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Scott N. Sanders,[1,2,*] Florian Mintert,[3,2] and Eric J. Heller[2]
[1]*Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*Harvard University, Cambridge, Massachusetts 02138, USA*
[3]*Albert-Ludwigs-Universität Freiburg, Hermann-Herder-Strasse 3, 79104 Freiburg, Germany*
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We investigate decoherence in atom interferometry due to scattering from a background gas and show that the supposition that residual coherence is due to near-forward scattering is incorrect. In fact, the coherent part is completely unscattered, although it is phase shifted. This recoil-free process leaves both the atom and the gas in an unchanged state, but allows for the acquisition of a phase shift. This is essential to understanding decoherence in a separated-arm atom interferometer, where a gas of atoms forms a refractive medium for a matter wave. Our work elucidates the actual microscopic, many-body, quantum-mechanical scattering mechanism that gives rise to prior phenomenological results for the phase shift and decoherence.

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I. INTRODUCTION

Can a propagating atom run the gauntlet through a gas of free atoms, interacting with all of them at long range, and still remain coherent, leaving the quantum state of every gas atom unchanged? The answer is yes, much of the time, depending on gas density, propagation distance, atom-atom collision cross sections, etc. (the answer is no, however, if the force is Coulombic). If this were not the case, the measurement in [1] of the refractive index of a gas of atoms for an atomic matter wave would not have worked. These measurements took place in a separated-arm atom interferometer, where one arm intersected a gas cell, filled with other atoms.

The separated-arm atom interferometer exploits superposition by splitting the spatial wave function of an atom into two wave packets that can be made to travel along separate paths and experience different interactions. It takes advantage precisely of the ability of quantum systems to exist in superposition states. Such an apparatus is, consequently, a highly sensitive detector of decoherence. In the language of decoherence theory, the atom passing through the interferometer is the system, and the free gas it interacts with is the "environment." "Leaving a trace of passing" in the gas is a which-way detection that causes decoherence and loss of fringe contrast when the arms of the interferometer are recombined. Any collision that had disturbed the state of an atom in the gas cell would have been a which-way measurement that reduced the interference fringe contrast of the interferometer.

It is not correct to attribute, as in [1], the residual coherence in the interference signal to near-forward scattering. Our intention is to describe the actual mechanism of the microscopic theory that determines the residual coherence of a matter wave undergoing scattering from a free gas. As collisions produce decoherence, we expect that the coherent part of the propagating wave should be determined using known atom-atom elastic quantum cross sections, computing the chance of avoiding a collision in the usual way. However, this leaves another question unanswered: if there is a large survival rate, avoiding any collisions, can the phase shift acquired by the coherent atom wavefunction be large compared to π?

It is well known that matter can act as a coherent, refracting medium for matter waves, as, for example, in the propagation of neutrons through condensed matter. In passing through a solid, neutrons may acquire large phase shifts relative to the vacuum and emerge coherently; to wit, consider neutron diffraction from a crystal; the elastic diffractive spots are *prima facie* evidence of coherent scattering from the crystal. However solids are rather rigid compared to a low-density gas, and it is therefore surprising perhaps that atoms passing through gaseous matter can also acquire large phase shifts without leaving a trace of their passing, since gas atoms are so easily perturbed. Our analysis is in the context of a separated arm atom interferometer, with one arm intersecting a cell containing a fixed density of gaseous atoms (Fig. 1). As we will see, a low-density gas is completely intolerant of any momentum transfer; momentum transfer will always lead to decoherence and reduction of interference fringe contrast.

The interference fringe contrast is defined precisely in [2] as $C=(I_{\max}-I_{\min})/(I_{\max}+I_{\min})$, where I, the count rate in their detector, is proportional to the probability density of detecting a projectile at a particular position on the screen in

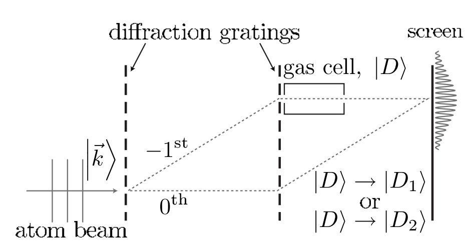

FIG. 1. A Mach-Zender interferometer with the gas cell serving as a which-way detector. The atom beam is coherently split into the two arms of the interferometer by the leftmost diffraction grating. The initial state of the gas is $|D\rangle$, which evolves into $|D_1\rangle$ or $|D_2\rangle$ depending on whether an atom from the beam passes through the cell. An interference pattern forms on the screen at right where the arms overlap.

*ssanders@post.harvard.edu

Fig. 1. The contrast of the interference pattern is reduced due to correlation with the environment [3]. This point of view is equivalent to a which-way detection, in which the state of the environment records partial or complete path information.

The relative phase acquired along the different paths of the interferometer is observed as a shift in the interference pattern that forms when the wave packets recombine [4,5]. Previous experiments have measured the interference pattern due to the presence of a free gas in only one arm of such an interferometer [1]. The other arm was physically separated from the gas, and did not interact with it. The experiments showed that, like light passing through glass, a matter wave passing through a dilute gas experiences a phase shift, with the dilute gas acting as a medium with an index of refraction for matter waves.

When produced by propagation through a free gas, the phase shift of the interference fringes is a probe of the atom-atom interactions, and was the focus of much theoretical work [1,6,7]. These treatments build upon the multiple-scattering theory derived in [8], and they neglect the possibility of recoil of the background atoms. Only the projectile is treated explicitly quantum-mechanically. The background gas creates a background potential, and decoherence is obtained phenomenologically by averaging the resulting scattered projectile wave function over different realizations of the potential. Our interest is to show that the fundamental source of decoherence in this system does not require an *ad hoc* averaging process. In order to properly understand the decoherence, however, we must take a substantial step beyond the case of a single-particle scattering from a distribution of potential centers. It is imperative that we address the full many-body scattering problem, in which the background gas possesses a quantum-mechanical state that is affected by scattering interactions with the projectile. This is critical because, in the absence of recoiling target particles, there would be no decoherence whatsoever that emerges naturally from the scattering theory. By eliminating the recoil of the targets, there remains no possibility of the gas recording the passage of the projectile and no which-way measurement.

Experiments have also been performed to measure the amount of decoherence experienced by an atom due to the scattering of photons from a laser [9,10] and to the scattering of atoms in a free background gas [2,11]. The decoherence is observed as a loss of contrast in the interference patterns formed.

The theoretical foundation of the analysis used to understand these experiments postulates that scattering events can be described as an instantaneous modification of the system-environment density matrix, $\rho_i \rightarrow \rho_f = T\rho_i T^\dagger$ [12]. The changes to the density matrix due to these scattering events may be explicitly added to the usual Heisenberg equation of motion. The additional term gives rise to decoherence of the system when the degrees of freedom of the environment are traced over.

The physical mechanism by which this process occurs, however, remains hidden in the *ad hoc* addition to the purely coherent evolution of the density matrix. The explanation of this process in the literature is incomplete because the decoherence phenomenon does not emerge directly as an outcome of the microscopic scattering process. It is our intention to make this connection explicit here, in order to lay a foundation for prior theoretical work.

The effect on a quantum particle due to a gas environment, treated as a Markovian reservoir in which only two-body scattering is considered, has been treated in a very general way by [13]. Our objective is rather different. We wish to show the origin of the phase shift on an atom wave function due to scattering from other atoms. We also seek an explanation of the surprising lack of sensitivity of a free particle as a which-way detector based on microscopic multiple scattering theory. We will, therefore, calculate the reduction in interference fringe contrast due to the presence of a free gas interacting with only one arm of a separated-arm atom interferometer. Our derivation shows how these processes emerge directly from microscopic quantum mechanical scattering and avoids the *ad hoc* modification of the Heisenberg equation of motion and the introduction of an average wave function. In fact, the coherent wave introduced in [8] *emerges* directly from our calculations, providing a justification for its use and bridging the gap in the literature between phenomenological results and the microscopic theory.

II. THEORETICAL PERSPECTIVE

Standard scattering theory suggests a naive argument that little or no coherence should remain after an atom passes through a column of gas. The usual expression for scattering in free space (in the center of mass frame) gives the wave function for the scattered atom as [14]

$$\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\left(e^{i\vec{k}\cdot\vec{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}\right). \tag{1}$$

The first term on the right-hand side of this equation is the unscattered incident wave, which preserves coherence but has no phase shift. A phase-shifted, coherent contribution cannot arise from this term. The second term on the right-hand side is the scattered wave, which corresponds to a momentum-conserving recoil of the target gas atom, except when scattering into the exactly forward direction. Scattering into the infinitesimal solid angle around the forward direction occurs with zero probability. Any finite recoil changes the state of a free target gas atom and constitutes a which-way measurement that ought to eliminate the possibility of observing interference. Only an infinitesimal fraction of the incident beam would interact with the free gas atom and remain coherent. The rest either is not scattered at all or decoheres completely.

Nonetheless, the experimental results [1] clearly demonstrate that atoms in the beam do interact with the background gas coherently because the phase shift that results from the interaction is observable in the interference pattern. The beam atoms are able to "scatter" off of the free gas atoms and acquire a phase shift, without touching the free gas and changing its quantum state at all.

A better approach to understanding the phase shift and the decoherence is to enclose the target gas in a box, confining it in three dimensions. We may then treat the interactions between a projectile and a gas of confined particles. The pro-

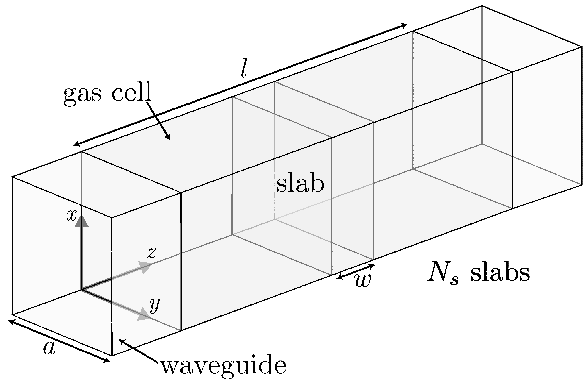

FIG. 2. Details of the gas cell appearing in Fig. 1. The cell has length l and is embedded in a waveguide. The jth slab, running from $z=(j-1)w$ to $z=jw$, is illustrated. The waveguide in which the projectile is confined is an infinite tube with a square cross section of dimensions $a\times a$.

jectile can be assumed to be unaffected by the walls of the box through which it passes, as we will eventually remove this artifice. The benefit of the box is immediate—there can be a finite amplitude associated with exactly "forward" scattering, in which the quantum state of the projectile and the target are unaffected by the interaction. A key point is that *this coherent amplitude automatically comes with a nonvanishing phase shift*. The argument of the complex amplitude gives rise to a phase shift, and its magnitude squared gives the probability of not disturbing the environment in any way, and thus leaving the system coherent. This result differs from free space scattering because there will in general be a finite flux of system atoms that acquire a phase shift and remain coherent. The coherent phase shift due to a single target atom will approach zero as the cross sectional area of the confining box is enlarged. This recalls the conundrum of (1); however, the phase shift does not vanish, even as the box is enlarged, if the column density of the gas remains constant. It is a crucial task here to consider this limit carefully.

We will solve the problem of scattering of a beam atom, the "system," or projectile, from a gas of atoms, the "environment." The latter are confined to a three-dimensional box. The beam atom itself will be confined to a waveguide that overlaps the gas cell (Fig. 2). In this way, the transverse modes of the beam atom eigenstates are discrete, as are the modes in all three directions of the gas atoms. We will assume that the beam atom does not feel the confining wall that defines the length of the gas cell. We can then study the interactions that lead to phase shifts of the beam atom without changing the discrete state of the gas atoms. It is precisely this recoilless interaction that gives rise to the coherent wave.

The imposition of a cell and a waveguide are reasonable in the context of the experiments [1], where the gas was in fact confined to a cell. The cell was macroscopically large, however, so our results must not depend on the size of our cell. The relevant experimental parameter is the column density of the gas. When we consider the limit of large cell dimensions, we will choose the number of gas atoms correspondingly, so that the column density remains fixed. We will find that our results are independent of the dimensions of the waveguide and gas cell and only depend on the column density. In the limit where the cross section of the waveguide is very large, our results explain the coherent interactions in free space that cause a phase shift on the atom beam, while leaving the background atoms completely untouched.

III. MULTIPLE SCATTERING DUE TO A FREE GAS

In a dilute gas, any scattering event which leads to recoil of a target atom, placing it in an orthogonal state, leads also to complete decoherence of the two-arm projectile density matrix. The orthogonal target atom state constitutes which-way evidence and coherence cannot persist. To calculate the total decoherence, we need only calculate the amplitude of the many-particle state that remains unchanged by the interactions, other than the acquisition of a phase shift. Over short enough distances traveled by the projectile, we may neglect multiple scattering altogether because the gas is dilute. If the projectile survives the interactions over a short distance by remaining in the initial state, then it is able to continue its journey toward the detector and scatter downstream. The projectile can have many sequential interactions with the gas atoms, so long as it remains in its initial state after each scattering event. In this way, it can accumulate a potentially large phase shift, even if the phase accumulated by a single scattering event is small. After passing through the entire cell, the amplitude of the initial state, which is coherent with the other arm of the interferometer, will also have been reduced due to scattering out of it.

Figure 1 shows the experimental configuration we are considering. The projectile passes coherently through the upper and lower arms of the interferometer. A low pressure gas is present in the upper arm.

We model the upper arm as an overlapping waveguide and gas cell (Fig. 2). We discretize the transverse states of the projectile atom by requiring that its wave function satisfy periodic boundary conditions on the surface of the wave guide. Similarly the states of the N gas atoms are discretized by requiring that they satisfy periodic boundary conditions on all the surfaces of the gas cell. The projectile and target gas atoms are otherwise free. The Hamiltonian describing this $(N+1)$-particle system, in the absence of interactions, is H_0, with eigenstates $|k,\vec{n}\rangle$. The components of $\vec{n}$ are the $3N+2$ discrete quantum numbers describing the transverse state of the projectile and the states of the N target atoms. k is the initial longitudinal wave number of the projectile.

For a dilute gas we neglect interactions between target atoms. The interaction potential between the projectile and the targets is a sum of binary terms. The projectile is labeled as the 0th particle, and the targets will be labeled 1 through N. The full interaction potential V is then

$$V=\sum_{i=1}^{N} V_{0i}. \quad (2)$$

V_{0i} gives the potential between the projectile and the ith target, and the full Hamiltonian is $H=H_0+V$. We will take the projectile to be initially in an eigenstate of the waveguide. Conservation of energy and momentum requires that if a target remains in its initial state, then so must the projectile.

The S matrix connects the initial many-body state $|k,\vec{n}_0\rangle$ with the asymptotic output channel $|\psi\rangle$ [15]

$$|\psi\rangle = S|k,\vec{n}_0\rangle. \tag{3}$$

$|\psi\rangle$ is the many-body state that emerges after interactions between the projectile and the gas are complete. We will refer to the diagonal element of the S matrix that gives the $|k,\vec{n}_0\rangle$ component of $|\psi\rangle$ as $S_{0,0}$,

$$|\psi\rangle = S_{0,0}|k,\vec{n}_0\rangle + (\text{orthogonal terms}). \tag{4}$$

The first term on the right-hand side of (4) is the only part of $|\psi\rangle$ that interferes with the other arm of the interferometer. The probability of finding the system plus environment in this state is the probability that the system will remain coherent and interfere with itself. The contrast of the interference fringes will be reduced by the factor $|S_{0,0}|$ [3]. In order to calculate the amplitude of the coherent state after interactions with the gas, we need to calculate the $S_{0,0}$ matrix element. This task is facilitated by subdividing the gas cell into thin slabs, and computing the contributions to $S_{0,0}$ from each slab.

Thin slab construction

The volume of the gas cell can be thought of as the composition of many adjacent, thin slabs, which are the regions of space formed by the surface of the waveguide and two of its cross sections, placed a distance w apart, as in Fig. 2. Imagine subdividing the gas cell into N_s such regions, so that $l = N_s w$. If we number the slabs, $j=1,2,3,\ldots$, beginning from the point of entry of the projectile into the gas cell, then slab j has the width and height of the waveguide, and runs from $z=(j-1)w$ to $z=jw$.

The total interaction potential can be rewritten in terms of the contribution from each slab,

$$V = \sum_{j=1}^{N_s} V^{(j)} = \sum_{j=1}^{N_s}\sum_{i=1}^{N} V_{0i}^{(j)}, \tag{5}$$

$$V_{0i}^{(j)} = V_{0i}\,\theta(\hat{z}_i - (j-1)w)\,\theta(jw - \hat{z}_i). \tag{6}$$

$\hat{z}_i$ is the z-position operator for the ith target atom. $V^{(j)}$ picks out the contribution to the total interaction potential due to a particular region of space. Summing over these contributions, we obtain the original interaction potential.

For each $V^{(j)}$, we will define a corresponding $S^{(j)}$, which is the S matrix due only to the interactions with the jth slab. Beginning with the first slab, we can compute the scattered state due only to that slab. If we then use that result as the incident state to the subsequent slab (again removing all other slabs), the state we will obtain after N_s such iterations is

$$|\psi^{(N_s)}\rangle = S^{(N_s)}S^{(N_s-1)}\cdots S^{(1)}|k,\vec{n}_0\rangle. \tag{7}$$

$|\psi^{(N_s)}\rangle$ is different from $|\psi\rangle$ in general because the wave function at earlier slabs is unaffected by subsequent slabs. This excludes the possibility that the projectile could backscatter but be recovered into the incident state by scattering a second time from an earlier slab; however, for a dilute gas this process is negligible, so we may safely approximate $|\psi^{(N_s)}\rangle \approx |\psi\rangle$.

The decoherence, which causes the interference fringe contrast to be reduced, is due to the reduced amplitude of the initial many-body state. The phase shift that is measured as a spatial shift in the observed interference fringes is given by the argument of that amplitude. Equivalently, the magnitude of the overlap of the final states of the free gas associated with each arm gives the decoherence and the argument of the overlap gives the phase shift. We denote the state of the many-body system by a single subscript, so that $|\phi_i\rangle = |k',\vec{n}\rangle$. The initial state is $|\phi_0\rangle = |k,\vec{n}_0\rangle$. The $|\phi_0\rangle$ component of the scattered state after interactions with the gas is given by

$$S_{0,0}|\phi_0\rangle \approx S_{0,i_{n-1}}^{(N_s)} S_{i_{n-1},i_{n-2}}^{(N_s-1)} \cdots S_{i_1,0}^{(1)}|\phi_0\rangle. \tag{8}$$

Repeated indices are implicitly summed over. The physical process that corresponds to each set of indices is scattering $|\phi_0\rangle \to |\phi_{i_1}\rangle \to |\phi_{i_2}\rangle \to \cdots \to |\phi_{i_{n-1}}\rangle \to |\phi_0\rangle$. When any of these intermediate states is not $|\phi_0\rangle$, we have argued that the projectile totally decoheres, so the contribution of these terms to the final coherent state amplitude can be neglected:

$$S_{0,0}|\phi_0\rangle \approx S_{0,0}^{(N_s)} S_{0,0}^{(N_s-1)} \cdots S_{0,0}^{(1)}|\phi_0\rangle. \tag{9}$$

The physical interpretation of this expression is that the probability amplitude for remaining in the initial state is reduced by each slab. Only this amplitude interferes with the other arm of the interferometer. $|S_{0,0}|^2$ is the probability that an atom in the beam will interfere with itself. The remaining fraction of the atomic beam contributes only to an incoherent background. The net result is that the interference fringe contrast is reduced by the factor $|S_{0,0}|$. The shift of the interference fringes compared to a vacuum is given by the argument of $S_{0,0}$.

We will now calculate these quantities by first calculating the $S_{0,0}^{(j)}$ matrix element due to scattering from a single slab. Then we may take the product in (9) to obtain $S_{0,0}$.

IV. CALCULATION OF THE S-MATRIX

The S matrix is the time evolution operator that takes a quantum state from the distant past, prior to a collision, into the distant future, after the collision; that is, $S = \lim_{t\to\infty} U(t,-t)$. It can be expressed in terms of the scattering matrix T as [16]

$$S = 1 - 2\pi i\,\delta(E - H_0)T, \tag{10}$$

where T is defined [17] by

$$T = V + \lim_{\epsilon\to 0} V\frac{1}{E - H_0 + i\epsilon}T. \tag{11}$$

The limit will not appear in what follows; it is understood that we must take the small ϵ limit. V and H_0 are the $(N+1)$-particle operators defined above. $S^{(j)}$ is the S matrix due to the potential $V^{(j)}$. Replacing V with $V^{(j)}$ in the definition of T gives $T^{(j)}$. We calculate $S^{(j)}_{0,0}$ by expanding $S^{(j)}|k,\vec{n}_0\rangle$ to find the coefficient on its $|k,\vec{n}_0\rangle$ component,

$$S^{(j)}|k,\vec{n}_0\rangle = |k,\vec{n}_0\rangle - 2\pi i\,\delta(E-H_0)T^{(j)}|k,\vec{n}_0\rangle. \tag{12}$$

In order to extract the $|k,\vec{n}_0\rangle$ component of the second term in (12), we insert a complete set of eigenstates of H_0 between the δ function and $T^{(j)}$,

$$\delta(E-H_0)T^{(j)}|k,\vec{n}_0\rangle = \int dk' \sum_{\vec{n}} \delta(E-E_{k',\vec{n}})|k',\vec{n}\rangle \times\langle k',\vec{n}|T^{(j)}|k,\vec{n}_0\rangle. \tag{13}$$

The terms of the sum with $\vec{n}\neq\vec{n}_0$ are orthogonal to $|k,\vec{n}_0\rangle$. They do not contribute to $S^{(j)}_{0,0}$. It is only necessary to consider the term $\vec{n}=\vec{n}_0$. There, the argument of the δ function is considerably simplified due to the cancellation of the energy contribution of the discrete quantum numbers. In that case, $E-E_{k',\vec{n}_0}=\frac{\hbar^2k^2}{2m}-\frac{\hbar^2k'^2}{2m}$, where m is the mass of the projectile. The integral over k' can then be performed easily to find that the coefficient on the $|k,\vec{n}_0\rangle$ component of $S^{(j)}|k,\vec{n}_0\rangle$ is

$$S^{(j)}_{0,0} = 1 - i\frac{2\pi m}{\hbar^2 k}\langle k,\vec{n}_0|T^{(j)}|k,\vec{n}_0\rangle. \tag{14}$$

The net effect of the gas on the amplitude of the initial state is obtained according to (9) as the product of the individual slab results,

$$S_{0,0} \approx \prod_{j=1}^{N_s}\left(1 - i\frac{2\pi m}{\hbar^2 k}\langle k,\vec{n}_0|T^{(j)}|k,\vec{n}_0\rangle\right). \tag{15}$$

$T^{(j)}$ is the full scattering matrix due to a single slab, including multiple scattering within the slab. In a dilute gas, sequential scattering from different targets is unlikely within a slab that is much thinner than the length of the gas cell. Neglecting multiple scattering *within* single slabs, the $(N+1)$-particle matrix element of $T^{(j)}$ reduces to a sum of two-particle matrix elements (see Appendix A),

$$\langle k,\vec{n}_0|T^{(j)}|k,\vec{n}_0\rangle \approx \sum_{i=1}^{N}\langle\chi,\varphi_i|t^{(j)}_{0i}|\chi,\varphi_i\rangle, \tag{16}$$

where χ designates the initial state of the projectile, φ_i designates the initial state of the ith particle, and $t^{(j)}_{0i}$ is the scattering matrix for the 0th and ith particles without any other atoms present,

$$t^{(j)}_{0i} = V^{(j)}_{0i} + V^{(j)}_{0i}\frac{1}{E_0+E_i-H_0-H_i+i\epsilon}t^{(j)}_{0i}. \tag{17}$$

The expression for $S_{0,0}$, excluding multiple scattering within individual slabs, is

$$S_{0,0} \approx \prod_{j=1}^{N_s}\left(1 - i\frac{2\pi m}{\hbar^2 k}\sum_{i=1}^{N}\langle\chi,\varphi_i|t^{(j)}_{0i}|\chi,\varphi_i\rangle\right). \tag{18}$$

This result gives the complex probability amplitude for the component of the projectile that remains coherent after interactions with the gas. We have explicitly taken into account multiple scattering. It remains to examine the limit in which the dimensions of the gas cell and waveguide become arbitrarily large. This will allow us to remove the artificial confinement depicted in Fig. 2. We find that the result is independent of the confinement and that a solution of the coherent wave equation emerges directly from these considerations, without invoking the concept of an average wave function [8]. Even for an arbitrarily large cell, the projectile may remain partially coherent after scattering from a completely free gas. This resolves the conflict between the experimental results and our expectations based on the usual expression for scattering in free space.

The initial state of each target and the transverse states of the projectile appropriate to the waveguide and gas cell are box-normalized plane waves. Along the z direction, the projectile remains a free particle. $|\chi,\phi_i\rangle$ in (18) will be denoted using the wave vectors of the projectile and target as $|\vec{k}_0,\vec{k}_i\rangle$. The normalization of $|\vec{k}_0,\vec{k}_i\rangle$ reflects the free nature of the projectile along the longitudinal axis of the waveguide,

$$\langle\vec{r}_0,\vec{r}_i|\vec{k}_0,\vec{k}_i\rangle = \frac{e^{i\vec{k}_0\cdot\vec{r}_0}}{a\sqrt{2\pi}}\frac{e^{i\vec{k}_i\cdot\vec{r}_i}}{a\sqrt{l}}. \tag{19}$$

When we convert this expression to center-of-mass coordinates, we must allocate the normalization constants,

$$\langle\vec{r}_{0i},\vec{R}_{0i}|\vec{k}_{0i},\vec{K}_{0i}\rangle = \frac{e^{i\vec{K}_{0i}\cdot\vec{R}_{0i}}}{a\sqrt{2\pi}}\frac{e^{i\vec{k}_{0i}\cdot\vec{r}_{0i}}}{a\sqrt{l}}, \tag{20}$$

where $\vec{R}_{0i}$ is the center-of-mass coordinate and $\vec{r}_{0i}=\vec{r}_0-\vec{r}_i$ is the relative coordinate of the 0th and ith particles. The center-of-mass momentum $\vec{K}_{0i}=\vec{k}_0+\vec{k}_i$ is normalized to the waveguide, and the relative momentum $\vec{k}_{0i}=[m_i/(m+m_i)]\vec{k}_0-[m/(m+m_i)]\vec{k}_i$ is normalized to the dimensions of the gas cell. m is the mass of the projectile and m_i is the mass of the ith target.

The potential $V^{(j)}_{0i}$ depends only on the relative coordinates of the projectile and the ith target, with the exception that it vanishes if the target coordinates lie outside of the jth slab. When the range of the potential is much smaller than the width of the slab, this has the effect of limiting the domain of the matrix element $\langle\vec{k}_0,\vec{k}_i|V^{(j)}_{0i}|\vec{k}_0,\vec{k}_i\rangle$ to the jth slab. As such, only when both particles are in the slab is there a contribution to the matrix element. This requires that the center-of-mass coordinate must also be in the slab. In principle, the domain of the relative coordinate that contributes to the matrix element depends on the position of the center-of-mass coordinate relative to the slab boundaries, but for local potentials we may take the domain of the relative coordinate $\vec{r}_{0i}$ to be all space, and replace $t^{(j)}_{0i}$ with t_{0i}. t_{0i} is obtained by replacing $V^{(j)}_{0i}$ with V_{0i} in (17). Using these assumptions, we can rewrite

$$\langle\chi,\varphi_i|t^{(j)}_{0i}|\chi,\varphi_i\rangle \approx \langle\vec{k}_{0i}|t_{0i}|\vec{k}_{0i}\rangle\langle\vec{K}_{0i}|\vec{K}_{0i}\rangle. \tag{21}$$

Taken over a slab, $\langle\vec{K}_{0i}|\vec{K}_{0i}\rangle=w/2\pi=l/(2\pi N_s)$. Substituting this result into (18) gives

$$S_{0,0} \approx \prod_{j=1}^{N_s} \left(1 - i\frac{2\pi m}{\hbar^2 k} \sum_{i=1}^{N} \langle \vec{k}_{0i} | t_{0i} | \vec{k}_{0i} \rangle \frac{l}{2\pi N_s} \right). \tag{22}$$

The expression under the product sign in (22) does not depend on the slab index j. We can express the matrix element of t_{0i} in terms of the forward scattering amplitude in the center-of-mass frame, $f(k_{0i},0)$, of the 0th and ith particles [18] and rewrite the sum over the particles as N times the average,

$$S_{0,0} \approx \left(1 + i2\pi \frac{N}{a^2} \frac{1}{N_s} \left\langle \frac{f(k_{0i},0)}{\mu_{0i} k/m} \right\rangle \right)^{N_s}. \tag{23}$$

$N/a^2 = \rho l$ is the column density of the gas. We have explicitly written μ_{0i} to indicate the reduced mass for each combination of the projectile and a target. In the case of a target gas comprised of a single species of atom, we will write μ for the reduced mass. The situation in which the projectile velocity dominates the target velocities permits us to simplify, $k_{0i} \approx (\mu/m)k$ [7,19]. As the number of slabs becomes large, and the width of each slab becomes small compared to the length of the gas cell, $S_{0,0}$ approaches

$$S_{0,0} \approx \exp\left(i2\pi\rho l \left\langle \frac{f(k_{0i},0)}{k_{0i}} \right\rangle \right). \tag{24}$$

We may take the dimensions of the waveguide and gas cell to be arbitrarily large under the condition that we also choose the number of target atoms so that the column density remains fixed. Equation (24) is valid in this free-space limit, and gives precisely a solution of the coherent wave equation when we take the incident projectile wave function to be a plane wave.

This central result accounts completely for the phase shift and persistence of coherence after multiple scattering with a dilute, many-body quantum-mechanical target of free particles. The probability of remaining in the coherent state decays as $|S_{0,0}|^2 = e^{-\rho\sigma l}$, where σ is the average quantum-mechanical scattering cross section,

$$\sigma = \left\langle \frac{4\pi}{k_{0i}} \mathrm{Im}[f(k_{0i},0)] \right\rangle. \tag{25}$$

The cross section is proportional to the imaginary part of the forward scattering amplitude, whereas the phase shift of the interference fringes is proportional to the real part,

$$\Delta\phi = \left\langle \frac{2\pi}{k_{0i}} \rho l \, \mathrm{Re}[f(k_{0i},0)] \right\rangle. \tag{26}$$

The ratio of the real and imaginary parts of the forward scattering amplitude, which is directly measured in interferometric measurements such as [1], characterizes the extent that the interference pattern can be shifted before it is washed out due to decoherence. For weak interactions—the typical situation in interferometry—the real part of the forward scattering amplitude is proportional to the interaction potential V, whereas the imaginary part is second order in V (see Appendix B). Consequently, the phase shift acquired by the projectile can be made large by increasing the column density of the gas, while the loss of contrast,

$$1 - |S_{0,0}| \approx 2\pi\rho l \, \mathrm{Im}\left(\left\langle \frac{f(k_{0i},0)}{k_{0i}} \right\rangle \right) \propto V^2, \tag{27}$$

remains smaller by a factor of the interaction strength. The difference in the dependence on the interaction strength clarifies the ability of a seemingly sensitive, free gas to generate a large phase shift on a projectile wave function due to scattering. This occurs essentially without loss of contrast if the target gas is sufficiently dilute and weakly interacting with the projectile.

V. LOW-ENERGY PROJECTILE: PSEUDOPOTENTIAL

It is well known that a collection of potential centers, which are assumed to form a uniform medium in a thin slab, can give rise to an index of refraction for matter waves [20]. Furthermore, [8] has shown that a finite collection of scattering centers can, when the scattered waves are appropriately averaged, act as a medium. The scattered wave is the so-called coherent wave, which suffers attenuation due to the averaging process. We have shown here that even a finite collection of recoiling quantum-mechanical particles in free space can act as a refractive medium. In addition, decoherence is a natural consequence of entanglement with the target particles.

In order to illustrate the broader context of our results, it is instructive to compare the phase shift we obtain for a special case of the interaction potential with the well-known results of pure potential scattering. When the projectile is moving slowly relative to the target atoms, only s-wave scattering needs to be considered, and we can model the interaction as a contact potential,

$$V_{0i} = V_0 \delta(\vec{r}_0 - \vec{r}_i). \tag{28}$$

Recall that the coefficient on the coherent state after interactions is given by

$$S_{0,0} \approx \prod_{j=1}^{N_s} \left(1 - 2\pi i \frac{m}{\hbar^2 k} \sum_{i=1}^{N} \langle \chi, \phi_i | T_{0i}^{(j)} | \chi, \phi_i \rangle \right). \tag{29}$$

For weak potentials, we may approximate $T_{0i}^{(j)}$ to first order in a Dyson series expansion as

$$T_{0i}^{(j)} \approx V_{0i}^{(j)}. \tag{30}$$

The matrix element $\langle \chi, \phi_i | V_{0i}^{(j)} | \chi, \phi_i \rangle$ is readily computed using box-normalized plane waves as before,

$$\langle \chi, \phi_i | V_{0i}^{(j)} | \chi, \phi_i \rangle = \frac{V_0}{2\pi a^2} \frac{1}{N_s}. \tag{31}$$

Substituting (31) into (29) gives

$$S_{0,0} \approx \exp\left[i\frac{2\pi}{k} \left(\frac{-mV_0}{2\pi\hbar^2} \right) \rho l \right]. \tag{32}$$

$S_{0,0}$ is a pure phase factor in this approximation. The gas acts as a medium with an index of refraction for the projectile matter wave, producing a phase shift $\phi = (2\pi/k)(-mV_0/2\pi\hbar^2)\rho l$.

A direct calculation [20] that ignores the quantum state of the gas atoms and treats them as potential centers leads to the phase shift

$$\phi = -\frac{2\pi}{k} a_0 \rho l, \tag{33}$$

where the scattering length a_0 can be determined from the solution to the δ potential scattering problem [21],

$$a_0 = \frac{mV_0/(2\pi\hbar^2)}{1 + ikmV_0/(2\pi\hbar^2)} \approx mV_0/(2\pi\hbar^2). \tag{34}$$

The approximation of a_0 is valid under the same conditions as our expansion of the T matrix. The first-order term in (29) gives precisely the result for the phase shift that is obtained due to potential scattering. If we were to keep terms up to second order in the expansion of $T_{0i}^{(j)}$, $S_{0,0}$ would also reduce the amplitude of the coherent state, giving rise to decoherence.

VI. CONCLUSIONS

We have demonstrated the correct mechanism by which a matter wave is coherently refracted by scattering interactions with a free gas. This work remedies the inconsistency in the literature due to the interpretation of the coherent part of the wave as the forward scattered fraction. Explicit treatment of the many-body quantum state of the targets, which has been ignored or treated phenomenologically in the previous literature, connects the microscopic scattering theory to the macroscopic decoherence. This is a significant extension of prior multiple scattering treatments, and it is absolutely necessary to obtain decoherence from the fundamental theory.

It was shown that the part of the projectile which does not become entangled with the target gas can be extracted by analyzing the scattering in a waveguide. The waveguide admits discrete scattering channels, and the complex amplitude of the coherent channel, in which the state of all the target atoms is completely unchanged, gives the attenuation and phase shift of the coherent beam. The finite probability that the atom may scatter from a free gas particle, but not change the state of the gas particle at all, is the quantum-mechanical scattering cross section. Nevertheless, the projectile may interact strongly enough with the target to acquire a measurable phase shift due to a recoiless interaction. The residual coherence and phase shift are independent of the dimensions of the waveguide, which permits the free-space limit of an arbitrarily large waveguide. The quantitative results for the phase shift and decoherence properly depend on the column density of the free gas but not on the dimensions of the confinement.

The results we have obtained for these experimental conditions are consistent with the potential scattering of [8], revealing the physical origin of that phenomenological treatment. Our work provides a crucial link, which has been unaddressed in the literature, between the microscopic scattering theory and reduction in fringe contrast of the interference pattern due to decoherence.

The projectile must avoid the cross sections of all the gas atoms in order to remain coherent after passing through the gas sample. We have shown above that it is possible for a projectile to do so and still acquire a large phase shift due to the weaker dependence of the cross section, $O(V^2)$, than the phase shift, $O(V)$, on the strength of the interaction potential. Physical insight into the dominance of the phase shift over the scattering cross section in this regime can be had by realizing, as suggested by [1], that it is small impact parameters that contribute to the cross section, and large impact parameters that contribute to the phase shift. It is precisely at large impact parameters, when scattering is avoided, that the phase shift varies as δ_L, the Lth partial wave phase shift, and the cross section varies as δ_L^2. In this region, δ_L is small, and the phase accumulates much faster than scattering occurs. A projectile that is passing through a dilute gas will interact at long range with the targets and is operating in this regime.

The requirement that a projectile evade the scattering cross sections of the targets as it skirts its way through the free gas becomes an impossibility for interaction potentials which have long-range forces. A particularly common example is the Coulomb potential, for which the total scattering cross section diverges. Such an interaction potential between the coherent projectile and a target comprised of free particles should completely suppress interference fringes, even for a very low-density target gas.

The techniques that we have developed to calculate the effect of a free gas as a which-way detector explain the surprising robustness of the spatial coherence in an atom interferometer to interactions with free particles. The calculations we have done also lay the groundwork for future investigations into the impact of other, many-body atomic systems on a coherent atom, due to scattering interactions.

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APPENDIX A: APPROXIMATION OF THE SCATTERING T MATRIX

We wish to exclude multiple scattering from the initial state diagonal matrix element of an $(N+1)$-particle scattering matrix T,

$$\langle T\rangle = \langle \chi, \phi_1, \ldots, \phi_N | T | \chi, \phi_1, \ldots, \phi_N\rangle. \tag{A1}$$

The Hamiltonian is a sum of operators acting only on the Hilbert spaces of the indicated particles,

$$H = H_0 + V = (H_0 + \cdots + H_N) + (V_{0i} + \cdots + V_{0N}). \tag{A2}$$

Recall that the definition [17] of the corresponding T matrix is

$$T = V + VG_0T, \tag{A3}$$

$$G_0 = \lim_{\epsilon\to 0} \frac{1}{E - H_0 + i\epsilon}. \quad (A4)$$

We will introduce the operators $T_{01}, \ldots, T_{0N}$ that satisfy

$$T = \sum_{i=1}^{N} T_{0i}, \quad (A5)$$

$$T_{0i} = V_{0i} + V_{0i}G_0T_{0i} + \sum_{j\neq i} V_{0i}G_0T_{0j}. \quad (A6)$$

It is only the third term on the right-hand side of (A6) that contributes to multiple scattering. The expression for T_{0i} that excludes multiple scattering is

$$T_{0i} \approx V_{0i} + V_{0i}G_0T_{0i}. \quad (A7)$$

This approximation of T_{0i} differs from the definition of the two-particle scattering matrix, t_{0i}, by the replacement of the $(N+1)$-particle operator G_0, with a two-particle operator g_{0i}:

$$t_{0i} = V_{0i} + V_{0i}g_{0i}t_{0i}, \quad (A8)$$

$$g_{0i} = \lim_{\epsilon\to 0} \frac{1}{(E_0+E_i) - (H_0+H_i) + i\epsilon}. \quad (A9)$$

Consider $i=1$, and note that

$$\langle \phi_2, \ldots, \phi_N | T_{01} | \phi_2, \ldots, \phi_N \rangle \approx V_{01} + V_{01}g_{01}\langle \phi_2, \ldots, \phi_N | T_{01} \times | \phi_2, \ldots, \phi_N \rangle. \quad (A10)$$

We have used

$$\langle \phi_2, \ldots, \phi_N | G_0 = \langle \phi_2, \ldots, \phi_N | g_{01}. \quad (A11)$$

(A10) is identical to (A8), so when multiple scattering is ignored we can identify

$$t_{01} \approx \langle \phi_2, \ldots, \phi_N | T_{01} | \phi_2, \ldots, \phi_N \rangle. \quad (A12)$$

This result is the same for any i. Summing the contributions due to each $\langle T_{0i}\rangle$ gives the approximation we desired,

$$\langle T \rangle \approx \sum_{i=1}^{N} \langle \chi, \phi_i | t_{0i} | \chi, \phi_i \rangle. \quad (A13)$$

APPENDIX B: EXPANSION OF THE SCATTERING AMPLITUDE

We seek the dependence of the real and imaginary parts of the forward scattering amplitude, $f(k_{0i},0)$, on the interaction potential V_{0i} between the projectile and the ith target. This may be accomplished by relating the scattering amplitude to the two-body scattering matrix t_{0i}, defined in (A8) [18],

$$\langle \vec{k}_{0i} | t_{0i} | \vec{k}_{0i} \rangle = \frac{-2\pi\hbar^2}{\mu a^2 l} f(k_{0i},0). \quad (B1)$$

$f(k_{0i},0)$ is the forward scattering amplitude in the center-of-mass frame of the projectile and target. μ is the reduced mass, a and l are the previously defined dimensions of the gas cell, and k_{0i} is the relative wave vector. Expanding t_{0i} in a Dyson series to second order gives

$$t_{0i} \approx V_{0i} + V_{0i}g_{0i}V_{0i}. \quad (B2)$$

Separating the real and imaginary parts of the two-body Green's function g_{0i} gives [22]

$$g_{0i} = \mathrm{P}\frac{1}{(E_0+E_i) - (H_0+H_i)} - i\pi\delta((E_0+E_i) - (H_0+H_i)). \quad (B3)$$

Substituting this result into (B2), we find that the real part of the two-body scattering matrix is first order in V_{0i}, whereas the imaginary part is second order,

$$\mathrm{Re}(t_{0i}) \approx V_{0i}, \quad (B4)$$

$$\mathrm{Im}(t_{0i}) \approx -\pi V_{0i}\delta((E_0+E_i) - (H_0+H_i))V_{0i}. \quad (B5)$$

Therefore, $\mathrm{Im}[f(k_{0i},0)]$ is a factor of the interaction potential smaller than $\mathrm{Re}[f(k_{0i},0)]$.

[1] J. Schmiedmayer, M. S. Chapman, C. R. Ekstrom, T. D. Hammond, S. Wehinger, and D. E. Pritchard, Phys. Rev. Lett. **74**, 1043 (1995).

[2] H. Uys, J. D. Perreault, and A. D. Cronin, Phys. Rev. Lett. **95**, 150403 (2005).

[3] S. M. Tan and D. F. Walls, Phys. Rev. A **47**, 4663 (1993).

[4] T. D. Hammond, M. S. Chapman, A. Lenef, J. Schmiedmayer, E. T. Smith, R. A. Rubenstein, D. A. Kokorowski, and D. E. Pritchard, Braz. J. Phys. **27**, 193 (1997).

[5] *Atom Interferometry*, edited by P. R. Berman (Academic Press, San Diego, 1997).

[6] R. C. Forrey, L. You, V. Kharchenko, and A. Dalgarno, Phys. Rev. A **54**, 2180 (1996).

[7] V. Kharchenko and A. Dalgarno, Phys. Rev. A **63**, 023615 (2001).

[8] M. Lax, Rev. Mod. Phys. **23**, 287 (1951).

[9] M. S. Chapman, T. D. Hammond, A. Lenef, J. Schmiedmayer, R. A. Rubenstein, E. Smith, and D. E. Pritchard, Phys. Rev. Lett. **75**, 3783 (1995).

[10] D. A. Kokorowski, A. D. Cronin, T. D. Roberts, and D. E. Pritchard, Phys. Rev. Lett. **86**, 2191 (2001).

[11] K. Hornberger, S. Uttenthaler, B. Brezger, L. Hackermüller, M. Arndt, and A. Zeilinger, Phys. Rev. Lett. **90**, 160401 (2003).

[12] M. Tegmark, Found. Phys. Lett. **6**, 571 (1993).

[13] K. Hornberger, Phys. Rev. Lett. **97**, 060601 (2006).

[14] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, MA, 1994), pp. 379–386.

[15] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley and Sons, New York, 1964), Chap. 3.

[16] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley and Sons, New York, 1964), Chap. 3, pp. 80–82.
[17] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, MA, 1994), p. 389.
[18] E. Merzbacher, *Quantum Mechanics* (John Wiley and Sons, New York, 1998), p. 524.
[19] R. C. Forrey, V. Kharchenko, and A. Dalgarno, J. Phys. B **35**, L261 (2002).
[20] E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950), pp. 201–202.
[21] K. Wódkiewicz, Phys. Rev. A **43**, 68 (1991).
[22] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley and Sons, New York, 1964), Chap. 3, p. 74.