An Analytical Model for Traffic Delays and the Dynamic User Equilibrium Problem

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In a transportation network, the dynamic user equilibrium is a model of traffic assignment in which each traveler noncooperatively seeks to minimize his travel time. To determine such equilibrium, it is important to evaluate the impact of congestion on a driver's travel time. This paper derives an analytical travel time function, based on the kinematic wave theory, and integrates it within a dynamic user equilibrium setting.

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Abstract

In urban transportation planning, it has become critical (1) to determine the travel time of a traveler and how it is affected by congestion, and (2) to understand how traffic distributes in a transportation network. In the first part of the paper, we derive an analytical function of travel time, based on the theory of kinematic waves. This travel time function integrates the traffic dynamics as well as the effects of shocks. Numerical examples demonstrate the quality of the analytical function, in comparison with simulated travel times. In the second part of the paper, we incorporate the travel time model within a dynamic user equilibrium (DUE) setting. We prove that the travel time function is continuous and strictly monotone under some conditions. We illustrate how the model applies to solve a large network assignment problem through a numerical example.

1 Introduction

Over the last 20 years, traffic congestion has grown dramatically, resulting in high environmental costs and productivity losses. According to the Texas Transportation Institute (2003) on traffic congestion in the United States, "the average delay for every person in the 75 urban areas studied climbed from 7 hours in 1982 to 26 hours in 2001. [...] The total congestion "invoice" for the 75 areas in 2001 came to $69.5 billion, which was the value of 3.5 billion hours of delay." All the proposed solutions (capacity increase, highway efficiency improvement, demand management) rely on an accurate prediction of traffic congestion. To this end, it has become critical: (1) to determine the travel time of a traveler and how it is affected by congestion, and (2) to understand how traffic distributes in a transportation network.

In this paper, we derive an analytical travel time function that integrates the traffic dynamics and the effects of shocks. Subsequently, we illustrate how this function can be employed to determine the routes that individual travelers take in a transportation network. In particular, we assume that each user in the system minimizes the time length of his or her own trip, depending on congestion, leading to a dynamic user equilibrium (DUE) in the transportation network.

Analysis of traffic flow can be microscopic or macroscopic. Microscopic models focus on the behavior of a single vehicle, reacting to other vehicles' behavior, and adopt a simulation approach (e.g., see Herman et al. 1959 and Mahut 2000). Macroscopic models rely on the aggregate behavior of vehicles, depending on surrounding aggregate traffic conditions. Most macroscopic models are based on the theory of kinematic waves in transportation, developed by Lighthill and Whitham (1955) and Richards (1956). In this paper, we derive a function of congestion delays from the theory of kinematic waves.

Over the past decade, most developments based on the theory of kinematic waves have focussed on modeling flow propagation rather than deriving an analytical travel time function. Newell (1993) modeled flow propagation by using curves of the cumulative number of vehicles at the road entrance and at the road exit. Travel times correspond to the horizontal difference between these two curves. Daganzo (1994, 1995a) proposed an efficient way of propagating flow, by splitting the road into cells. See also Velan (2000) for enhancements of this model. His model, called the Cell Transmission Model, is not only very efficient for simulating the transportation network but can also be formulated as a linear optimization problem (Ziliaskopoulos 2000). While the Cell Transmission Model relies on a triangular flow-density curve, other simulation models rely on a quadratic flow-density curve (see Daganzo 1995b, and Khoo et al. 2002). In contrast, very few models have derived directly an analytical travel time function. Based on the theory of kinematic waves, Perakis (2000) and Kachani and Perakis (2001) proposed polynomial and exponential travel time functions.

On the other hand, most of the DUE models rely on an analytical function for travel time. This allows studying the convergence behavior of algorithms (e.g., see Friesz et al. 1993, Ran and Boyce 1994). However, it is unclear which analytical travel time function should be used. The most generic travel time function depends on the number of cars, the inflow and the outflow (see Ran and Boyce 1994). Although Daganzo (1995c) claimed that a travel time function should depend only on the number of cars on the road, Lin and Lo (2000) pointed a paradoxical situation with such a function. On the other hand, Carey et al. (2003) introduced a generic travel time function depending on the average flow rate, satisfying FIFO and other desirable properties. Based on the simplified kinematic wave model proposed by Newell (1993), Kuwahara and Akamatsu (2001) derived an analytical function of the instantaneous travel time in order to solve the dynamic user equilibrium. However, the travel time they derived does not represent the actual (or experienced) travel time, unless traffic conditions remain constant.

In this paper, we extend the work by Perakis (2000) and Kachani and Perakis (2001). In particular, we propose a methodology for deriving a polynomial travel time function on a single stretch of road, based on the theory of kinematic waves. This travel time model only depends on the traffic conditions at the entrance and at the exit of the road and is therefore less memory-intensive than the Cell Transmission Model. Furthermore, we complement the work by Kuwahara and Akamatsu (2001) by analyzing the dynamic user equilibrium with experienced instead of instantaneous travel times. The main contributions of our model are the following:

1. We develop a methodology for determining travel times analytically that applies to both triangular and quadratic flow-density curves.

2. Our model establishes the connection between the theory of kinematic waves and the theory of deterministic queues (see Newell 1982).

3. We introduce a travel time function that integrates the first-order traffic dynamics, shocks and queue spillovers.

4. From our numerical experiments, it appears that the travel time function is consistent with the results obtained by simulation.

5. We establish that the travel time function satisfies properties such as continuity, monotonicity and FIFO.

6. We incorporate the travel time model within a dynamic user equilibrium (DUE) model.

The paper is organized as follows. In Section 2, we review the theory of kinematic waves. In Section 3, we propose a general methodology for deriving a travel time function, and illustrate it within two particular models of flow-density (quadratic and triangular). In Section 4, we discuss the properties of the travel time function that we derived. In Section 5, we embed our travel time function into a more general dynamic user equilibrium problem and illustrate our model through a numerical example. Finally, we conclude by outlining some directions for further research.

2 Review of the theory of kinematic waves

In this section, we review the hydrodynamic theory of traffic flow, proposed by Lighthill and Whitham (1955) and Richards (1956). From a macroscopic point of view, the flow of traffic on a stretch of a road can be modeled as the flow of a fluid in a pipe. Accordingly, traffic flow is animated with waves, moving backwards or forwards.

Because of the dynamic nature of traffic, we work on a time-space plane. The fundamental traffic variables to describe traffic conditions on a road are:

- the flow rate, $f(x,t)$, which is the number of vehicles per hour passing location x at time t,
- the rate of density (or concentration), $k(x,t)$, which is the number of vehicles per mile, at location x at time t, and
- the instantaneous velocity, $u(x,t)$, which is the speed of vehicles passing location x at time t.

Assuming a homogeneous road, these quantities are respectively bounded from above by f^{max}, k^{max}, and u^{max}.

Most models assume a one-to-one relation between the speed and the density, i.e., $u(x,t) = u(k(x,t))$. This relation has the additional property that when the density is zero, the speed is equal to the free-flow speed u^{max}, and when the density is equal to the jam density, $k = k^{max}$, the speed is zero.

Notice that the definition of the fundamental traffic variables implies that $f(x,t) = k(x,t)u(x,t)$. Therefore, the flow and the density are related through the so called *fundamental diagram*:

$$f(x,t) = k(x,t)u(k(x,t)), \quad \forall x, t. \tag{1}$$

Depending on the assumed speed-density relation $u(k(x,t))$, the fundamental diagram can have different shapes. In what follows, we analyze the two most common shapes, namely the quadratic and the triangular fundamental diagrams.

Greenshields (1935) modeled the vehicle velocity as a linear function of density, i.e., $u(k) = u^{max}(1 - k/k^{max})$. Based on this function, the fundamental diagram has a quadratic shape, as in Richards (1956):

$$f(x,t) = k(x,t)u^{max}(1 - k(x,t)/k^{max}). \tag{2}$$

Newell (1993) proposed a triangular curve with a left slope of $1/u_0$ and a right slope of $-1/w_0$. The change of slope occurs when $k = k(f^{max})$, where $k(f^{max})$ is the density associated with the road capacity.

$$f(x,t) = \begin{cases} k(x,t)/u_0 & \text{if } k(x,t) \leq k(f^{max}) = f^{max}u_0, \\ (k^{max} - k(x,t))/w_0 & \text{otherwise.} \end{cases} \tag{3}$$

In addition to (1), the traffic variables are related through a conservation law, stated as the following partial differential equation:

$$\frac{\partial k(x,t)}{\partial t} + \frac{df(k)}{dk}\frac{\partial k(x,t)}{\partial x} = 0. \tag{4}$$

This conservation law describes the fact that on a single stretch of road, no cars are lost.

In the time-space plane, a level curve of density is the set of points (x,t) such that $k(x,t)$ remains constant. Since the density uniquely determines the flow according to (2) or (3), the flow $f(k)$ and by consequence $df(k)/dk$ also remain constant along a level curve of density. Therefore, if $k(x,t)$ remains constant, (4) reduces to the equation of a straight line, with slope df/dk. In other words, the level curves of density are straight lines, called characteristic lines.

Characteristic lines represent the propagation of traffic density in the time-space plane. We say that density propagates as a wave, with wave speed df/dk. The wave speed (i.e., the slope of the characteristic line), df/dk, is positive when the traffic is light ($k \leq k(f^{max})$) and negative when the traffic is heavy ($k \geq k(f^{max})$). Accordingly, macroscopic traffic conditions will propagate forwards when traffic is light and backwards against traffic when traffic is heavy. In fact, in many dynamical systems, particles (i.e., vehicles) do not move in the same direction as waves. Haberman (1977) illustrates this phenomenon by a horizontal rope, attached at one side, and vertically moved by a person at the other side. Although waves propagate on the rope from the person side to the other side, particles of the rope only move up and down. In transportation, when a car brakes suddenly, say because of a pedestrian crossing, the car behind it will also brake short time after but before reaching the pedestrian location, and similarly for the following cars. As a result, a wave, identified by the braking lights, will propagate backwards while cars are still propagating forwards.

If two characteristic lines intersect, the density around the point of intersection is discontinuous. The set of points of intersection is called a shock wave, and represents a sudden change in traffic conditions (e.g., due to an accident or a downstream bottleneck capacity). The shock wave may propagate backwards (e.g., when a queue is increasing) or forwards (e.g., when a queue is decreasing). More details and examples of the theory can be found in Haberman (1977).

3 An analytical derivation of the travel time function

In this section, we propose a general methodology for evaluating the travel time of a vehicle entering a homogeneous road of length L at time t_0, based on the theory of kinematic waves. In particular, our methodology applies to the triangular and quadratic flow-density curves.

We assume that sensors have been set up at the entrance and the exit of the road, to record the cumulative number of vehicles and the traffic density. In Section 4, we will develop a model of flow propagation, so that only the sensors at the entrance are required to evaluate the travel time of vehicles.

3.1 General framework

Because of the discontinuity in density induced by shocks, the kinematic wave model may be quite hard to solve. In what follows, we introduce three assumptions that simplify the model.

3.1.1 Assumptions

Assumption A1 - At most one shock. We assume that there is at most one shock on the road. In the original model by Lighthill and Whitham (1955) and Richards (1956), a shock may result from the focusing of two forward waves, two backward waves, or one forward and one backward wave. However, as argued by Newell (1993), only the latter type of shocks is observed in reality. This assumption allows us to divide a road into two segments, separated by the shock wave: on the first segment, the traffic flow has a low density, whereas on the second, it has a high density. If there is no shock, the second segment has zero length, while if heavy traffic conditions back up to the road entrance, the first segment has zero length. As we will show later, Assumption **A1** always holds with a fundamental diagram that is either triangular or quadratic (as long as Assumption **A2**, that we describe below, is satisfied).

Assumption A2 - Linear density. We assume that the second-order variation of density is locally negligible (notice this is always true for small road lengths). Accordingly, the density at location x at time t can be approximated by

$$k(x,t) = k(\xi,t) + B(\xi,t)(x-\xi), \tag{5}$$

where $\xi = 0$, if the traffic conditions at (x,t) are light, and $\xi = L$, if they are heavy. We denote by $B(\xi,t)$ the rate of evolution of the density at time t at the road entrance (if $\xi = 0$) or at the road exit (if $\xi = L$). In Appendix A, we establish a relationship between the rate of evolution of density with respect to space, $B(\xi,t)$, and the rate of evolution of the density with respect to time, $\partial k(\xi,t)/\partial t$.

To ensure that the estimated density is below $k(f^{max})$ if the traffic conditions at (x,t) are light, we require $0 \leq k(0,t) + B(0,t)L \leq k(f^{max})$. Similarly, if the traffic conditions at (x,t) are heavy, we require $k(f^{max}) \leq k(L,t) - B(L,t)L \leq k^{max}$.

Assumption A3 - Bounded variations of density. In addition, we require the variations of density to be bounded. This assumption allows us to neglect high order terms in the travel time function, as we will show in the proofs of Theorems 1 and 2. Mathematically, we impose the following condition:

$$|B(\xi,t)| < (k^{max} - k(\xi,t))^2/(5Lk^{max}), \text{ for } \xi = 0 \text{ or } L. \tag{6}$$

If the evolution of traffic flow is highly variable, we can relax (6) by considering smaller road lengths.

3.1.2 Methodology

From Assumption **A1**, the road can be decomposed into two segments, separated by the shock wave. Therefore, before a vehicle reaches the shock wave, its travel time depends only on light traffic conditions, while after the shock wave, its travel time depends only on heavy traffic conditions. As a result, as in Ran et al. (1997), we can compute the total travel time as the sum of

- the travel time to go from the entrance to the shock wave (under light conditions), and
- the travel time to go from the shock wave to the road exit (under heavy traffic conditions).

In particular, we consider a vehicle that starts its trip at time t_0, on a road of length L. We denote by $\tau(x)$ its travel time to reach location x.

Shock location. A shock is a discontinuity in the traffic flow. Rather than working with flows, Newell (1993) introduced the concept of cumulative number of vehicles passing through location x by time t, $F(x,t)$. The partial derivatives of $F(x,t)$ correspond to the density and the flow rates, i.e., $\partial F(x,t)/\partial x = -k(x,t)$ and $\partial F(x,t)/\partial t = f(x,t)$.

Along a characteristic line passing through (x_0, t_0) with slope df/dk, the rate of evolution of $F(x,t)$ with respect to x is equal to:

$$\frac{dF(x,t)}{dx} = -k(x,t) + f(x,t)(\frac{df(k)}{dk})^{-1}. \tag{7}$$

Along a characteristic line, variables k, $f(k)$ and $df(k)/dk$ remain constant, implying that dF/dx also remains constant. Therefore, the knowledge of $F(x_0,t_0)$ and $k(x_0,t_0)$ completely determines the cumulative number of vehicles at each point on the characteristic line.

If the characteristic line has positive slope, $k(x_0,t_0)$ is approximated by $k(0,t_0) + B(0,t_0)x_0$, according to (5). Similarly, $F(x_0,t_0)$ is approximated by $F(0,t_0) + (\partial F(0,t_0))/(\partial x)x_0 = F(0,t_0) - k(0,t_0)x_0$. As a result, the cumulative number of cars at (x,t) is a function of the traffic conditions at the entrance, i.e., a function of $k(0,t_0)$, $B(0,t_0)$ and $F(0,t_0)$. In this case, we denote by $A(x,t)$ the cumulative number of vehicles passing x by time t, instead of $F(x,t)$. We use this notation in order to emphasize the reference to the cumulative number of arrivals on the road.

Similarly, if the characteristic line has negative slope, the cumulative number of vehicles at (x,t) is a function of the traffic conditions at the exit of the road. In this case, we denote by $D(x,t)$ the cumulative number of vehicles passing x by time t, instead of $F(x,t)$, in order to emphasize the reference to the cumulative number of departures from the road.

If there was no shock, all characteristic lines would never intersect, and the cumulative number of vehicles would be uniquely determined at every point. However, in the presence of shocks, two characteristic lines could potentially intersect at point (x,t). In such a case, the cumulative number of vehicles at (x,t) would have two different values, namely $A(x,t)$ and $D(x,t)$. Newell (1993) argued that the correct value for the cumulative number of vehicles passing x by time t is the minimum between $A(x,t)$ and $D(x,t)$, and that the intersection, $A(x,t) = D(x,t)$, determines the shock.

Plugging the vehicle's trajectory $(x, t_0 + \tau(x))$ into the shock wave equation, $A(x,t) = D(x,t)$, gives rise to the point $(\hat{x}, t_0 + \hat{\tau})$ at which the vehicle goes through the shock.

Travel time function in light/heavy traffic. As mentioned above, the total travel time can be decomposed as the sum of the travel times before and after the shock.

From the fundamental diagram (1), the instantaneous vehicle's velocity at (x,t) is the ratio between the flow and the density. Accordingly, the vehicle's trajectory $(x, t_0 + \tau(x))$ evolves as follows:

$$\frac{d\tau(x)}{dx} = \frac{1}{u(x, t_0 + \tau(x))} = \frac{k(x, t_0 + \tau(x))}{f(x, t_0 + \tau(x))}. \tag{8}$$

From the flow-density curve ((2) or (3)), $f(x, t_0 + \tau(x))$ can be expressed as a function of $k(x, t_0 + \tau(x))$. Therefore, the right hand side of (8) depends only on the density.

If traffic is light, we approximate the instantaneous velocity (8) from the traffic conditions at $(0, t_0)$, when the vehicle entered the road. Along a characteristic line, the density remains constant, and $k(x, t_0 + \tau(x)) = k(x_0, t_0)$, where x_0 is the ordinate at t_0 of the characteristic line passing through $(x, t_0 + \tau(x))$, as shown in the left part of Figure 1.

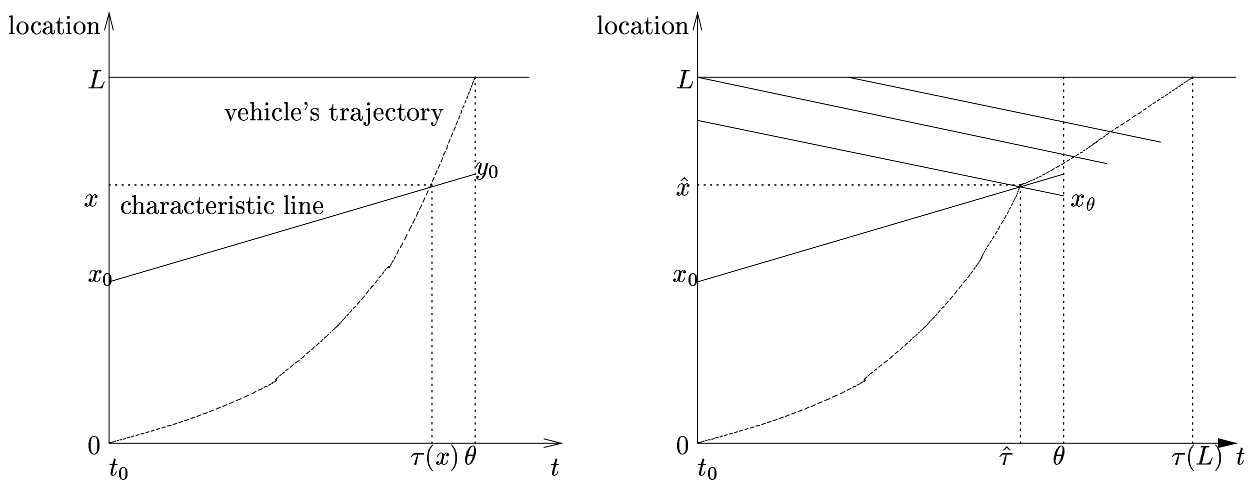

Figure 1: Travel time without and with shock (see left and right figures respectively)

Using Assumption **A2**, $k(x_0, t_0) = k(0, t_0) + B(0, t_0)x_0$. Plugging the density function into (8), we obtain an ordinary differential equation (ODE). This ODE, together with the initial condition $\tau(0) = 0$, can be solved through a power series expansion (see Edwards and Penney 1985). Under Assumption **A3**, the ratio between two successive terms in the series is bounded from above by 1, and the series converges.

Using the travel time function under light traffic, we obtain a lower bound on the total travel time, θ, as if there was no shock. However, because of the shock, the total travel time will be significantly larger, as illustrated in Figure 1.

When traffic is heavy, we would like to approximate the instantaneous velocity (8) in reference to the traffic conditions prevailing at the road exit when the vehicle departs the road. However, we do not know the departure time of the vehicle, and we consider instead the lower bound θ, as if there was no shock. Since the density remains constant along a characteristic line, $k(x, t_0+\tau(x)) = k(x_\theta, t_0+\theta)$, where x_θ is the ordinate at $t_0+\theta$ of the characteristic line passing through $(x, t_0+\tau(x))$. Using Assumption **A2**, $k(x_0, t_0+\theta) = k(L, t_0+\theta) - B(L, t_0+\theta)(L-x_\theta)$. Plugging this approximation into (8), we obtain an ODE. Using the boundary condition that the vehicle passes through the shock location, i.e., $\tau(\hat{x}) = \hat{\tau}$, this ODE can be solved with a power series expansion to give rise to a functional form for the travel time.

In what follows, we apply this general methodology to the triangular and the quadratic fundamental diagrams.

3.2 Triangular fundamental diagram

With a triangular diagram, the speed of waves is either $1/u_0$ if they move forwards, or $1/w_0$ if they move backwards. As a result, all characteristic lines in the same region (light or heavy traffic) have the same slope, and Assumption **A1** always holds.

3.2.1 Shock location

Along the vehicle's trajectory, the cumulative number of vehicles based on the prevailing traffic conditions at the entrance, $A(x,t)$, remains constant. Indeed, when traffic is light, the vehicle is moving at the same speed as the wave conveying the entering flow, $\tau(x) = xu_0$. Therefore,

$$\begin{aligned} A(x, t_0+\tau(x)) &= A(0,t_0) + \left.\frac{dA(\xi, t_0)}{d\xi}\right|_{\xi=0} x, \\ &= A(0,t_0) + (-k(0,t_0) + f(0,t_0)u_0)x && \text{from (7)}, \\ &= A(0,t_0) && \text{from (2)}. \end{aligned}$$

On the other hand, the cumulative number of vehicles depending on the traffic conditions at the exit, $D(x,t)$, varies along the vehicle trajectory. In contrast to the light traffic region, the vehicle's trajectory crosses several backward waves.

As already mentioned, the cumulative number of vehicles in the heavy traffic region, $D(x,t)$, is evaluated in reference to the traffic conditions at $(L, t_0+\theta)$, where θ would have been the travel time of the vehicle if there was no shock (see Figure 1). With a triangular flow-density relationship, the speed of the vehicle is constant in light traffic, i.e., $\tau(x) = u_0 x$; therefore, $\theta = u_0 L$.

The ordinate at $t_0+\theta$ of a characteristic line passing through point $(x, t_0+\tau(x))$ is $x_\theta = x - (\theta - \tau(x))/w_0$.

Because $dD/d\xi$ remains constant along a characteristic line, from (7), the cumulative number of vehicles based on the traffic conditions at the exit varies as follows:

$$D(x, t_0 + \tau(x)) = D(x_\theta, t_0 + \theta) + \left.\frac{dD(\xi, t_0 + \theta)}{d\xi}\right|_{\xi = x_\theta} (x - x_\theta). \tag{9}$$

Furthermore, assuming a nearly constant density in the heavy traffic region, $D(x_\theta, t_0 + \theta) \approx D(L, t_0 + \theta) + k(L, t_0 + \theta)(L - x_\theta)$. Replacing $dD/d\xi$ in (9) with (7) gives rise to

$$\begin{aligned} D(x, t_0 + \tau(x)) &= D(L, t_0 + \theta) + k(L, t_0 + \theta)(L - x_\theta) + (-k(x_\theta, t_0 + \theta) - f(x_\theta, t_0 + \theta)w_0)(x - x_\theta), \\ &= D(L, t_0 + \theta) + k(L, t_0 + \theta)(L - x_\theta) - k^{max}(x - x_\theta), \end{aligned}$$

where the last equality comes from the backward wave speed definition, $w_0 f(x_\theta, t_0 + \theta) = k^{max} - k(x_\theta, t_0 + \theta)$.

At the shock location, $x_\theta = \hat{x} - (L - \hat{x})u_0/w_0$, and the cumulative number of vehicles becomes

$$D(\hat{x}, t_0 + \tau(\hat{x})) = D(L, t_0 + \theta) + k(L, t_0 + \theta)(L - \hat{x})(1 + \frac{u_0}{w_0}) - k^{max}(L - \hat{x})\frac{u_0}{w_0}.$$

The vehicle will pass through the shock when $A(\hat{x}, t_0 + \tau(\hat{x})) = D(\hat{x}, t_0 + \tau(\hat{x}))$, i.e., at location

$$\hat{x} = L - \frac{A(0, t_0) - D(L, t_0 + \theta)}{k(L, t_0 + \theta) - (k^{max} - k(L, t_0 + \theta))\frac{u_0}{w_0}}. \tag{10}$$

3.2.2 Travel time function

The travel time to go from the road entrance to the shock is $u_0\hat{x}$. This follows since the vehicle's speed is constant when traffic is light. Similarly, if there was no shock, the travel time would be $\theta = u_0 L$.

On the other hand, the travel time to go from $\hat{x}$ to the road exit is obtained by plugging $f(x, t_0 + \tau(x)) = 1/w_0(k^{max} - k(x, t_0 + \tau(x)))$ into (8).

Since the characteristic line passing through $(x, t_0 + \tau(x))$ intersects the time line $t_0 + \theta$ at $x_\theta = x - (Lu_0 - \tau(x))/w_0$, and the density remains constant along a characteristic line, $k(x, t_0 + \tau(x)) = k(x_\theta, t_0 + \theta)$. Moreover, from (5), $k(x_\theta, t_0 + \theta) = k(L, t_0 + \theta) - B(L, t_0 + \theta)(L - x_\theta)$. Updating (8) gives rise to

$$\frac{d\tau(x)}{dx} = \frac{(k(L, t_0 + \theta) - B(L, t_0 + \theta)(L - x_\theta))w_0}{k^{max} - (k(L, t_0 + \theta) - B(L, t_0 + \theta)(L - x_\theta))}. \tag{11}$$

As shown in Theorem 1, solving this differential equation gives rise to a closed form solution of the travel time function. The static parameters of the obtained travel time are the positive and negative wave speeds u_0 and w_0 (which can be derived from the free flow speed, the jam density, and the road capacity, according to (3)), and the road length L. On the other hand, the travel time also depends on the dynamic evolution of traffic, namely the cumulative number of vehicles at the road entrance and at the road exit, $A(0,t_0)$ and $D(L,t_0+\theta)$, the associated densities, $k(0,t_0)$ and $k(L,t_0+\theta)$, and the rate of evolution of the latter with respect to distance, $B(L,t_0+\theta)$. However, as shown in Subsection 4.1, the required data can be limited to the knowledge of $A(0,t_0)$, while the other parameters can be estimated from a model of flow propagation.

Theorem 1. *Under Assumptions* **A2** *and* **A3**, *the triangular diagram (3) gives rise to the following travel time for a vehicle entering the road at time t_0:*

$$\begin{aligned}\tau &= \hat{x}u_0 + \frac{(k(L,t_0+\theta)w_0 - B(L,t_0+\theta)(L-\hat{x})(u_0+w_0))w_0}{(k^{max}-k(L,t_0+\theta))w_0 + B(L,t_0+\theta)(w_0+u_0)(L-\hat{x})}(L-\hat{x}) \\ &\quad + O(\frac{Lk(L,t_0+\theta)}{50u^{max}(k^{max}-k(L,t_0+\theta))}),\end{aligned} \tag{12}$$

where $\hat{x}$ is defined in (10).

Proof. The travel time to go from the road entrance to the shock location is $u_0\hat{x}$. On the other hand, the vehicle's trajectory in the heavy traffic region is given by differential equation (11). Solving the differential equation, together with the boundary condition $\tau(\hat{x}) = u_0\hat{x}$ and using a power series solution gives rise to (12).

This power series is guaranteed to converge as long as Assumption **A3** holds. Indeed, the ratio of two successive terms is bounded above by

$$\left|\frac{2B(L,t_0+\theta)k^{max}(L-\hat{x})}{(k^{max}-k(L,t_0+\theta)+B(L,t_0+\theta)(L-\hat{x})(1+\frac{u_0}{w_0}))^2}\right|, \tag{13}$$

and the series converges if the ratio is bounded above by 1.

If the numerator is nonnegative (i.e., $B(L,t_0+\theta)\geq 0$), we can remove the absolute value. By rearranging the terms, it is easy to see that requiring that the ratio (13) is less than 1 is equivalent to requiring that a quadratic equation of the form $a(B(L,t_0+\theta))^2 + bB(L,t_0+\theta)+c$ is nonnegative, for some parameters a, b and c. This quadratic function is convex and has two positive roots. Therefore, if $B(L,t_0+\theta)$ is not larger than the smallest root, then ratio (13) will be less than 1.

The discriminant of the quadratic equation is of the form $\sqrt{b^2-4ac} < |b|(1-2ac/|b|)$. Considering the upper bound of the discriminant in the smallest root gives rise to the following lower bound on the smallest root:

$$\frac{(k^{max}-k(L,t_0+\theta))^2}{2(L-\hat{x})(k(L,t_0+\theta)(1+\frac{u_0}{w_0})-k^{max}\frac{u_0}{w_0})}.$$

This is itself bounded below by $(k^{max} - k(L, t_0 + \theta))^2/(2Lk^{max})$. Hence, the upper bound of $B(L, t_0 + \theta)$ defined in (6) guarantees the series to converge.

If the numerator is negative (i.e., $B(L, t_0 + \theta) < 0$), a similar reasoning applies, and we omit it for the sake of brevity.

□

If $B(L, t_0 + \theta) = 0$, there is no variation in density, and the travel time function (12) reduces to

$$\tau = u_0 L + \frac{A(0, t_0) - D(L, t_0 + \theta)}{f(L, t_0 + \theta)}, \tag{14}$$

which is the sum of the free-flow travel time and the waiting time in a vertical queue. If in addition $A(0, t_0) = D(L, t_0 + \theta)$, all vehicles that enter the road at or prior to t_0 leave the road at or prior to $t_0 + \theta$, and the travel time reduces to the free flow travel time, $\tau = u_0 L = \theta$.

3.2.3 Numerical comparison

Daganzo (1994) proposed a method using discrete simulation under a triangular relation between the flow and the density. This algorithm is available through a user-friendly software program, called NETCELL (Cayford et al. 1997). Netcell builds the curves of cumulative number of vehicles at the entrance and at the exit of the road. The travel time of a vehicle corresponds to the horizontal difference between those two curves.

Figure 2 illustrates that the analytical function (12) integrates all traffic dynamics simulated using Netcell. In this example, we considered a quadratic entering flow rate, $f(0, t) = 1600 - 6400(t/3600 - 0.5)^2$ vehicles/hour, for $t = 1, .., 3600$ seconds. The flow-density relation is symmetric triangular, with $u_0 = w_0 = 1/u^{max} = 1/40$ hour/mile, $k^{max} = 200$ vehicles/mile, $f^{max} = 4000$ vehicles/hour. The road has a length of 4 miles and has a bottleneck at its end authorizing only 1400 vehicles/hour to exit the road.

We propagated the flow according to the theory of deterministic queues, as explained in Section 4, in order to build the curve of the cumulative number of vehicles at the exit. According to (3), the density at the road exit, $k(L, t+\theta)$ is $f(L, t+\theta)u_0$, if $D(L, t+\theta) = A(0, t)$, and $k^{max} - f(L, t+\theta)w_0$, if $D(L, t + \theta) < A(0, t)$. The rate of evolution, $B(\xi, t)$ is approximated by $-(k(\xi, t) - k(\xi, t - 1))k^{max}/(u^{max}(k^{max} - 2k(\xi, t)))$, $\xi = 0$ or L (see Appendix A). Finally, we considered a cubic series for the travel time function (12).

Regardless of the transient effects at the beginning of the simulation, the analytical travel time function practically coincides with the Cell Transmission Model, as illustrated in Figure 2. This illustrates how the two models integrate the same traffic dynamics, since both are based on the

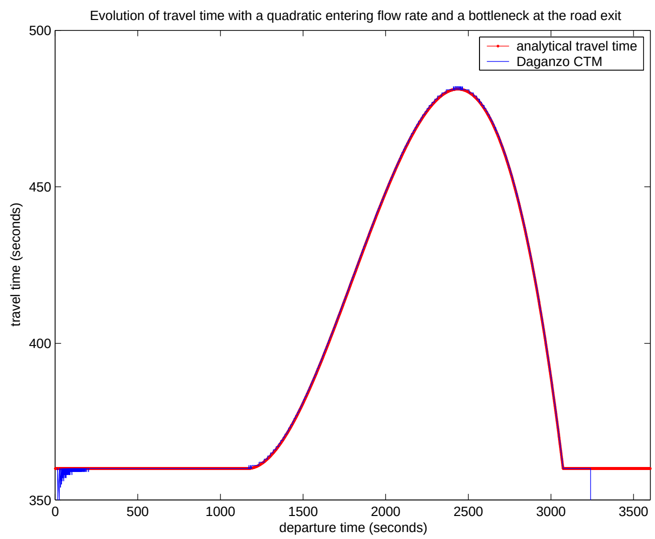

Figure 2: Comparison between the analytical travel time function and the travel time simulated with Netcell

theory of kinematic waves. We performed many different numerical tests (with or without shock, low/high entering flow, queue spillover), and all displayed the same behavior as the example we study in Figure 2.

In summary, in this subsection, we analyzed the traffic delays experienced by a traveler on a single stretch of road, under a triangular fundamental diagram. In particular, under Assumptions **A2** and **A3**,

- we proposed a closed-form solution of the shock location, assuming constant density in the heavy traffic region,
- we derived a closed-form solution of the travel time, capturing the traffic dynamics, based on the solution of a single ordinary differential equation, and
- we compared the analytical travel time with Daganzo's Cell Transmission Model, revealing that the same traffic dynamics were integrated into our model.

3.3 Quadratic fundamental diagram

Under a quadratic diagram, the slope of a characteristic line passing through a point (x, t) is no longer constant but depends on the density at (x, t). In particular, $df/dk = u^{max}(1-2k(x,t)/k^{max})$. In Appendix B, we show that Assumption **A1** always holds, as long as Assumption **A2** is satisfied.

3.3.1 Shock location

Assuming a constant density in the light traffic region implies that the cumulative number of vehicles depending on the prevailing conditions at the road entrance, $A(x,t)$, remains constant along the vehicle's trajectory. Let us denote by x_0 the ordinate at t_0 of the characteristic line passing through $(x, t_0+\tau(x))$. If the density is constant, $k(x_0,t_0)=k(0,t_0)$, the slope of the characteristic line is $u^{max}(1-2k(0,t_0)/k^{max})$ and $x_0 = x - \tau(x)u^{max}(1-2k(0,t_0)/k^{max})$. Therefore,

$$\begin{aligned}
A(x,t_0+\tau(x)) &= A(x_0,t_0) + \left.\frac{dA(\xi,t_0)}{d\xi}\right|_{\xi=x_0}(x-x_0) \\
&= A(0,t_0) - k(0,t_0)x_0 + (-k(0,t_0) + \frac{f(0,t_0)}{u^{max}(1-2k(0,t_0)/k^{max})})(x-x_0) \quad \text{from (7)}, \\
&= A(0,t_0), \quad \text{from (2)}.
\end{aligned}$$

Similarly, $D(x,t)$, is defined with respect to the traffic conditions at the exit $(L, t_0+\theta)$. Let y be the ordinate at $t_0+\theta$ of the characteristic line passing through $(x,t_0+\tau(x))$. Assuming that the density is constant in the heavy traffic region, the characteristic line has a slope of $u^{max}(1-2k(L,t_0+\theta)/k^{max})$ and $x_\theta = x + (t-\tau(x))u^{max}(1-2k(L,t_0+\theta)/k^{max})$. From (7),

$$\begin{aligned}
D(x,t_0+\tau(x)) &= D(x_\theta,t_0+\theta) + \left.\frac{dD(\xi,t_0+\theta)}{d\xi}\right|_{\xi=x_\theta}(x-x_\theta) \\
&= D(L,t_0+\theta) + k(L,t_0+\theta)(L-x_\theta) \\
&\quad +(-k(L,t_0+\theta) + \frac{f(L,t_0+\theta)}{u^{max}(1-2k(L,t_0+\theta)/k^{max})})(x-x_\theta).
\end{aligned}$$

At the shock location, $\tau(\hat{x}) = L/(u^{max}(1-k(0,t_0)/k^{max}))$, if the density is constant in the light traffic region. Plugging $\tau(\hat{x})$ into x_θ defines the cumulative number of vehicles at the shock location as follows:

$$D(\hat{x},t_0+\tau(\hat{x})) = D(L,t_0+\theta) + k(L,t_0+\theta)(L-\hat{x})\frac{k(L,t_0+\theta)-k(0,t_0)}{k^{max}-k(0,t_0)}.$$

A shock occurs when $A(\hat{x},t_0+\tau(\hat{x})) = D(\hat{x},t_0+\tau(\hat{x}))$, i.e., at location

$$\hat{x} = L - (A(0,t_0) - D(L,t_0+\theta))\frac{k^{max}-k(0,t_0)}{k(L,t_0+\theta)(k(L,t_0+\theta)-k(0,t_0))}. \qquad (15)$$

3.3.2 Travel time function

From the quadratic diagram, we substitute $f(x,t_0+\tau(x)) = k(x,t_0+\tau(x))u^{max}(1-k(t_0+\tau(x))/k^{max})$ into (8).

A characteristic line passing through $(x, t_0 + \tau(x))$ intersects the time origin t_0 at

$$x_0 = \frac{x - u^{max}\tau(x)(1 - 2k(0,t_0)/k^{max})}{1 - 2u^{max}\tau(x)B(0,t_0)/k^{max}}.$$

Because the density remains constant along a characteristic line, $k(x, t_0 + \tau(x)) = k(x_0, t_0)$. According to Assumption **A2**, $k(x_0, t_0) = k(0, t_0) + B(0, t_0)x_0$. Substituting this function of density into (8) gives rise to an ODE. This ODE, together with the initial condition $\tau(0) = 0$, can be solved with a power series solution, giving rise to

$$\tau(x) = \frac{x}{u^{max}(1 - k(0,t_0)/k^{max})} + \frac{1}{2}\frac{B(0,t_0)k(0,t_0)x^2}{(k^{max})^2 u^{max}(1 - k(0,t_0)/k^{max})^3} + O(\frac{L}{50u^{max}}). \tag{16}$$

This power series converges under Assumption **A3** since the ratio of two successive terms is bounded above by $\left|2B(0,t)L(k^{max}/(k^{max} - k(0,t))^2)\right|$, which is less than 1 if (6) holds. Notice that if there is no variation in density, i.e., $B(0,t) = 0$, the travel time (16) reduces to the ratio of distance and speed.

In the following theorem, we integrate the effects of a shock into the travel time function. As with the triangular flow-density curve, the resulting travel time function has a closed form, and depends on static road features (maximum speed u^{max}, jam density k^{max} and length L) and on the dynamic evolution of, namely the cumulative number of vehicles at the entrance and at the exit, $A(0, t_0)$ and $D(L, t_0 + \theta)$, the associated densities, $k(0, t_0)$ and $k(L, t_0 + \theta)$, and their rate of evolution, $B(0, t_0)$ and $B(L, t_0 + \theta)$.

Theorem 2. *Under Assumptions* **A2** *and* **A3**, *the quadratic diagram (2) gives rise to the following travel time for a vehicle entering the road at time* t_0*:*

$$\begin{aligned} \tau &= \hat{\tau} + \frac{k^{max} + 2u^{max}(\theta - \hat{\tau})B(L, t_0 + \theta)}{u^{max}(k^{max} - k(L, t_0 + \theta) + B(L, t_0 + \theta)(L - \hat{x} + u^{max}(\theta - \hat{\tau})))}(L - \hat{x}) \\ &\quad + O(\frac{Lk(L, t_0 + \theta)}{50u^{max}(k^{max} - k(L, t_0 + \theta))}), \end{aligned} \tag{17}$$

where $\hat{x}$ *is defined in (15),* $\theta = \tau(L)$, $\hat{\tau} = \tau(\hat{x})$ *according to (16).*

Proof. We obtain an ODE representing the instantaneous velocity of a vehicle in a similar fashion as in the case of light traffic. However, the boundary condition is defined at the shock location, $\tau(\hat{x}) = \hat{\tau}$, where $\tau(\hat{x})$ is the time to reach the shock, using (16). Using a power series solution to solve the differential equation, together with the boundary condition, gives rise to (17). The proof of convergence of this power series is similar to the proof of Theorem 1, under Assumption **A3**. We omit it for the sake of brevity. □

Similarly to the triangular fundamental diagram, if the dynamic effects are negligible, i.e. $B(0,t_0) = B(L, t_0+\theta) = 0$, the travel time function reduces to the sum of the free flow travel time and the waiting time in a vertical queue, that is

$$\tau = \frac{L}{u^{max}(1 - k(0,t_0)/k^{max})} + \frac{A(0,t_0) - D(L, t_0+\theta)}{f(L, t_0+\theta)}. \tag{18}$$

Notice that ignoring the variations of density before and after the shock does not reduce the travel time function to the instantaneous travel time. Indeed, although density is approximated as constant before and after the shock, the shock wave keeps moving; as a result, travel time (18) is a first-order approximation of the actual (or experienced) travel time.

3.3.3 Numerical comparison

We compared the analytical travel time (17) with several travel time functions that were proposed in the literature, obtained either via simulation or via an analytical derivation.

Under a quadratic diagram, Kachani and Perakis (2001) derived two functions (polynomial and exponential) of travel time. On the other hand, Khoo et al. (2002) simulated the flow propagation along Godunov's scheme for calculating the numerical flux. They used two different methods, called hyperbolic PDE and iterative position update methods. Finally, Daganzo (1995b) proposed a approximation of the kinematic wave model with finite difference equations (FDE). The travel time is the horizontal difference between the cumulative departures and the cumulative arrivals on the road.

In Figure 3, we consider the same quadratic entering flow rate as in Subsection 3.2.3. The road has a length of 4 miles, a maximum speed $u^{max} = 40$ miles/hour, a maximum density $k^{max} = 200$ cars/mile, and hence, a capacity of $f^{max} = 2000$ cars/hour. There is no exit bottleneck. As illustrated in the figure, the analytical travel time function (17) is very close to what was obtained from the simulations.

On the other hand, with the introduction of a bottleneck at the exit of the road, we only compared (17) with Daganzo's FDE model. The simulation results obtained by Khoo et al. were not available to us in this case, and the analytical travel time functions proposed by Kachani and Perakis are only valid under light traffic. Figure 4 illustrates the case of a road with an exit bottleneck of 1500 cars/hour. Similarly to the case with a triangular diagram, we propagated the traffic flow according to a model of deterministic queuing (see Section 4). Comparing this figure with Figure 3 illustrates the asymmetry in the travel time function induced by the bottleneck.

However, when the entering traffic flow is close to the road capacity, the value of $B(0,t)$ may become very large, and eventually violate Assumption **A3**. Figure 5 depicts the travel time functions

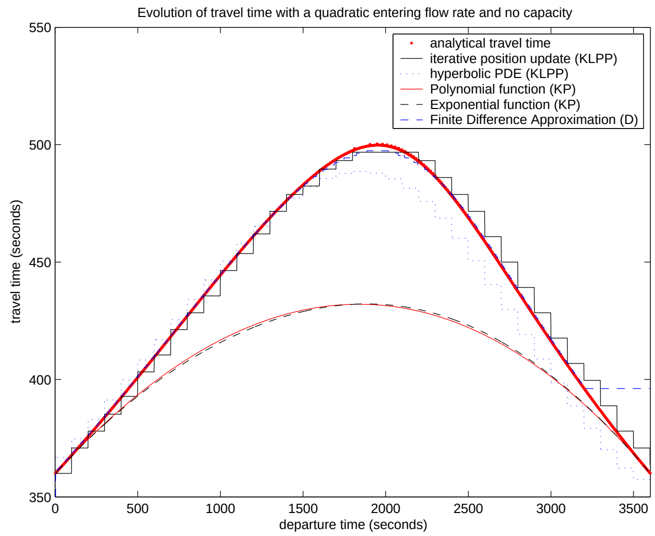

Figure 3: Comparison between the analytical travel time functions in Kachani and Perakis (2001), the simulated travel times in Khoo et al. (2003), and the finite difference approximation in Daganzo (1995b)

when Assumption **A3** holds, as well as when Assumption **A3** is violated. We considered a larger entering flow, $f(0,t) = 1990 - 6400(t/3600 - 0.5)^2$ vehicles/hour, for $t = 1, .., 3600$ seconds.

However, even when Assumption **A3** holds, the analytical travel time tends to overestimate the travel time simulated with the FDE model. In fact, the first-order approximation of density defined in Assumption **A2** only holds locally. Moreover, it is typical that the analytical function overestimates the simulated travel time. Since the forward waves tend to fan out when the entering flow is increasing, and to contract when it is decreasing, the assumption of linear evolution of density overemphasizes the effects of high density (close to $k(f^{max})$), leading to an overestimation of the travel time.

Therefore, to improve the accuracy of the approximation, one should consider smaller road lengths. Figure 6 shows how the approximation improves by considering a road length of 0.25 mile instead of 4 miles.

In summary, in this subsection, we analyzed the traffic delays experienced by a traveler on a single stretch of road, under a quadratic fundamental diagram. In particular, under Assumptions **A2** and **A3**,

- we proposed a closed-form solution of the shock location, assuming constant density in the light and in the heavy traffic region,
- we derived a closed-form solution of the travel time, capturing the traffic dynamics, based on

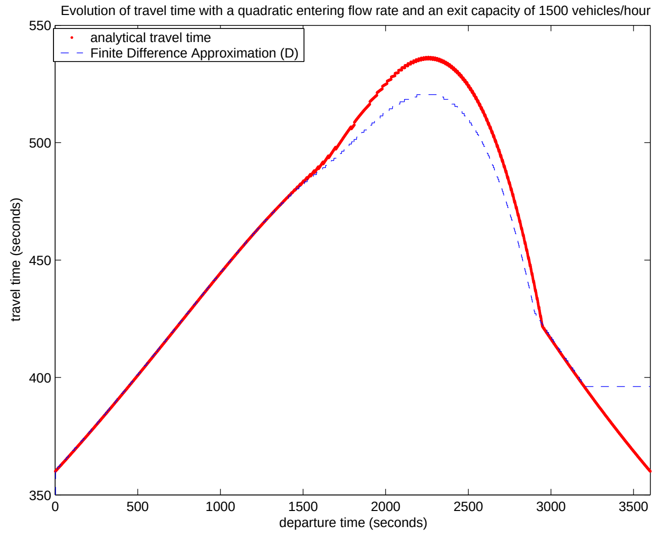

Figure 4: Comparison between the analytical travel time function and the finite difference approximation in Daganzo (1995b), when there is an exit bottleneck capacity of 1500 vehicles/hour

the solution of a single ordinary differential equation,

- we highlighted the quality of prediction of the analytical travel time in comparison to simulations, and
- we noticed that Assumptions **A2** and **A3** hold in general if we consider small road lengths.

4 Properties of the travel time function

In this section, we analyze some common properties of the two travel time functions that we derived in Section 3, assuming $B(0,t_0) = B(L,t_0+\theta) = 0$, namely (14) and (18). First, we build a consistent model for flow propagation. Second, we prove that the travel time function is continuous, monotone and satisfies the FIFO property under reasonable conditions.

In what follows, we neglect the location index from $A(0,t)$ and $D(L,t)$ and consider $A(t)$ and $D(t)$ instead, since, by definition, these quantities are always evaluated at the entrance and at the exit respectively.

4.1 Flow propagation

Whether the fundamental diagram is triangular or quadratic, the travel time ((14) or (18)) is the sum of:

- the time to travel a distance L at the free flow speed, and

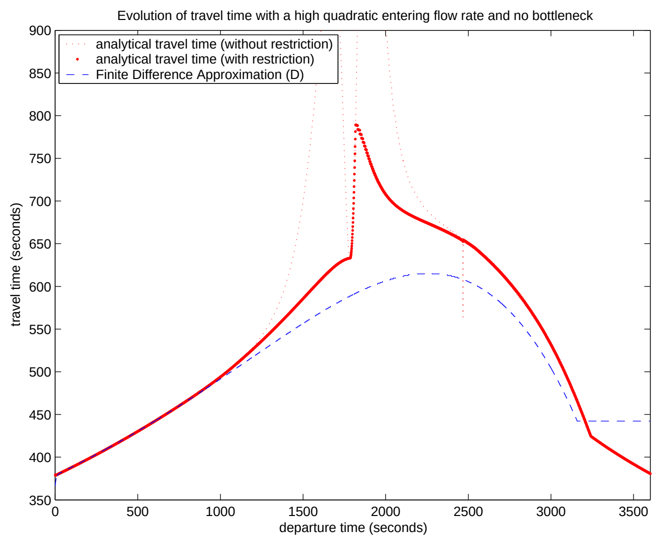

Figure 5: Comparison between the analytical travel time function and the Finite Difference Approximation in Daganzo (1995b), when the dynamic effects are important

- the time to wait in queue for available downstream capacity, $\frac{A(t_0)-D(t_0+\theta)}{f(L,t_0+\theta)}$.

Such a travel time function can also be derived from a queuing model where the arrival and service rates, although time-varying, are deterministic (e.g., see Newell 1982 and Han 2003). This connection between the theory of kinematic waves and the theory of deterministic queues provides us with a rationale for propagating flow. In Section 3, we assumed that two traffic sensors were set up along the road, one at the entrance and one at the exit. With a consistent model for flow propagation, one only needs one sensor at the entrance.

In what follows, we describe a procedure for propagating flow according to the theory of deterministic queues. Assume that, at period t, there is an entering flow $\bar{f}(0,t)$ on a road of length L with an exit capacity of Q.

If there was no queue, the flow would propagate in $\theta(t)$ periods. In particular, $\theta(t) = u_0 L$ under the triangular flow-density diagram or $L/(u^{max}(1-k(0,t)/k^{max}))$ under the quadratic diagram. Therefore, the potential cumulative number of vehicles at the road exit at time $t+\theta(t)$ is equal to the cumulative number of vehicles that passed the entrance at or prior to t, i.e., $\bar{D}(t+\theta(t)) = A(t)$. However, because of capacity restrictions at the exit, the cumulative number of vehicles that have actually passed the exit, $D(t+\theta(t))$, will be less than the potential $\bar{D}(t+\theta(t))$, as illustrated in Figure 7. In fact, only a flow of Q vehicles can depart the road. As a result, the actual cumulative number of cars at the exit, $D(t)$, is the minimum between the potential number of vehicles, $\bar{D}(t)$, and what is authorized to depart the road, $\lim_{\delta\to 0} D(t-\delta)+\delta Q$.

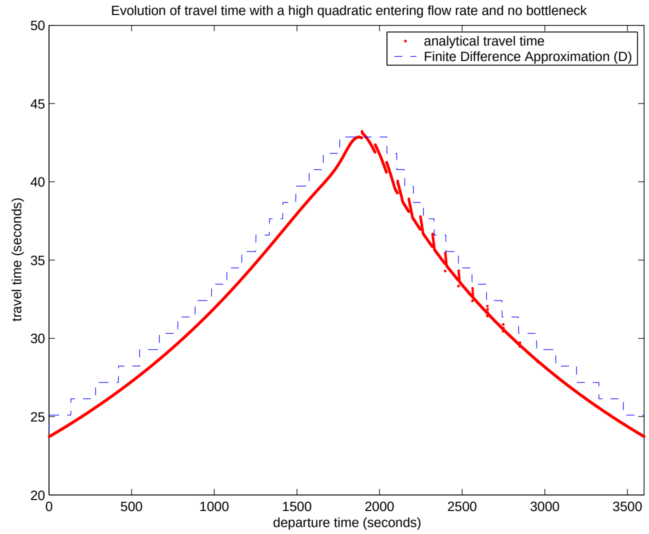

Figure 6: Comparison between the analytical travel time function and the finite difference approximation in Daganzo (1995b), when the dynamic effects are important, but the road length is small

Similarly, the flow may be constrained to wait at the entrance of the road. Let us assume that the road has capacity f^{max} vehicles/hour and can accept at most $k^{max}L$ vehicles at the same time. If the entering flow $\bar{f}(0,t)$ exceeds capacity, we assume that the vehicles in excess wait in an infinite size parking lot at the entrance of the road (i.e., a vertical queue) for available capacity. The cumulative number of vehicles that wish to enter the road at time t, $\bar{A}(t)$, is the sum of all potential entering flows up to t, i.e. $\bar{A}(t) = \int_{s \leq t} \bar{f}(0,s)$. On the other hand, the actual cumulative number of vehicles that have passed the entrance by time t, $A(t)$, is the minimum between what is willing to enter, $\bar{A}(t)$, and what is authorized to enter, $\min\{\lim_{\delta \to 0} A(t-\delta) + \delta f^{max}, D(t) + k^{max}L\}$.

From a computational perspective, we work with discrete time periods, with finite horizon T. Algorithm 1 propagates flow under the triangular fundamental diagram. The case for the quadratic diagram is very similar, and we omit it for the sake of brevity.

By applying this algorithm, one obtains the curves of cumulative number of vehicles at the entrance and at the exit of the road, $A(t)$ and $D(t)$, as well as the travel time, without extensive memory requirements. It is important to evaluate the travel time in the course of the algorithm, so that the value of $D(t)$ is not altered by future flows. In the numerical experiments reported in Section 3, we implemented this model of flow propagation to compute the travel time.

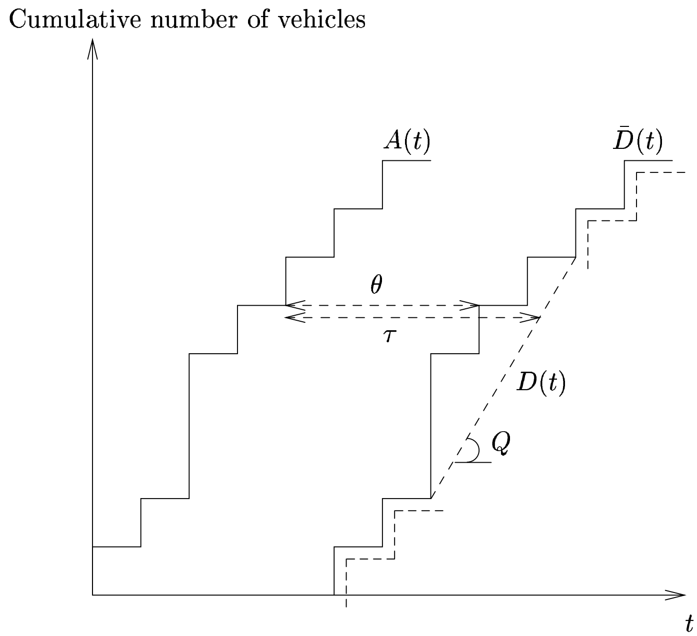

Figure 7: Evolution of the cumulative number of vehicles at the entrance ($A(t)$) and at the exit ($D(t)$) with an exit capacity restriction.

4.2 Continuity and Monotonicity

Continuity and strict monotonicity are desirable properties for the travel time function for solving the dynamic traffic equilibrium problem. In particular, the DUE problem can be formulated as a variational inequality over a compact set (see Nagurney 1993). If the travel time function is continuous, the variational inequality has a solution. If the travel time function is strictly monotone, the variational inequality has at most one solution. Since the traffic quantities at the road exit (i.e., $D(t), f(L,t)$) depend on the entering flow, the characterization of the travel time functions ((14) and (18)) relies on the flow propagation descibed in the previous subsection.

Theorem 3. *In a continuous-time setting, the travel time functions ((14) and (18)) are continuous. In a discrete-time setting, only (14) is continuous.*

Proof. In a continuous-time setting, it is obvious from the flow propagation that all functions $\theta(t), A(t), D(t+\theta)$ and $f(L,t+\theta)$ are continuous in $f(0,t)$. Therefore, the travel times are also continuous. In a discrete-time setting, $\theta(t)$ is rounded to the nearest integer. In the case of a triangular fundamental diagram, the value of θ does not depend on the flow, and $D(t+\theta), f(L,t+\theta)$ remain constant after an increase in flow, whereas $A(t)$ changes continuously. On the other hand, with a quadratic diagram, a variation in flow may lead to a discontinuous change of θ, and hence, a discontinuous change in $D(t+\theta)$. □

Algorithm 1 Flow Propagation for the triangular fundamental diagram

for $t = 1$ to T **do**
  $D(t) = 0, A(t) = 0, \bar{A} = 0$
end for
for $t = 1$ to T **do**
  $\bar{A} \leftarrow \bar{A} + \bar{f}(0,t)$
  $A(t) \leftarrow \min\{\bar{A}, A(t-1) + f^{max}, D(t) + k^{max}L\}$
  $f_0 = A(t) - A(t-1)$
  $\theta = \text{round}(u_0 L)$ (rounded to the nearest integer)
  $\bar{D} = A(t)$
  for $s = t + \theta$ to T **do**
    $D(s) \leftarrow \min\{D(s-1) + Q, \bar{D}\}$
  end for
  $f_L = D(t+\theta) - D(t+\theta-1)$
  $\tau(t) = u_0 L + \frac{A(t) - D(t+\theta)}{f_L}$ or (12)
end for

Theorem 4. *In a continous-time setting, the travel time function (14) is monotone and (18) is strictly monotone. In a discrete-time setting, the travel time function (14) is monotone.*

Proof. Proving the (strict) monotonicity of a function is equivalent to showing that its Jacobian matrix is positive semi-definite (positive definite) (see Nagurney 1993). With a triangular fundamental diagram, an increase of flow will keep $\theta(t)$ unchanged. If there is no queue in period t, a variation in $f(0,t)$ will have no impact on $\tau(t)$ (because $\tau(t) = \theta(t) = u_0 L$). In addition, since the flow is not delayed in a queue, it will not affect the travel times associated with subsequent periods. As a result, the entire row of the Jacobian matrix is zero. If a queue appears in period t for S periods, an increase in flow in period t will affect the travel times of the next S periods. In particular, for $0 \leq s \leq S$, $dA(t+s)/df = 1$ while $D(t+s+\theta(t+s))$ remains unchanged (because $\theta(t+s)$ is constant). Therefore, $d\tau(t+s)/df = 1/Q$, where Q is the exit capacity. As an illustration, if there are 5 periods, and a queue appears in period 2 until period 4, the Jacobian matrix becomes:

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1/Q & 1/Q & 1/Q & 0 \\ 0 & 0 & 1/Q & 1/Q & 0 \\ 0 & 0 & 0 & 1/Q & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Considering the quadratic equation $\mathbf{x}'(\mathbf{J} + \mathbf{J}')\mathbf{x}$, for $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]' \neq \mathbf{0}$, where $\mathbf{x}'$ ($\mathbf{J}'$) denotes the transpose of the vector $\mathbf{x}$ (matrix $\mathbf{J}$), we obtain $(x_2 + x_3 + x_4)^2 + x_2^2 + x_3^2 + x_4^2 \geq 0$. Extending the same argument, it is easy to see that $\mathbf{J}$ is positive semi-definite (Notice that, if there is no queue at all, the Jacobian matrix is zero.) As a result, the travel time function (14) is monotone.

With a quadratic fundamental diagram, the value of θ does depend on the flow. In a continuous-time setting, and if there is no shock at time t, $d\tau(t)/df = d\theta/df = (df/dk)(L/(u^{max}k^{max}(1 - k(0,t)/k^{max})^2)) > 0$. Since there is no queue at t, the travel time of the following periods is not influenced by $f(0,t)$, and the off-diagonal elements of the row are zero. If a shock appears in period t for S periods, $dA(t)/df = dA(t+s)/df = 1$ for $1 \leq s \leq S$, while $dD(t+\theta(t))/df = Q(d\theta/df)$ and $dD(t+s+\theta(t+s))/df = 0$. As a result, $d\tau(t)/df = (d\theta/df) + 1/Q(1 - Q(d\theta/df)) = 1/Q$ and $d\tau(t+s)/df = 1/Q$, for $1 \leq s \leq S$. Considering a discretized version of the Jacobian matrix for the same example as above (queue in "periods" 2 to 4 only), we obtain

$$\mathbf{J} = \begin{bmatrix} d\theta/df & 0 & 0 & 0 & 0 \\ 0 & 1/Q & 1/Q & 1/Q & 0 \\ 0 & 0 & 1/Q & 1/Q & 0 \\ 0 & 0 & 0 & 1/Q & 0 \\ 0 & 0 & 0 & 0 & d\theta/df \end{bmatrix}.$$

This matrix is positive definite, proving that the travel time function (18) is strictly monotone in a continuous time setting. However, in a discrete-time setting, θ is rounded to the nearest integer and may change discontinuously. Therefore, $D(t+\theta)$ may either remain constant or increase by Q with an increase of flow. If $D(t+\theta)$ remains constant, matrix $\mathbf{J}$ will be positive definite; however, if $D(t+\theta)$ increases by Q, this may not be true. As a result, in a discrete-time setting, the travel time is not guaranteed to be monotone. □

Notice that the Jacobian matrix is upper triangular, which is consistent to Chen and Hsueh (1998)'s findings. According to the principle of causality (see Daganzo 1995b), the travel time can only be affected by current or previous flows.

4.3 First-In-First-Out (FIFO)

Finally, we prove that the travel time functions ((14) and (18)) satisfy the FIFO property, under some additional restriction on the magnitude of $B(0,t)$. If FIFO holds, a vehicle that departed later cannot arrive earlier.

Theorem 5. *If there is no shock, the travel time functions (14) and (18) satisfy the FIFO property, as long as Assumption* **A3** *holds. In the case of shocks, FIFO holds if* $B(0,t) \leq f(L, t + \theta(t))dk(0,t)/df$.

Proof. Satisfying FIFO means that the derivative of the travel time with respect to time, i.e., $d\tau/dt$ is greater than or equal to -1. The chain rule implies that $d\tau/dt = (d\tau/dk)(dk/dt)$. From Appendix A, we know that $dk/dt = -B(0,t)df(0,t)/dk$. Furthermore, according to the proof of Theorem 4,

$d\tau/dk = 1/f(L, t+\theta)$ if there is a shock. Replacing these expressions gives rise to the theorem statement. □

The FIFO property gives consistency to the model, especially when one computes the path travel times as the sum of the link travel times.

5 Integration within a DUE problem

5.1 Path formulation of the DUE

The model for traffic delays can be embedded within a DUE setting. In this subsection, we propose a standard path formulation of a discrete-time DUE, defined on a time-expanded transportation network (see Ran and Boyce 1994).

Let us consider a network with a set of arcs A, a set of paths P, over a time horizon of T periods. Let W be the set of all origin-destination (OD) pairs. For a particular OD pair $w \in W$, let $P^w \subset P$ be the set of paths linking this origin to this destination, and let $d^w(t)$ be the given demand for period t. A DUE based on Wardrop's first principle satisfies the following property: if the flow on path $p \in P^w$ at time t, $f_p(t)$, is positive, the associated travel time $\tau_p(t)$ is minimal. Let $\pi^w(t)$ be the smallest travel time for the OD pair w at time t. Mathematically, Wardrop's first principle can be formulated as follows:

$$\tau_p(t) \begin{cases} = \pi^w(t) & \text{if } f_p(t) > 0 \\ \geq \pi^w(t) & \text{if } f_p(t) = 0 \end{cases} \quad \forall p \in P^w, \forall w \in W. \tag{19}$$

Stated differentely, according to Wardrop's first principle, each traveler non-cooperatively seeks to minimize his travel time. If path p consists of the m arcs $\{a_1, a_2, ..., a_m\}$, the path travel time for a vehicle starting its trip at time t, $\tau_p(t)$, is defined recursively as

$$\tau_p(t) = \tau_{a_1}(t) + \tau_{a_2}(t + \tau_{a_1}(t)) + ... + \tau_{a_m}(t + \tau_{a_1}(t) + \tau_{a_2}(t + \tau_{a_1}(t)) + ...), \tag{20}$$

where τ_{a_i}, $i = 1, ..., m$, are the arc travel time functions defined in (12) or (17), depending upon the assumed curve in the fundamental diagram.

In the following, we denote by $\mathbf{f}$ the vector of path flows for all periods, i.e.,

$$\mathbf{f} = (f_1(1), f_2(1), ..., f_{|P|}(1), f_1(2), ..., f_1(T), ..., f_{|P|}(T)),$$

associated with a vector of path travel times $\tau(\mathbf{f})$. A vector of path flows is feasible if it is nonnegative and if it satsifes the demand, that is belongs to the following polyhedron:

$$\mathcal{K} = \{\mathbf{f} : \sum_{p \in P^w} f_p(t) = d^w(t), \forall w \in W, \mathbf{f} \geq \mathbf{0}\}.$$

Wardrop's first principle can be expressed as the following variational inequality (see Nagurney 1993):

$$\tau^*(\mathbf{f})'[\mathbf{f} - \mathbf{f}^*] \geq 0, \quad \forall \mathbf{f} \in \mathcal{K}. \tag{21}$$

However, there is no guarantee for the existence and the uniqueness of a solution to this variational inequality in a discrete-time setting. Even if the arc travel time function (12) is continuous in a discrete-time setting (see Theorem 3), the path travel times can be discontinuous, since (20) involves discrete time indices. This problem would not occur with a continuous-time formulation; however, the variational inequality would have infinite dimension. On the other hand, even if the Jacobian matrix of the arc travel times (17) is positive definite in a continuous-time setting (see Theorem 4), the Jacobian matrix of the path travel times is much more complex to analyze, since it relies on the path flow pattern (notice that if a path flow increases, it may affect the travel time of another path flow that shares some arcs in common) and the network structure (the capacity at the exit of an arc, $Q(t)$, depends on the flow exiting adjacent arcs). Further research is needed to characterize the structure of the Jacobian matrix of the path travel times. Alternatively, one could consider a formulation of the variational inequality in terms of arcs.

5.2 Path flow disaggregation

Formulating (21) relies on mapping path flows into path travel times. However, in Section 3, we only mapped link flows into link travel times. In order to use our previous results, we need to develop a procedure for disaggregating path flows into link flows. Once we know the path flows, we can easily compute the link travel times from (12) or (17), and obtain the path travel times through (20).

In fact, we propagate the path flows exactly in the same way as we explained in Subsection 4.1. Some additional care is nonetheless required. On the one hand, the flow should be disaggregated by paths to keep track of their final destination. On the other hand, one needs an aggregate value of flow to deal with capacity restrictions. An additional complexity comes from the prioritization of flows in case of capacity restrictions. In practice, priorities can be modified by adjusting the traffic controls (signals, stops, roundabouts). Even if the model is more complex when dealing with an entire network and multiple origin-destinations, it remains the same in essence (see Appendix C for more details).

5.3 Numerical example

To validate the adequacy of our model for solving DUE problems, we applied it to the transportation network test example of Sioux Falls (LeBlanc 1975) with time-varying demands. As shown in Figure 8, this transportation network consists of 24 nodes, 76 arcs and contains cycles. As input data, we used the daily capacities and node coordinates found on the website of Bar-Gera (2002) and the free-flow times proposed by Han (2003). From these data, we derived the lengths, free-flow speeds, hourly capacities (assuming 9 operating hours in a day) and densities of all arcs (assuming the quadratic flow-density relationship (2)). Table 1 presents for each link the origin node (O), the destination node (D), the maximum speed (in miles/hour), the density (in vehicles/mile), and the length (in miles).

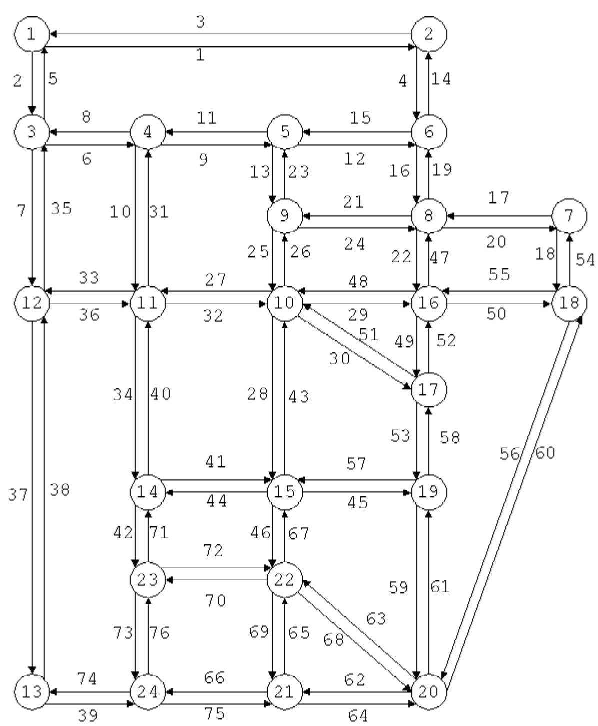

Figure 8: Sioux Falls Network

We considered 92 different OD pairs, namely all trips originating from nodes 1, 2, 3 and 4 to all other nodes, during 5 consecutive minutes. For each OD pair, we assumed that the traffic demand has a triangular distribution, starting from 0, reaching a maximum after two minutes, and falling back to 0 after four minutes. We arbitrarily defined the demand peak as the average hourly traffic, computed from the daily traffic data (Bar-Gera 2002) and assuming 9 operating hours.

To solve the dynamic user equilibrium problem on the Sioux Falls network, we implemented in

Table 1: Sioux Falls Network Data

Link	O	D	u^{max}	k^{max}	L	Link	O	D	u^{max}	k^{max}	L
1	1	2	71.0	162.1	4.3	39	13	24	31.6	71.7	1.3
2	1	3	27.6	376.6	1.1	40	14	11	51.3	42.3	2.1
3	2	1	71.0	162.1	4.3	41	14	15	28.4	80.2	1.4
4	2	6	22.1	99.7	1.1	42	14	23	23.7	92.5	0.9
5	3	1	27.6	376.6	1.1	43	15	10	34.2	175.6	2.1
6	3	4	31.6	240.9	1.3	44	15	14	28.4	80.2	1.4
7	3	12	47.3	219.7	1.9	45	15	19	52.6	123.0	1.6
8	4	3	31.6	240.9	1.3	46	15	22	31.6	135.2	0.9
9	4	5	71.0	111.3	1.4	47	16	8	18.9	118.4	0.9
10	4	11	31.6	69.1	1.9	48	16	10	39.5	54.7	1.6
11	5	4	71.0	111.3	1.4	49	16	17	47.3	49.1	0.9
12	5	6	39.5	55.7	1.6	50	16	18	52.6	166.3	1.6
13	5	9	18.9	234.7	0.9	51	17	10	23.0	96.5	1.8
14	6	2	22.1	99.7	1.1	52	17	16	47.3	49.1	0.9
15	6	5	39.5	55.7	1.6	53	17	19	55.2	38.8	1.1
16	6	8	47.3	46.0	0.9	54	18	7	47.3	219.7	0.9
17	7	8	52.6	66.2	1.6	55	18	16	52.6	166.3	1.6
18	7	18	47.3	219.7	0.9	56	18	20	113.6	91.6	4.5
19	8	6	47.3	46.0	0.9	57	19	15	52.6	123.0	1.6
20	8	7	52.6	66.2	1.6	58	19	17	55.2	38.8	1.1
21	8	9	15.8	142.2	1.6	59	19	20	55.2	40.2	2.2
22	8	16	18.9	118.4	0.9	60	20	18	113.6	91.6	4.5
23	9	5	18.9	234.7	0.9	61	20	19	55.2	40.2	2.2
24	9	8	15.8	142.2	1.6	62	20	21	26.3	85.5	1.6
25	9	10	31.6	195.9	0.9	63	20	22	40.4	55.8	2.0
26	10	9	31.6	195.9	0.9	64	21	20	26.3	85.5	1.6
27	10	11	28.4	156.4	1.4	65	21	22	63.1	36.8	1.3
28	10	15	34.2	175.6	2.1	66	21	24	47.3	45.9	1.4
29	10	16	39.5	54.7	1.6	67	22	15	31.6	135.2	0.9
30	10	17	23.0	96.5	1.8	68	22	20	40.4	55.8	2.0
31	11	4	31.6	69.1	1.9	69	22	21	63.1	36.8	1.3
32	11	10	28.4	156.4	1.4	70	22	23	35.5	62.6	1.4
33	11	12	21.0	103.7	1.3	71	23	14	23.7	92.5	0.9
34	11	14	51.3	42.3	2.1	72	23	22	35.5	62.6	1.4
35	12	3	47.3	219.7	1.9	73	23	24	63.1	35.8	1.3
36	12	11	21.0	103.7	1.3	74	24	13	31.6	71.7	1.3
37	12	13	142.0	81.0	4.3	75	24	21	47.3	45.9	1.4
38	13	12	142.0	81.0	4.3	76	24	23	63.1	35.8	1.3

C++ a dynamic version of the Frank-Wolfe algorithm. At every iteration we solved a linear optimization problem, calling the CPLEX library, to solve the path formulation of the DUE, described in Subsection 5.1. Since the number of paths between two nodes is exponential, we generated the used paths dynamically. The Frank-Wolfe algorithm for solving static variational inequality problems is fairly standard in the literature (e.g., see Nagurney 1993). As a result, we outline the algorithm very briefly in Algorithm 2.

Algorithm 2 A dynamic Frank-Wolfe algorithm for solving (21) with paths generation

Choose $\epsilon > 0$
$m \leftarrow 0$,
$P^w \leftarrow \emptyset$, $\forall w \in W$, $\mathbf{f}^m \leftarrow \emptyset$
$\tau^m = \tau(\mathbf{0})$ (free flow travel times)
for all $w \in W$ **do**
  p =shortest path w.r.t. τ^m
  $P^w \leftarrow P^w \cup \{p\}, \mathbf{f}^m \leftarrow \mathbf{f}^m \cup f_p$
  $f_p^m(t) = d^w(t), \forall t = 1, ..., T$
end for
repeat
  $m \leftarrow m + 1$
  $\tau^m = \tau(\mathbf{f}^m)$ (see Appendix C)
  for all $w \in W$ **do**
    p =shortest path w.r.t. τ^m
    $P^w \leftarrow P^w \cup \{p\}, \mathbf{f}^m \leftarrow \mathbf{f}^m \cup f_p$
    $f_p^m(t) = 0, \forall t = 1, ..., T$
  end for
  $\mathbf{f}^* = \arg\max_{\mathbf{f}}\{(\tau^m)'\mathbf{f} : \sum_{p \in P^w} f_p^w(t) = d^w(t), \forall w \in W, \mathbf{f} \geq 0\}$
  Use binary search to find $\mathbf{f}^{m+1} = \lambda\mathbf{f}^m + (1-\lambda)\mathbf{f}^*$, $0 \leq \lambda \leq 1$, such that $\tau(\mathbf{f}^{m+1})'(\mathbf{f}^m - \mathbf{f}^*) \geq 0$
until $\tau(\mathbf{f}^m)'(\mathbf{f}^m - \mathbf{f}^*) < \epsilon$

The shortest path generation at iteration m is based on the current travel times τ^m; in the first iteration, it is based on the free-flow travel times. A more sophisticated procedure can also be used to generate the shortest paths dynamically, namely with the use of a gap function (see for instance Ziliaskopoulos 2003).

The key issue in implementing this algorithm is the evaluation of the path travel times that relies on a procedure described in Appendix C. Even if the procedure for disaggregating path flows into link flows may seem cumbersome, it runs pretty fast. In fact, after 160 seconds on a Pentium 1.5 GHz with 1 GB Ram, $\tau(\mathbf{f}^m)'(\mathbf{f}^m - \mathbf{f}^*) < 0.1$ (i.e., the optimal solution is nearly reached). As shown in Figure 9, the algorithm converges really fast at the beginning but has some difficulties to stabilize, a well-known feature of the Frank-Wolfe algorithm (see Bertsekas 1995).

Table 2 displays the optimal used paths for all trips originating from Node 1 at times 1, 2 and 3

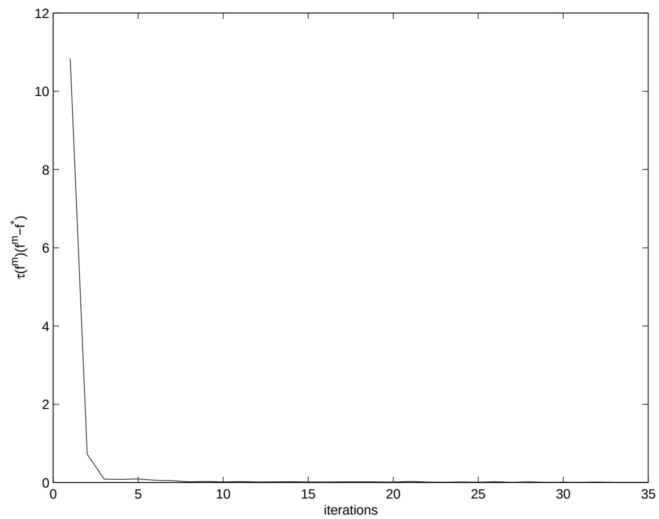

Figure 9: Convergence Plot of the Frank-Wolfe Algorithm

(periods associated with positive entering flow). Time t represents the path travel time (in minutes) of a flow starting its trip in period t, $t = 1, 2, 3$; a blank entry for a given period means that the path is not used in that period.

Table 2 emphasizes the importance of addressing the dynamic nature of traffic: for trips with destination nodes 6, 7, 8, 16, 18 and 20, a secondary path is used only temporarily. In addition, we can observe that the principle of optimality is maintained dynamically. For example, if one of the shortest paths for OD $\{1, 8\}$ in period 3 is $(1,3),(3,4),(4,5),(5,6),(6,8)$, one of the shortest paths for OD $\{1, 6\}$ in period 3 is $(1,3),(3,4),(4,5),(5,6)$. This observation may be used to speed up the resolution of the DUE.

6 Conclusions

In this paper, we proposed a methodology for deriving an analytical travel time function based on the theory of kinematic waves. In paticular, we derived a travel time function under the triangular and a quadratic fundamental diagrams. The derived travel time function integrates the traffic dynamics as well as shock waves. Moreover, we illustrated numerically that this travel time function is very consistent with the simulated travel times proposed in the literature. In addition, our model for traffic delays establishes a connection between the theory of kinematic waves and the theory of deterministic queues. This suggests an efficient procedure for propagating flow in a transportation network.

With the advent of advanced transportation management systems (ATMS), a lot of attention

Table 2: Optimal used paths for trips originating from Node 1

OD	Path	Time 1	Time 2	Time 3
{1,2}	(1,2)	4.7	5.8	6.9
{1,3}	(1,3)	3.8	5.6	6.5
{1,4}	(1,3),(3,4)	6.5	8.7	9.1
{1,5}	(1,3),(3,4),(4,5)	7.9	9.9	10.3
{1,6}	(1,2),(2,6)	8.7	11.4	12.8
	(1,3),(3,4),(4,5),(5,6)			12.7
{1,7}	(1,2),(2,6),(6.8),(8,7)	12.5	14.4	15.7
	(1,3),(3,4),(4,5),(5,6),(6,8),(8,7)			15.7
{1,8}	(1,2),(2,6),(6.8)	10.6	12.6	14.0
	(1,3),(3,4),(4,5),(5,6),(6,8)			14.0
{1,9}	(1,3),(3,4),(4,5),(5.9)	11.9	13.8	13.7
{1,10}	(1,3),(3,4),(4,5),(5.9),(9,10)	13.9	15.6	15.5
	(1,3),(3,12),(12,11),(11,10)	13.9	15.6	15.8
{1,11}	(1,3),(3,4),(4,11)	10.7	12.6	12.8
	(1,3),(3,12),(12,11)	10.7	12.6	12.8
{1,12}	(1,3),(3,12)	6.3	8.4	9.0
{1,13}	(1,3),(3,12),(12,13)	8.3	10.2	10.8
{1,14}	(1,3),(3,4),(4,11),(11,14)	13.3	15.0	15.2
	(1,3),(3,12),(12,11),(11,14)	13.3	15.0	15.2
{1,15}	(1,3),(3,4),(4,11),(11,14),(14,15)	16.3	18.0	18.2
	(1,3),(3,12),(12,13), (13,24),(24,21),(21,22),(22,15)	16.9	17.4	18.0
{1,16}	(1,2),(2,6),(6.8),(8,16)	14.3	15.8	17.0
	(1,3),(3,4),(4,5),(5,6),(6,8),(8,16)			17.0
{1,17}	(1,3),(3,4),(4,5),(5.9),(9,10),(10,17)	19.1	20.6	20.5
	(1,3),(3,12),(12,11),(11,10),(10,17)	19.1	20.6	20.8
{1,18}	(1,2),(2,6),(6.8),(8,7),(7,18)	13.7	15.6	17.0
	(1,3),(3,4),(4,5),(5,6),(6,8),(8,7),(7,18)			17.0
{1,19}	(1,3),(3,4),(4,5),(5.9),(9,10),(10,17),(17,19)	20.3	21.8	21.7
	(1,3),(3,12),(12,13), (13,24),(24,21),(21,22),(22,15),(15,19)	18.7	19.2	19.8
{1,20}	(1,2),(2,6),(6.8),(8,7),(7,18),(18,20)	16.1	18.0	19.3
	(1,3),(3,12),(12,13), (13,24),(24,21),(21,20)		18.0	18.6
{1,21}	(1,3),(3,12),(12,13),(13,24),(24,21)	13.9	14.4	15.0
{1,22}	(1,3),(3,12),(12,13),(13,24),(24,21),(21,22)	15.0	15.6	16.2
{1,23}	(1,3),(3,12),(12,13),(13,24),(24,23)	13.3	13.8	14.4
{1,24}	(1,3),(3,12),(12,13),(13,24)	12.0	12.6	13.2

has been devoted to solving the dynamic user equilibrium problem. We showed that the travel time function that we derived can be incorporated within a DUE setting, since it is (strictly) monotone and satisfies the FIFO property under some reasonable conditions. As an illustration, we applied our model to the Sioux Falls network, emphasizing the dynamic nature of the traffic assignment.

Further research is necessary to develop a mathematical formulation of the flow propagation proposed in Appendix C. With such a formulation, one would be able to solve directly a link formulation of the DUE, instead of a path formulation. Such a link formulation would take advantage of the continuity and the (strict) monotonicity of the travel time functions derived in Section 4. Moreover, the proposed model for delays can be incorporated into more general or alternative traffic assignment problems as the system equilibrium, the stochastic DUE, or the DUE with departure time choices. Finally, alternative algorithms can be used to solve the DUE problem, such as projection methods or simplicial decomposition.

Appendix

Appendix A - On Assumption A2

Although in the paper we focus on the density evolution at the road entrance, $B(0,t)$, a similar line of reasoning can be applied to the road exit. Let us consider the characteristic line $(x, t_0 + dk(0,t_0)/df x)$ passing through the origin. Since the density remains constant along a characteristic line, its total derivative with respect to x is zero. That is,

$$\frac{dk(x, t_0 + \frac{dk(0,t_0)}{df}x)}{dx} = \frac{dk(0,t_0)}{dx} = 0.$$

On the other hand, the total derivative can be expressed in terms of partial derivatives, by using the chain rule:

$$\begin{aligned} \frac{dk(0, t_0 + \frac{dk(0,t_0)}{df}0)}{dx} &= \frac{\partial k(0,t_0)}{\partial x} + \frac{\partial k(0,t_0)}{\partial t}\frac{dk(0,t_0)}{df} \\ &= B(0,t_0) + \frac{\partial k(0,t_0)}{\partial t}\frac{dk(0,t_0)}{df}. \end{aligned}$$

Therefore,

$$B(0,t_0) = -\frac{\partial k(0,t_0)}{\partial t}\frac{dk(0,t_0)}{df}.$$

The latter expression allows us to express $B(0,t)$ in terms of the rate of evolution of density with respect to time. This is very useful from a computational point of view: typically, in a dynamic traffic assignment problem, one knows or one chooses the entering flows at every period of time.

By converting the flow into density, we obtain the rate of evolution of density with respect to time, and hence $B(0,t)$. If such a relation did not exit, we would have to position additional sensors at some intermediate points of the road to evaluate the parameter $B(0,t)$.

Appendix B - Existence of at most one shock under a quadratic diagram

Potentially, a shock could result from the focusing of two even forward waves, or two even backward waves. We will impose a condition eliminating these cases. This will imply that there is at most one shock on every road, occurring when forward waves intersect backward waves.

Let us first consider two characteristic lines associated with forward waves, one emanating from the origin 0, and the other emanating from some arbitrary location y, with respective positive slopes defined according to Assumption **A2**. That is,

$$\frac{df(k(0,t))}{dk} = u^{max}(1 - 2au^{max}k(0,t)), \text{ and } \frac{df(k(y,t))}{dk} = u^{max}(1 - 2au^{max}(k(0,t) + B(0,t)y)).$$

A shock occurs at the intersection of these two characteristic lines, at location $\hat{x} = (1 - 2au^{max}k(0,t))/(2au^{max}B(0,t))$. Since the shock location is independent of y, we conclude that all forward characteristic lines intersect each other at the same location.

Therefore, no local shock occurs if $\hat{x} > L$, that is, if $B(0,t) \leq (k^{max}/2 - k(0,t))/L$. This is automatically satisfied since, from Assumption **A2**, we require that $0 \leq k(0,t) + B(0,t)L \leq k^{max}/2$.

Similarly, for backwards moving waves, we consider two characteristic lines, one emanating from the destination L, and the other emanating from some location $0 < y < L$. No local shock occurs if the shock location is below 0 (before the road entrance), i.e., $B(L,t) \leq (k(L,t) - k^{max}/2)/L$. This is automatically satisfied from Assumption **A2**, i.e., when $k^{max}/2 \leq k(L,t) - B(L,t)L \leq k^{max}$.

Appendix C - Flow propagation in a transportation network with multiple origin-demand pairs

The flow propagation consists in converting a vector of path flows $f_p(t)$ into a vector of link flows, $f_a(0,t)$ and $f_a(L_a,t)$, so that the travel times on each link can be computed through (12) or (17). The associated path travel times are computed by summing up these link travel times through (20).

We first introduce some additional notations. Let $B(p)$ be the first arc on path p and $E(p)$ be the last arc on path p. Let $U(a)$ be the set of arcs upstream of arc a, and $D(a)$ be the set of arcs downstream of a, for some $a \in A$. Algorithm 3 details the implementation of the flow propagation in a network and is explained in more details below.

We apply the procedure iteratively at every period of time, and within a certain period, for every arc of the network (considered in any order), so that cycles are authorized in the transportation network. The entering flow on a particular arc is the minimum between the potential exiting flow of upstream arcs and the road capacity. Once the entering flow has been computed, all quantities under interest are updated, both at the entrance of the arc and at the exit of the upstream arcs. Finally, the flow is propagated to the end of the arc and will be candidate for exiting the arc θ periods later, as in Subsection 4.1.

Algorithm 3 Network flow propagation for the triangular fundamental diagram

```
Input: f_p(t) (path flows)
Initialization
for all a ∈ A do
  for t = 1 to T do
    Global Variables: f_a, f̄_a, f_{a,p}, ∀p ∈ P : a ∈ p
    Potential Entering Flow (a,t)
    Actual Entering Flow (a,t)
    for all p ∈ P such that a ∈ p do
      f_{a,p} = f̄_{a,p} min{1, f_a / f̄_a} (disaggregation)
      Quantity Update at the Entrance (a,t,p)
      Quantity Update at the Exit (a,t,p)
    end for
  end for
end for
Output: A_{a,p}(t), D_{a,p}(t) ∀a ∈ A
```

Initialization At the beginning of the algorithm, all quantities are initalized to zero, namely the potential cumulative number of vehicles at the entrance and at the exit, $\bar{A}_a(t)$ and $\bar{D}_a(t)$, and the actual cumulative number of vehicles at the entrance and at the exit, $A_a(t)$ and $D_a(t)$, $\forall a \in A, t = 1, ..., T$.

Algorithm 4 Initialization

```
for all a ∈ A do
  for t = 1 to T do
    for all p ∈ P such that a ∈ p do
      A_{a,p}(t) = 0, Ā_{a,p}(t) = 0, D_{a,p}(t) = 0, D̄_{a,p}(t) = 0
      U_a = {u : ((u, a) ∈ A}
    end for
  end for
end for
```

In what follows, we focus on a particular time period t and a particular arc a. We illustrate our procedure with the example displayed in Figure 10. In this network example, four paths transit through node 2, each with a flow of 10 vehicles: 1. (1,2); 2. (1,2),(2,3); 3. (1,2),(2,4); 4. (2,3). We consider a capacity of 20 vehicles/hour for arc u and 10 vehicles/hour for arc a.

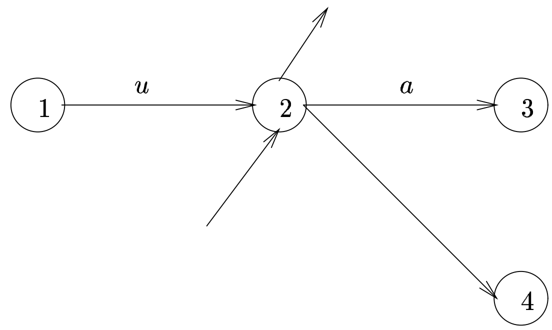

Figure 10: Example of flow propagation on arc b. The paths are: 1. (1,2); 2. (1,2),(2,3); 3. (1,2),(2,4); 4. (2,3).

Potential Entering Flow The entering potential flow on arc a is the sum of the flow incoming to the transportation network (if a is the beginning of a path p, i.e., if $a = B(p)$) and the transient flow, coming from upstream arcs.

Consider an upstream arc $u \in U(a)$. We assume that the flow that exits the transportation network at the end of arc u has priority over the transient flow. Therefore, the upstream exit capacity Q_u alloted to the transient flow is less than the arc capacity, when some flow reaches its final destination. On the other hand, the potential total transient flow is the difference between the potential cumulative number of vehicles and the actual cumulative number of vehicles. In the example, the potential flow at the exit of arc u that is in transit is the sum of the flows of Paths 2 and 3, i.e., $10 + 10 = 20$. However, since the 10 vehicles that leave the network (Path 1) have priority over the transient vehicles (Paths 2 and 3), the effective capacity for the transient vehicles is 20-10=10.

For every path p, the potential entering flow is the difference between the potential cumulative number of vehicles at the entrance and the actual cumulative number of vehicles. In the case of upstream capacity restriction, the incoming flow is weighted in proportion to its relative capacity utilization. In the example, there are 10 vehicles that wish to enter the network (Path 4). On the other hand, there are 20 vehicles that wish to exit arc u (Paths 2 and 3), but only 10 are authorized to transit because of upstream capacity restrictions. As a result, the potential flow at the entrance of arc a is $10 + (10(10/20) + 10(10/20)) = 20$ vehicles.

Actual Entering Flow The entering flow can be restricted by capacity or space limit (considering the number of vehicles currently traveling on the arc). In the example, 20 vehicles wish to enter arc a, but arc a's capacity is 10 vehicles per hour. Therefore, the actual flow is 10 vehicles.

Algorithm 5 Potential Entering Flow (a,t)

for all $u \in U_a$ **do**
  $Q_u = f_u^{max} - \sum_{p:u=E(p)}(D_{u,p}(t) - D_{u,p}(t-1))$ (remaining upstream capacity)
  $\bar{f}_u = \sum_{p:u\neq E(p)}(\bar{D}_{u,p}(t) - D_{u,p}(t-1))$ (transient upstream flow)
end for
for all $p \in P$ such that $a \in p$ **do**
  $\bar{A}_{a,p}(t) = \bar{A}_{a,p}(t-1) + f_p(t)$
  if $a = B(p)$ **then**
    $\bar{f}_{a,p} = (\bar{A}_{a,p}(t) - A_{a,p}(t-1))$
  else
    for all $u \in U(a)$ s.t. $u \in p$ **do**
      $\bar{f}_{a,p} = (\bar{A}_{a,p}(t) - A_{a,p}(t-1)) \min\{1, \frac{Q_u}{\bar{f}_u}\}$
    end for
  end if
end for
$\bar{f}_a = \sum_{p:a=B(p)} \bar{f}_{a,p} + \sum_{p:u\in U(a),u,a\in p} \bar{f}_{a,p}$

Algorithm 6 Actual Entering Flow (a,t)

$f_a = \min\{\bar{f}_a, f_a^{max}, k_a^{max} L_a - \sum_{p:a\in p}(A_{a,p}(t-1) - D_{a,p}(t))\}$

Quantity Update at the Entrance We can arbitrarily assume that all travel patterns have the same priority. Therefore, the actual flow is disaggregated for every path in proportion to their relative importance, and all quantities are updated accordingly. In the example, the potential flows at the entrance of arc a were 10 vehicles for Path 4, and 5 vehicles (10(10/20)) for both Paths 2 and 3. Since only 10 vehicles are authorized to enter arc a because of capacity restrictions, the resulting actual flows are 5 vehicles for Path 4 and 2.5 vehicles for both Paths 2 and 3.

Algorithm 7 Quantity Update at the Entrance (a,t,p)

$A_{a,p}(0,t) \leftarrow A_{a,p}(0,t) + f_{a,p}$
for all $u \in U(a)$ s.t. $u \in p$ **do**
  $D_{u,p}(0,t) \leftarrow D_{u,p}(0,t) + f_{a,p}$
end for

Quantity Update at the Exit Once the flow $f_a(0,t)$ is known, the free-flow travel time θ can be computed. The actual entering flows $f_{a,p}(0,t)$, $\forall p : a \in p$, are candidates for exiting the road θ periods later, and the quantities at the exit are updated accordingly.

Algorithm 8 Quantity Update at the Exit (a,t,p)

$\theta = u_0 L$
for $s = t + \theta$ to T **do**
  $\bar{D}_{a,p}(0,s) \leftarrow \bar{D}_{a,p}(0,s) + f_{a,p}$
  if $a = E(p)$ **then**
    $D_{a,p}(0,s) \leftarrow D_{a,p}(0,s) + f_{a,p}$
  else
    for all $d \in D(a)$ s.t. $d \in p$ **do**
      $\bar{A}_{d,p}(0,s) \leftarrow \bar{A}_{d,p}(0,s) + f_{a,p}$
    end for
  end if
end for

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