots\\ \tilde{\pi}^*(P_m)-P_m\end{pmatrix} - A\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix}\right) - A^{-1}\begin{pmatrix}P_0\\ \vdots\\ P_m\end{pmatrix}; \tag{3.4.2.8}$$

the last term is just $-A^{-1}\cdot\chi_1(V_{J^+})$. We need to bound the first two terms.

By (3.4.2.7), $I - A^{-1}A$ has norm $\leq \eta^a$. Hence, in the non-logarithmic case, the first term in (3.4.2.8) has norm $\leq \eta^{2a}$; in the logarithmic case the first term in (3.4.2.8) has norm $\leq \eta^{2a}$, except for the first row, which has norm $\leq \eta^{2a+1}$. By the definition of A and Theorem 3.3.2.9, the second term in (3.4.2.8) has norm $\leq \eta^{2a}$ in the non-logarithmic case; it has norm $\leq \eta^{2a}$ in the logarithmic case, except for the first row, which has norm $\leq \eta^{2a+1}$.

Since we want χ_2 to be the inverse of χ_1, we define recursively by

$$\chi_2\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} = -A^{-1}\begin{pmatrix}V_0\\ \vdots\\ V_m\end{pmatrix} + \chi_2\begin{pmatrix}\Lambda_0\\ \vdots\\ \Lambda_m\end{pmatrix},$$

where Λ_{J^+} denotes the sum of the first two terms in (3.4.2.8). Since Λ_{J^+} have strictly smaller norms than δ_{J^+} and Λ_{J^+} are in the ideal (δ_{J^+}), one can plug the image of $\chi_2(\delta_{J^+})$ back into $\chi_2(\Lambda_{J^+})$ and iterate this substitution. This construction will converge to a well-defined continuous homomorphism χ_2 which is an inverse of χ_1. Moreover, from the construction, one can see that

$$|\chi_2(\delta_j)|_{\mathcal{R}_1} \leq \eta^a, \text{ for all } j \in J^+, \eta \in [\eta_0, 1),$$
$$|\chi_2(\delta_0)|_{\mathcal{R}_{1,\log}} \leq \eta^{a+1} \text{ and } |\chi_2(\delta_j)|_{\mathcal{R}_{1,\log}} \leq \eta^a \text{ for all } j \in J, \eta \in [\eta_0, 1).$$

Therefore, we have two continuous homomorphisms χ_1 and χ_2, being inverse to each other; this concludes the proof. □

Remark 3.4.2.9. The isomorphisms constructed in Theorem 3.4.2.2 are canonical in the sense that they match up U_{J^+} on the both sides. However, slight perturbations of the isomorphisms will continue to be isomorphic. This point will be important when studying the mixed characteristic case.

3.4.3 Comparison theorems

In this subsection, we wrap up the argument and prove the comparison between the arithmetic conductors and the differential conductors. As a reminder, we do not impose Hypotheses 3.2.2.1 and 3.3.3.1 in this subsection.

Theorem 3.4.3.1. *Let K be a complete discretely valued field of equal characteristic $p > 0$ and let G_K be its absolute Galois group. For a p-adic representation $\rho : G_K \to GL(V_\rho)$ of finite local monodromy, the arithmetic Artin conductor* $\mathrm{Art}(\rho)$ *of ρ coincides with the differential Artin conductor* $\mathrm{Art}_{\mathrm{dif}}(\rho)$*; the arithmetic Swan conductor* $\mathrm{Swan}(\rho)$ *coincides with the differential Swan conductor* $\mathrm{Swan}_{\mathrm{dif}}(\rho)$.

Proof. It suffices to prove for irreducible representations, as all the conductors are additive. Since all the conductors remain the same if we pass to the completion of the unramified closure of K (Proposition 2.2.2.11(4), Theorem 3.2.3.5(2)), we may assume that the residue field κ_K is separably closed; hence ρ factors through the Galois group of a finite totally ramified extension L/K as $\rho : G_K \twoheadrightarrow \mathrm{Gal}(L/K) \hookrightarrow \mathrm{GL}(V_\rho)$ with the second map injective. Moreover, we may assume that L/K is wildly ramified because the theorem is known when L/K is tamely ramified (Proposition 2.2.2.11(6) and Theorem 3.2.3.5(3)). In other words, we may assume Hypothesis 3.3.3.1. In particular, $b(L/K) > 1$ and $b_{\log}(L/K) > 0$.

Next, we want to reduce to the case when the p-basis of κ_K is finite. By Construction 2.3.3.3, one can choose lifted p-basis of L so that all but finitely many of them are actually in K. Let $(c_i)_{i \in I}$ be a subset of those elements in the lifted p-basis which lie in K. Denote $\widetilde{K} = K(c_i^{1/p^n}, i \in I, n \in \mathbb{N})^\wedge$ and $\widetilde{L} = L\widetilde{K}$. We claim that $\mathcal{O}_{\widetilde{L}} = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{\widetilde{K}}$. Indeed, after base change to $\widetilde{K}$, the value groups do not change: $|\widetilde{K}^\times| = |K^\times|$. Thus, $[|\widetilde{L}^\times| : |\widetilde{K}^\times|] \geq [|L^\times| : |K^\times|]$. On the other hand, the residue field extension of $\widetilde{L}/\widetilde{K}$ has degree at least the same as κ_L/κ_K because $\bar{c}_{J\setminus I}$ are not in the residue field of $\widetilde{K}$. But we know that the degree of the extension does not increase. Therefore, we have equality on both naïve ramification degrees and degrees of residue field extension. It is then clear that $\mathcal{O}_{\widetilde{L}} = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{\widetilde{K}}$, as the right hand side contains the uniformizer of the left hand side and both sides are isomorphic modulo that uniformizer. Therefore, by Proposition 2.2.2.11(4), $b(\widetilde{L}/\widetilde{K}) = b(L/K)$.

On the differential conductors side, [Ked07a, Lemma 3.5.4] (the non-log case follows by similar argument) shows that we can consider only finitely many elements in the p-basis and the differential conductors are unchanged after making inseparable field extension with respect to other elements in the p-basis.

To sum up, we can make an inseparable extension so that all conductors do not change, and we are reduced to the case where Hypothesis 2.3.2.1 holds.

Now, we will prove the comparison theorem for the Swan conductors and the proof for the Artin conductors follows verbatim, except replacing Swan by Art and $a > 0$ by $a > 1$ and dropping all the log's in the subscripts.

Since ρ is irreducible, $\mathrm{Swan}(\rho) = b_{\log}(L/K) \cdot \dim V_\rho$. Recall that in Subsection 3.2.1, we can associate to ρ a differential module $\mathcal{E}_\rho$ over $\mathcal{R}_F^{\eta_0} \otimes_{\mathbb{Z}_q} \mathcal{O}$ for some $\eta_0 \in (0,1)$. As the representation ρ is irreducible, $\mathcal{E}_\rho$ has a unique ramification break $b_{\log}(\mathcal{E}_\rho)$. So the differential Swan conductor of ρ is $\mathrm{Swan}_{\mathrm{dif}}(\rho) = b_{\log}(\mathcal{E}_\rho) \cdot \dim V_\rho$. Therefore, to conclude, it suffices to show that $b_{\log}(L/K) = b_{\log}(\mathcal{E}_\rho)$.

This follows from the following equivalence relations.

$a > b_{\log}(\mathcal{E}_\rho)$

$\Longleftrightarrow$ for any (or some) extension L'/L with naïve ramification degree e', $\pi_0^{\mathrm{geom}}\Big(Z_{L'}^{\geq \eta_0^{1/ee'}} \times_{Z_L^{\geq \eta_0^{1/e}}} TS^{a, \geq \eta_0}_{L/K \backslash L, \log}\Big) = [L:K]$ when $\eta_0 \to 1^-$ (Theorem 3.3.4.6)

$\Longleftrightarrow \pi_0^{\mathrm{geom}}(AS^a_{L/K,\log}) = [L:K]$ (Theorem 3.4.1.3)

$\Longleftrightarrow a > b_{\log}(L/K)$,

where a is a rational number. □

Remark 3.4.3.2. In an early version of this paper, Theorem 3.4.3.1 is stated for representations with finite image. Andrea Pulita pointed out that this could be extended to the finite local monodromy case by a standard argument as in the proof.

Corollary 3.4.3.3. (1) (Hasse-Arf Theorem) *Let K be a complete discretely valued field of equal characteristic $p > 0$, let G_K be its absolute Galois group, and let $\rho : G_K \to GL(V_\rho)$ be a p-adic representation of finite local monodromy. Then the Artin conductor* $\mathrm{Art}(\rho)$ *and the Swan conductor* $\mathrm{Swan}(\rho)$ *are integers.*

(2) *Let K be a complete discretely valued field of equal characteristic $p > 0$. Then the subquotients* $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ *(resp.* $\mathrm{Fil}^a_{\log} G_K/\mathrm{Fil}^{a+}_{\log} G_K$*) of the arithmetic ramification filtrations are elementary p-abelian groups if $a \in \mathbb{Q}_{>1}$ (resp. $a \in \mathbb{Q}_{>0}$) and are trivial if $a \notin \mathbb{Q}$.*

Proof. It follows from Theorems 3.2.3.5 and 3.4.3.1. □

Corollary 3.4.3.4. *Let K be a complete discretely valued field of equal characteristic $p > 0$ and let ρ be a representation of G_K of finite local monodromy. Then* $\mathrm{Art}_B(\rho) = \mathrm{Art}(\rho)$.

Proof. We combine Theorem 3.4.3.1 and Proposition 3.2.4.8 to verify the condition in Proposition 2.4.2.1. The corollary follows from that. □

Corollary 3.4.3.5. *Conjecture 2.2.2.21 is true when K is of equal characteristic* $p > 0$.

Proof. We use Theorem 3.4.3.1 to translate Proposition 3.2.5.8 into the language of arithmetic ramification filtration. □

Chapter 4

Ramification Theory for Local Fields: Mixed Characteristic Case

Plan of this chapter

In this chapter, we will prove Hasse-Arf Conjecture 2.2.2.17 in the mixed characteristic case, except for some special cases. The proof follows closely the strategy outlined in Section 2.3.

In Section 4.1, we set up the framework for the proof. In Subsection 4.1.1, we define the function ψ_K we mentioned earlier in Fake-assumption 2.3.4.11. In Subsection 4.1.2, we prove the $AS = TS$ theorem. In Subsection 4.1.3, we prove that the map Π in Construction 2.3.6.1 is étale. In Subsection 4.1.4, we translate the question about the ramification breaks into a question about the intrinsic radii of convergence. In Subsection 4.1.5, we discuss a variant of thickening spaces.

In Section 4.2, we prove the main Theorem 4.2.3.5. This is achieved by proving that the ramification break is invariant under adding a generic p-th root of p-basis. The core of the proof is Theorem 4.2.1.7.

In Section 4.3, we study the logarithmic part of the Hasse-Arf Theorem 2.2.2.19. In Subsection 4.3.1, we deduce the integrality of Swan conductors from that of Artin conductors by tame base change. In Subsections 4.3.2 and 4.3.3, we use a trick of Kedlaya to prove that the subquotients of the logarithmic filtration (on the wild ramification group) are abelian groups killed by p.

4.1 Construction of spaces

In this section, we construct a series of spaces and study their relations; in particular, we prove that the Abbes-Saito spaces are the same as the thickening spaces, and translate the question on ramification breaks to a question on generic radii of differential modules.

4.1.1 The ψ_K-function and thickening spaces

In this subsection, we first define a function (*not* a homomorphism) $\psi_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$, which is an approximation to the deformation of the uniformizer π_K and a lifted p-basis as in Fake-assumption 2.3.4.11. Then, we introduce the thickening spaces for the extension L/K (See Fake-definition 2.3.5.1 for motivation).

Hypothesis 4.1.1.1. Throughout this section, unless otherwise specified, we assume that K is a complete discretely valued field of mixed characteristic $(0, p)$, with *separably closed* and *imperfect* residue field. Let L be a finite Galois extension of K of naïve ramification degree $e = e_{L/K}$. Assume that K admits a *finite* lifted p-basis b_J and fix a uniformizer π_K of K.

Construction 4.1.1.2. Let $r \in \mathbb{N}$ and $h \in \mathcal{O}_K^\times$. An *$r$-th p-basis decomposition* of h is to write h as

$$h = \sum_{e_J=0}^{p^r-1} b_J^{e_J}\Big(\sum_{n=0}^{\infty}\Big(\sum_{n'=0}^{\lambda_{(r),e_J,n}} \alpha_{(r),e_J,n,n'}^{p^r}\Big)\pi_K^n\Big) \tag{4.1.1.3}$$

for some $\alpha_{(r),e_J,n,n'} \in \mathcal{O}_K^\times \cup \{0\}$ and some $\lambda_{(r),e_J,n} \in \mathbb{Z}_{\geq 0}$. Such expressions always exist but are not unique. For $r' > r$, we can express each of $\alpha_{(r),e_J,n,n'}$ in (4.1.1.3) using an $(r'-r)$-th p-basis decomposition and then rearrange the formal sum to obtain an r'-th p-basis decomposition. For $h \in \mathcal{O}_K^\times$, we say that an r'-th p-basis decomposition is *compatible* with the r-th p-basis decomposition in (4.1.1.3) if it can be obtained in the above sense.

We define the function $\psi_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_{J^+}]\!]$ as follows: for $h \in \mathcal{O}_K^\times \backslash \{1\}$, we fix a compatible system of r-th p-basis decomposition for all $r \in \mathbb{N}$, and define

$$\psi_K(h) = \lim_{r \to +\infty} \sum_{e_J=0}^{p^r-1} (b_J + \delta_J)^{e_J} \Big(\sum_{n=0}^{\infty} \big(\sum_{n'=0}^{\lambda_{(r),e_J,n}} \alpha^{p^r}_{(r),e_J,n,n'} \big) (\pi_K + \delta_0)^n \Big). \qquad (4.1.1.4)$$

This expression converges by the compatibility of the p-basis decompositions. Define $\psi_K(1) = 1$, which corresponds to the naïve compatible system of p-basis decomposition of the element 1. For $h \in \mathcal{O}_K \backslash \{0\}$, write $h = \pi_K^s h_0$ for $s \in \mathbb{N}$ and $h_0 \in \mathcal{O}_K^\times$. Define $\psi_K(h) = (\pi_K + \delta_0)^s \psi'_K(h_0)$, where $\psi'_K(h_0)$ is the limit in (4.1.1.4) with respect to a compatible system of p-basis decompositions of h_0 (which does not have to be the same as the one that defines $\psi_K(h_0)$). Finally, we define $\psi_K(0) = 0$.

Most of the time, it is more convenient to view ψ_K as a function on $\mathcal{O}_K$ which takes value in the larger ring $\mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$.

We naturally extend ψ_K to polynomial rings or formal power series rings with coefficients in $\mathcal{O}_K$ by applying ψ_K termwise.

Notation 4.1.1.5. For the rest of the chapter, let $\mathcal{R}_K = \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$.

Caution 4.1.1.6. The map ψ_K is *not* a homomorphism, nor is it canonically defined. This is because one cannot "deform" the uniformizer in the mixed characteristic case. Also, since K will not be absolutely unramified in application, lifted p-basis may not deform freely either. (See also Fake-assumption 2.3.4.11 and the following discussion.) However, Proposition 4.1.1.8 below says that ψ_K is approximately a homomorphism.

Definition 4.1.1.7. For two $\mathcal{O}_K$-algebras R_1 and R_2 and an ideal I of R_2, an *approximate homomorphism modulo* I is a function $f : R_1 \to R_2$ such that for $h_1 \in \pi_K^{a_1} R_1$ and $h_2 \in \pi_K^{a_2} R_2$ with $a_1, a_2 \in \mathbb{Z}_{\geq 0}$, $\psi_K(h_1 h_2) - \psi_K(h_1)\psi_K(h_2) \in \pi_K^{a_1+a_2} I$ and $\psi_K(h_1 + h_2) - \psi_K(h_1) - \psi_K(h_2) \in \pi_K^{\min\{a_1,a_2\}} I$.

Moreover, if R'_1 and R'_2 are two $\mathcal{O}_K$-algebras, a diagram of functions

$$\begin{array}{ccc} R'_1 & \xrightarrow{f'} & R'_2 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ R_1 & \xrightarrow{f} & R_2 \end{array}$$

is called *approximately commutative modulo I* if for $h \in \pi_K^a R_1'$, $g'(f'(h)) - f(g(h)) \in \pi_K^a I$.

Proposition 4.1.1.8. *For $h \in \mathcal{O}_K$, we have $\psi_K(h) - h \in (\delta_{J^+}) \cdot \mathcal{O}_K[\![\delta_{J^+}]\!]$. Modulo $I_K = p(\delta_0/\pi_K, \delta_J)\mathcal{R}_K$, $\psi_K(h)$ does not depend on the choice of the compatible system of p-basis decompositions. Moreover, ψ_K is an approximate homomorphism modulo I_K.*

Proof. First, $\psi_K(h) - h \in (\delta_{J^+}) \cdot \mathcal{O}_K[\![\delta_{J^+}]\!]$ is obvious from the construction. Next, we observe that when $p^r > \beta_K$, in any r-th p-basis decomposition for $h \in \mathcal{O}_K^\times$, the sum $\sum_{n'=0}^{\lambda_{(r),e_J,n}} \alpha_{(r),e_J,n,n'}^{p^r} \pi_K^n$ for any e_J and n in (4.1.1.3) is well-defined modulo p. So, the ambiguity of defining ψ_K lies in I_K.

For $h_1, h_2 \in \mathcal{O}_K^\times$, the formal sum or product of compatible systems of p-basis decompositions of h_1 and h_2 are just some compatible systems of p-basis decompositions of $h_1 + h_2$ or h_1h_2. Thus, $\psi_K(h_1) + \psi_K(h_2)$ and $\psi_K(h_1)\psi_K(h_2)$ are the same as $\psi_K(h_1 + h_2)$ and $\psi_K(h_1h_2)$ modulo I_K, respectively. The statement for general elements in $\mathcal{O}_K$ follows from this. □

Remark 4.1.1.9. From Proposition 4.1.1.8, we see that the ideal case is when $\beta_K \gg 1$. In contrast, when $\beta_K = 1$, $I_K = (\delta_0, p\delta_J)$. The above proposition does not give us much information about ψ_K. This is why we are not able to prove Conjecture 2.2.2.17 in the absolutely unramified and non-logarithmic case. Compare Remark 2.2.2.20.

Hypothesis 4.1.1.10. For the rest of the section, assume that K is not absolutely unramified, i.e., $\beta_K \geq 2$.

The following is an analogue of Corollary 2.3.4.10, which will enable us to invoke Lemma 2.3.4.15 in Lemma 4.1.2.1.

Lemma 4.1.1.11. *Let $h \in \mathcal{O}_K$. Denote $dh = \bar{h}_0 d\pi_K + \bar{h}_1 db_1 + \cdots + \bar{h}_m db_m$ when viewed as a differential in $\Omega^1_{\mathcal{O}_K/\mathbb{Z}_p} \otimes_{\mathcal{O}_K} \kappa_K$. Then $\psi_K(h) - h \equiv \bar{h}_0\delta_0 + \cdots + \bar{h}_m\delta_m$ modulo $(\pi_K) + (\delta_0/\pi_K, \delta_J)^2$ in $\mathcal{R}_K$.*

Proof. For an r-th p-basis decomposition ($r \geq 1$) as in (4.1.1.3), we have, modulo the ideal $(\pi_K) + (\delta_{J^+})(\delta_0/\pi_K, \delta_J)$,

$$\begin{aligned}
\psi_K(h) - h &\equiv \sum_{e_J=0}^{p^r-1} \sum_{n=0}^{\infty} \sum_{n'=0}^{\lambda_{(r),e_J,n}} \Big((b_J + \delta_J)^{e_J} \alpha^{p^r}_{(r),e_J,n,n'} (\pi_K + \delta_0)^n - b_J^{e_J} \alpha^{p^r}_{(r),e_J,n,n'} \pi_K^n \Big) \\
&\equiv \sum_{e_J=0}^{p^r-1} \sum_{n=0}^{\infty} \sum_{n'=0}^{\lambda_{(r),e_J,n}} \alpha^{p^r}_{(r),e_J,n,n'} b_J^{e_J} \pi_K^n \Big(\frac{n\delta_0}{\pi_K} + \frac{e_1\delta_1}{b_1} + \cdots + \frac{e_m\delta_m}{b_m} \Big) \\
&\equiv \bar{h}_0 \delta_0 + \cdots + \bar{h}_m \delta_m.
\end{aligned}$$

Taking limit does not break the congruence relation. □

Definition 4.1.1.12. Denote $\mathcal{S}_K = \mathcal{R}_K\langle u_{J^+}\rangle$. For $\omega \in \frac{1}{e}\mathbb{N} \cap [1, \beta_K]$, we say a set of elements $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ *has error gauge* $\geq \omega$ if $R_0 \in (N^\omega \delta_0, N^{\omega+1}\delta_J) \cdot \mathcal{S}_K$ and $R_j \in (N^{\omega-1}\delta_0, N^\omega \delta_J) \cdot \mathcal{S}_K$ for all $j \in J$. We say that (R_{J^+}) is *admissible* if it has error gauge ≥ 1.

Definition 4.1.1.13. Let $a > 1$. We define the *standard (non-logarithmic) thickening space (of level a)* $TS^a_{L/K,\psi_K}$ of L/K to be the rigid space associated to

$$\mathcal{O}^a_{TS,L/K,\psi_K} = K\langle \pi_K^{-a}\delta_{J^+}\rangle\langle u_{J^+}\rangle / \big(\psi_K(p_{J^+})\big).$$

For $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ admissible, we define the *(non-logarithmic) thickening space (of level a)* $TS^a_{L/K,R_{J^+}}$ to be the rigid space associated to

$$\mathcal{O}^a_{TS,L/K,R_{J^+}} = K\langle \pi_K^{-a}\delta_{J^+}\rangle\langle u_{J^+}\rangle / \big(\psi_K(p_{J^+}) + R_{J^+}\big).$$

Similarly, for $a > 0$, we define the *standard logarithmic thickening space (of level a)* $TS^a_{L/K,\log,\psi_K}$ of L/K to be the rigid space associated to

$$\mathcal{O}^a_{TS,L/K,\log,\psi_K} = K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle\langle u_{J^+}\rangle / \big(\psi_K(p_{J^+})\big).$$

For $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ admissible, we define the *logarithmic thickening space (of level* *a)* $TS^a_{L/K,\log,R_{J^+}}$ to be the rigid space associated to

$$\mathcal{O}^a_{TS,L/K,\log,R_{J^+}} = K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle\langle u_{J^+}\rangle \big/ \big(\psi_K(p_{J^+}) + R_{J^+}\big).$$

Denote $TS_{L/K,R_{J^+}} = \cup_{a>0} TS^a_{L/K,\log,R_{J^+}}$. Then we have the following natural Cartesian diagram for $a > 0$.

$$\begin{array}{ccccc}
TS^{a+1}_{L/K,R_{J^+}} & \hookrightarrow & TS^a_{L/K,\log,R_{J^+}} & \hookrightarrow & TS_{L/K,R_{J^+}} \\
\downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} \\
A_K^{m+1}[0,\theta^{a+1}] & \hookrightarrow & A_K^1[0,\theta^{a+1}] \times A_K^m[0,\theta^a] & \hookrightarrow & A_K^1[0,\theta) \times A_K^m[0,1)
\end{array}$$

Remark 4.1.1.14. Error gauge is supposed to measure how "standard" a thickening space is. Unfortunately, a standard thickening space itself depends on a very non-canonical function ψ_K. The upshot is that, by Proposition 4.1.1.8, the notion of having error gauge $\geq \omega$ does not depend on the choice of ψ_K if $\omega \in [1, \beta_K]$; note that the terms in p_0 are all divisible by π_K, except u_0^e.

Remark 4.1.1.15. The reason of introducing non-standard thickening spaces (or rather thickening spaces which do not have error gauge $\geq \beta_K$) is, as we will show later, that adding a generic p-th root results in the error gauge of (R_{J^+}) dropping by one; the comparison Theorem 4.1.2.2 guarantees that as long as (R_{J^+})'s are admissible, the thickening spaces still compute the same ramification break. On the same issue, if $\beta_K = 1$, we can not afford to drop the error gauge; this is why we are not able to prove Conjecture 2.2.2.17 in the absolutely unramified and non-logarithmic case (see also Remark 4.1.1.9).

Notation 4.1.1.16. Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. We extend Δ to mean the composite

$$\mathcal{S}_K \big/ \big(\psi_K(p_{J^+}) + R_{J^+}\big) \xrightarrow{\text{mod } (\delta_0/\pi_K, \delta_J)} \mathcal{O}_K\langle u_{J^+}\rangle/(p_{J^+}) \xrightarrow[\simeq]{\Delta} \mathcal{O}_L.$$

We remark that $\psi_K(p_{J^+}) - p_{J^+} + R_{J^+}$ are in fact contained in the ideal of $\mathcal{S}_K$ generated by δ_{J^+}. We denote the composition of Δ and the reduction $\mathcal{O}_L \twoheadrightarrow \kappa_L$ by $\overline{\Delta}$.

Lemma 4.1.1.17. *Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. Then*

$$\{u_{J^+}^{e_{J^+}} | e_j \in \{0, \dots, p^{r_j} - 1\} \text{ for all } j \in J, \text{ and } e_0 \in \{0, \dots, e-1\}\} \tag{4.1.1.18}$$

form a basis of $\mathcal{S}_K/(\psi_K(p_{J^+}) + R_{J^+})$ over $\mathcal{R}_K$. As a consequence, they form a basis of $\mathcal{O}^a_{TS,L/K,R_{J^+}}$ over $K\langle\pi_K^{-a}\delta_{J^+}\rangle$ for $a > 1$ and a basis of $\mathcal{O}^a_{TS,L/K,\log,R_{J^+}}$ over $K\langle\pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle$ for $a > 0$. In particular, the morphism $\Pi : TS_{L/K,R_{J^+}} \to A^1_K[0,\theta) \times A^m_K[0,1)$ is finite and flat.

Proof. Given an element $h \in \mathcal{S}_K/(\psi_K(p_{J^+}) + R_{J^+})$, we first take a representative $\tilde{h} \in \mathcal{S}_K$ in $\mathcal{S}_K$. Then we can simplify it by iteratively replacing u_0^e and $u_j^{p^{r_j}}$ by $u_0^e - \psi_K(p_0) - R_0$ and $u_j^{p^{r_j}} - \psi_K(p_j) - R_j$ for $j \in J$, respectively. This procedure converges and gives an element with the power of u_0 smaller than e and power of u_j smaller than p^{r_j} for $j \in J$. □

4.1.2 $AS = TS$ theorem

As explained in Subsection 2.3.5, the essential step which links the arithmetic conductors and the differential conductors is to establish Fake-theorem 2.3.5.2, which asserts that the lifted Abbes-Saito spaces are isomorphic to the thickening spaces.

Remember that we continue to assume Hypotheses 4.1.1.1 and 4.1.1.10 here.

Lemma 4.1.2.1. *Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. We have*

$$\det\left(\frac{\partial(\psi_K(p_i) - p_i + R_i)}{\partial\delta_j}\right)_{i,j\in J^+}\Big|_{\delta_{J^+}=0} \in \left(\mathcal{O}_K\langle u_{J^+}\rangle/(p_{J^+})\right)^\times = \mathcal{O}_L^\times.$$

Proof. It is enough to prove that the matrix is of full rank modulo π_L. By Lemma 4.1.1.11 and the admissibility of R_{J^+}, modulo π_L, the first row will be all zero except the first element which is $\overline{\mathfrak{d}(c_1, \dots, c_m)} \in \kappa_L^\times$ defined in Construction 2.3.3.3. Hence, we need only to look at

$$\left(\frac{\partial(\bar{\psi}_K(\bar{p}_i) - \bar{p}_i)}{\partial\delta_j}\right)_{i,j\in J}\Big|_{\delta_{J^+}=0};$$

this is an element in $\kappa_L^\times$ by Lemma 2.3.4.15 (whose condition is verified by Lemma 4.1.1.11).

□

Theorem 4.1.2.2. *If $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ is admissible, we have isomorphisms of K-algebras*

$$\mathcal{O}^a_{AS,L/K} \simeq \mathcal{O}^a_{TS,L/K,R_{J^+}} \quad \text{if} \quad a > 1,$$
$$\mathcal{O}^a_{AS,L/K,\log} \simeq \mathcal{O}^a_{TS,L/K,\log,R_{J^+}} \quad \text{if} \quad a > 0.$$

Proof. The proof is similar to Theorem 3.4.2.2. We will match up u_{J^+} in both rings.

First, $\{u_{J^+}^{e_{J^+}} | e_j \in \{0, \ldots, p^{r_j} - 1\}$ for all $j \in J$, and $e_0 \in \{0, \ldots, e-1\}\}$ forms a basis of $\mathcal{O}^a_{AS,L/K}$ (resp. $\mathcal{O}^a_{AS,L/K,\log}$) over $K\langle \pi_K^{-a} V_{J^+}\rangle$ (resp. $K\langle \pi_K^{-a-1} V_0, \pi_K^{-a} V_J\rangle$) as a finite free module. Given

$$h = \sum_{e_{J^+}, e'_{J^+}} \alpha_{e_{J^+}, e'_{J^+}} V_{J^+}^{e_{J^+}} u_{J^+}^{e'_{J^+}} \in \mathcal{O}^a_{AS,L/K} \text{ (resp. } \mathcal{O}^a_{AS,L/K,\log})$$

written in this basis, where $\alpha_{e_{J^+}, e'_{J^+}} \in K$, we define

$$|h|_{AS,a} = \max_{e_{J^+}, e'_{J^+}} \left\{ |\alpha_{e_{J^+}, e'_{J^+}}| \cdot \theta^{ae_0 + \cdots + ae_m + e'_0/e} \right\}$$
$$\text{(resp. } |h|_{AS,\log,a} = \max_{e_{J^+}, e'_{J^+}} \left\{ |\alpha_{e_{J^+}, e'_{J^+}}| \cdot \theta^{(a+1)e_0 + ae_1 + \cdots + ae_m + e'_0/e} \right\}).$$

It is clear that $\mathcal{O}^a_{AS,L/K}$ (resp. $\mathcal{O}^a_{AS,L/K,\log}$) is complete for this norm. The requirement $a > 1$ in the non-logarithmic case guarantees that when substituting u_0^e by $u_0^e - p_0 - V_0$, the norm does not increase.

Similarly, by Lemma 4.1.1.17, $\{u_{J^+}^{e_{J^+}} | e_j \in \{0, \ldots, p^{r_j} - 1\}$ for all $j \in J$, and $e_0 \in \{0, \ldots, e-1\}\}$ also forms a basis of $\mathcal{O}^a_{TS,L/K,R_{J^+}}$ (resp. $\mathcal{O}^a_{TS,L/K,\log,R_{J^+}}$) over $K\langle \pi_K^{-a} \delta_{J^+}\rangle$ (resp. $K\langle \pi_K^{-a-1} \delta_0, \pi_K^{-a} \delta_J\rangle$) as a finite free module. Given

$$h = \sum_{e_{J^+}, e'_{J^+}} \alpha_{e_{J^+}, e'_{J^+}} \delta_{J^+}^{e_{J^+}} u_{J^+}^{e'_{J^+}} \in \mathcal{O}^a_{TS,L/K,R_{J^+}} \text{ (resp. } \mathcal{O}^a_{TS,L/K,\log,R_{J^+}})$$

written in this basis, where $\alpha_{e_{J+},e'_{J+}} \in K$, we define

$$|h|_{TS,a} = \max_{e_{J+},e'_{J+}} \left\{ |\alpha_{e_{J+},e'_{J+}}| \cdot \theta^{ae_0+\cdots+ae_m+e'_0/e} \right\}$$
$$(\text{resp. } |h|_{TS,\log,a} = \max_{e_{J+},e'_{J+}} \left\{ |\alpha_{e_{J+},e'_{J+}}| \cdot \theta^{(a+1)e_0+ae_1+\cdots+ae_m+e'_0/e} \right\}).$$

It is clear that $\mathcal{O}^a_{TS,L/K,R_{J+}}$ (resp. $\mathcal{O}^a_{TS,L/K,\log,R_{J+}}$) is complete for this norm. The requirement $a > 1$ in the non-logarithmic case guarantees that when substituting u_0^e by $u_0^e - \psi_K(p_0) - R_0$, the norm does not increase.

Define $\chi_1 : \mathcal{O}^a_{AS,L/K} \to \mathcal{O}^a_{TS,L/K,R_{J+}}$ (resp. $\chi_1 : \mathcal{O}^a_{AS,L/K,\log} \to \mathcal{O}^a_{TS,L/K,\log,R_{J+}}$) by sending u_{J+} to u_{J+} and hence V_j to $p_j(u_{J+}) = p_j(u_{J+}) - \psi_K(p_j(u_{J+})) - R_j$ for all $j \in J^+$. We need to verify the convergence condition for all V_j. Indeed, Proposition 4.1.1.8 and the admissibility of R_{J+} imply that

$$|p_j - \psi_K(p_j)|_{TS,a} \leq \theta^a, \quad |R_j|_{TS,a} \leq \theta^a \text{ for all } j \in J^+$$
$$(\text{resp. } |p_j - \psi_K(p_j)|_{TS,\log,a} \leq \begin{cases} \theta^{a+1} & j = 0 \\ \theta^a & j \in J \end{cases}, \; |R_j|_{TS,\log,a} \leq \begin{cases} \theta^{a+1+1/e} & j = 0 \\ \theta^{a+1/e} & j \in J \end{cases}).$$

Now we define the inverse χ_2 of χ_1. Obviously, one should send u_{J+} back to u_{J+}. We need to define $\chi_2(\delta_{J+})$. By Lemma 4.1.2.1,

$$A = (A_{ij})_{i,j\in J^+} := \left(\frac{\partial(\psi_K(p_i) + R_i)}{\partial \delta_j}\right)_{i,j\in J^+}\Big|_{\delta_{J+}=0}$$
$$\in \mathrm{GL}_{m+1}(\mathcal{O}_L) \cong \mathrm{GL}_{m+1}\big(\mathcal{O}_K\langle u_{J+}\rangle/(p_{J+})\big).$$

Let A^{-1} denote the inverse matrix in $\mathrm{GL}_{m+1}\big(\mathcal{O}_K\langle u_{J+}\rangle/(p_{J+})\big)$, whose entries are written as polynomials in u_{J+} (using the basis (4.1.1.18)). Thus,

$$A^{-1} \cdot A - I \in \mathrm{Mat}_{m+1}\big((\delta_{J+}) \cdot \mathcal{O}^a_{TS,L/K,R_{J+}}\big) \text{ (resp. } \mathrm{Mat}_{m+1}\big((\delta_{J+}) \cdot \mathcal{O}^a_{TS,L/K,\log,R_{J+}}\big)\text{)},$$
(4.1.2.3)

where I is the $(m+1)\times(m+1)$ identity matrix. Now, we write

$$\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} = (I - A^{-1}A)\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} - A^{-1}\left(\begin{pmatrix}\psi_K(p_0)-p_0+R_0\\ \vdots\\ \psi_K(p_m)-p_m+R_m\end{pmatrix} - A\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix}\right) - A^{-1}\begin{pmatrix}p_0\\ \vdots\\ p_m\end{pmatrix}; \tag{4.1.2.4}$$

the last term is just $-A^{-1}\cdot\chi_1(V_{J^+})$. We need to bound the first two terms.

By (4.1.2.3), $I - A^{-1}A$ has norm $\leq \theta^a$. Hence, in the non-logarithmic case, the first term in (4.1.2.4) has norm $\leq \theta^{2a}$; in the logarithmic case the first term in (4.1.2.4) has norm $\leq \theta^{2a}$, except for the first row, which has norm $\leq \theta^{2a+1}$. By the definition of A, the second term in (4.1.2.4) has entries in $(\delta_{J^+})(\delta_0/\pi_K, \delta_J)\cdot\mathcal{O}^a_{TS,L/K,R_{J^+}}$, except for the first row, which is in $(\delta_0/\pi_K, \delta_J)^2\cdot\mathcal{O}^a_{TS,L/K,R_{J^+}}$. Hence, in the non-logarithmic case, the term has norm $\leq \theta^{2a-1}$; in the logarithmic case, the term has norm $\leq \theta^{2a}$, except for the first row, which has norm $\leq \theta^{2a+1}$.

Since we want χ_2 to be the inverse of χ_1, we define recursively by

$$\chi_2\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} = -A^{-1}\begin{pmatrix}V_0\\ \vdots\\ V_m\end{pmatrix} + \chi_2\begin{pmatrix}\Lambda_0\\ \vdots\\ \Lambda_m\end{pmatrix},$$

where Λ_{J^+} denotes the sum of the first two terms in (4.1.2.4). Since Λ_{J^+} have strictly smaller norms than δ_{J^+} and Λ_{J^+} are in the ideal (δ_{J^+}), one can plug the image of $\chi_2(\delta_{J^+})$ back into $\chi_2(\Lambda_{J^+})$ and iterate this substitution. This construction will converge to a continuous homomorphism χ_2, which is an inverse of χ_1. Moreover, from the construction, one can see that

$$|\chi_2(\delta_j)|_{AS,a} \leq \theta^a, \text{ for all } j\in J^+,$$
$$|\chi_2(\delta_0)|_{AS,\log,a} \leq \theta^{a+1} \text{ and } |\chi_2(\delta_j)|_{AS,\log,a} \leq \theta^a \text{ for all } j\in J.$$

Therefore, we have two continuous homomorphisms χ_1 and χ_2, being inverse to each other; this concludes the proof. □

Remark 4.1.2.5. An alternative way to understand this theorem is to think of the thickening spaces as perturbations of the morphisms $AS^a_{L/K} \to A^{m+1}_K[0,\theta^a]$ and $AS^a_{L/K,\log} \to A^1_K[0,\theta^{a+1}] \times A^m_K[0,\theta^a]$. Abbes-Saito spaces will behave better under base change using the new morphisms.

4.1.3 Étaleness of thickening spaces

In this subsection, we will study a variant of [AS02, Theorem 7.2] and [AS03, Corollary 4.12].

Remember that Hypotheses 4.1.1.1 and 4.1.1.10 are still in force.

Definition 4.1.3.1. Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot S_K$ be an admissible subset. Let $ET_{L/K,R_{J^+}}$ be the rigid analytic subspace of $A^1_K[0,\eta) \times A^m_K[0,1)$ over which the morphism Π defined in Definition 4.1.1.13 is étale. When there is no ambiguity of R_{J^+}, we may omit it from the notation by writing $ET_{L/K}$ instead.

Theorem 4.1.3.2. *Let $b(L/K)$ be the highest non-logarithmic ramification break of L/K. There exists $\epsilon \in (0, b(L/K)-1)$ such that for any $(R_{J^+}) \subset (\delta_{J^+})\cdot S_K$ admissible, $A^{m+1}_K[0,\theta^{b(L/K)-\epsilon}] \subseteq ET_{L/K,R_{J^+}}$.*

Proof. Recall from [AS02, Proposition 7.3] that

$$\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} = \oplus_{i=1}^r \mathcal{O}_L/\pi_L^{\alpha_i}\mathcal{O}_L \text{ with } \alpha_i < e(b(L/K)-\epsilon) \tag{4.1.3.3}$$

for some $\epsilon > 0$ and $r \in \mathbb{N}$. It does not hurt to take $\epsilon < b(L/K) - 1$. Let $\mathcal{J} = \big(\partial(\psi_K(p_i) + R_i)/\partial u_j\big)_{i,j\in J^+}$ be the Jacobian matrix of $TS^a_{L/K,R_{J^+}}$ over $A^{m+1}_K[0,\theta^a]$, whose entries are elements in $\mathcal{O} = \mathcal{O}_K\langle\pi_K^{-a}\delta_{J^+}\rangle\langle u_{J^+}\rangle/\big(\psi_K(p_i)+R_i\big)$.

Let $a \geq b(L/K) - \epsilon$ and $P = (\delta_{J^+}) \in A^1_K[0,\theta^a]$ be any point. Suppose the thickening space is not étale at P. Then the relative differential $\Omega^1_{TS^a_{L/K,R_{J^+}}/A^{m+1}_K[0,\theta^a]}$ have a constituent isomorphic to $K(P)$ at P, where $K(P)$ is the residue field at P. This implies that Coker $(\mathcal{O} \xrightarrow{\mathcal{J}} \mathcal{O})$ has a torsion-free constituent at P.

One the other hand, at P, $|\delta_j| \leq \theta^a$ for $j \in J^+$. Hence,

$$\begin{aligned} \mathcal{J} \bmod \pi_K^a &\equiv \left(\partial p_i/\partial u_j\right)_{i,j\in J^+} \bmod \pi_K^a, \\ \operatorname{Coker}(\mathcal{O} \xrightarrow{\mathcal{J}} \mathcal{O}) \otimes \mathcal{O}/\pi_K^a &= \operatorname{Coker}(\mathcal{O} \xrightarrow{(\partial p_i/\partial u_j)} \mathcal{O}) \otimes \mathcal{O}/\pi_K^a, \end{aligned}$$

which should not have a direct summand $\mathcal{O}_L/\pi_K^a\mathcal{O}_L$ according to (4.1.3.3) because $a > \alpha_i$ for all i. Contradiction. We have the étaleness as stated. □

Remark 4.1.3.4. Theorem 4.1.3.2 (as well as Theorem 4.1.3.6 later) states that the étale locus $ET_{L/K,R_{J^+}}$ is a bit larger than the locus where $TS^a_{L/K,R_{J^+}}$ (resp. $TS^a_{L/K,\log,R_{J^+}}$) becomes a geometrically disjoint union of $[L:K]$ discs.

The following lemma is an easy fact about logarithmic relative differentials. This is not a good place to introduce the whole theory of logarithmic structure. For a systematic account of logarithmic structures and log-schemes, one may consult [KS04, Section 4] and [Kat89b].

Lemma 4.1.3.5. *If we provide $\mathcal{O}_L$ and $\mathcal{O}_K$ with the canonical log-structures $\pi_L^{\mathbb{N}} \hookrightarrow \mathcal{O}_L$ and $\pi_K^{\mathbb{N}} \hookrightarrow \mathcal{O}_K$, respectively, then the logarithmic relative differentials*

$$\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}(\log/\log) = \bigoplus_{j\in J} \mathcal{O}_L du_j \oplus \mathcal{O}_L \frac{du_0}{u_0} \Big/ \Big(d(p_J), \frac{d(p_0)}{\pi_K}, \frac{d\pi_K}{\pi_K}, dx \textit{ for } x \in \mathcal{O}_K\Big).$$

Theorem 4.1.3.6. *Let $b_{\log}(L/K)$ be the highest logarithmic ramification break of L/K. Then there exists $\epsilon \in (0, b_{\log}(L/K))$ such that, for any $(R_{J^+}) \subset (\delta_{J^+}) \cdot S_K$ admissible, $A^1_K[0, \theta^{b_{\log}(L/K)+1-\epsilon}] \times A^m_K[0, \theta^{b_{\log}(L/K)-\epsilon}] \subseteq ET_{L/K,R_{J^+}}$.*

Proof. The proof is similar to Theorem 4.1.3.2 except that we need to invoke [AS03, Proposition 4.11(2)] to give a bound on $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}(\log/\log)$; the explicit description of $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}(\log/\log)$ in Lemma 4.1.3.5 singles out δ_0 and gives rise to the smaller radius θ^{a+1}. □

4.1.4 Construction of differential modules

In this subsection, we set up the framework of interpreting ramification filtrations by differential modules.

As a reminder, we keep Hypotheses 4.1.1.1 and 4.1.1.10.

Construction 4.1.4.1. Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. By Lemma 4.1.1.17, $\Pi : \Pi^{-1}(ET_{L/K}) \to ET_{L/K}$ is finite and étale. We call $\mathcal{E} = \Pi_*(\mathcal{O}_{\Pi^{-1}(ET_{L/K})})$ *a differential module associated to* L/K; it is defined over $ET_{L/K}$ and given by

$$\nabla : \mathcal{E} \to \Pi_*\big(\Omega^1_{\Pi^{-1}(ET_{L/K})/K}\big) \simeq \mathcal{E} \otimes_{\mathcal{O}_{ET_{L/K}}} \Omega^1_{ET_{L/K}/K} = \mathcal{E} \otimes_{\mathcal{O}_{ET_{L/K}}} \Big(\bigoplus_{j \in J^+} \mathcal{O}_{ET_{L/K}} d\delta_j\Big).$$

Also, we use $F_{\theta^{a_0},\dots,\theta^{a_m}}$ to denote the completion of $K(\delta_{J^+})$ with respect to the $(\theta^{a_0}, \dots, \theta^{a_m})$-Gauss norm. We can define the action of differential operators $\partial_j = \partial/\partial\delta_j$ for $j \in J^+$ on $\mathcal{E}$ and talk about intrinsic radius $IR(\mathcal{E} \otimes F_{\theta^{a_0},\dots,\theta^{a_m}})$ as in Definition 1.1.6.3 if $A^1_K[0, \theta^{a_0}] \times \cdots \times A^1_K[0, \theta^{s_m}] \subseteq ET_{L/K}$.

Notation 4.1.4.2. We use $IR(\mathcal{E}; a_{J^+})$ to denote $IR(\mathcal{E} \otimes F_{\theta^{a_0},\dots,\theta^{a_m}})$ for short. If $a_1 = \cdots = a_m = a$, we further simplify the notation to be $IR(\mathcal{E}; a_0, \underline{a})$. If furthermore, $a_0 = a$, we will just simply write $IR(\mathcal{E}, \underline{a})$ for short.

We call some attention on the following result extracted from the theory of differential modules.

Proposition 4.1.4.3. *Let* $a_{J^+} \subset \mathbb{R}$ *be a tuple and let* $\mathcal{E}$ *be a* ∂_{J^+}*-differential module on* $A^1_K[0, \theta^{a_0}] \times \cdots \times A^1_K[0, \theta^{a_m}]$. *Then*

(a) (Continuity) The function $\log_\theta IR(\mathcal{E}; s_{J^+})$ *is continuous for* $s_j \in [a_j, +\infty)$ *and* $j \in J^+$.

(b) (Monotonicity) Let $s_j \geq s'_j \geq a_j$ *for all* $j \in J^+$. *Then* $IR(\mathcal{E}; s_{J^+}) \geq IR(\mathcal{E}; s'_{J^+})$.

(c) (Zero Loci) The subset $Z(\mathcal{E}) = \{s_{J^+} \in [a_0, +\infty) \times \cdots \times [a_m, +\infty) | IR(\mathcal{E}; s_{J^+}) = 1\}$ *is transrational polyhedral (see Definition 1.3.1.6).*

Proof. Statements (a) and (c) follow from Theorem 1.3.3.9. For (b), by drawing zig-zag lines parallel to axes linking the two points s_{J^+} and s'_{J^+}, it suffices to consider the case when $s_j = s'_j$ for $j \in J^+ \backslash \{j_0\}$ and $s_{j_0} \geq s'_{j_0}$ for some $j_0 \in J^+$. In this case, we may base change to the completion of $K(\delta_{J^+\backslash\{j_0\}})$ with respect to the $s_{J^+\backslash\{j_0\}}$-Gauss norm. The result follows from Theorem 1.2.4.4. □

Theorem 4.1.4.4. *The following statements are equivalent for $a > 1$ (resp. $a > 0$):*

(1) The highest non-logarithmic (resp., logarithmic) ramification break satisfies $b(L/K) \leq a$ (resp. $b_{\log}(L/K) \leq a$);

(2) For any (some) admissible $(R_{J^+}) \subset \mathcal{S}_K$ and any rational number $a' > a$,

$$\#\pi_0^{\mathrm{geom}}(TS^{a'}_{L/K,R_{J^+}}) = [L:K] \ (\textit{resp. } \#\pi_0^{\mathrm{geom}}(TS^{a'}_{L/K,\log,R_{J^+}}) = [L:K]\ ).$$

(3) For any (some) admissible $(R_{J^+}) \subset \mathcal{S}_K$, $A_K^{m+1}[0,\theta^a] \subseteq ET_{L/K,R_{J^+}}$ (resp. $A^1_K[0,\theta^{a+1}] \times A^m_K[0,\theta^a] \subseteq ET_{L/K,R_{J^+}}$) and the intrinsic radius of $\mathcal{E}$ over $A_K^{m+1}[0,\theta^a]$ (resp. $A^1_K[0,\theta^{a+1}] \times A^m_K[0,\theta^a]$) is maximal:

$$IR(\mathcal{E};\underline{a}) = 1 \ (\textit{resp. } IR(\mathcal{E};a+1,\underline{a}) = 1).$$

Proof. The proof is similar to Theorem 3.3.4.6. It is the mixed characteristic version of Fake-theorem 2.3.6.5.

(1) $\Leftrightarrow$ (2) is immediate from Theorem 4.1.2.2.

(2) $\Rightarrow$ (3): For any rational number $a' > a$, (2) implies that for some finite extension K' of K, $TS^{a'}_{L/K,R_{J^+}} \times_K K'$ (resp. $TS^{a'}_{L/K,\log,R_{J^+}} \times_K K'$) has $[L:K]$ connected components and is hence force to be $[L:K]$ copies of $A^{m+1}_{K'}[0,\theta^{a'}]$ (resp. $A^1_{K'}[0,\theta^{a'+1}] \times A^m_{K'}[0,\theta^{a'}]$) because Π is finite and flat; in particular, Π is étale there. Therefore, $\mathcal{E} \otimes_K K'$ is a trivial differential module over $A^{m+1}_{K'}[0,\theta^{a'}]$ (resp. $A^1_{K'}[0,\theta^{a'+1}] \times A^m_{K'}[0,\theta^{a'}]$). As a consequence,

$$IR(\mathcal{E};\underline{a'}) = IR(\mathcal{E}\otimes K';\underline{a'}) = 1 \text{ (resp. } IR(\mathcal{E};a'+1,\underline{a'}) = IR(\mathcal{E}\otimes_K K';a'+1,\underline{a'}) = 1\text{)}.$$

Statement (3) follows from the continuity of intrinsic radii in Proposition 4.1.4.3(a).

(3) $\Rightarrow$ (2): (3) implies that, for any rational number $a' > a$, $\mathcal{E}$ is a trivial differential module on $A_K^{m+1}[0, \theta^{a'}]$ (resp. $A_K^1[0, \theta^{a'+1}] \times A_K^m[0, \theta^{a'}]$). Indeed, we have a bijection

$$H_\nabla^0(A_K^{m+1}[0, \theta^{a'}], \mathcal{E}) \xrightarrow{\cong} \mathcal{E}|_{\delta_{J^+}=0} \text{ (resp. } H_\nabla^0(A_K^1[0, \theta^{a'+1}] \times A_K^m[0, \theta^{a'}], \mathcal{E}) \xrightarrow{\cong} \mathcal{E}|_{\delta_{J^+}=0} \text{),} \tag{4.1.4.5}$$

whose inverse is given by Taylor series. This is in fact a ring isomorphism by basic properties of Taylor series. The left hand side of (4.1.4.5) is a subring of $\mathcal{O}^{a'}_{TS,L/K,R_{J+}}$ (resp. $\mathcal{O}^{a'}_{TS,L/K,\log,R_{J+}}$); the right hand side is just $K\langle u_{J+}\rangle/(p_{J+}) \simeq L$. Thus, after the extension of scalars from K to L, we can lift the idempotent elements in $L \otimes_K L \simeq \prod_{g \in G_{L/K}} L_g$ to idempotent elements in $\mathcal{O}^{a'}_{TS,L/K,R_{J+}} \otimes_K L$ (resp. $\mathcal{O}^{a'}_{TS,L/K,\log,R_{J+}} \otimes_K L$). This proves (2). □

Corollary 4.1.4.6. *Given the differential module $\mathcal{E}$ over $ET_{L/K}$ with respect to some admissible subset $(R_{J+}) \subset (\delta_{J+}) \cdot \mathcal{S}_K$, we have*

$$\begin{aligned} b(L/K) &= \min\left\{s \mid A_K^{m+1}[0, \theta^s] \subseteq ET_{L/K} \text{ and } IR(\mathcal{E}; \underline{s}) = 1\right\}, \text{ and} \\ b_{\log}(L/K) &= \min\left\{s \mid A_K^1[0, \theta^{s+1}] \times A_K^m[0, \theta^s] \subseteq ET_{L/K} \text{ and } IR(\mathcal{E}; s+1, \underline{s}) = 1\right\}. \end{aligned}$$

In other words, $b(L/K)$ (resp. $b_{\log}(L/K)$) corresponds to the intersection of the boundary of $Z(\mathcal{E})$ with the line defined by $s_0 = \cdots = s_m$ (resp. $s_0 - 1 = s_1 = \cdots = s_m$).

Proof. It is obvious from Theorem 4.1.4.4 and Proposition 4.1.4.3. □

4.1.5 Recursive thickening spaces

In this subsection, we introduce a generalization of thickening spaces. This will give us some freedom when changing the base field.

In this subsection, we continue to assume Hypotheses 4.1.1.1 and 4.1.1.10.

Construction 4.1.5.1. This is a variant of Construction 2.3.3.3. First, filter the (inseparable) extension κ_L/κ_K by elementary p-extensions

$$\kappa_K = \kappa_0 \subsetneq \kappa_1 \subsetneq \cdots \subsetneq \kappa_r = \kappa_L,$$

where for each $\lambda = 1, \ldots, r$, $\kappa_\lambda = \kappa_{\lambda-1}(\bar{\mathfrak{c}}_\lambda)$ with $\bar{\mathfrak{c}}_\lambda^p = \bar{\mathfrak{b}}_\lambda \in \kappa_{\lambda-1}$. Denote $\Lambda = \{1, \ldots, r\}$. Pick lifts $\mathfrak{c}_\Lambda$ of $\bar{\mathfrak{c}}_\Lambda$ in $\mathcal{O}_L$. Let $e = e_0, \ldots, e_{r_0} = 1$ be a strictly decreasing sequence of integers such that $e_i \mid e_{i-1}$ for $1 \leq i \leq r_0$. Set $I = \{1, \ldots, r_0\}$. For each $i \in I$, pick an element $\pi_{L,i}$ in $\mathcal{O}_L$ with valuation e_i; in particular, we take $\pi_{L,r_0} = \pi_L$. It is easy to see that $(\mathfrak{c}_\Lambda, \pi_{L,I})$ generate $\mathcal{O}_L$ over $\mathcal{O}_K$. So we have an isomorphism

$$\Delta : \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda \rangle / \mathfrak{J} \xrightarrow{\sim} \mathcal{O}_L,$$

sending $\mathfrak{u}_{0,i} \mapsto \pi_{L,i}$ for $i \in I$ and $\mathfrak{u}_\lambda \mapsto \mathfrak{c}_\lambda$ for $\lambda \in \Lambda$, where $\mathfrak{J}$ is some proper ideal and we use the same Δ as in Construction 2.3.3.3. Moreover,

$$\left\{ \mathfrak{u}_{0,I}^{\mathfrak{e}_{0,I}} \mathfrak{u}_\Lambda^{\mathfrak{e}_\Lambda} \middle| \mathfrak{e}_{0,i} \in \{0, \ldots, \frac{e_{i-1}}{e_i} - 1\} \text{ for all } i \in I \text{ and } \mathfrak{e}_\lambda \in \{0, \ldots, p-1\} \text{ for all } \lambda \in \Lambda \right\} \tag{4.1.5.2}$$

form a basis of $\mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda \rangle / \mathfrak{J}$ as a free $\mathcal{O}_K$-module, which we refer later as the *standard basis.*

We provide $\mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda]$ with the following norm: for $h = \sum_{\mathfrak{e}_{0,I}, \mathfrak{e}_\Lambda} \alpha_{\mathfrak{e}_{0,I}, \mathfrak{e}_\Lambda} \mathfrak{u}_{0,I}^{\mathfrak{e}_{0,I}} \mathfrak{u}_\Lambda^{\mathfrak{e}_\Lambda}$ with $\alpha_{\mathfrak{e}_{0,I}, \mathfrak{e}_\Lambda} \in \mathcal{O}_K$, we set

$$|h| = \max_{\mathfrak{e}_{0,I}, \mathfrak{e}_\Lambda} \{ |\alpha_{\mathfrak{e}_{0,I}, \mathfrak{e}_\Lambda}| \cdot \theta^{(\mathfrak{e}_{0,1} \cdot e_1 + \cdots + \mathfrak{e}_{0,r_0} \cdot e_{r_0})/e} \}.$$

For $a \in \frac{1}{e}\mathbb{Z}_{\geq 0}$, we use $\mathfrak{M}^a$ to denote the set consisting of elements in $\mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda]$ with norm $\leq \theta^a$; it is in fact an ideal.

In $\mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda \rangle / \mathfrak{J}$, we can write $\mathfrak{u}_{0,i}^{e_{i-1}/e_i}$ for $i \in I$ and $\mathfrak{u}_\Lambda^p$ in terms of the basis (4.1.5.2). This gives a set of generators of $\mathfrak{J}$:

$$\begin{aligned}
\mathfrak{p}_{0,1} &\in \mathfrak{u}_{0,1}^{e/e_1} - \mathfrak{d}_1 \pi_K + \mathfrak{M}^{1+1/e} \cdot \mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda], \\
\mathfrak{p}_{0,i} &\in \mathfrak{u}_{0,i}^{e_{i-1}/e_i} - \mathfrak{d}_i \mathfrak{u}_{0,i-1} + \mathfrak{M}^{(e_{i-1}+1)/e} \cdot \mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda], \ i \in I \backslash \{1\}, \\
\mathfrak{p}_\lambda &\in \mathfrak{u}_\lambda^p - \tilde{\mathfrak{b}}_\lambda + \mathfrak{M}^{1/e} \cdot \mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda],
\end{aligned}$$

where $\mathfrak{d}_I$ are some elements in $\mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda]$ whose images under Δ are invertible in $\mathcal{O}_L$, and for each λ, $\tilde{\mathfrak{b}}_\lambda$ is some element in $\mathcal{O}_K[\mathfrak{u}_1, \ldots, \mathfrak{u}_{\lambda-1}]$ whose image under Δ

reduces to $\bar{\mathfrak{b}}_\lambda \in k_{\lambda-1}$ modulo π_L.

We say that $\mathfrak{p}_\lambda$ *corresponds* to the extension $\kappa_\lambda/\kappa_{\lambda-1}$.

Definition 4.1.5.3. As in Definition 4.1.1.12, we define $\mathfrak{S}_K = \mathcal{R}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle = \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$. For $\omega \in \frac{1}{e}\mathbb{N}\cap[1, \beta_K]$, we say that a set of elements $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+})\cdot \mathfrak{S}_K$ *has error gauge* $\geq \omega$ if $\mathfrak{R}_{0,i} \in (\mathfrak{N}^{\omega-1+e_i/e}\delta_0, \mathfrak{N}^{\omega+e_i/e}\delta_J)\cdot \mathfrak{S}_K$ for $i \in I$ and $\mathfrak{R}_\lambda \in (\mathfrak{N}^{\omega-1}\delta_0, \mathfrak{N}^{\omega}\delta_J)\cdot\mathfrak{S}_K$ for $\lambda \in \Lambda$. The subset $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+})\cdot\mathfrak{S}_K$ is *admissible* if it has error gauge ≥ 1.

Let $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+})\cdot\mathfrak{S}_K$ be admissible. For $a > 1$, we define the *(non-logarithmic) recursive thickening space (of level a)* $TS^a_{L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$ to be the rigid space associated to

$$\mathcal{O}^a_{TS,L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} = K\langle \pi_K^{-a}\delta_{J^+}\rangle\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle / \big(\psi_K(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi_K(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda\big).$$

For $a > 0$, we define the *logarithmic recursive thickening space (of level a)* $TS^a_{L/K,\log,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$ to be the rigid space associated to

$$\mathcal{O}^a_{TS,L/K,\log,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} = K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle / \big(\psi_K(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi_K(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda\big).$$

We still use Δ to denote the natural homomorphism

$$\mathfrak{S}_K/\big(\psi_K(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi_K(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda\big) \xrightarrow{\bmod (\delta_0/\pi_K, \delta_J)} \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/(\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda) \xrightarrow[\simeq]{\Delta} \mathcal{O}_L;$$

we use $\overline{\Delta}$ to denote the composition with the reduction $\mathcal{O}_L \to \kappa_L$.

Lemma 4.1.5.4. *Let* $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+})\cdot\mathfrak{S}_K$ *be admissible. Then* (4.1.5.2) *forms a basis of* $\mathfrak{S}_K/\big(\psi_K(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi_K(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda\big)$ *as a free* $\mathcal{R}_K$*-module, which we refer later as the* standard basis. *As a consequence, they form a basis of* $\mathcal{O}^a_{TS,L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$ *(resp.* $\mathcal{O}^a_{TS,L/K,\log,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$*) as a free module over* $K\langle \pi_K^{-a}\delta_{J^+}\rangle$ *(resp.* $K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle$*).*

Proof. Same as Lemma 4.1.1.17. □

Example 4.1.5.5. The construction of the thickening spaces in Definition 4.1.1.13 is a special case of the above construction. If we start with a uniformizer π_L, a p-basis

c_J, and relations p_{J^+} in Construction 2.3.3.3, the following dictionary translates the information to fit in Construction 4.1.5.1.

$$\begin{aligned}
\pi_{L,I} &\longleftrightarrow \pi_L \quad (I = \{1\}),\\
\mathfrak{c}_\Lambda &\longleftrightarrow c_1, c_1^p, \dots, c_1^{p^{r_1-1}}, c_2, c_2^p, \dots, c_m^{p^{r_m-1}},\\
\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda &\longleftrightarrow \text{the ones determined by } \mathfrak{c}_\Lambda \text{ and } \pi_{L,I},\\
\mathfrak{R}_{0,I} &\longleftrightarrow R_0,\\
\mathfrak{R}_\lambda &\longleftrightarrow R_j \text{ when } \lambda \text{ corresponds to some } c_j^{p^{r_j-1}}, \text{ and 0 otherwise.}
\end{aligned}$$

Moreover, this construction preserves the error gauge.

Conversely, we have the following.

Proposition 4.1.5.6. *Let $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+}) \cdot \mathfrak{S}_K$ be admissible with error gauge $\geq \omega \in \frac{1}{e}\mathbb{N} \cap [1, \beta_K]$. Then, for any choices of c_J and π_L as in Construction 2.3.3.3, there exists an $\mathcal{R}_K$-isomorphism*

$$\Theta : \mathcal{S}_K/\big(\psi_K(p_{J^+}) + R_{J^+}\big) \xrightarrow{\sim} \mathfrak{S}_K/\big(\psi_K(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi_K(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda\big), \qquad (4.1.5.7)$$

for some admissible R_{J^+} with error gauge $\geq \omega$, such that Θ mod $(\delta_0/\pi_K, \delta_J)$ induces the identity map if we identify both side with $\mathcal{O}_L$ via Δ. This gives isomorphisms between the recursive thickening spaces and thickening spaces.

$$TS^a_{L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} \simeq TS^a_{L/K,R_{J^+}} \; (a > 1) \quad \textit{and} \quad TS^a_{L/K,\log,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} \simeq TS^a_{L/K,\log,R_{J^+}} \; (a > 0)$$

Proof. For each $j \in J$, express c_j as a polynomial $\tilde{\mathfrak{c}}_j$ in $\mathfrak{u}_{0,I}$ and $\mathfrak{u}_\Lambda$ with coefficients in $\mathcal{O}_K$ via $\Delta^{-1} : \mathcal{O}_L \xrightarrow{\sim} \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/(\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda)$, and set $\Theta(u_j) = \psi_K(\tilde{\mathfrak{c}}_j)$. We also set $\Theta(u_0) = \mathfrak{u}_{0,r_0}$. It is then obvious that for $a \in \frac{1}{e}\mathbb{Z}_{\geq 0}$, $\Theta(N^a \cdot \mathcal{S}_K) \subset \mathfrak{N}^a \cdot \mathfrak{S}_K$.

We need to determine R_{J^+}. For each fixed $j_0 \in J^+$, since $\Delta(p_{j_0}(u_{J^+})) = 0$, we can write

$$p_{j_0}(\mathfrak{u}_{0,r_0}, \tilde{\mathfrak{c}}_J) = \sum_{i \in I} \mathfrak{h}_{0,i}\mathfrak{p}_{0,i} + \sum_{\lambda \in \Lambda} \mathfrak{h}_\lambda \mathfrak{p}_\lambda, \quad \text{in } \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$$

for some $\mathfrak{h}_{0,i}, \mathfrak{h}_\lambda \in \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$ for $i \in I$ and $\lambda \in \Lambda$. Moreover, when $j_0 = 0$, we can require $\mathfrak{h}_{0,i} \in \mathfrak{N}^{1-e_{i-1}/e} \cdot \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$, and $\mathfrak{h}_\lambda \in \mathfrak{N}^1 \cdot \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$ for $i \in I$ and $\lambda \in \Lambda$. Thus,

$$\begin{aligned}
-\Theta(R_{j_0}) &= \psi_K(p_{j_0})(\Theta(u_{J^+})) \\
&= \sum_{i\in I} \psi_K(\mathfrak{h}_{0,i})\psi_K(\mathfrak{p}_{0,i}) + \sum_{\lambda\in\Lambda} \psi_K(\mathfrak{h}_\lambda)\psi_K(\mathfrak{p}_\lambda) + \mathfrak{E} \\
&= \sum_{i\in I} \psi_K(\mathfrak{h}_{0,i})(-\mathfrak{R}_{0,i}) + \sum_{\lambda\in\Lambda} \psi_K(\mathfrak{h}_\lambda)(-\mathfrak{R}_\lambda) + \mathfrak{E} \\
&\in \begin{cases} (\mathfrak{N}^\omega\delta_0, \mathfrak{N}^{\omega+1}\delta_J)\cdot \mathfrak{S}_K & j_0 = 0 \\ (\mathfrak{N}^{\omega-1}\delta_0, \mathfrak{N}^{\omega}\delta_J)\cdot \mathfrak{S}_K & j_0 \in J \end{cases},
\end{aligned}$$

where $\mathfrak{E} \in (\mathfrak{N}^{\beta_K}\delta_0, \mathfrak{N}^{(\beta_K+1)}\delta_J)\cdot\mathfrak{S}_K$ if $j_0 = 0$ and $\mathfrak{E} \in (\mathfrak{N}^{(\beta_K-1)}\delta_0, \mathfrak{N}^{\beta_K}\delta_J)\cdot\mathfrak{S}_K$ if $j_0 \in J$; they correspond to the error terms coming from ψ_K failing to be a homomorphism (See Proposition 4.1.1.8).

Thus, we can find polynomials $q_0, \dots, q_m \in \mathcal{O}_K[u_{J^+}]$ such that

$$q_0 \in \begin{cases} N^\omega \cdot \mathcal{S}_K & j_0 = 0 \\ N^{\omega-1}\cdot \mathcal{S}_K & j_0 \in J \end{cases} \qquad q_1, \dots, q_m \in \begin{cases} N^{\omega+1}\cdot \mathcal{S}_K & j_0 = 0 \\ N^{\omega}\cdot \mathcal{S}_K & j_0 \in J \end{cases},$$

$$\Theta(R_j - q_0\delta_0 - \cdots - q_m\delta_m) \in \begin{cases} (\delta_0/\pi_K, \delta_J)(\mathfrak{N}^\omega\delta_0, \mathfrak{N}^{\omega+1}\delta_J)\cdot\mathfrak{S}_K & j_0 = 0 \\ (\delta_0/\pi_K, \delta_J)(\mathfrak{N}^{\omega-1}\delta_0, \mathfrak{N}^{\omega}\delta_J)\cdot\mathfrak{S}_K & j_0 \in J \end{cases}.$$

Further, we can similarly clear up the coefficients for $\delta_j\delta_{j'}$ for $j, j' \in J$. Repeating this approximation gives the expressions for R_{J^+}. They clearly have error gauge $\geq \omega$.

The surjectivity of Θ follows from the surjectivity modulo $(\delta_0/\pi_K, \delta_J)$, which is the identity via Δ. Moreover, a surjective morphism between two finite free modules of the same rank over a Noetherian base is automatically an isomorphism. The theorem is proved. □

Remark 4.1.5.8. The isomorphism Θ is not unique. Basically, $\Theta(u_0) \bmod (\mathfrak{N}^\omega\delta_0, \mathfrak{N}^{\omega+1}\delta_J)\cdot \mathfrak{S}_K$ and $\Theta(u_j) \bmod (\mathfrak{N}^{\omega-1}\delta_0, \mathfrak{N}^\omega\delta_J)\cdot\mathfrak{S}_K$ for $j \in J$ are fixed; any lifts of them will give a desired isomorphism (with different (R_{J^+})).

Lemma 4.1.5.9. *Let $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+}) \cdot \mathfrak{S}_K$ be admissible. Then an element*

$$h \in \mathfrak{S}_K / \big(\psi_K(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi_K(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda\big)$$

is invertible if and only if $\Delta(h) \in \mathcal{O}_L^\times$. In particular, $\mathfrak{u}_{0,r_0}^e/\pi_K$ is invertible.

Proof. The necessity is obvious. To see the sufficiency, we construct the inverse of h directly. Let $h^{(-1)}$ be a lift of $\Delta(h^{-1}) \in \mathcal{O}_L^\times$ in $\mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$. We have $\Delta(1-h^{(-1)}h) = 0$ and hence $1 - h^{(-1)}h = g \in (\delta_{J^+}) \cdot \mathfrak{S}_K$. Thus,

$$\frac{1}{h} = \frac{h^{(-1)}}{1-g} = h^{(-1)} \cdot (1 + g + g^2 + \cdots).$$

The series converges to the inverse of h. □

4.2 Non-logarithmic Hasse-Arf theorems

4.2.1 Base change for generic p-th roots

In this subsection, we prove the key technical Theorem 4.2.1.7.

We continue to assume Hypotheses 4.1.1.1 and 4.1.1.10. When proving the main theorem, we will assume a technical Hypothesis 4.2.1.6, which is satisfied by any recursive thickening space coming from a thickening space by Example 4.1.5.5.

Notation 4.2.1.1. For this subsection, Fix $j_0 \in J$ and $n \in \mathbb{N}$ coprime to p. As in Definition 2.3.2.7, let $K(x)^\wedge$ be the completion of $K(x)$ with respect to the 1-Gauss norm and let K' be the completion of the maximal unramified extension of $K(x)^\wedge$. Let $\widetilde{K} = K'((b_{j_0} + x\pi_K^n)^{1/p})$ and $\widetilde{L} = L\widetilde{K}$. Denote $\beta_{j_0} = (b_{j_0} + x\pi_K^n)^{1/p}$ for simplicity. We put in an extra n here to ease the deduction later.

Lemma 4.2.1.2. *If $\bar{b}_{j_0}^{1/p} \notin \kappa_L$, the ramification break $b(\widetilde{L}/\widetilde{K}) = b(L/K)$.*

Proof. Since $\kappa_{\widetilde{L}} = \kappa_{\widetilde{K}}\kappa_L$, we have $\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$; the Lemma follows from Proposition 2.2.2.11(4'). □

So we need to deal with the non-trivial case when $\bar{b}_{j_0}^{1/p} \in \kappa_L$.

Notation 4.2.1.3. Denote $\mathcal{R}_{\widetilde{K}} = \mathcal{O}_{\widetilde{K}}[\![\eta_0/\pi_K, \eta_J, \eta_{m+1}]\!]$. Applying Construction 4.1.1.2 to $\widetilde{K}$ gives a function $\psi_{\widetilde{K}} : \mathcal{O}_{\widetilde{K}} \to \mathcal{R}_{\widetilde{K}}$, which is an approximate homomorphism modulo the ideal $I_{\widetilde{K}} = p(\eta_0/\pi_K, \eta_{J\cup\{m+1\}}) \cdot \mathcal{R}_K$.

Lemma 4.2.1.4. *There exists a unique continuous $\mathcal{O}_K$-homomorphism $f^* : \mathcal{R}_K \to \mathcal{R}_{\widetilde{K}}$ such that $f^*(\delta_j) = \eta_j$ for $j \in J^+\backslash\{j_0\}$ and $f^*(\delta_{j_0}) = (\beta_{j_0}+\eta_{j_0})^p - (x+\eta_{m+1})(\pi_K + \eta_0)^n - b_{j_0}$. It gives an approximately commutative diagram modulo $I_{\widetilde{K}}$.*

$$\begin{array}{ccc} \mathcal{O}_K & \xrightarrow{\psi_K} & \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!] = \mathcal{R}_K \\ \big\downarrow & & \big\downarrow f^* \\ \mathcal{O}_{\widetilde{K}} & \xrightarrow{\psi_{\widetilde{K}}} & \mathcal{O}_{\widetilde{K}}[\![\eta_0/\pi_K, \eta_{J\cup\{m+1\}}]\!] = \mathcal{R}_{\widetilde{K}} \end{array} \tag{4.2.1.5}$$

For $a > 1$, f^* gives a morphism $f : A^{m+2}_{\widetilde{K}}[0, \theta^a] \to A^{m+1}_K[0, \theta^a]$.

Proof. It follows immediately from Proposition 4.1.1.8. □

Hypothesis 4.2.1.6. For the next theorem, we assume that in Construction 4.1.5.1, there exists $\lambda_0 \in \Lambda$ such that the field extension $\kappa_{\lambda_0}/\kappa_{\lambda_0-1}$ is given by $\kappa_{\lambda_0} = \kappa_{\lambda_0-1}(\bar{b}_{j_0}^{1/p})$ and $\bar{\mathfrak{c}}_{\lambda_0} = \bar{b}_{j_0}^{1/p}$.

Theorem 4.2.1.7. *Assume Hypothesis 4.2.1.6 and keep the notation as above. Moreover, assume that $\beta_K \geq n+1$. Let $a > 1$ and $\omega \geq n+1$. Let $TS^a_{L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$ be a recursive thickening space with error gauge $\geq \omega$. Then $TS^a_{L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} \times_{A^{m+1}_K[0,\theta^a],f} A^{m+2}_{\widetilde{K}}[0, \theta^a]$ is a recursive thickening space for $\widetilde{L}/\widetilde{K}$ with error gauge $\geq \omega - n$.*

The reader may skip this proof when reading this paper for the first time. Roughly speaking, the argument presented here is a combination of the arguments in Propositions 4.1.5.6 and 4.3.1.4, but in a more complicated way.

Proof. <u>**Step 1:**</u> Find the generators of $\mathcal{O}_{\widetilde{L}}/\mathcal{O}_{\widetilde{K}}$.

The difficulty comes from that $\pi_{L,I}, \mathfrak{c}_\Lambda$ do not generate $\mathcal{O}_{\widetilde{L}}$ over $\mathcal{O}_{\widetilde{K}}$ (although they do generate $\widetilde{L}$ over $\widetilde{K}$). We need to change the generator $\mathfrak{c}_{\lambda_0}$ to an element which either gives

Case A: the inseparable extension $\kappa_{\widetilde{L}}/\kappa_L(\bar{x})^{\mathrm{sep}}$ which happens when $\widetilde{L}/\widetilde{K}$ has naïve ramification degree e; or

Case B: a ramified extension of naïve ramification degree p which happens when $\widetilde{L}/\widetilde{K}$ has naïve ramification degree ep.

Denote $L' = LK'$, which has residue field $\kappa_{L'} = \kappa_L(\bar{x})^{\mathrm{sep}}$. Then, we have $\mathcal{O}_{L'} = \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathcal{O}_L$. Hence, $\mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L \cong \mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_{K'}} \mathcal{O}_{L'} \subseteq \mathcal{O}_{\widetilde{L}}$. We may extend the valuation $v_{L'}(\cdot)$ to $\widetilde{L}$ by allowing rational valuations in Case B. Let $\beta_{j_0} - \mu$ for $\mu \in \mathcal{O}_{L'}$ be an element achieving the maximal valuation under $v_{L'}(\cdot)$ among $\beta_{j_0} + \mathcal{O}_{L'}$.

Claim: we have $\alpha = v_{L'}(\beta_{j_0} - \mu) \leq en/p$ and

in case A, the reduction of $\tilde{\mathfrak{c}}_{\lambda_0} = \pi_L^{-\alpha}(\beta_{j_0} - \mu)$ in $\kappa_{\widetilde{L}}$ generate $\kappa_{\widetilde{L}}$ over $\kappa_{L'}$ (we also set $d = 1$ by convention);

in case B, $v_{\widetilde{L}}(\pi_L^{-[\alpha]}(\beta_{j_0} - \mu)) = d/p$ for some $d \in \{1, \dots, p-1\}$, in which case, we fix a d-th root $\pi_{\widetilde{L},r_0+1}$ of $\pi_L^{-[\alpha]}(\beta_{j_0} - \mu)$; it generates the extension $\mathcal{O}_{\widetilde{L}}/\mathcal{O}_{L'}$.

Proof of the Claim: We have the norm $\mathbf{N}_{\widetilde{L}/L'}(\mu - \beta_{j_0}) = \mu^p - (b_{j_0} + x\pi_K^n)$. Since there is no $\mu \in \mathcal{O}_{L'}$ that can kill the $x\pi_K^n$ term (note $\beta_K \geq n+1$), $v_{L'}(\mathbf{N}_{\widetilde{L}/L'}(\beta_{j_0} - \mu)) \leq en$ and the first statement of the claim follows. When $\alpha \notin \mathbb{N}$, we are forced to fall in Case B and the claim is obvious. Assume for contradiction that $\alpha \in \mathbb{N}$ and the reduction of $\tilde{\mathfrak{c}}_{\lambda_0}$ lies in $\kappa_{L'}$. Then there exists $\mu' \in \mathcal{O}_{L'}$ such that $\mu'/\pi_L^\alpha \equiv \tilde{\mathfrak{c}}_{\lambda_0}$ (mod $\mathfrak{m}_{\widetilde{L}}$). But then $\beta_{j_0} - \mu - \mu'$ will have bigger valuation, which contradicts our choice of μ. This proves the claim.

Step 2: Find the generating relations.

By previous step, we can write

$$\mathcal{O}_{\widetilde{K}}\langle \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{v}}\rangle / (\tilde{\mathfrak{p}}_{0,I}, \tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{q}}) \simeq \mathcal{O}_{\widetilde{L}}.$$

by sending $\tilde{\mathfrak{u}}_{0,I}$ to $\mathfrak{c}_{0,I}$, $\tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}$ to $\mathfrak{c}_{\Lambda\backslash\lambda_0}$, and $\tilde{\mathfrak{v}}$ to $\tilde{\mathfrak{c}}_{\lambda_0}$ in Case A and $\pi_{\widetilde{L},r_0+1}$ in Case B, where the relations $\mathfrak{p}_{0,I}, \tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{q}}$ can be obtained using Construction 4.1.5.1. Now, we link these relations to the relations $\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda$ for $\mathcal{O}_L/\mathcal{O}_K$. We first lift the isomorphism

$$\tilde{\chi} : \widetilde{K}\langle \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{v}}\rangle / (\tilde{\mathfrak{p}}_{0,I}, \tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{q}}) \simeq \widetilde{L} \cong \widetilde{K} \otimes_{\mathcal{O}_K} \mathcal{O}_L \simeq \widetilde{K}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle / (\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda)$$

to a homomorphism $\chi : \mathcal{O}_{\widetilde{K}}\langle \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{v}}\rangle \to \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle[\frac{1}{\mathfrak{u}_{0,r_0}}]$ sending $\tilde{\mathfrak{u}}_{0,I}$ to $\mathfrak{u}_{0,I}$, $\tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}$ to $\mathfrak{u}_{\Lambda\backslash\lambda_0}$, and $\tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}$ to the lift of $\bar{\chi}(\tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}})$ using the standard basis defined in Construction 4.1.5.1. Then $\mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\chi(\tilde{\mathfrak{p}}_{0,I}), \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\chi(\tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0})$ and $\mathfrak{u}_{0,r_0}^{p[\alpha]}\chi(\tilde{\mathfrak{q}})$ are contained in the ideal $(\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda)\mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$.

<u>**Step 3:**</u> Explicate the goal.

We are going to establish an $\mathcal{R}_{\widetilde{K}}$-isomorphism $\widetilde{\chi} : \widetilde{\mathcal{A}} \xrightarrow{\sim} \mathcal{A}$, where

$$\mathcal{A} = \mathfrak{S}_K / \big(\psi_K(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi_K(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda\big) \otimes_{\mathcal{R}_K, f^*} \mathcal{R}_{\widetilde{K}}\Big[\frac{1}{p}\Big], \tag{4.2.1.8}$$

$$\widetilde{\mathcal{A}} = \mathfrak{S}_{\widetilde{K}}\Big[\frac{1}{p}\Big] / \big(\psi_{\widetilde{K}}(\tilde{\mathfrak{p}}_{0,I}) + \widetilde{\mathfrak{R}}_{0,I}, \psi_{\widetilde{K}}(\tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0}) + \widetilde{\mathfrak{R}}_{\Lambda\backslash\lambda_0}, \psi_{\widetilde{K}}(\tilde{\mathfrak{q}}) + \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}\big). \tag{4.2.1.9}$$

Here, $\mathfrak{S}_{\widetilde{K}} = \mathcal{R}_{\widetilde{K}}\langle \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{v}}\rangle$ and we can define $\mathfrak{N}_{\widetilde{K}}^a$ for $a \in \frac{1}{ep}\mathbb{N}$ similarly to Construction 4.1.5.1; the ring homomorphism $\widetilde{\chi}$ is given (and determined) by $\widetilde{\chi}(\tilde{\mathfrak{u}}_{0,I}) = \mathfrak{u}_{0,I}$, $\widetilde{\chi}(\tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}) = \mathfrak{u}_{\Lambda\backslash\lambda_0}$, and $\widetilde{\chi}(\tilde{\mathfrak{v}}) = \psi_{\widetilde{K}}(\chi(\tilde{\mathfrak{v}}))$; the set $\widetilde{\mathfrak{R}}_{0,I}, \widetilde{\mathfrak{R}}_{\Lambda\backslash\lambda}, \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}$ will be admissible with error gauge $\geq \omega - n$ so that $\widetilde{\chi}$ is an isomorphism.

Such an isomorphism $\widetilde{\chi}$ will be sufficient to prove the main theorem.

<u>**Step 4:**</u> Bound the error gauge. We first determine $\widetilde{\mathfrak{R}}_{0,I}, \widetilde{\mathfrak{R}}_{\Lambda\backslash\lambda_0}, \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}$. We proceed similarly to Proposition 4.1.5.6. To write this argument uniformly, we first divide into the following four cases.

Case (a): Denote $\tilde{\mathfrak{p}} = \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\tilde{\mathfrak{p}}_{0,i_0}$ for some $i_0 \in I$ and $\widetilde{\mathfrak{R}} = \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\widetilde{\mathfrak{R}}_{0,I}$;

Case (b): Denote $\tilde{\mathfrak{p}} = \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\tilde{\mathfrak{p}}_\lambda$ for $\lambda \in \Lambda\backslash\{\lambda_0\}$ and $\widetilde{\mathfrak{R}} = \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\widetilde{\mathfrak{R}}_\lambda$;

Case (c): Denote $\tilde{\mathfrak{p}} = \mathfrak{u}_{0,r_0}^{p[\alpha]}\tilde{\mathfrak{q}}$ and $\widetilde{\mathfrak{R}} = \mathfrak{u}_{0,r_0}^{p[\alpha]}\widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}$, assuming we are in Case A;

Case (d): Denote $\tilde{\mathfrak{p}} = \mathfrak{u}_{0,r_0}^{p[\alpha]}\tilde{\mathfrak{q}}$ and $\widetilde{\mathfrak{R}} = \mathfrak{u}_{0,r_0}^{p[\alpha]}\widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}$, assuming we are in Case B;

By Step 2,

$$\bar{\chi}(\tilde{\mathfrak{p}}) = \sum_{i\in I} \mathfrak{h}_{0,i}\mathfrak{p}_{0,i} + \sum_{\lambda\in\Lambda} \mathfrak{h}_\lambda \mathfrak{p}_\lambda,$$

for some $\mathfrak{h}_{0,i}, \mathfrak{h}_\lambda \in \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$ for $i \in I, \lambda \in \Lambda$. Moreover, in Case (a) for some $i_0 \in I$, we can require $\mathfrak{h}_{0,i} \in \mathfrak{N}_K^{\max\{(e_{i_0-1}-e_{i-1})/e,0\}} \cdot \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$, and $\mathfrak{h}_\lambda \in \mathfrak{N}_K^{e_{i_0-1}/e} \cdot \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$

for $i \in I, \lambda \in \Lambda$; in Case (d), we can require $\mathfrak{h}_\lambda \in \mathfrak{N}_K^{1/e} \cdot \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda \rangle$ for $\lambda \in \Lambda$. Thus,

$$
\begin{aligned}
-\widetilde{\chi}(\widetilde{\mathfrak{R}}) = \widetilde{\chi}(\psi_{\widetilde{K}}(\widetilde{\mathfrak{p}})) &= \sum_{i \in I} \psi_{\widetilde{K}}(\mathfrak{h}_{0,i}) \psi_{\widetilde{K}}(\mathfrak{p}_{0,i}) + \sum_{\lambda \in \Lambda} \psi_{\widetilde{K}}(\mathfrak{h}_\lambda) \psi_{\widetilde{K}}(\mathfrak{p}_\lambda) + \mathfrak{E} \\
&= \sum_{i \in I} \psi_{\widetilde{K}}(\mathfrak{h}_{0,i})(-\mathfrak{R}_{0,i}) + \sum_{\lambda \in \Lambda} \psi_{\widetilde{K}}(\mathfrak{h}_\lambda)(-\mathfrak{R}_\lambda) + \mathfrak{E} \\
&\in \begin{cases} (\mathfrak{N}^{\omega-1+e_{i_0}-1/e}\eta_0, \mathfrak{N}^{\omega+e_{i_0}-1/e}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{case (a)}, \\ (\mathfrak{N}^{\omega-1}\eta_0, \mathfrak{N}^{\omega}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{case (b) or (c)}, \\ (\mathfrak{N}^{\omega-1+1/e}\eta_0, \mathfrak{N}^{\omega+1/e}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{case (d)}, \end{cases}
\end{aligned}
$$

where the error term $\mathfrak{E}$ coming from ψ_K failing to be a homomorphism (See Proposition 4.1.1.8) can be bounded as

$$
\mathfrak{E} \in \begin{cases} (\mathfrak{N}^{\beta_K}\eta_0, \mathfrak{N}^{\beta_K+1}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{case (a)}, \\ (\mathfrak{N}^{\beta_K-1}\delta_0, \mathfrak{N}^{\beta_K}\delta_J) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{case (b) or (c)}, \\ (\mathfrak{N}^{\beta_K}\eta_0, \mathfrak{N}^{\beta_K+1}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{case (d)}. \end{cases}
$$

Thus, we can find polynomials $\tilde{\mathfrak{r}}_0, \dots, \tilde{\mathfrak{r}}_{m+1} \in \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\setminus\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] \twoheadrightarrow \mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ such that

$$
\tilde{\mathfrak{r}}_0 \in \begin{cases} \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e+e_{i_0}-1} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\setminus\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{case (a)}, \\ \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\setminus\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{case (b) or (c)}, \\ \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e+1} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\setminus\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{case (d)}; \end{cases}
$$

$$
\tilde{\mathfrak{r}}_1, \dots, \tilde{\mathfrak{r}}_{m+1} \in \begin{cases} \tilde{\mathfrak{u}}_{0,r_0}^{\omega e+e_{i_0}-1} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\setminus\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{case (a)}, \\ \tilde{\mathfrak{u}}_{0,r_0}^{\omega e} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\setminus\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{case (b) or (c)}, \\ \tilde{\mathfrak{u}}_{0,r_0}^{\omega e+1} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\setminus\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{case (d)}; \end{cases}
$$

$$
\begin{aligned}
&\widetilde{\chi}(\widetilde{\mathfrak{R}} - \tilde{\mathfrak{r}}_0\eta_0 - \cdots - \tilde{\mathfrak{r}}_{m+1}\eta_{m+1}) \\
&\in \begin{cases} (\eta_0/\pi_K, \eta_{J\cup\{m+1\}})(\mathfrak{N}^{\omega-1+e_{i_0}-1/e}\eta_0, \mathfrak{N}^{\omega+e_{i_0}-1/e}\eta_{J\cup\{m+1\}}) \cdot (\mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}}) & \text{case (a)}, \\ (\eta_0/\pi_K, \eta_{J\cup\{m+1\}})(\mathfrak{N}^{\omega-1}\eta_0, \mathfrak{N}^{\omega}\eta_{J\cup\{m+1\}}) \cdot (\mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}}) & \text{case (b) or (c)}, \\ (\eta_0/\pi_K, \eta_{J\cup\{m+1\}})(\mathfrak{N}^{\omega-1+1/e}\eta_0, \mathfrak{N}^{\omega+1/e}\eta_{J\cup\{m+1\}}) \cdot (\mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}}) & \text{case (d)}. \end{cases}
\end{aligned}
$$

Further, we can similarly clear up the coefficients of $\eta_j\eta_{j'}$ for $j, j' \in J^+ \cup \{m+1\}$. Repeating this approximation gives the expression of $\widetilde{\mathfrak{R}}$. From this and $\alpha \leq en/p$, we can obtain $\widetilde{\mathfrak{R}}_{0,I}, \widetilde{\mathfrak{R}}_{\Lambda\backslash\lambda_0}, \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}} \in (\eta_{J^+\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}}$ such that

$$\begin{aligned}
\widetilde{\mathfrak{R}}_{0,i_0} &\in (\tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e+e_{i_0-1}-en}\eta_0, \tilde{\mathfrak{u}}_{0,r_0}^{\omega e+e_{i_0-1}-en}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}}, \quad i_0 \in I, \\
\widetilde{\mathfrak{R}}_{\lambda} &\in (\tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e-en}\eta_0, \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-en}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}}, \quad \lambda \in \Lambda\backslash\lambda_0 \\
\widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}} &\in \begin{cases} (\tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e-en}\eta_0, \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-en}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}} & \text{in Case A} \\ (\tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e-en+1}\eta_0, \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-en+1}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}} & \text{in Case B} \end{cases}
\end{aligned}$$

They have error gauge $\geq \omega - n$.

Step 5: Prove that $\tilde{\chi}$ is an isomorphism.

To prove that $\tilde{\chi}$ is an isomorphism, it suffices to show the surjectivity, as both $\mathcal{A}'$ and $\mathcal{A}$ are finite free modules over $\mathcal{R}_{\widetilde{K}}[\frac{1}{p}]$ of the same rank. Since (4.1.5.2) forms a basis of $\mathcal{A}$ over $\mathcal{R}_{\widetilde{K}}[\frac{1}{p}]$, we need only to show that $\mathfrak{u}_{0,I}$ and $\mathfrak{u}_{\Lambda}$ are in the image of $\tilde{\chi}$. This is obvious for $\mathfrak{u}_{0,I}$ and $\mathfrak{u}_{\Lambda\backslash\lambda_0}$. For $\mathfrak{u}_{\lambda_0}$, we first find an element in $\mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] \twoheadrightarrow \mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ whose image under $\tilde{\chi}$ is $\mathfrak{u}_{\lambda_0}$. Then we use the similar approximation in Step 4 to find an element in $\mathcal{A}'$ whose image under $\tilde{\chi}$ is exactly $\mathfrak{u}_{\lambda_0}$. This finishes the proof. □

Remark 4.2.1.10. We expect that when ω and hence β_K is "large" compared to $[L : K]$, Theorem 4.2.1.7 is also valid if we add a generic p^∞-th root (Defintion 2.3.2.7); this amounts to control the discrepancy between $\mathcal{O}_{\widetilde{L}}$ and $\mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$. Hence, in this case, one can obtain a comparison theorem between the arithmetic Artin conductor and Borger's Artin conductor as in Subsection 2.4.2.

4.2.2 A digression on differential modules

We study some basic properties of intrinsic radii of convergence under certain base changes, in particular, the off-centered tame base change and the off-centered Frobenius pullback.

Construction 4.2.2.1. Let K be a nonarchimdedean field and let $\pi_K \in K$ be an element with $|\pi_K| = \theta < 1$. Denote $X = A^1_K[0, \theta^a]$. Fix $n \in \mathbb{N}$ prime to p and fix $x_0 \in K$ such that $|x_0| = \theta^b > \theta^a$ ($b < a$). In particular, the point $\delta_0 = -x_0$ is not in the disc X. Denote $K_n = K(x_0^{1/n})$, where we fix an n-th root $x_0^{1/n}$ of x_0.

Consider the K-homomorphism $f_n^* : K\langle \pi_K^{-a}\delta_0 \rangle \to K_n\langle \pi_K^{-a+b(n-1)/n}\eta_0 \rangle$, sending δ_0 to

$$(x_0^{1/n} + \eta_0)^n - x_0 = x_0^{(n-1)/n}\eta_0 \Big(\sum_{i=0}^{n-1} \binom{n}{i+1} \big(\frac{\eta_0}{x_0^{1/n}}\big)^i \Big),$$

where the term in the bracket on the right has norm 1 and invertible because $|x_0^{1/n}| > |\eta_0|$. Hence f_n^* extends continuously to a homomorphism $F_a \to F'_{a-b(n-1)/n}$, where $F'_{a-b(n-1)/n}$ is the completion of $K_n(\eta_0)$ with respect to the $\theta^{a-b(n-1)/n}$-Gauss norm.

Also, f_n^* gives a morphism of rigid K-spaces $f_n : Z = A^1_{K_n}[0, \theta^{a-b(n-1)/n}] \to X = A^1_K[0, \theta^a]$. It is finite and étale because the branching locus is at $\delta_0 = -x_0$, outside the disc X. Thus, for a differential module $\mathcal{E}$ on X, its pull back $f_n^*\mathcal{E}$ is a differential module over Z via

$$f_n^*\mathcal{E} \xrightarrow{f_n^*\nabla} f_n^*\Big(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X d\delta_0\Big) \longrightarrow f_n^*\mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_Z d\eta_0,$$

where the last homomorphism is given by $d\delta_0 \mapsto n(x_0^{1/n} + \eta_0)^{n-1} d\eta_0$.

Proposition 4.2.2.2. *Keep the notation as above. We have*

$$IR_{\partial_{\eta_0}}(f_n^*\mathcal{E}; a - b(n-1)/n) = IR_{\partial_0}(\mathcal{E}; a).$$

Proof. The proof is essentially the same as [Ked05a, Lemma 5.11] or [Ked**, Proposition 9.7.6]. Lemma 1.1.2.16 gives the following commutative diagram

$$\begin{array}{ccc} F_a & \xrightarrow{f^*_{\mathrm{gen},0}} & F_a[\![\pi_K^{-a}T_0]\!]_0 \\ \downarrow{\scriptstyle f_n^*} & & \downarrow{\scriptstyle \tilde{f}_n^*} \\ F'_{a-b(n-1)/n} & \xrightarrow{f'^*_{\mathrm{gen},0}} & F'_{a-b(n-1)/n}[\![\pi_K^{-a+b(n-1)/n}T'_0]\!]_0 \end{array}$$

where $\tilde{f}_n^*$ extends f_n^* by sending T_0 to $(x_0^{1/n} + \eta_0 + T'_0)^n - (x_0^{1/n} + \eta_0)^n$.

We claim that for $r \in [0,1]$, $\tilde{f}_n$ induces an isomorphism between

$$F'_{a-b(n-1)/n} \times_{f_n^*, F_a} \left(A^1_{F_a}[0, r\theta^a)\right) \cong A^1_{F'_{a-b(n-1)/n}}[0, r\theta^{a-b(n-1)/n}).$$

Indeed, if $|T'_0| < r\theta^{a-b(n-1)/n} < \theta^{b/n}$, then

$$|T_0| = |(x_0^{1/n}+\eta_0+T'_0)^n-(x_0^{1/n}+\eta_0)^n| = |nT'_0(x_0^{1/n}+\eta_0)^{n-1}| < r\theta^{a-b(n-1)/n}.(\theta^{b/n})^{n-1} = r\theta^a.$$

Conversely, if $|T_0| < r\theta^a$, we define the inverse map by the binomial series

$$T'_0 = (x_0^{1/n} + \eta_0)\cdot\Big[-1+\Big(1+\frac{T_0}{(x_0^{1/n}+\eta_0)^n}\Big)^{1/n}\Big] = \sum_{i=1}^{\infty}\binom{1/n}{i}\frac{T_0^i}{(x_0^{1/n}+\eta_0)^{ni-1}}.$$

The series converges to an element with norm $< r\theta^{a-b(n-1)/n}$.

Therefore, Proposition 1.1.2.18 implies that for $r \in [0,1]$,

$$\begin{aligned} &IR_{\partial_0}(\mathcal{E}; a) \geq r \\ \Leftrightarrow\ & f^*_{\mathrm{gen},0}(\mathcal{E} \otimes_{\mathcal{O}_X} F_a) \text{ is trivial over } A^1_{F_a}[0, r\theta^a) \\ \Leftrightarrow\ & \tilde{f}_n^* f^*_{\mathrm{gen},0}(\mathcal{E}\otimes F_a) = f'^*_{\mathrm{gen},0}\big(f_n^*\mathcal{E}\otimes F'_{a-\frac{b(n-1)}{n}}\big) \text{ is trivial over } A^1_{F'_{a-b(n-1)/n}}[0, r\theta^{a-\frac{b(n-1)}{n}}) \\ \Leftrightarrow\ & IR_{\partial_{\eta_0}}(f_n^*\mathcal{E}; a - b(n-1)/n) \geq r. \end{aligned}$$

The proposition follows. □

Similarly, we can study a type of off-centered Frobenius.

Construction 4.2.2.3. Let $b > 0$ and $0 < a < \min\{-\log_\theta p + b, pb\}$ and let $\beta \in K$ be an element of norm 1. Let L be the completion of $K(x)$ with respect to the θ^a-Gauss norm.

Let $f : Z = A^1_L[0,\theta^b] \to A^1_K[0,\theta^a]$ be the morphism given by $f^* : \delta_0 \mapsto (\beta + \eta_0)^p - \beta^p + x$. By our choices of a and b, the leading term of $f^*(\delta_0)$ is x, which is transcendental over K. Hence f^* extends continuously to a homomorphism $F_a \to F'_b$, where F'_b is the completion of $L(\eta_0)$ with respect to the θ^b-Gauss norm. Moreover, $f^*\Omega^1_X \cong \Omega^1_Z$ as the branching locus is at $\eta_0 = -\beta$, outside the disc. Thus $f^*\mathcal{E}$ becomes a differential module over $Z = A^1_L[0, \theta^b]$ via

$$f^*\mathcal{E} \xrightarrow{f^*\nabla} f^*\Big(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X d\delta_0\Big) \longrightarrow f^*\mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_Z d\eta_0,$$

where the second homomorphism is given by $d\delta_0 \mapsto p(\beta + \eta_0)^{p-1} d\eta_0$.

Proposition 4.2.2.4. *Keep the notation as above. We have*

$$IR_{\partial_0}(f^*\mathcal{E}; b) \geq IR_{\partial_{\eta_0}}(\mathcal{E}; a).$$

Proof. This proof is very similar to Lemma 1.1.4.16. We start with the following commutative diagram from Lemma 1.1.2.16.

$$\begin{array}{ccc} F_a & \xrightarrow{f^*_{\mathrm{gen},0}} & F_a[\![\pi_K^{-a} T_0]\!]_0 \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \tilde{f}^*} \\ F'_b & \xrightarrow{f'^*_{\mathrm{gen},0}} & F'_b[\![\pi_K^{-b} T'_0]\!]_0 \end{array}$$

where $\tilde{f}^*$ extends f^* by sending T_0 to $(\beta + \eta_0 + T'_0)^p - (\beta + \eta_0)^p$.

For $r \in [0, 1]$, by Lemma 1.1.4.17, $|T'_0| < r\theta^a$ implies that $|T_0| < \max\{r^p\theta^{pa}, p^{-1}r\theta^a\} < r\theta^b$.

Therefore, Proposition 1.1.2.18 implies that

$$\begin{aligned} & IR_{\partial_0}(\mathcal{E}; a) \geq r \\ \Leftrightarrow\ & f^*_{\mathrm{gen},0}(\mathcal{E} \otimes_{\mathcal{O}_X} F_a) \text{ is trivial over } A^1_{F_a}[0, r\theta^a) \\ \Rightarrow\ & \tilde{f}^* f^*_{\mathrm{gen},0}(\mathcal{E} \otimes_{\mathcal{O}_X} F_a) = f'^*_{\mathrm{gen},0}(f^*\mathcal{E} \otimes_{\mathcal{O}_Z} F'_b) \text{ is trivial over } A^1_{F'_b}[0, r\theta^b) \\ \Leftrightarrow\ & IR_{\partial_{\eta_0}}(f^*\mathcal{E}; b) \geq r. \end{aligned}$$

The proposition follows. □

4.2.3 Non-logarithmic Hasse-Arf theorem

In this subsection, we apply Theorem 4.2.1.7 to obtain the Hasse-Arf Theorem 4.2.3.5 for non-logarithmic ramification filtrations.

We assume Hypotheses 4.1.1.1 until stating the last theorem. As a reminder, Hypothesis 4.2.1.6 is no longer assumed till the end of the paper.

Notation 4.2.3.1. Keep the notation as in Construction 2.3.3.3. Fix $j_0 \in J$ and $n \in \mathbb{N}$. Let $\widetilde{K} = K'((b_{j_0}+x\pi_K^n)^{1/p})$ as in Notation 4.2.1.1. Denote $\beta_{j_0} = (b_{j_0}+x\pi_K^n)^{1/p}$ for simplicity.

Lemma 4.2.3.2. *Assume $p \nmid n$ and $\beta_K \geq n$. Let $a_{J^+} \subset \mathbb{R}_{>0}$ and $a_0 = a_{j_0} = a_{m+1} > \max\{\frac{n-1}{p-1}, 1\}$. Define $a'_j = a_j$ for $j \in J^+ \backslash \{j_0\}$ and $a'_{j_0} = a_{j_0} + n - 1$. The morphism f^* defined in Lemma 4.2.1.4 restricts to a morphism*

$$f : A^1_{\widetilde{K}}[\theta^{a_0}, \theta^{a_0}] \times \cdots \times A^1_{\widetilde{K}}[\theta^{a_{m+1}}, \theta^{a_{m+1}}] \to A^1_K[\theta^{a'_0}, \theta^{a'_0}] \times \cdots \times A^1_K[\theta^{a'_m}, \theta^{a'_m}].$$

In other words, we change the j_0-th radius from a_{j_0} to $a_{j_0} + n - 1$.

Proof. It suffices to verify that if $|\eta_0| = |\eta_{j_0}| = |\eta_{m+1}| = \theta^{a_0}$, then $|\delta_j| = \theta^{a_0+n-1}$; indeed

$$\delta_{j_0} = \left((\beta_{j_0} + \eta_{j_0})^p - \beta_{j_0}^p\right) - x\left((\pi_K + \eta_0)^n - \pi_K^n\right) + \eta_{m+1}(\pi_K + \eta_0)^n,$$

which has norm θ^{a_0+n-1} because the second term does and other terms have smaller norms. □

Lemma 4.2.3.3. *Keep the notation as in the previous lemma. Let $\mathcal{E}$ be a differential module over $A^1_K[0, \theta^{a'_0}] \times \cdots \times A^1_K[0, \theta^{a'_m}]$, then $IR(f^*\mathcal{E}; a_{J^+}) = IR(\mathcal{E}; a'_{J^+\cup\{m+1\}})$.*

Proof. The morphism f^* induces the homomorphism on the differentials: $d\delta_j \mapsto d\eta_j$ for $j \in J^+ \backslash \{j_0\}$ and $d\delta_{j_0} \mapsto p(\beta_{j_0} + \eta_{j_0})^{p-1} d\eta_{j_0} + (\pi_K + \eta_0)^n d\eta_{m+1} + n(x + \eta_{m+1})(\pi_K +$

$\eta_0)^{n-1}d\eta_0$. Thus,

$$\begin{aligned}
\partial'_j|_{f^*\mathcal{E}} &= \partial_j|_{\mathcal{E}}, \quad j \in J\backslash\{j_0\}, \\
\partial'_{j_0}|_{f^*\mathcal{E}} &= p(\beta_{j_0}+\eta_{j_0})^{p-1}\partial_{j_0}|_{\mathcal{E}}, \\
\partial'_{m+1}|_{f^*\mathcal{E}} &= (\pi_K+\eta_0)^n \cdot \partial_{j_0}|_{\mathcal{E}}, \\
\partial'_0|_{f^*\mathcal{E}} &= \partial_0|_{\mathcal{E}} + n(x+\eta_{m+1})(\pi_K+\eta_0)^{n-1}\cdot\partial_{j_0}|_{\mathcal{E}},
\end{aligned}$$

where $\partial'_j = \partial/\partial\eta_j$ for $j = 0, \ldots, m+1$. Thus,

$$\begin{aligned}
IR_j(f^*\mathcal{E}; a_{J^+\cup\{m+1\}}) &= IR_j(\mathcal{E}; a'_{J^+}) \quad \forall j \in J\backslash\{j_0\}, \\
IR_{j_0}(f^*\mathcal{E}; a_{J^+\cup\{m+1\}}) &\leq IR_{j_0}(\mathcal{E}; a'_{J^+}), \\
IR_{m+1}(f^*\mathcal{E}; a_{J^+\cup\{m+1\}}) &= \theta^n \cdot IR_{j_0}(\mathcal{E}; a'_{J^+}), \\
IR_0(f^*\mathcal{E}; a_{J^+\cup\{m+1\}}) &= \min\left\{IR_0(\mathcal{E}, a'_{J^+}), IR_{j_0}(\mathcal{E}; a'_{J^+})\right\},
\end{aligned}$$

where the second inequality follows from Proposition 4.2.2.4 and the last equality holds by Proposition 4.2.2.2 because x is transcendental over K. It follows that $IR(\mathcal{E}; a'_{J^+}) = IR(f^*\mathcal{E}; a_{J^+\cup\{m+1\}})$. ☐

Theorem 4.2.3.4. *Let L/K be a finite Galois extension satisfying Hypotheses 4.1.1.1 and 4.1.1.10. The highest non-logarithmic ramification break of L/K is invariant under the operation of adding a generic p-th root.*

Proof. Adding a generic p-th root corresponds to setting $n = 1$ in the notation in this subsection. Fix a choice of ψ_K in Construction 4.1.1.2. Let $TS^a_{L/K,\psi_K}$ be the standard thickening space for L/K. By Example 4.1.5.5, we can turn this standard thickening space into a recursive thickening space (with error gauge $\geq \beta_K$). By Theorem 4.2.1.7, $TS^a_{L/K,\psi_K} \times_{A_K^{m+1}[0,\theta^a],f} A_{\widetilde{K}}^{m+2}[0,\theta^a]$ is a recursive thickening space for $\widetilde{L}/\widetilde{K}$ with error gauge $\geq \beta_K - 1$, which is isomorphic to some thickening space for $\widetilde{L}/\widetilde{K}$ by Proposition 4.1.5.6.

Let $\mathcal{E}$ be the differential module over $A_K^{m+1}[0,\theta^a]$ coming from $TS^a_{L/K,\psi_K}$. Then the differential module $f^*\mathcal{E}$ is associated to $\widetilde{L}/\widetilde{K}$. Applying Lemma 4.2.3.3 (to the case $n = 1$) gives $IR(f^*\mathcal{E};\underline{s}) = IR(\mathcal{E};\underline{s})$ for $s \geq b(L/K) - \epsilon$ with $\epsilon > 0$ as in Theorem 4.1.3.2. The theorem follows from Theorem 4.1.4.4. □

Combining Theorem 4.2.3.4 and Proposition 2.3.2.13, we have the following.

Theorem 4.2.3.5. *Let K be a complete discretely valued field of mixed characteristic $(0,p)$ which is not absolutely unramified. Let $\rho : G_K \to GL(V_\rho)$ be a representation with finite local monodromy. Then,*

(1) $\mathrm{Art}(\rho)$ is a non-negative integer;

(2) the subquotients $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ are trivial if $a \notin \mathbb{Q}$ and are abelian groups killed by p if $a \in \mathbb{Q}_{>1}$.

4.3 Logarithmic Hasse-Arf theorem

4.3.1 Integrality for Swan conductors

In this subsection, we will deduce the integrality of Swan conductors from that of Artin conductors (Theorem 4.2.3.5). We will use the fact that the logarithmic ramification breaks behave well under tame base changes.

We will keep Hypothesis 3.2.2.1 until we state Theorem 4.3.1.14.

Notation 4.3.1.1. Let $n \in \mathbb{N}$ such that $n \equiv 1 (\bmod ep)$. Define $K_n = K(\pi_K^{1/n})$ and $L_n = LK_n$. Since K_n and L are linearly independent over K, $\mathrm{Gal}(L_n/K_n) = \mathrm{Gal}(L/K)$. We take the uniformizer of K_n and L_n to be $\pi_{K_n} = \pi_K^{1/n}$ and $\pi_{L_n} = \pi_L/\pi_{K_n}^{(n-1)/e}$, respectively.

Notation 4.3.1.2. Denote $\mathcal{R}_{K_n} = \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!]$. Applying Construction 4.1.1.2 to K_n gives an approximate homomorphism $\psi_{K_n} : \mathcal{O}_{K_n} \to \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!]$.

Lemma 4.3.1.3. *There exists a unique continuous $\mathcal{O}_K$-homomorphism $f_n^* : \mathcal{R}_K \to \mathcal{R}_{K_n}$ sending δ_0 to $(\pi_{K_n}+\eta_0)^n - \pi_K$ and δ_j to η_j for $j \in J$. This gives an approximately* *commutative diagram modulo* $I_{K_n} = p(\eta_0/\pi_{K_n}, \eta_J) \cdot \mathcal{R}_{K_n}$*:*

$$\begin{array}{ccc}
\mathcal{O}_K & \xrightarrow{\psi_K} & \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!] \\
\big\downarrow & & \big\downarrow f_n^* \\
\mathcal{O}_{K_n} & \xrightarrow{\psi_{K_n}} & \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!]
\end{array}$$

Proof. Follows from Proposition 4.1.1.8. □

Proposition 4.3.1.4. *Fix* $a > 0$. *Let* $TS^a_{L/K,\log,\psi_K}$ *be the standard logarithmic thickening space. Then the space*

$$X = TS^a_{L/K,\log,\psi_K} \times_{(A^1_K[0,\theta^{a+1}]\times A^m_K[0,\theta^a]),f_n} \left(A^1_{K_n}[0,\theta^{a+1/n}] \times A^m_{K_n}[0,\theta^a]\right)$$

is a logarithmic thickening space for L_n/K_n *with error gauge* $\geq n\beta_K - (n-1)$*; in particular, it is admissible.*

Proof. First, we have

$$\mathcal{S}_K \otimes_{\mathcal{O}_K} K_n \cong \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!][\frac{1}{p}]\langle u_{J^+}\rangle / \left(f_n^*(\psi_K(p_{J^+}))\right).$$

Now we consider a construction of the logarithmic thickening space of L_n/K_n, using the same c_J as the ones for L/K and π_{L_n} in Notation 4.3.1.1. Therefore, the ideal $\mathcal{I}_{L_n/K_n}$ is generated by p'_{J+} and $p'_0/\pi_{K_n}^{n-1}$, where the prime means to substitute u_0 with $\pi_{K_n}^{(n-1)/e}u'_0$.

Lemma 4.3.1.3 implies that

$$\psi_{K_n}(p'_0/\pi_{K_n}^{n-1}) - f_n^*(\psi_K(p'_0))/(\pi_{K_n} + u'_0)^{n-1} \in \pi_{K_n}^{-n+1}(\pi_{K_n}^{n\beta_K-1}\eta_0, p\eta_J) \cdot \mathcal{S}_{K_n}, \quad (4.3.1.5)$$

where $\mathcal{S}_{K_n} = \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!]\langle u'_0, u_J\rangle$. Hence,

$$\begin{aligned}
\mathcal{S}_K \otimes_{\mathcal{O}_K} K_n &\cong \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!][\frac{1}{p}]\langle u'_0, u_J\rangle / \left(f_n^*(\psi_K(p'_0)), f_n^*(\psi_K(p'_J))\right) \\
&= \mathcal{S}_{K_n}[\frac{1}{p}] / \left(f_n^*(\psi_K(p'_0))/(\pi_{K_n} + \eta_0)^{n-1}, f_n^*(\psi_K(p'_J))\right)
\end{aligned}$$

gives rise to logarithmic thickening spaces for L_n/K_n with error gauge $\geq n\beta_K-(n-1)$; note that K_n/K being tamely ramified of ramification degree n gives a different normalization on error gauge. $\square$

Proposition 4.3.1.6. *There exists $N \in \mathbb{N}$ and $\alpha_{L/K} \in [0,1]$ such that, for all integers $n > N$ congruent to 1 modulo ep, we have*

$$n \cdot b_{\log}(L/K) = b(L_n/K_n) - \alpha_{L/K}.$$

Proof. By Construction 4.2.2.1, f_n^* gives a finite étale morphism $f_n : A^1_{K_n}[0, \theta^{1/n}) \times A^m_{K_n}[0,1) \to A^1_K[0,\theta) \times A^m_K[0,1)$ for $a > 0$. Let $\mathcal{E}$ denote the differential module associated to L/K coming from a standard logarithmic thickening space. By Proposition 4.3.1.4, $f_n^*\mathcal{E}$ is a differential module associated to L_n/K_n. In particular,

$$ET_{L_n/K_n} \supseteq ET_{L/K} \times_{A^1_K[0,\theta)\times A^m_K[0,1), f_n} A^1_{K_n}[0,\theta^{1/n}) \times A^m_{K_n}[0,1) =: f_n^*(ET_{L/K})$$

The morphism f_n is an off-centered tame base change, as discussed in Subsection 4.2.2. By Proposition 4.2.2.2, for $s_{J^+} \subset \mathbb{R}$ such that $A^1_K[0,\theta^{s_0}] \times \cdots \times A^1_K[0,\theta^{s_m}] \subset ET_{L/K}$, we have $IR(f_n^*\mathcal{E}; s_{J^+}) = IR(\mathcal{E}; s_0 + \frac{n-1}{n}, s_J)$. Thus, by Corollary 4.1.4.6,

$$\begin{aligned} b(L_n/K_n) &= n \cdot \min\left\{s \,\middle|\, A^{m+1}_{K_n}[0,\theta^s] \subseteq ET_{L_n/K_n} \text{ and } IR(f_n^*\mathcal{E}; \underline{s}) = 1\right\} \\ &= n \cdot \min\left\{s \,\middle|\, A^{m+1}_{K_n}[0,\theta^s] \subseteq f_n^*(ET_{L/K}) \text{ and } IR(f_n^*\mathcal{E}; \underline{s}) = 1\right\} \qquad (4.3.1.7) \\ &= n \cdot \min\left\{s \,\middle|\, A^1_K[0,\theta^{s+(n-1)/n}] \times A^m_K[0,\theta^s] \subseteq ET_{L/K} \text{ and } IR(\mathcal{E}; s + \frac{n-1}{n}, \underline{s}) = 1\right\}, \end{aligned}$$

where the second equality holds because we will see in a moment that the minimal of s can be achieved inside $ET_{L/K}$.

Applying Proposition 4.1.4.3(c) to $\mathcal{E}$, we know the locus $Z(\mathcal{E}) = \{(s_{J^+}) | IR(\mathcal{E}; s_{J^+}) = 1\}$ is transrational polyhedral in a neighborhood of $[b_{\log}(L/K), +\infty)^{m+1}$, namely, where $\mathcal{E}$ is defined. Hence, in a neighborhood of $s_1 = b_{\log}(L/K)$, the intersection of the boundary of Z with the surface defined by $s_1 = \cdots = s_m$ is of the form

$$s_0 - \alpha' s_1 = b_{\log}(L/K) + 1 - \alpha' b_{\log}(L/K),$$

where α' is the slope; $\alpha' \in [-\infty, 0]$ by the monotonicity Proposition 4.1.4.3(c). When $n \gg 0$, it is clear that the line $s \mapsto (s + \frac{n-1}{n}, s, \ldots, s)$ hits the boundary of Z at $s = b_{\log}(L/K) + 1/(n(1-\alpha'))$. This justifies the equality in (4.3.1.7). It follows that

$$b(L_n/K_n) = n \cdot b_{\log}(L/K) + 1/(1-\alpha');$$

the different normalizations for ramification filtrations on G_K and G_{K_n} give the extra factor n. □

Remark 4.3.1.8. With more careful calculation, one may prove the above proposition and Proposition 4.3.1.11 below for any n sufficiently large and coprime to p.

Notation 4.3.1.9. Assume $p > 2$. Let (b_J) be a p-basis of K; it naturally gives a p-basis of K_n. Let $K_n(x_J)^\wedge$ denote the completion of $K_n(x_J)$ with respect to the $(1, \ldots, 1)$-Gauss norm, and let K'_n denote the completion of the maximal unramified extension of $K_n(x_J)^\wedge$. Set

$$\widetilde{K}_n = K'_n\big((b_J + x_J \pi_{K_n}^2)^{1/p}\big), \quad \widetilde{L}_n = \widetilde{K}_n L.$$

Denote $\beta_j = (b_j + x_j \pi_{K_n}^2)^{1/p}$ for $j \in J$. By Lemma 4.2.1.4, we have a continuous $\mathcal{O}_{K_n}$-homomorphism $\tilde{f} : \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!] \to \mathcal{O}_{\widetilde{K}_n}[\![\xi_0/\pi_{K_n}, \xi_J, \xi'_J]\!]$ such that $\tilde{f}^*(\eta_0) = \xi_0$ and $\tilde{f}^*(\eta_j) = (\beta_j + \xi_j)^p - (x_j + \xi'_j)(\pi_{K_n} + \xi_0)^2 - b_j$ for $j \in J$. For $a > 1$, it gives rise to $\tilde{f} : A_{\widetilde{K}_n}^{2m+1}[0, \theta^a] \to A_{K_n}^{m+1}[0, \theta^a] \hookrightarrow A_{K_n}^1[0, \theta^a] \times A_{K_n}^m[0, \theta^{a-1/n}]$, where the last morphism is the natural inclusion of affinoid subdomain.

Proposition 4.3.1.10. *Assume $p > 2$, $\beta_K \geq \frac{2m+n}{n}$, and $a > 1$. Let X be as in Proposition 4.3.1.4. Then the space*

$$X \times_{(A_{K_n}^1[0, \theta^{a+1/n}] \times A_{K_n}^m[0, \theta^a]), \tilde{f}} A_{\widetilde{K}_n}^{2m+1}[0, \theta^{a+1/n}]$$

is a thickening space for $\widetilde{L}_n/\widetilde{K}_n$ with error gauge $\geq n\beta_K - 2m - n + 1$; in particular, it is admissible.

Proof. It immediately follows from Proposition 4.3.1.6 and applying Theorem 4.2.1.7 m times. $\square$

Proposition 4.3.1.11. *Assume $p > 2$ and $\beta_K \geq 2$. There exists $N \in \mathbb{N}$ such that, for all integers $n > N$ congruent to* 1 *modulo ep, we have*

$$n \cdot b_{\log}(L/K) - 1 = b(\widetilde{L}_n/\widetilde{K}_n) - 2\alpha_{L/K}, \tag{4.3.1.12}$$

where $\alpha_{L/K}$ is the same *as in Proposition 4.3.1.6.*

Proof. We continue with the notation from Proposition 4.3.1.6. Previous proposition implies that $\tilde{f}^* f_n^* \mathcal{E}$ is a differential module associated to $\widetilde{L}_n/\widetilde{K}_n$ when $n > m$. By applying Lemma 4.2.3.3 m times, we have $IR(\tilde{f}^* f_n^* \mathcal{E}; \underline{s}) = IR(f_n^* \mathcal{E}; s, \underline{s + \frac{1}{n}})$. By Proposition 4.2.2.2, it further equals $IR(\mathcal{E}; s + \frac{n-1}{n}, \underline{s + \frac{1}{n}})$. By the same argument as in Theorem 4.3.1.6, we deduce our result with the same $\alpha_{L/K}$. $\square$

Remark 4.3.1.13. When $p = 2$, we study $\widetilde{K}_n = K'_n\big((b_J + x_J \pi_{K_n}^3)^{1/p}\big)$ instead; the same argument above proves the proposition with (4.3.1.12) replaced by

$$n \cdot b_{\log}(L/K) - 2 = b(L_n/K_n) - 3\alpha_{L/K}.$$

For the following theorem, we do not impose any hypothesis on K.

Theorem 4.3.1.14. *Let K be a complete discretely valued field of mixed characteristic $(0, p)$ and let $\rho : G_K \to GL(V_\rho)$ be a representation with finite local monodromy. Then* $\mathrm{Swan}(\rho)$ *is a non-negative integer if $p \neq 2$ and is in $\frac{1}{2}\mathbb{Z}$ if $p = 2$.*

Proof. First, as in the proof of Proposition 2.3.2.13, we may reduce to the case when ρ is irreducible and factors through a finite Galois extension L/K, for which Hypothesis 4.1.1.1 hold. In this case, $\mathrm{Swan}(\rho) = b_{\log}(L/K) \cdot \dim \rho$.

By Proposition 2.2.2.11(4), we have $\mathrm{Swan}(\rho|_{K_n}) = n \cdot \mathrm{Swan}(\rho)$ for any $K_n = K(\pi_K^{1/n})$ with $\gcd(n, ep) = 1$. We need only to prove $\mathrm{Swan}(\rho|_{K_n}) \in \mathbb{Z}$ for two coprime n's satisfying $\gcd(n, ep) = 1$, and the statement for $\mathrm{Swan}(\rho)$ will follow immediately. In particular, we may assume that $\beta_K \geq 2$.

When $p > 2$, we repeat the same argument again. There exist n_1, n_2 satisfying the condition of Propositions 4.3.1.6 and 4.3.1.11 and $\gcd(n_1, n_2) = 1$. Thus, by the non-logarithmic Hasse-Arf Theorem 4.2.3.5,

$$n_1 \mathrm{Swan}(\rho) + \alpha_{L/K} \dim \rho \in \mathbb{Z}, \qquad n_1 \mathrm{Swan}(\rho) + 2\alpha_{L/K} \dim \rho \in \mathbb{Z};$$
$$n_2 \mathrm{Swan}(\rho) + \alpha_{L/K} \dim \rho \in \mathbb{Z}, \qquad n_2 \mathrm{Swan}(\rho) + 2\alpha_{L/K} \dim \rho \in \mathbb{Z}.$$

This implies immediately that $\alpha_{L/K} \dim \rho \in \mathbb{Z}$; hence, $\mathrm{Swan}(\rho) \in \mathbb{Z}$.

When $p = 2$, a similar argument using Remark 4.3.1.13 gives $\mathrm{Swan}(\rho) \in \frac{1}{2}\mathbb{Z}$. □

Remark 4.3.1.15. When $p = 2$, we expect the integrality of Swan conductors in the case K is the composition of a discrete completely valued field with perfect residue field and an absolutely unramified complete discrete valuation field. In this case, we can factor ψ_K as $\mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K]\!] \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$ with the second map a *homomorphism*. This fact may allow us to show that $\alpha_{L/K}$ is either 0 or 1 depending on whether ∂_0 dominates.

We do not know if the integrality is true for $p = 2$ in general.

4.3.2 An example of wildly ramified base change

In this subsection, we explicitly calculate an example, which we will use in the next subsection. This example was first introduced in [Ked07a, Proposition 2.7.11]. We retain Hypotheses 4.1.1.1 and 4.1.1.10.

Lemma 4.3.2.1. *Let K_* be the finite extension of K generated by a root of*

$$T^p + \pi_K T^{p-1} = \pi_K. \tag{4.3.2.2}$$

Then K_* is Galois over K. Moreover the logarithmic ramification break $b_{\log}(K_*/K) = 1$.

Proof. Let $h(T) = T^p - \pi_K T^{p-1} - \pi_K$ and ϖ a root of h. It is clear that ϖ is a uniformizer of K_*.

$$\begin{aligned}
h(\varpi+T) &= (\varpi+T)^p + \pi_K(\varpi+T)^{p-1} - \pi_K \\
&= T^p + p\big(\varpi T^{p-1} + \cdots + \varpi^{p-1}T\big) \\
&\quad + \pi_K\big(T^{p-1} + (p-1)\varpi T^{p-2} + \cdots + (p-1)\varpi^{p-2}T\big), \\
h(\varpi+\varpi^2 T) &= \varpi^{2p}T^p + \pi_K\big(\varpi^{2p-2}T^{p-1} + (p-1)\varpi^{2p-1}T^{p-2} + \cdots + (p-1)\varpi^p T\big) \\
&\quad + p\big(\varpi^{2p-1}T^{p-1} + \cdots + \varpi^{p+1}T\big) \\
&= \pi_K^2\big((1-\varpi^{p-1})^2T^p + \varpi^{p-2}(1-\varpi^{p-1})T^{p-1} + \cdots + (p-1)(1-\varpi^{p-1})T\big) \\
&\quad + p\pi_K(1-\varpi^{p-1})\big(\varpi^{p-1}T^{p-1} + \cdots + \varpi T\big).
\end{aligned}$$

We see that $h(\varpi+\varpi^2T)/\pi_K^2$ is congruent to $T^p - T$ modulo ϖ. By Hensel's lemma, it splits completely in K_*. Hence, K_*/K is Galois. Moreover, the valuation of the difference between two distinct roots is 2. This implies that $b_{\log}(K_*/K) = 1$. □

Notation 4.3.2.3. Denote the roots of $h(T) = T^p + \pi_K T^{p-1} - \pi_K$ by $\varpi = \varpi_1, \dots, \varpi_p$.

For $a > 0$, the standard logarithmic thickening space $TS^a_{K_*/K,\log,\psi_K}$ for K_*/K is given by

$$\mathcal{O}^{a+1}_{TS,K_*/K,\log,\psi_K} = K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J, z\rangle \big/ \big(z^p + (\pi_K+\delta_0)z^{p-1} - (\pi_K+\delta_0)\big).$$

Lemma 4.3.2.4. *Assume $a > 1$. Then the standard logarithmic thickening space $TS^a_{K_*/K,\log,\psi_K} \times_K K_*$ is isomorphic to the product of $A^m_{K_*}[0,\theta^a]$ with the disjoint union of p discs $|z-\varpi_\gamma| \leq \theta^{a-(p-2)/p}$ for $\gamma = 1,\dots,p$.*

Proof. We can rewrite $z^p + (\pi_K+\delta_0)z^{p-1} - (\pi_K+\delta_0)$ as

$$\prod_{\gamma=1}^{p}(z-\varpi_\gamma) = \delta_0(1-z^{p-1}). \tag{4.3.2.5}$$

Since $|z| \leq 1$, the right hand side of (4.3.2.5) has norm $\leq \theta^{a+1} < \theta^2$. On the left hand side, for $\gamma \neq \gamma' \in \{1,\dots,p\}$, $|\varpi_\gamma - \varpi_{\gamma'}| = \theta^{2/p}$. This forces one of $|z-\varpi_{\gamma_0}|$ to be strictly smaller than the others, for some $\gamma_0 \in \{1,\dots,p\}$. Thus, $|z-\varpi_{\gamma_0}| =$

$|\delta_0|/(\theta^{2/p})^{p-1} = \theta^{a-(p-2)/p}$. $\square$

Notation 4.3.2.6. For $\gamma = 1, \dots, p$, we define the K_*-homomorphism $f_\gamma^* : \mathcal{O}_K[\![\delta_0/\pi_K]\!] \to \mathcal{O}_{K_*}[\![\eta_0/\varpi_\gamma]\!]$ by sending δ_0 to

$$\frac{(\varpi_\gamma + \eta_0)^p}{1 - (\varpi_\gamma + \eta_0)^{p-1}} - \pi_K = \sum_{n=0}^{\infty} \left((\varpi_\gamma + \eta_0)^{p+n(p-1)} - \varpi_\gamma^{p+n(p-1)} \right). \tag{4.3.2.7}$$

Lemma 4.3.2.8. *For $a > 1$, f_γ^* induces a K-morphism $f_\gamma : A^1_{K_*}[0, \theta^{a-(p-2)/p}] \to A^1_K[0, \theta^{a+1}]$, which is an isomorphism when we tensor the target with K_* over K. Moreover, if we use F_{a+1} and $F^*_{a-(p-2)/p}$ to denote the completion of $K(\delta_0)$ and $K_*(\eta_0)$ with respect to the θ^{a+1}-Gauss norm and $\theta^{a+(p-2)/p}$-Gauss norm, respectively, then f_γ^* extends to a homomorphism $F_{a+1} \to F^*_{a-(p-2)/p}$.*

Proof. The statement follows from the fact that the leading term in (4.3.2.7) is $(2p-1)\varpi_\gamma^{2p-2}\eta_0$. $\square$

Proposition 4.3.2.9. *Assume $a > 1$. Let $\mathcal{E}$ be a differential module over $A^1_K[0, \theta^{a+1}]$. For each $\gamma \in \{1, \dots, p\}$, this gives a differential module $f_\gamma^*\mathcal{E}$ over $A^1_{K_*}[0, \theta^{a-(p-2)/p}]$. Then we have*

$$IR_0(f_\gamma^*\mathcal{E}; a - (p-2)/p) = IR_0(\mathcal{E}; a+1).$$

Proof. The proof is similar to Proposition 4.2.2.2. By Lemma 4.3.2.8, we have the following commutative diagram

$$\begin{array}{ccc} F_{a+1} & \xrightarrow{f^*_{\mathrm{gen}}} & F_{a+1}[\![\pi_K^{-a-1}T_0]\!]_0 \\ \downarrow{\scriptstyle f_\gamma^*} & & \downarrow{\scriptstyle f_\gamma^*} \\ F^*_{a-(p-2)/p} & \xrightarrow{f^*_{\mathrm{gen}}} & F^*_{a-(p-2)/p}[\![\varpi_\gamma^{-pa+p-2}T'_0]\!]_0 \end{array}$$

where we extend f_γ^* by $f_\gamma^*(T_0) = \dfrac{(\varpi_\gamma + \eta_0 + T'_0)^p}{1 - (\varpi_\gamma + \eta_0 + T'_0)^{p-1}} - \dfrac{(\varpi_\gamma + \eta_0)^p}{1 - (\varpi_\gamma + \eta_0)^{p-1}}$.

We claim that for $r \in [0, 1)$, f_γ^* induces an isomorphism between

$$F^*_{a-(p-2)/p} \times_{f_\gamma^*, F_{a+1}} \left(A^1_{F_{a+1}}[0, r\theta^{a+1}) \right) \simeq A^1_{F^*_{a-(p-2)/p}}[0, r\theta^{a-(p-2)/p}).$$

Indeed, if $|T_0'| < r\theta^{a-(p-2)/p}$, then

$$\begin{aligned}
T_0 &= \frac{(\varpi_\gamma + \eta_0 + T_0')^p}{1-(\varpi_\gamma + \eta_0 + T_0')^{p-1}} - \frac{(\varpi_\gamma + \eta_0)^p}{1-(\varpi_\gamma + \eta_0)^{p-1}} \\
&= \big((\varpi_\gamma + \eta_0 + T_0')^p - (\varpi_\gamma + \eta_0)^p\big) + \big((\varpi_i + \eta_0 + T_0')^{2p-1} - (\varpi_\gamma + \eta_0)^{2p-1}\big) + \cdots \\
&\in (2p-1)(\varpi_\gamma + \eta_0)^{2p-2}T_0' + \big((\varpi_\gamma + \eta_0)^{2p-1}T_0', T_0'^p\big)\mathcal{O}_{K_*}\langle \varpi_\gamma^{-pa+p-2}\eta_0\rangle[\![\varpi_\gamma^{-pa+p-2}T_0']\!]
\end{aligned}$$

Hence, $|T_0| = \theta^{(2p-2)/p} \cdot |T_0'| < r\theta^a$.

Conversely, if $|T_0| < r\theta^a$, we rewrite the above equation as

$$T_0' \in \frac{1}{(2p-1)(\varpi_\gamma + \eta_0)^{2p-2}}T_0 + (\varpi_\gamma T_0') \cdot \mathcal{O}_{K_*}\langle \varpi_\gamma^{-pa+p-2}\eta_0\rangle[\![\varpi_\gamma^{-pa+p-2}T_0']\!]. \quad (4.3.2.10)$$

We substitute (4.3.2.10) back into itself recursively. The equation converges to a T_0', which is an inverse.

Therefore, Proposition 1.1.2.18 implies that for $r \in [0, 1)$,

$$\begin{aligned}
& IR_0(\mathcal{E}; a+1) \leq r \\
\Leftrightarrow\ & f_{\mathrm{gen}}^*(\mathcal{E} \otimes F_{a+1}) \text{ is trivial on } A^1_{F_{a+1}}[0, r\theta^{a+1}) \\
\Leftrightarrow\ & \tilde{f}_\gamma^* f_{\mathrm{gen}}^*(\mathcal{E} \otimes F_{a+1}) = f_{\mathrm{gen}}^*\big(f_\gamma^*\mathcal{E} \otimes F^*_{a-\frac{p-2}{p}}\big) \text{ is trivial on } A^1_{F^*_{a-\frac{p-2}{p}}}[0, r\theta^{a-\frac{p-2}{p}}) \\
\Leftrightarrow\ & IR_0(f_\gamma^*\mathcal{E}; a-(p-2)/p) \leq r.
\end{aligned}$$

The proposition follows. □

Construction 4.3.2.11. Fix a p-basis (b_J) of K; it naturally gives a p-basis of K_*. Fix a choice of $\psi_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$ as in Construction 4.1.1.2. We will use the method in Construction 4.1.1.2 to define $\psi_{K_*,\gamma}$ for $\gamma = 1, \dots, p$ such that the following diagram *commutes*.

$$\begin{array}{ccc}
\mathcal{O}_K & \xrightarrow{\psi_K} & \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!] \\
\downarrow & & \downarrow f_\gamma^* \\
\mathcal{O}_{K_*} & \xrightarrow{\psi_{K_*}} & \mathcal{O}_{K_*}[\![\eta_0/\varpi_\gamma, \delta_J]\!]
\end{array} \quad (4.3.2.12)$$

For any element $h \in \mathcal{O}_{K_*}$, first write $h = \sum_{i=0}^{p-1} h_i \varpi_\gamma^i$ where $h_i \in \mathcal{O}_K$. As in Construction 4.1.1.2, write each of h_i as $h_i^\circ \pi_K^{e_i}$ for $e_i = v_K(h_i)$ and $h_i^\circ \in \mathcal{O}_K$; chose a compatible system of r-th p-basis decomposition of h_i° as

$$h_i^\circ = \sum_{e_J=0}^{p^r-1} b_J^{e_J}\Big(\sum_{n=0}^{\infty}\Big(\sum_{n'=0}^{\lambda_{i,(r),e_J,n}} \alpha_{i,(r),e_J,n,n'}^{p^r}\Big)\pi_K^n\Big)$$

for some $\alpha_{i,(r),e_J,n,n'} \in \mathcal{O}_K^\times \cup \{0\}$ and some $\lambda_{i,(r),e_J,n} \in \mathbb{Z}_{\geq 0}$. We choose the system of r-th p-basis decomposition of $h/\varpi_\gamma^{v_{K_*}(h)}$ to be

$$\frac{h}{\varpi_\gamma^{v_{K_*}(h)}} = \frac{1}{\varpi_\gamma^{v_{K_*}(h)}}\sum_{i=0}^{p-1}\varpi_\gamma^i \sum_{e_J=0}^{p^r-1} b_J^{e_J}\Big(\sum_{n=0}^{\infty}\Big(\sum_{n'=0}^{\lambda_{i,(r),e_J,n}} \alpha_{i,(r),e_J,n,n'}^{p^r}\Big)(\varpi_\gamma^{p-1}+\varpi_\gamma^{2p-1}+\cdots)^{n+e_i}\Big)$$

and define $\psi_{K_*,\gamma}(h)$ to be the limit for $r \to +\infty$ of

$$\sum_{i=0}^{p-1}(\varpi_\gamma+\eta_0)^i \sum_{e_J=0}^{p^r-1}(b_J+\delta_J)^{e_J}\Big(\sum_{n=0}^{\infty}\Big(\sum_{n'=0}^{\lambda_{i,(r),e_J,n}} \alpha_{i,(r),e_J,n,n'}^{p^r}\Big)\big((\varpi_\gamma+\eta_0)^{p-1}+\cdots\big)^{n+e_i}\Big).$$

This gives a $\psi_{K_*,\gamma}$ defined in the way of Construction 4.1.1.2; the diagram (4.3.2.12) is commutative.

Hypothesis 4.3.2.13. For the rest of this subsection, let L/K_* be a finite Galois extension satisfying Hypotheses 4.1.1.1 and 4.1.1.10 and such that L/K is Galois.

Proposition 4.3.2.14. *Let $a > 1$. Then there exists admissible $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ such that the logarithmic thickening space for L/K, after extension of scalars from K to K_*, is isomorphic to a disjoint union of p (different) logarithmic thickening spaces for L/K_*:*

$$TS^a_{L/K,\log,R_{J^+}} \times_K K_* \xrightarrow{\sim} \coprod_{\gamma=1}^{p} TS^{pa-p+1}_{K_*/K,\log,\psi_{K_*,\gamma}}.$$

Proof. Write $\mathcal{O}_{K_*}\langle u_{J^+}\rangle/(p_{J^+}) = \mathcal{O}_L$ using Construction 2.3.3.3. Since $\mathcal{O}_K\langle z\rangle/(z^p + \pi_K z^{p-1} - \pi_K) = \mathcal{O}_{K_*}$, we may replace the coefficients in p_{J^+} by elements in $\mathcal{O}_K\langle z\rangle$ with degree $\leq p-1$ in z, denoting the result polynomials by p'_{J^+}. Thus by Lemma 4.3.2.4

and the commutativity of (4.3.2.12),

$$\prod_{\gamma=1}^{p} K_*\langle \varpi_\gamma^{-pa+p-2}\eta_0, \varpi_\gamma^{-pa+p-1}\eta_J\rangle\langle u_{J+}\rangle / (\psi_{K_*,\gamma}(p_{J+}))$$
$$\cong K_*\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle\langle u_{J+}, z\rangle / \big(\psi_K(p'_{J+}), z^p + (\pi_K+\delta_0)z^{p-1} - (\pi_K+\delta_0)\big),$$

where the latter one is a successive logarithmic thickening space for L/K, base changed to K_*. By Proposition 4.1.5.6, this successive logarithmic thickening space is isomorphic to a logarithmic thickening space $TS^a_{L/K,\log,R_{J+}}$ for L/K for some admissible subset $R_{J+} \subset (\delta_{J+}) \cdot \mathcal{S}_K$. □

Corollary 4.3.2.15. *Let $\mathcal{E}_{L/K}$ be the differential module over $A^1_K[0,\theta^{a+1}] \times A^m_K[0,\theta^a]$ coming from $TS^a_{L/K,\log,R_{J+}}$. For $\gamma \in \{1,\dots,p\}$, let $\mathcal{E}_{L/K_*,\gamma}$ be the differential module over $A^1_{K_*}[0,\theta^{a-(p-2)/p}] \times A^m_{K_*}[0,\theta^{a-(p-1)/p}]$ coming from $TS^{ap-p+1}_{L/K_*,\log,\psi_{K_*,\gamma}}$. Then $\mathcal{E}_{L/K}\otimes_K K_* \simeq \bigoplus_{\gamma=1}^{p} f_{\gamma*}\mathcal{E}_{L/K_*,\gamma}$.*

Proof. It follows from Lemma 4.3.2.4 and Proposition 4.3.2.14. □

4.3.3 Subquotients of logarithmic ramification filtration

In this subsection, we prove Theorem 4.3.3.3 that the subquotients $\mathrm{Fil}^a_{\log}G_K/\mathrm{Fil}^{a+}_{\log}G_K$ of logarithmic ramification filtration are abelian groups killed by p if $a \in \mathbb{Q}_{>0}$ and are trivial if $a \notin \mathbb{Q}$. This uses the tricky base change discussed in previous subsection.

We assume Hypothesis 4.3.2.13 until we state the main Theorem 4.3.3.3.

Notation 4.3.3.1. Fix $\gamma \in \{1,\dots,p\}$. Let (b_J) be a finite p-basis of K. It naturally gives a p-basis of K_*. Denote by $K(x_J)^\wedge$ the completion of $K(x_J)$ with respect to the $(1,\dots,1)$-Gauss norm and by K' the completion of the maximal unramified extension of $K(x_J)^\wedge$. Write $K'_* = K_*K'$ and $L' = K'_*L$. Set

$$\widetilde{K}_\gamma = K'_*((b_J + x_J\varpi_\gamma^{p-1})^{1/p}).$$

Denote $\beta_J = (b_J + x_J\varpi_\gamma^{p-1})^{1/p}$ for simplicity. Take the uniformizer and a set of lifted p-basis of $\widetilde{K}_\gamma$ to be ϖ_γ and $\{\beta_J, x_J\}$, respectively.

Situation 4.3.3.2. We have the following diagram of field extensions:

$$\begin{array}{ccccc} L & \text{——} & L' & \text{——} & \widetilde{L}_\gamma \\ | & & | & & | \\ K_* & \text{——} & K'_* & \text{——} & \widetilde{K}_\gamma \\ | & & | & & \\ K & \text{——} & K' & & \end{array}$$

Note that $(\widetilde{K}_\gamma)_{\gamma=1,\dots,p}$ are extensions of K'_* conjugate over K'. The ramification filtrations on $G_{\widetilde{K}_\gamma}$ are stable under the conjugate action of $\mathrm{Gal}(K'_*/K')$. Precisely, for any $b \geq 0$ and $g \in \mathrm{Gal}(K'_*/K')$, $g\mathrm{Fil}^b_{\log} G_{\widetilde{K}_\gamma} g^{-1} = \mathrm{Fil}^b_{\log} G_{g(\widetilde{K}_\gamma)}$ and $g\mathrm{Fil}^b G_{\widetilde{K}_\gamma} g^{-1} = \mathrm{Fil}^b G_{g(\widetilde{K}_\gamma)}$ inside $G_{K'}$. In particular, since L'/K' and hence $\widetilde{L}_\gamma/\widetilde{K}_\gamma$ is Galois, $b(\widetilde{L}_\gamma/\widetilde{K}_\gamma)$ and $b_{\log}(\widetilde{L}_\gamma/\widetilde{K}_\gamma)$ do not depend on $\gamma = 1, \dots, p$.

For the following theorem, we do not impose any hypothesis on the field K.

Theorem 4.3.3.3. *Let K be a complete discretely valued field of mixed characteristic $(0, p)$. Let G_K be its Galois group. Then the subquotients $\mathrm{Fil}^a_{\log} G_K / \mathrm{Fil}^{a+}_{\log} G_K$ of the logarithmic ramification filtration are trivial if $a \notin \mathbb{Q}$ and are abelian groups killed by p if $a \in \mathbb{Q}_{>0}$.*

Proof. We will proceed as in the proof of Theorem 4.2.3.5. Fix $a > 0$. Let L be a finite Galois extension of K with Galois group $G_{L/K}$ with an induced ramification filtration. We may assume that $\mathrm{Fil}^{a+}_{\log} G_{L/K}$ is the trivial group but $\mathrm{Fil}^a_{\log} G_{L/K}$ is not. We may also assume Hypothesis 4.1.1.1. Furthermore, by Proposition 2.2.2.11(4), we are free to make a tame base change and assume that $a = b_{\log}(L/K) > 1$ and $p\beta_K \geq m(p-1)+1$. Finally, we may replace L by LK_* since $b_{\log}(K_*/K) = 1$ by Lemma 4.3.2.1. We need to show that $\mathrm{Fil}^a_{\log} G_{L/K}$ is an abelian group killed by p if $a \in \mathbb{Q}_{>1}$ and is trivial if $a \notin \mathbb{Q}$.

We claim that each of the logarithmic ramification breaks $b > 1$ of L/K will become a non-log ramification break $bp-p+2$ on $\widetilde{L}_1/\widetilde{K}_1$. In other words, $\mathrm{Fil}^b_{\log} G_{L/K} \subseteq \mathrm{Fil}^{pb-p+2} G_{\widetilde{L}_\gamma/\widetilde{K}_\gamma}$ for any $\gamma \in \{1, \dots, p\}$ and $b > 1$. (It does not matter which γ we choose as they give the same answer by Situation 4.3.3.2.) Then the theorem is a direct consequence of the non-logarithmic Hasse-Arf theorem 4.2.3.5(2).

To prove the claim, it suffices to prove the highest ramification breaks as the others will follow from the calculation of the other L's.

For each $\gamma \in \{1, \dots, p\}$, there exists a unique continuous $\mathcal{O}_{K_*}[\![\eta_0/\varpi_\gamma]\!]$-homomorphism $\tilde{f}_\gamma^* : \mathcal{O}_{K_*}[\![\eta_0/\varpi_\gamma, \delta_J]\!] \to \mathcal{O}_{\widetilde{K}_\gamma}[\![\eta_0/\varpi_\gamma, \eta_J, \eta'_J]\!]$ such that $\tilde{f}_\gamma^*\delta_j = (\beta_j+\eta_j)^p - (x_j+\eta'_j)(\varpi_\gamma+\eta_0)^{p-1} - b_j$ for $j \in J$. For $a > 1$, $\tilde{f}_\gamma^*$ gives a morphism $\tilde{f}_\gamma : A^{2m+1}_{\widetilde{K}_\gamma}[0, \theta^a] \to A^{m+1}_{K_*}[0, \theta^a]$.

Let $TS^a_{L/K_*,\psi_{K_*,\gamma}}$ be the standard thickening space for L/K_* and $\psi_{K_*,\gamma}$. We have a Cartesian diagram

$$\begin{array}{ccccc}
 & & TS^a_{L/K_*,\psi_{K_*,\gamma}} & \xleftarrow{\tilde{f}_\gamma} & TS^a_{L/K_*,\psi_{K_*,\gamma}} \times_{A^{m+1}_{K_*}[0,\theta^a],\tilde{f}_\gamma} A^{2m+1}_{\widetilde{K}_\gamma}[0, \theta^a] \\
 & \swarrow & \downarrow \Pi & & \downarrow \Pi \\
A^1_{K_*}[0, \theta^{a+\frac{2p-2}{p}}] \times A^m_{K_*}[0, \theta^a] & \xleftarrow{f_\gamma} & A^{m+1}_{K_*}[0, \theta^a] & \xleftarrow{\tilde{f}_\gamma} & A^{2m+1}_{\widetilde{K}_\gamma}[0, \theta^a]
\end{array}$$

By applying Theorem 4.2.1.7 m times, $TS^a_{L/K_*,\psi_{K_*,\gamma}} \times_{A^{m+1}_{K_*}[0,\theta^a],\tilde{f}_\gamma} A^{2m+1}_{\widetilde{K}_\gamma}[0, \theta^a]$ is an admissible recursive non-logarithmic thickening space (of error gauge $\geq p\beta_K - m(p-1) \geq 1$), which is isomorphic to an admissible non-logarithmic thickening space for $\widetilde{L}_\gamma/\widetilde{K}_\gamma$ by Proposition 4.1.5.6. Thus $\tilde{f}_\gamma^*\mathcal{E}_{L/K_*,\gamma}$ is a differential module associated to $\widetilde{L}_\gamma/\widetilde{K}_\gamma$.

By Proposition 4.3.2.9 and Lemma 4.2.3.3, we have

$$IR(\tilde{f}_\gamma^*\mathcal{E}_{L/K_*,\gamma}; \underline{s}) = IR\Big(\mathcal{E}_{L/K_*,\gamma}; s, \underline{s + \frac{p-2}{p}}\Big) = IR\Big((f_\gamma)_*\mathcal{E}_{L/K_*,\gamma}; s+\frac{2p-2}{p}, \underline{s + \frac{p-2}{p}}\Big).$$

The claim follows by Corollaries 4.3.2.15 and 4.1.4.6. $\square$

Chapter 5

Towards Global Ramification Theory

This chapter is dedicated to describe a project on understanding the global ramification situation using the tool of nonarchimedean differential modules.

In Section 5.1, we lay out the big picture of the project. More precisely, Subsection 5.1.1 describes the analogy of three basic objects that we are studying. Subsection 5.1.2 explains different micro-local versions of the global objects. Subsection 5.1.3 includes all the conjectures and plans of attack.

In Section 5.2, we work out the toroidal case following [Ked07+b].

5.1 Description of a project

We outline the basic structure of the project studying ramification theory using nonarchimedean differential modules. This section will be of survey type. We will not include the definition of all the terminology, rather we refer to other papers.

5.1.1 What objects are we talking about here?

We introduce the main objects of the ramification theory.

Let k be a field. Let X be a projective smooth variety over k and let $D = \cup_{i=1}^{n} D_i$

be a divisor on X with simple normal crossings, where D_i are irreducible components. Let $U = X \backslash D$ denote the complement.

Suppose that $\mathcal{F}$ is one of the following:

(a) a locally free coherent sheaf on U with an integrable connection, when $\operatorname{char} k = 0$;

(b) an F-isocrystal on U overconvergent along D, when $\operatorname{char} k = p > 0$;

(c) a lisse $\overline{\mathbb{Q}}_l$-sheaf on U, where l is a prime number different from $\operatorname{char} k$.

Remark 5.1.1.1. The analogy among the three objects listed above is known for a long time. One can further extend the analogy to the following table.

$\operatorname{char} k = 0$	vector bundles with flat connections	holonomic algebraic D-modules
$\operatorname{char} k = p$	overconvergent F-isocrystals	overholonomic arithmetic D-modules
$\operatorname{char} k \neq l$	lisse $\overline{\mathbb{Q}}_l$-sheaves	constructible $\overline{\mathbb{Q}}_l$-sheaves

Remark 5.1.1.2. The case (c) is equivalent to the category of $\overline{\mathbb{Q}}_l$-representations of the fundamental group $\pi_1(U, \bar{\eta})$, where $\bar{\eta}$ is the geometric generic point of U. Also, there is a fully-faithful functor from the category of $\overline{\mathbb{Q}}_p$ representations of $\pi_1(U, \bar{\eta})$ with finite local monodromy into the category of all objects in (b) (See [Tsu02]. So, in some sense, we are really studying representations of the fundamental groups of a scheme.

Let η_i be the generic point of an irreducible component D_i of D. Use K_{η_i} to denote the completion of the fraction field $\operatorname{Frac}\mathcal{O}_U$ with respect to the norm given by η_i. We have one of the following:

(a) Irregularity (will be defined in Definition 5.2.1.1), denoted temporarily by $\operatorname{Swan}(\mathcal{F}, D_i) = \operatorname{Irr}(\mathcal{F} \otimes K_{\eta_i})$ for notational convenience;

(b) the (differential) Swan conductor $\operatorname{Swan}(\mathcal{F}, D_i) = \operatorname{Swan}(\mathcal{F} \otimes K_{\eta_i})$, defined as in Definition 1.2.8.3;

(c) the Swan conductor $\mathrm{Swan}(\mathcal{F}, D_i)$ associated to the Galois representation $\mathrm{Gal}(K_{\eta_i}^{\mathrm{sep}}/K_{\eta_i}) \to \pi_1(U, \eta) \to \mathrm{GL}(V_{\mathcal{F}})$, where the latter homomorphism is the representation associated to the lisse sheaf $\mathcal{F}$.

Remark 5.1.1.3. We in fact have two more objects which are analogous to the three cases above. One is representations of the fundamental group $\pi_1(X, \eta_X)$ of an arithmetic scheme, i.e. a scheme over $\operatorname{Spec}\mathbb{Z}$. We use (d) to denote this case. The other analogue is p-adic étale sheaves over a proper smooth *semistable* variety over $\operatorname{Spec}\mathbb{Q}_p$; we use may use Faltings' almost étale theory to study the ramification. (See [Col03+].) We use (e) to denote this case.

We are interested in the behavior of these conductors over the variety X and the its relation to the Euler characteristic of $\chi_c(U, \mathcal{F})$, where the latter is defined as follows.

(a) We have the de Rham cohomology of $\mathcal{F}$: $H_{\mathrm{dR}}(X, j_*\mathcal{F}) = \mathbb{H}(X, j_*(\mathcal{F} \otimes \Omega_X^\bullet))$ defined by the hypercohomology of the pushforward along $j : U \to X$ of the de Rham complex

$$\mathcal{F} \otimes \Omega_U^\bullet = \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega_U^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{F} \otimes \Omega_U^{\dim X}.$$

The Euler characteristic is defined as the alternating sum

$$\chi_c(U, \mathcal{F}) \stackrel{\text{def}}{=} \sum_{i=0}^{2\dim X} (-1)^i \dim H_{\mathrm{dR}}^i(X, j_*\mathcal{F}).$$

(b) We have the (compactly supported) rigid cohomology $H_{\mathrm{rig},c}^\bullet(U, \mathcal{F})$ defined as in [Brt96+]. We take the Euler characteristic to be

$$\chi_c(U, \mathcal{F}) \stackrel{\text{def}}{=} \sum_{i=0}^{2\dim X} (-1)^i \dim H_{\mathrm{rig},c}^i(U, \mathcal{F}).$$

(c) We have the compactly supported étale cohomology $H_{\text{ét},c}^\bullet(U, \mathcal{F})$. We take the

Euler characteristic to be

$$\chi_c(U, \mathcal{F}) \stackrel{\text{def}}{=} \sum_{i=0}^{2\dim X} (-1)^i \dim H^i_{\text{ét},c}(U, \mathcal{F}).$$

Question 5.1.1.4. How can we calculate Euler characteristic $\chi_c(U, \mathcal{F})$ from the ramification information as in the Grothendieck-Ogg-Shavarevich formula (2.1.1.1) (described in Subsection 2.1.1)? The formula also has a p-adic analogue for overconvergent F-isocrystals. See [Ked06b, Theorem 4.4.1]

Remark 5.1.1.5. The real difficulty of extending the Grothendieck-Ogg-Shavarevich formula (2.1.1.1) to higher dimensional case is that merely the information of $\mathrm{Swan}(\mathcal{F}, D_i)$ for irreducible components D_i's of D is not enough. In particular, if we blow up the intersection $D_i \cap D_j$ of two irreducible components. Then no ramification information along the exceptional divisor is recorded in the data $\mathrm{Swan}(\mathcal{F}, D_i)$'s. This leads to our study of the subject of the next subsection.

5.1.2 Micro-local variation of Swan conductors

In this subsection, we describe different levels of micro-local variation of Swan conductors.

Construction 5.1.2.1. It would be more helpful to keep in mind the Riemann-Zariski construction for valuation space of a variety. Precisely, let X be an integral scheme. Define the *Riemann-Zariski space*, denoted by $\mathrm{RZ}(X)$, to be the inverse limit $\varprojlim_{(\phi', X')} X'$ over all *birational proper* morphisms $\phi' : X' \to X$. The topology on $RZ(X)$ is given by the inverse limit topology where each of X' is equipped with the Zariski topology.

It is a well-known fact that $\mathrm{RZ}(X)$ parameterizes all (multi-indexed) valuations of $\mathrm{Frac}(\mathcal{O}_X)$ which are centered on X. For more details, please consult [Ked08].

A codimension one point $x \in \mathrm{RZ}(X)$ is the inverse limit starting from the generic point $\eta_{D'}$ of an integral divisor D' on some X' for which $\mathcal{O}_{X', \eta_{D'}}$ is a discrete valuation ring. (Note that the proper transform of $\eta_{D'}$ under any further birational proper morphism will continue to be a point with the prescribed property.) From the valuation theory point of view, the codimension one points correspond to the case when the valuations which take value in $\mathbb{Z}$. We use $\mathrm{RZ}^1(X)$ to denote all codimensional one points in $\mathrm{RZ}(X)$; they form a dense subset of $\mathrm{RZ}(X)$.

Given one of the three situations from previous subsection, we may ask what is the ramification situation at each codimension one point of the Riemann-Zariski space, that is to base change to the completion of $\mathrm{Frac}(\mathcal{O}_X)$ with respect to the corresponding norm. We can also ask how the ramification globally varies on the Riemann-Zariski space. The variation of Swan conductors should be "continuous" in an appropriate sense, although the Swan conductors are not defined over all points on the Riemann-Zariski space $\mathrm{RZ}(X)$.

The following is a list of variation questions we may ask.

(α) Complete discretely valued fields: The most local level is represented by codimension one points $\mathrm{RZ}^1(X)$. They in fact form the basic building blocks of the theory since $\mathrm{RZ}^1(X)$ is dense in $RZ(X)$. This is just simply the ramification filtration on a complete discretely valued field, which is the the main topic of Chapters 2–4. We proved the Hasse-Arf theorem regarding the integrality of the Swan conductors.

(β) Higher dimensional local fields: This also corresponds to a point on the Riemann-Zariski space which is typically a limit of codimensional one points. (For more details, one may consult the symposium [FK00].) This captures the limit of Swan conductors for points in $\mathrm{RZ}^1(X)$.

(γ) Over regular local rings: Given a point $x \in X$, we can consider a subset $\mathrm{RZ}(X, x) \subset \mathrm{RZ}(X)$ consists of valuations v such that $v(f) \geq 0$ for all $f \in \mathcal{O}_{X,x}$. The variation of Swan conductors on (a subset of) $\mathrm{RZ}(X, x) \cap \mathrm{RZ}^1(X)$ will be studied in detail in Section 5.2.

(δ) Global variation: We will discuss in detail what we expect in this case.

5.1.3 What is expected to be true?

In this subsection, we make a series of conjectures on the variation of Swan conductors as well as the Euler-characteristic formulas.

Conjecture 5.1.3.1. *Let K be a higher dimensional local field (see [FK00, Section 1] for definition). Then there exists a multi-indexed ramification filtration* $\mathrm{Fil}_{\log}^{a_1,\dots,a_n} G_K$ *of normal subgroups on the Galois group indexed by* $(\mathbb{Z}^n)_{\geq(0,\dots,0)}$ *with lexicographic order such that*

(a) *it degenerates to the arithmetic ramification filtration if we view K simply as a complete discretely valued field and ignore the finer part $a_2, \dots, a_n$ of the filtration.*

(b) *it is compatible with a filtration on the Milnor K-group via the higher class field theory. (See [FK00] for the terms.)*

Remark 5.1.3.2. The multi-index is expected to depend on the choice of a system of uniformizers of K, but the order of the subgroups does not.

Notation 5.1.3.3. Let X be a scheme and let U be an open subscheme of X. Let $\mathfrak{R}_{X,U}$ denote the ordered monoid of all the $\mathcal{O}_X$-subsheaves $\mathcal{I}$ of $\mathrm{Frac}\mathcal{O}_X$ such that $\mathcal{O}_X \subset \mathcal{I}$ and $\mathcal{I}|_U = \mathcal{O}_U$, where the monoid structure is given by fraction ideal multiplication and the ordering is given by inclusion, i.e. $\mathcal{I} \preceq \mathcal{I}'$ if $\mathcal{I} \subseteq \mathcal{I}'$.

Let $\mathfrak{R}_{X,U}^{\mathbb{Q}}$ denote the proxy for $\mathfrak{R}_{X,U}[1/n; n \in \mathbb{Z}]$; more precisely, it consists of pairs $(\mathcal{I}, n)$ with $\mathcal{I} \in \mathfrak{R}_{X,U}$ and $n \in \mathbb{N}$ and two pairs $(\mathcal{I}, n)$ and $(\mathcal{I}', n')$ are considered same if $\mathcal{I}^{n'm} = \mathcal{I}'^{nm}$ for some $m \in \mathbb{N}$. We usually use $\mathcal{I}^{1/n}$ to denote the pair $(\mathcal{I}, n) \in \mathfrak{R}_{X,U}^{\mathbb{Q}}$. The ordering on $\mathfrak{R}_{X,U}$ extends to $\mathfrak{R}_{X,U}^{\mathbb{Q}}$ by setting $\mathcal{I}^{1/n} \preceq \mathcal{I}'^{1/n'}$ if $\mathcal{I}^{n'm} \subseteq \mathcal{I}'^{nm}$ for some $m \in \mathbb{N}$.

Conjecture 5.1.3.4. *Let X be a scheme and let U be an open subscheme of X. Let $\mathcal{F}$ be one of the three options in Subsection 5.1.1 or a representation of $\pi_1(U)$ which is unramified on U. Then there exists an* irregularity sheaf $\mathcal{I} \subset \mathfrak{R}_{X,U}$ *as an $\mathcal{O}_X$-subsheaf such that at each codimension one point $\eta \in X'$ for some birational proper morphism $X' \to X$, $\mathcal{I} \otimes \mathcal{O}_{X',\eta}$ has valuation* $-\mathrm{Swan}(\mathcal{F}, \eta)$ *in* $\mathrm{Frac}(\mathcal{O}'_X)$.

Then irregularities described above induce a canonical ramification descending filtration $\mathrm{Fil}^{\mathcal{I}^{1/n}}\pi_1(U)$ *on the fundamental group of* U*, indexed by* $\mathcal{I}^{1/n} \in \mathfrak{R}_{X,U}^{\mathbb{Q}}$ *for some* n*, where descending means that if* $\mathcal{I}^{1/n} \preceq \mathcal{I}'^{1/n'}$ *then* $\mathrm{Fil}^{\mathcal{I}'^{1/n'}}\pi_1(U) \subseteq \mathrm{Fil}^{\mathcal{I}^{1/n}}\pi_1(U)$.

Furthermore, if we define $\mathrm{Fil}^{\mathcal{I}^{1/n}+}\pi_1(U)$ *to be the closure of* $\cap_{\mathcal{I}^{1/n} \prec \mathcal{J}^{1/m}} \mathrm{Fil}^{\mathcal{J}^{1/m}}\pi_1(U)$*, then for* $\mathcal{I}^{1/n} \succeq 0$*,* $\mathrm{Fil}^{\mathcal{I}^{1/n}}\pi_1(U)/\mathrm{Fil}^{\mathcal{I}^{1/n}+}\pi_1(U)$ *is abelian and we have a refined Swan conductor homomorphism*

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{Fil}^{\mathcal{I}^{1/n}}\pi_1(U)/\mathrm{Fil}^{\mathcal{I}^{1/n}+}\pi_1(U), k^\times) \to \mathcal{I}^{1/n}\Omega_X^1(\log D) \otimes_{\mathcal{O}_X} k.$$

Remark 5.1.3.5. The first part of this conjecture is in fact verified for case (a) in [Ked09+].

Conjecture 5.1.3.6. *Keep the notation as in Conjecture 5.1.3.4 and we assume that we are in case (a)–(c) in Subsection 5.1.1. Then we have* $\mathcal{B}_1, \dots, \mathcal{B}_d \in \mathfrak{R}_{X,U} \otimes_{\mathbb{Z}} \mathbb{Q}$ *which has valuation (after tensoring the valuation by* $\mathbb{Q}$*) equals the subsidiary breaks of* $\mathcal{F}$ *at each codimension one point of* $\mathrm{RZ}^1(X)$*. Moreover, the Euler characteristic* $\chi_c(U, \mathcal{F})$ *can be computed by*

$$\chi_c(U, \mathcal{F}) = \sum_{i=1}^{d} \#\big(c(\mathcal{O}_X^1(\log D)) \cap (1 + (\mathcal{B}_i/\mathcal{O}_X))^{-1}\big)_{\dim 0},$$

where $c(\mathcal{O}_X^1(\log D))$ *is the total Chern class of the logarithmic differential sheaf,* $\mathcal{B}_i/\mathcal{O}_X$ *is viewed as an element in the rational Grothendieck group of coherent modules over* $\mathcal{O}_X$*, the intersection is taken over the rational Chow ring, and we finally take the dimension* 0 *part. (See [Ful98] for more details on intersection theory.)*

5.2 Toroidal variation

In this subsection, we collect some results regarding the toroidal variation of refined Swan conductor.

5.2.1 Vector bundles with connections

Definition 5.2.1.1. Let K be a complete discretely valued field of *residual* characteristic zero, equipped with differential operators $\partial_1, \dots, \partial_m$ of rational type. Let V be a ∂_J-differential modules on K of rank d. The irregularity of V is defined to be $\mathrm{Irr}(V) = \sum_{i=1}^{d} \log_{\pi_K} \mathcal{IR}(V; i)$

For the rest of this subsection, we assume that k is a field of characteristic zero.

Notation 5.2.1.2. For $n \geq m \geq 0$ integers, put $R_{n,m} = k[\![x_1, \dots, x_n]\!][x_1^{-1}, \dots, x_m^{-1}]$. A ∇-module on $R_{n,m}$ is a differential module with respect to $\partial_i = \frac{d}{dx_i}$.

Denote $S = \{(r_1, \dots, r_n) \in [0, +\infty)^n : r_1 + \cdots + r_n = 1\}$. For $r \in S$, write F_r for the completion of $\mathrm{Frac} R_{n,m}$ with respect to the $(e^{-r_1}, \dots, e^{r_n})$-Gauss norm. For M a ∇-module of rank d over $R_{n,m}$ and $r \in S \cap \mathbb{Q}^n$, define $\mathrm{Irr}(M, r) = \mathrm{Irr}(M \otimes F_r)/\mathrm{denom}$, where denom is the least common multiplier of the denominators of $r_1, \dots, r_n$.

Theorem 5.2.1.3. *Let M be a ∇-module of rank d over $R_{n,m}$. Then $\mathrm{Irr}(M, r)$ extends by continuity to a function on S which can be written as $\max_{j=1}^{h}\{\lambda_j(r)\}$ for some integral affine functionals $\lambda_1, \dots, \lambda_h$. In particular, $\mathrm{Irr}(M, r)$ is continuous, convex, and piecewise affine.*

Proof. It follows from Theorem 1.3.3.9. □

5.2.2 Solvable overconvergent isocrystals

In this subsection, we state a theorem from [Ked07+b] regarding the variation of Swan conductors in case (b) of Subsection 5.1.1.

Hypothesis 5.2.2.1. Let k be a perfect field of characteristic p. let $\overline{X}$ be a smooth irreducible k-variety. Let $D_1, \dots, D_n$ be smooth irreducible divisors on $\overline{X}$ meeting transversely at a closed point x. Choose a local coordinates $t_1, \dots, t_n$ at x such that t_i vanishes along D_i. Put $D = D_1 \cup \cdots \cup D_n$ and $X = \overline{X} \backslash D$. Let $\mathcal{F}$ be an F-isocrystal of rank d on X overconvergent along D.

Notation 5.2.2.2. Denote $S = \{(r_1, \dots, r_n) | r_1 + \cdots + r_n = 1\}$. For $r = (r_1, \dots, r_n) \in S \cap \mathbb{Q}^n$, we have a valuation $v_r(\cdot)$ on $\mathrm{Frac}(\mathcal{O}_X)$ given by the restriction from the $(e^{-r_1}, \dots, e^{-r_n})$-Gauss valuation on $\mathrm{Frac}(k\llbracket t_1, \dots, t_n \rrbracket)$. It corresponds to a smooth divisor D_r on a proper variety X' birational over X. We denote $\mathrm{Swan}(\mathcal{F}, r) = \mathrm{Swan}(\mathcal{F}, D_r)/\mathrm{denom}$, where denom is the least common multiplier of the denominators of $r_1, \dots, r_n$.

Theorem 5.2.2.3. *Keep the notation as above. Then* $\mathrm{Swan}(\mathcal{F}, r)$ *extends by continuity to a function on* S *which can be written as* $\max_{j=1}^{h}\{\lambda_j(r)\}$ *for some integral affine functionals* $\lambda_1, \dots, \lambda_h$. *In particular,* $\mathrm{Swan}(\mathcal{F}, r)$ *is continuous, convex, and piecewise affine.*

Proof. We refer to [Ked07+b, Section 3.4] for the proof. But the main ingredient is Theorem 1.3.3.9. □

5.2.3 Lisse ℓ-adic sheaves

In this subsection, we state a theorem from [Ked07+b] regarding the variation of Swan conductors in case (b) of Subsection 5.1.1.

Hypothesis 5.2.3.1. Let k be a perfect field of characteristic p. let $\overline{X}$ be a smooth irreducible k-variety. Let $D_1, \dots, D_n$ be smooth irreducible divisors on $\overline{X}$ meeting transversely at a closed point x. Choose a local coordinates $t_1, \dots, t_n$ at x such that t_i vanishes along D_i. Put $D = D_1 \cup \cdots \cup D_n$ and $X = \overline{X} \backslash D$. Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}_l}$-sheaf of rank d on U, which corresponds to an l-adic representation $\pi_1(U) \to \mathrm{GL}(V)$.

Notation 5.2.3.2. Denote $S = \{(r_1, \dots, r_n) | r_1 + \cdots + r_n = 1\}$. For $r = (r_1, \dots, r_n) \in S \cap \mathbb{Q}^n$, we have a valuation $v_r(\cdot)$ on $\mathrm{Frac}(\mathcal{O}_X)$ given by the restriction from the $(e^{-r_1}, \dots, e^{-r_n})$-Gauss valuation on $\mathrm{Frac}(k\llbracket t_1, \dots, t_n \rrbracket)$; denote the completion to be $F_{r_1, \dots, r_n}$. It corresponds to a smooth divisor D_r on a proper variety X' birational over X. We denote $\mathrm{Swan}(\mathcal{F}, r) = \mathrm{Swan}(\mathcal{F}, D_r)/\mathrm{denom}$, where denom is the least common multiplier of the denominators of $r_1, \dots, r_n$, and $\mathrm{Swan}(\mathcal{F}, D_r)$ is the Swan conductor for the representation

$$\mathrm{Gal}(F^{\mathrm{sep}}_{r_1,\dots,r_n}/F_{r_1,\dots,r_n}) \to \pi_1(U) \to \mathrm{GL}(V).$$

Theorem 5.2.3.3. *Keep the notation as above. Then* $\mathrm{Swan}(\mathcal{F}, r)$ *extends by continuity to a function on* S *which can be written as* $\max_{j=1}^{h}\{\lambda_j(r)\}$ *for some integral affine functionals* $\lambda_1, \dots, \lambda_h$. *In particular,* $\mathrm{Swan}(\mathcal{F}, r)$ *is continuous, convex, and piecewise affine.*

Proof. This is proved in [Ked07+b, Theorem 5.2.1]. □

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