VIII. STATISTICAL COMMUNICATION THEORY

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A. A THEOREM CONCERNING NOISE FIGURES

The single frequency noise figure of a one-tube amplifier, such as that shown in Fig. VIII-1, depends upon the coupling of the tube to the signal source. There exists an optimum coupling network for which the noise figure assumes a minimum value. This minimum value is then the greatest lower bound of the noise figure of such an amplifier. Similarly, for an amplifier using more than one tube, the noise figure will depend upon the interconnection of the tubes and the source. In general, for such an amplifier, the noise from any noise source reaches the output through more than one transmission path. Hence it seems possible that, through an appropriate interconnection of the tubes, cancellation effects among the transmissions through these paths may result in a very low noise figure; in particular, a noise figure lower than the greatest lower bound for the one-tube amplifier. In the past, attempts have been made to find circuits which would, in such a way, achieve these low noise figures.

This report is concerned with the greatest lower bound of the noise figure of a very broad class of systems employing n tubes or other amplifying devices, which we will refer to as UNITS.

The theorem concerns the single frequency noise figure of systems T belonging to the class τ, having the general form shown in Fig. VIII-2. The general system T belonging to τ consists of n UNITS and the coupling network C. The system is driven by a signal source S; its output is the voltage V shown in Fig. VIII-2. The units 1 through n and the signal source S are considered to be given and are unalterable.

For the systems T at the frequency f of the noise-figure calculations:

A. The noise generated within the system is such that it can be represented, as far as noise-figure calculations at the frequency f are concerned, by ensembles of constant-voltage and/or current generators of this frequency. We shall refer to these sine-wave generators as noise generators.

B. As far as its output terminals are concerned, the signal source S can be represented by the equivalent circuit shown in Fig. VIII-2, in which I_{NS} is the equivalent signal source noise generator.

C. The coupling network C consists of linear passive bilateral elements and possibly contains noise generators.

D. UNIT j consists of a single controlled source $\mathcal{V}_j$ (arbitrarily chosen as a voltage source) and the networks N_j and M_j. The constraint between the controlled source $\mathcal{V}_j$ and the control voltage E_j is linear. N_j consists of linear bilateral elements and noise generators. Its input connections consist of one or more terminal pairs. M_j consists

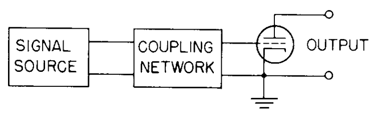

Fig. VIII-1
One-tube amplifier.

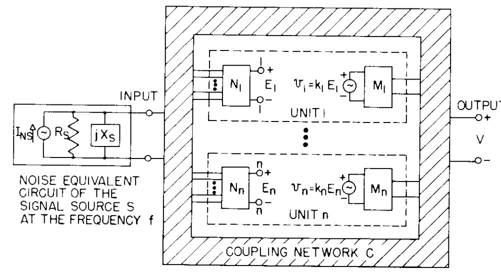

Fig. VIII-2
System T.

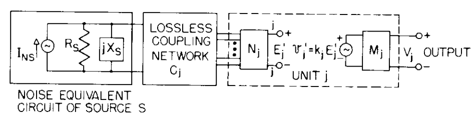

Fig. VIII-3
System T_j.

of linear passive bilateral elements and is driven by the controlled source $\mathcal{V}_j$. M_j does not contain any noise generators; all noise generated in it is represented, as far as its output terminal pair is concerned, by an equivalent noise generator in N_j.

E. With the signal source as excitation, the system output V, with all of the controlled sources $\mathcal{V}_j$'s replaced by short circuits, must be negligibly small compared to the output that occurs with all of the controlled sources operating.

F. The condition stated in E must also hold for the excitation consisting of the simultaneous action of all the noise generators in the system.

G. The noise generators in the system and signal source can be grouped into n + 2 groups according to their location. Noise sources are located in the n networks N_j, in the signal source, and possibly in the coupling network C. No correlation is permitted among the noise sources of different groups. However, correlation may exist among the noise sources located within any group.

Having defined the class τ of systems which we shall consider, we now state the theorem.

THEOREM: "Let the only constraint upon the interconnection of n given UNITS 1, ..., n be that the resulting system T must belong to the class τ. Then the greatest lower bound of the single-frequency noise figure of T, considering all allowable interconnections of the n UNITS, is equal to the noise figure of a system T_k (of the form shown in Fig. VIII-3) consisting of the "best UNIT" of the n UNITS coupled to the source through an optimum coupling network."

By the term "optimum" we mean optimum with respect to the noise figure of the system formed by the coupling network and the particular UNIT. The expression "best UNIT" refers to one of the UNITS 1, . . . , n which, when coupled to the source through an optimum coupling network (Fig. VIII-3), yields a noise figure at least as small as the noise figure of any of the n UNITS similarly connected with their optimum coupling networks.

By letting the UNITS be the noise equivalent circuits of triodes the theorem can be applied to n-tube systems, and a lower bound can be established for the noise figure of such systems in terms of the minimum noise figure attainable with one tube and an optimum coupling network.

This theorem was presented at the I. R. E. National Convention, March 21, 1955. Its proof, along with examples, will appear in full in the Convention Record.

A. G. Bose, S. D. Pezaris

B. SECOND-ORDER CORRELATION FUNCTIONS

1. Properties

Let $f_1(t)$ be a function that satisfies these three conditions:

I. The autocorrelation function $\phi_{11}(\tau)$ exists and is continuous at $\tau = 0$.

II. The autocorrelation function of $f_1^2(t)$ exists for all values of τ.

III. The second-order autocorrelation function $\phi^*_{111}(\tau_1, \tau_2)$ exists.

We can then show that the second-order autocorrelation function of $f_1(t)$ is continuous everywhere.

For the proof let us consider

$$|\phi^*_{111}(\tau_1, \tau_2) - \phi^*_{111}(\tau_1 + k_1, \tau_2 + k_2)| \tag{1}$$

in which $\phi^*_{111}(\tau_1, \tau_2)$ is defined as

$$\phi^*_{111}(\tau_1, \tau_2) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t)\, f_1(t + \tau_1)\, f_1(t + \tau_2)\, dt \tag{2}$$

and the abbreviated expression is

$$\phi^*_{111}(\tau_1, \tau_2) = \overline{f_1(t)\, f_1(t + \tau_1)\, f_1(t + \tau_2)} \tag{3}$$

To simplify the work that follows, let

$$a = f_1(t),\ b = f_1(t + \tau_1),\ c = f_1(t + \tau_2),\ d = f_1(t + \tau_1 + k_1),\ e = f_1(t + \tau_2 + k_2) \tag{4}$$

and the expression given as Eq. 1 be A. Accordingly, Eq. 1 is

$$A = |\overline{abc} - \overline{ade}| \tag{5}$$

which can be put into the form

$$A = |\overline{ac(b-d)} + \overline{ae(b-d)} + \overline{adc} - \overline{abe}| \tag{6}$$

Since for any numbers α, β, γ, we have the inequality

$$|\alpha+\beta+\gamma| \leqslant |\alpha| + |\beta| + |\gamma| \tag{7}$$

it follows that for Eq. 6

$$A \leqslant |\overline{ac(b-d)}| + |\overline{ae(b-d)}| + |\overline{adc} - \overline{abe}| \tag{8}$$

Since

$$|\overline{ac(b-d)}| \leqslant \overline{|ac(b-d)|} \tag{9}$$

and

$$|\overline{ae(b-d)}| \leqslant \overline{|ae(b-d)|} \tag{10}$$

we may write

$$A \leqslant \overline{|ac(b-d)|} + \overline{|ae(b-d)|} + |\overline{adc} - \overline{abe}| \tag{11}$$

The three right-hand members of this equation will be considered one by one. For the first term we need the proof that $\overline{|b-d|^2}$ exists. Since

$$\overline{|b-d|^2} = \overline{b^2} + \overline{d^2} - \overline{2bd} \tag{12}$$

and $\overline{bd}$ is

$$\overline{bd} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t+\tau_1)\, f_1(t+\tau_1+k_1)\, dt$$

$$= \phi_{11}(k_1) \tag{13}$$

which exists by condition I, we have

$$\overline{|b-d|^2} = 2\left[\phi_{11}(0) - \phi_{11}(k_1)\right] \tag{14}$$

showing that Eq. 12 exists. Furthermore, by condition II, $\overline{|ac|^2}$ exists because

$$\overline{|ac|^2} = \overline{a^2c^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1^2(t)\, f_1^2(t+\tau_2)\, dt \tag{15}$$

Now, by the Schwartz inequality

$$\overline{|ac(b-d)|} \leqslant \left[\overline{|ac|^2}\ \overline{|b-d|^2}\right]^{1/2} = \left\{2\overline{|ac|^2}\left[\phi_{11}(0) - \phi_{11}(k_1)\right]\right\}^{1/2} \tag{16}$$

By similar reasoning, the second right-hand term of Eq. 11 satisfies the inequality

$$\overline{|ae(b-d)|} \leqslant \left\{2\overline{|ae|^2}\left[\phi_{11}(0) - \phi_{11}(k_1)\right]\right\}^{1/2} \tag{17}$$

In considering the last term of Eq. 11, let us write

$$|\overline{adc} - \overline{abe}| = |\overline{adc} - \overline{abc} + \overline{abc} - \overline{abe}| \tag{18}$$

from which we obtain the inequality

$$|\overline{adc} - \overline{abc}| \leqslant |\overline{adc} - \overline{abc}| + |\overline{abc} - \overline{abe}| \tag{19}$$

$$\leqslant |\overline{ac(d-b)}| + |\overline{ab(c-e)}| \tag{20}$$

Application of inequalities similar to those of Eqs. 9 and 10 leads to

$$|\overline{adc} - \overline{abc}| \leqslant \overline{|ac(d-b)|} + \overline{|ab(c-e)|} \tag{21}$$

in which the first right-hand term is the same as Eq. 16, and by the Schwartz inequality the last term of Eq. 21 is

$$\overline{|ab(c-e)|} \leqslant \left\{2\overline{|ab|^2}\left[\phi_{11}(0) - \phi_{11}(k_2)\right]\right\}^{1/2} \tag{22}$$

Combining Eqs. 16, 17, and 22, we find that Eq. 11 becomes

$$A \leqslant 2\left\{2\overline{|ac|^2}\left[\phi_{11}(0) - \phi_{11}(k_1)\right]\right\}^{1/2}$$

$$+ \left\{2\overline{|ae|^2}\left[\phi_{11}(0) - \phi_{11}(k_1)\right]\right\}^{1/2} + \left\{2\overline{|ab|^2}\left[\phi_{11}(0) - \phi_{11}(k_2)\right]\right\}^{1/2} \tag{23}$$

Inasmuch as condition II assures that $\overline{|ac|^2}$, $\overline{|ae|^2}$, and $\overline{|ab|^2}$ are finite for all values of τ_1 and τ_2, and condition I enables us to make $|\phi_{11}(0) - \phi_{11}(k_1)|$ and $|\phi_{11}(0) - \phi_{11}(k_2)|$ as small as we please by a proper choice of k_1 and k_2, it follows that, given an $\epsilon > 0$, k_1 and k_2 can be so chosen that

$$A = |\phi^*_{111}(\tau_1,\tau_2) - \phi^*_{111}(\tau_1 + k_1, \tau_2 + k_2)| < \epsilon \tag{24}$$

This proves that $\phi^*_{111}(\tau_1,\tau_2)$ is continuous for all values of τ_1 and τ_2 under conditions I, II, and III.

J. Y. Hayase

2. Calculations

The second-order autocorrelation function $\phi_{111}(\tau_1, \tau_2)$ of a periodic wave consisting of the positive half cycles of a sine wave described in the Quarterly Progress Report, April 15, 1954 (p. 51), has been obtained, and inaccuracies in the previously reported results have been corrected.

Figure VIII-4 shows the τ_1, τ_2-plane of the second-order autocorrelation function of the clipped sine wave. In Δ OAB and Δ OFA of the plane the expressions for the function are as follows:

Δ OAB:

$$\phi_{111}(\tau_1, \tau_2) = \frac{2A^3}{3\pi} \cos^3 \frac{\tau_1}{2} \cos \frac{\tau_1 + 2\tau_2}{2} \tag{1}$$

Δ OFA:

$$\phi_{111}(\tau_1, \tau_2) = \frac{A^3}{3\pi} (\cos \tau_1 + \cos \tau_2) \cos^2 \frac{\tau_1 + \tau_2}{2} \tag{2}$$

The function is zero in the shaded areas of the plane.

Knowing the function in Δ OAB, Δ OFA, and the shaded regions, we can determine

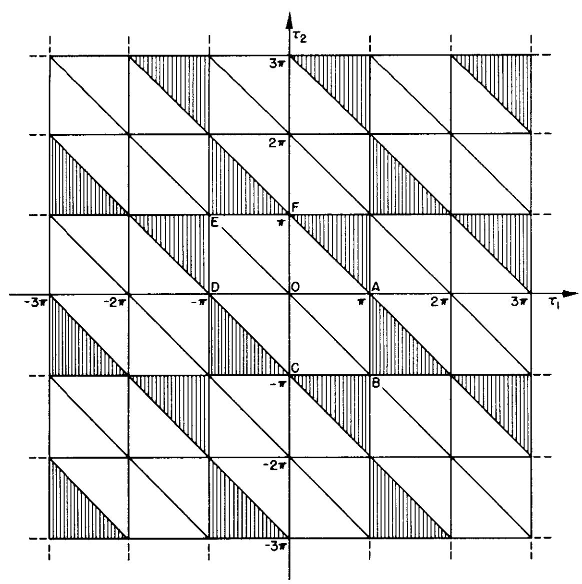

Fig. VIII-4

τ_1, τ_2-plane of the second-order autocorrelation function of the clipped sine wave. The function is zero in the shaded areas.

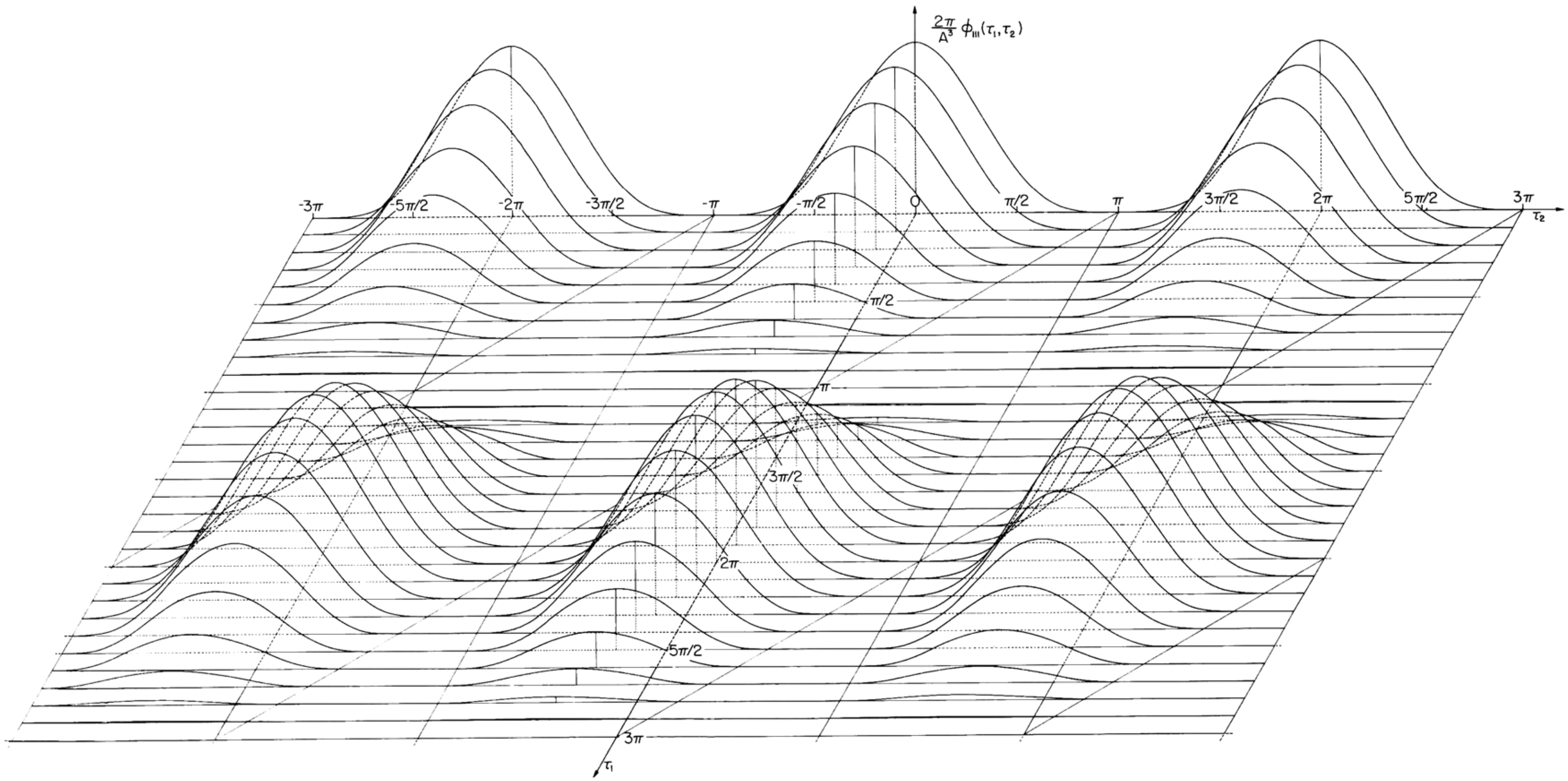

Fig. VIII-5

The second-order autocorrelation function (multiplied by $2\pi/A^3$) of the clipped sine wave.

the function in the entire τ_1, τ_2-plane. Since expression 2 is unchanged by replacing the variables τ_1 and τ_2 by $-\tau_1$ and $-\tau_2$, respectively, the function in Δ OCD is obtained by a 180° rotation of that in Δ OFA about the center O. By actual calculation, it can be shown that the expression for the function in ΔOCB is identical to Eq. 1 when τ_1 and τ_2 are interchanged. Hence the function in ΔOCB and ΔOAB are symmetrical about the line $\tau_1 = -\tau_2$. Inasmuch as expression 1 remains the same for $-\tau_1$ and $-\tau_2$, the function in Δ ODE is determined by rotating that in Δ OAB about O by 180°. Similarly, in Δ OFE, the function is obtained by a 180° rotation of that in Δ OCB about O.

Having determined the function in polygon ABCDEF, we can show geometrically that the remainder of the function is found by translating that in the polygon into the other polygons shown in Fig. VIII-4. Figure VIII-5 gives a plot of the function.

Although the expression for the autocorrelation function in the Quarterly Progress Report, April 15, 1954 (p. 51), when simplified, is the same as Eq. 1 of this report, the figure (Fig. X-2, p. 52) in the April report is incorrect, and should be replaced by Fig. VIII-5 of this report.

J. Y. Hayase

3. Measurements

In the method which has been used for measuring second-order autocorrelation functions (1, 2) the correlator inputs are reversed when measurements go from the first quadrant into the fourth quadrant. A new method has been introduced. It consists of stepping τ_2 from minus to plus for a given τ_1 and repeating the procedure for various values of τ_1, thereby obtaining the function for both quadrants without reversing inputs. This is done by adjusting the correlator in such a way that at the beginning of the measurement the "B" samples are taken before the "A" samples, and then the "B" samples are advanced in steps into the $+\tau_2$ region.

The block diagram for the new method is shown in Fig. VIII-6. The inputs to the correlator are

$$\left.\begin{aligned} g_1(t) &= f(t - \tau_1)\, f(t) \\ g_2(t) &= f(t) \end{aligned}\right\} \tag{1}$$

and the correlator measures

$$\begin{aligned} \phi_{111}(\tau_1, \tau_2) &= \overline{g_1(t)\, g_2(t + \tau_2)} = \overline{f(t - \tau_1)\, f(t)\, f(t + \tau_2)} \\ &= \overline{f(t)\, f(t + \tau_1)\, f(t + \tau_1 + \tau_2)} \end{aligned} \tag{2}$$

In the old method after measuring the function in the first quadrant the inputs were

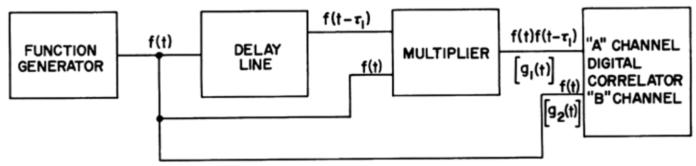

Fig. VIII-6

Block diagram for measurement of second-order autocorrelation functions.

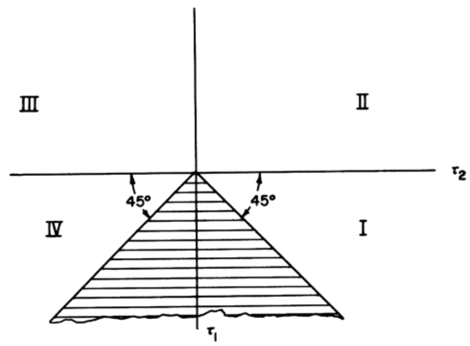

Fig. VIII-7

Region of required measurement of second-order autocorrelation functions.

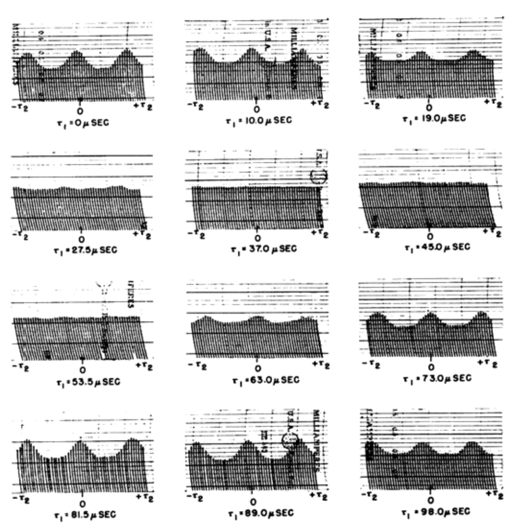

Fig. VIII-8

Second-order autocorrelation function for clipped sine wave, first and fourth quadrants.

reversed, giving the function

$$\phi_{111}(\tau_1, -\tau_2) = \overline{g_1(t)\, g_2(t+\tau_2)} = \overline{f(t)\, f(t+\tau_2)\, f(t-\tau_1+\tau_2)}$$
$$= \overline{f(t)\, f(t+\tau_1)\, f(t+\tau_1-\tau_2)} \qquad (3)$$

which is $\phi_{111}(\tau_1, \tau_2)$ in the fourth quadrant.

As a result of the symmetry properties discussed in a previous article (3), $\phi_{111}(\tau_1, \tau_2)$ need only be obtained in the shaded region shown in Fig. VIII-7 provided that the second-order probability distribution of the time series f(t) is equal to that of the time series f(-t). For these time series all other regions of $\phi_{111}(\tau_1, \tau_2)$ can be obtained by different orientations of this region. The time of measurement is cut down and reversing of inputs is now unnecessary.

The results in a measurement of the second-order autocorrelation of a clipped sine wave (4) are shown in Fig. VIII-8. Calculations of this function are given in Section VIII-B2.

A. G. Bose, K. L. Jordan

References

1. A. G. Bose and Y. W. Lee, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., Jan. 15, 1954, p. 41.
2. W. B. Smith, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., Jan. 15, 1955, p. 49.
3. Y. W. Lee et al., Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., Oct. 15, 1954, p. 63.
4. Fig. X-1, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., April 15, 1954, p. 52.

C. SOME PROPERTIES OF EXPERIMENTAL AUTOCORRELATION FUNCTIONS

It will be shown that there exist some definite restrictions on the shape and rates of change in the vicinity of the origin of measured autocorrelation functions. The conditions under which these restrictions apply to the autocorrelation functions of the more general class of time series used in theoretical analysis will also be shown.

Limiting the following discussion to measured autocorrelation functions implies that only those time series that can occur in the laboratory will be considered. Let f(t) be such a measurable time series. Since physically there is no such thing as an infinite voltage or current, f(t) is bounded. Also, since applying f(t) to a reactive element cannot give rise to infinite voltages or currents, $f'(t) = df/dt$ must also exist. Additional restrictions are implied by the requirement that f(t) be physical, but for the purposes of this discussion the two properties mentioned above are sufficient.

The autocorrelation function of f(t) is given by

$$\phi(\tau) = \overline{f(t)\, f(t+\tau)}$$

where the bar indicates average with respect to time. Formally, the first derivative of $\phi(\tau)$ is

$$\phi'(\tau) = \overline{f(t)\, f'(t+\tau)}$$

Writing $u = t + \tau$,

$$\phi'(\tau) = \overline{f(u-\tau)\, f'(u)}$$

whence

$$\phi''(\tau) = \overline{-f'(u-\tau)\, f'(u)}$$

or

$$\phi''(\tau) = \overline{-f'(t)\, f'(t+\tau)} \qquad (1)$$

Thus the physical nature of f(t) (as defined above) immediately implies the existence of ϕ, ϕ', and ϕ''.

It is well known that, for all τ, $|\phi(\tau)| \leqslant |\phi(0)|$. This relation is also true without absolute value signs. Since from Eq. 1, $\phi''(\tau)$ also has the form of an autocorrelation function, $|\phi''(\tau)| \leqslant |\phi''(0)|$ follows immediately. In this case, however, because of the minus sign in Eq. 1, $\phi''(\tau) \geqslant \phi''(0)$. Furthermore, since $\phi(\tau)$ is even and $\phi'(\tau)$ continuous, $\phi'(0) = 0$, implying that any measured autocorrelation function has a flat top at $\tau = 0$. These relations will be useful in establishing the following theorem.

Theorem. If $\phi'(\tau)$ (and therefore $\phi(\tau)$) is continuous in the closed interval $[0,\tau]$ and differentiable in the open interval $(0,\tau)$, there exists a positive finite constant k that satisfies the expression

$$\phi(0) - \phi(\tau) \leqslant k\tau^2$$

Note, in particular, that these hypotheses are satisfied by any measurable f(t), since for such a time series $\phi''(\tau)$ exists for all τ, whence $\phi'(\tau)$ is necessarily continuous and differentiable for all τ.

Proof. Since $\phi(\tau)$ is differentiable in $[0,\tau]$, the mean value theorem asserts that there exists an η $(0 < \eta < \tau)$ that satisfies

$$\phi(\tau) - \phi(0) = \phi'(\eta)\,(\tau - 0) = \phi'(\eta)\,\tau \qquad (2)$$

Furthermore, since $\phi'(\tau)$ is itself differentiable in $[0,\tau]$, there exists a ξ $(0 < \xi < \eta)$ for which

$$\phi'(\eta) - \phi'(0) = \phi''(\xi)\,(\eta-0) = \phi''(\xi)\,\eta$$

or, since $\phi'(0) = 0$,

$$\phi'(\eta) = \phi''(\xi)\,\eta$$

whence Eq. 2 becomes

$$\phi(\tau) - \phi(0) = \phi''(\xi)\,\eta\tau; \qquad 0 < \xi < \eta < \tau \tag{3}$$

Noting that $\phi(0) \geqslant \phi(\tau)$, Eq. 3 may be rewritten

$$|\phi(\tau) - \phi(0)| = \phi(0) - \phi(\tau) = |\phi''(\xi)|\,\eta\tau$$

But, as noted above, $|\phi''(\tau)| \leqslant |\phi''(0)|$ for all τ. Therefore

$$\phi(0) - \phi(\tau) \leqslant |\phi''(0)|\,\eta\tau \leqslant |\phi''(0)|\,\tau^2$$

since $\eta < \tau$. Thus, setting $k = |\phi''(0)|$, we have

$$\phi(0) - \phi(\tau) \leqslant k\tau^2 \qquad \text{for all } \tau \tag{4}$$

which was to be shown.

Condition 4 places a restriction on the rapidity with which $\phi(\tau)$ may depart from its value at the origin. If $\phi(\tau)$ is normalized so that $\phi(0) = 1$, and if we recall that $|\phi(\tau)| \leqslant |\phi(0)|$, then any measured autocorrelation is restricted to lie in a region such as that shown shaded in Fig. VIII-9. Evidently, for $|\tau| > \tau_o = (2/k)^{1/2}$ the condition of Eq. 4 becomes useless. In fact, its usefulness is limited to the region in the vicinity of $\tau = 0$.

The theorem may be extended to time series somewhat more general than measurable time series. The important requirement, to make the theorem meaningful, is that

$$k = |\phi''(0)|$$

be finite; for this it is necessary that $\phi'(\tau)$ be continuous at $\tau = 0$. To determine

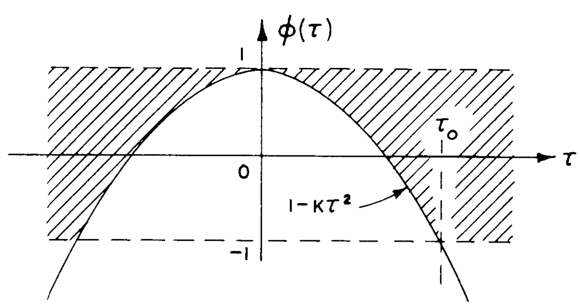

Fig. VIII-9
The region (shaded) in which the points of a measured $\phi(\tau)$ must lie.

whether or not this requirement is met by an arbitrary time series, the following theorem, proved in collaboration with Mr. Powers of this group, is useful.

Theorem. A sufficient condition that $\phi'(\tau)$ be continuous at $\tau = 0$ is that

$$\int_{-\infty}^{\infty} \omega^2 \, \Phi(\omega) \, d\omega \quad \text{exists.}$$

As usual, $\Phi(\omega)$ is the power-density spectrum of the time series.

Proof. By definition, $\phi'(\tau)$ will be continuous at the origin if, given $\epsilon > 0$, there exists a $\delta > 0$ that satisfies the condition

$$\left|\phi'(\tau) - \phi'(0)\right| < \epsilon$$

for all $|\tau| \leqslant \delta$. Now

$$\left|\phi'(\tau) - \phi'(0)\right| = \left| - \int_{-\infty}^{\infty} \omega \, \Phi(\omega) \sin \omega\tau \, d\omega - 0 \right| = \left| \int_{-\infty}^{\infty} \omega \, \Phi(\omega) \sin \omega\tau \, d\omega \right|$$

$$= \left| \tau \int_{-\infty}^{\infty} \omega^2 \, \Phi(\omega) \frac{\sin \omega\tau}{\omega\tau} \, d\omega \right| \leqslant \delta \left| \int_{-\infty}^{\infty} \omega^2 \, \Phi(\omega) \frac{\sin \omega\tau}{\omega\tau} \, d\omega \right|$$

$$\leqslant \delta \int_{-\infty}^{\infty} \left| \omega^2 \, \Phi(\omega) \frac{\sin \omega\tau}{\omega\tau} \right| d\omega = \delta \int_{-\infty}^{\infty} \omega^2 \, \Phi(\omega) \left| \frac{\sin \omega\tau}{\omega\tau} \right| d\omega$$

since both ω^2 and $\Phi(\omega)$ are always positive real. But since

$$\left| \frac{\sin \omega\tau}{\omega\tau} \right| \leqslant 1$$

$$\left|\phi'(\tau) - \phi'(0)\right| \leqslant \delta \int_{-\infty}^{\infty} \omega^2 \, \Phi(\omega) \, d\omega$$

Thus if the integral on the right exists, and equals A, say,

$$\left|\phi'(\tau) - \phi'(0)\right| \leqslant A\delta$$

$$< \epsilon \qquad \text{for all } \delta < \frac{\epsilon}{A}$$

which was to be shown.

R. E. Wernikoff

D. MEASUREMENT OF INDUSTRIAL PROCESS BEHAVIOR

Practical determination of transfer functions by correlation involves several basic difficulties. These are (a) input time functions are sometimes not statistically stationary; (b) the correlation functions measured in practice do not approach final values rapidly enough to allow Fourier transforms to be computed accurately; (c) solution of the integral equation

$$\phi_{io}(\tau) = \int_0^\infty h(\lambda)\ \phi_{ii}(\tau-\lambda)\ d\lambda$$

for the impulse response, h(t), presents serious difficulties in practice; (d) whereas definitions of correlation functions require integrations between infinite limits, only a finite length of data is available in practice. Simple methods for overcoming these difficulties are being developed and applied to the heat exchange system described in the Quarterly Progress Report, October 15, 1954, pages 75-76.

The nonstationary nature of the input time function imposes no restriction on the accuracy with which the transfer function can be determined if a change is made in the definition of the correlation functions. If the input θ_i and the output θ_o of a stable linear system with impulse response h(t) are replaced by the step approximations shown in Fig. VIII-10, the convolution integral

$$\theta_o(t) = \int_0^\infty h(\lambda)\ \theta_i(t-\lambda)\ d\lambda \qquad (1)$$

is replaced by the summation

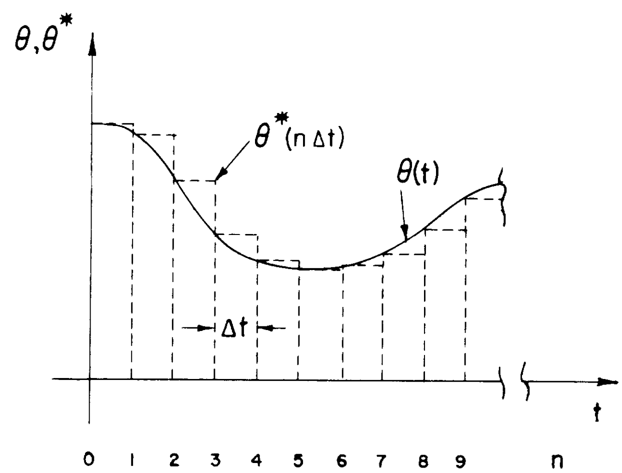

Fig. VIII-10
A step approximation to a continuous time function.

$$\theta_o^*(n\Delta t) = \theta_i^*(n\Delta t)\int_0^{\Delta t} h(\lambda)\, d\lambda + \theta_i^*([n-1]\,\Delta t)\int_{\Delta t}^{2\Delta t} h(\lambda)\, d\lambda + \dots$$

$$+ \theta_i^*([n-K]\,\Delta t)\int_{K\Delta t}^{(K+1)\Delta t} h(\lambda)\, d\lambda \qquad (2)$$

where the θ^* denotes a step approximation. It is assumed that $h(\lambda) = 0$ for $\lambda > (K+1)\,\Delta t$ and $n \geqslant K + 1$. We now define an experimentally determined "autocorrelation function" as

$$\phi_{ii}^*(m\,\Delta t) = \frac{1}{N}\sum_{n=m_1}^{N-m_2} \theta_i^*(n\,\Delta t)\;\theta_i^*([n+m]\,\Delta t) \qquad (3)$$

and an experimentally determined "crosscorrelation function" as

$$\phi_{io}^*(m\Delta t) = \frac{1}{N}\sum_{n=m_1}^{N-m_2} \theta_i^*(n\Delta t)\;\theta_o^*([n+m]\,\Delta t) \qquad (4)$$

where $N\Delta t$ is the total length of the experimental data. It should be noted that the "autocorrelation function" computed according to Eq. 3 is not an even function, but that Eqs. 3 and 4 reduce to the usual definitions when Δt is small, and $N\Delta t$ is large.

Replacing θ_o^* in Eq. 4 by Eq. 2, we obtain

$$\phi_{io}^*(m\Delta t) = \frac{1}{N}\sum_{n=m_1}^{N-m_2} \theta_i^*(n\Delta t)\sum_{k=0}^{K} \theta_i^*(n+m-k)\,\Delta t\; I(k\Delta t) \qquad (5)$$

where $I(k\,\Delta t)$ has been written for

$$\int_{k\,\Delta t}^{(k+1)\,\Delta t} h(\lambda)\, d\lambda$$

Interchanging the order of summation gives

$$\phi_{io}^*(m\,\Delta t) = \sum_{k=0}^{K} \phi_{ii}^*(m-k)\,\Delta t\; I(k\,\Delta t) \qquad (-m_1 + K < m < m_2) \qquad (6)$$

This result is independent of the statistical properties of θ_i^* and requires only that N be larger than $m_1 + m_2$. The primary effect of increasing N is a reduction in errors caused by disturbances which add to the output. Equation 6 can be expanded into a set of simultaneous equations that can, at least in principle, be solved for each $I(k\Delta t)$. The solution of Eq. 6 can be used to form a step approximation to h(t); a Fourier transformation gives $H(\omega)$. In Fig. VIII-11 the transfer function $H(\omega)$ obtained by transforming the solution of Eq. 6 is compared with the results of conventional measurements in which a sinusoidal variation in input temperature was introduced.

If the autocorrelation function of the input to the system is an impulse (the input is white noise), the solution of the set of Eq. 6 is not required because the crosscorrelation function between input and output has a form identical with the impulse response of the system. The proper use of filters allows the same results to be obtained as if the system were excited by white noise without altering the actual excitation of the system.

Identical linear filters are connected to the input and output of the system under study, as shown in Fig. VIII-12. Each filter has a transfer function $H_f(\omega)$. For sinusoidal excitation

$$\Theta_1 = H_f \Theta_i; \quad \Theta_2 = H\, H_f \Theta_i = H\, \Theta_1 \tag{7}$$

Equation 7 implies that θ_1 and θ_2, as well as ϕ_{11} and ϕ_{12}, are related through h(t) in precisely the same way as the input and output of the system itself:

$$\theta_2(t) = \int_0^\infty h(\lambda)\ \theta_1(t-\lambda)\ d\lambda \tag{8}$$

$$\phi_{12}(\tau) = \int_0^\infty h(\lambda)\ \phi_{11}(\tau-\lambda)\ d\lambda \tag{9}$$

The power density spectrum of θ_1 is

$$\Phi_{11} = |H_f|^2\ \Phi_{ii} \tag{10}$$

Ideally, $|H_f|^2$ would be designed to make Φ_{11} a constant independent of frequency. In practice the lack of exact knowledge of Φ_{ii} and the presence of noise in the recording equipment limit the frequency range over which Φ_{11} can be held constant. The transfer function H_f need not be realized in a physical filter, but can be incorporated in the digital computer program used to compute ϕ_{11} and ϕ_{12}.

The method for processing experimental data is diagrammed in Fig. VIII-13(a); the resulting ϕ_{ii} and ϕ_{11} are shown in Fig. VIII-14. The convolution corresponding to the first stage of digital filtering shown in Fig. VIII-13(a) was performed on I. B. M. equipment by the M. I. T. Office of Statistical Services. An examination of ϕ_{11} shows that it contains an oscillating component which should be removed by a further stage of digital

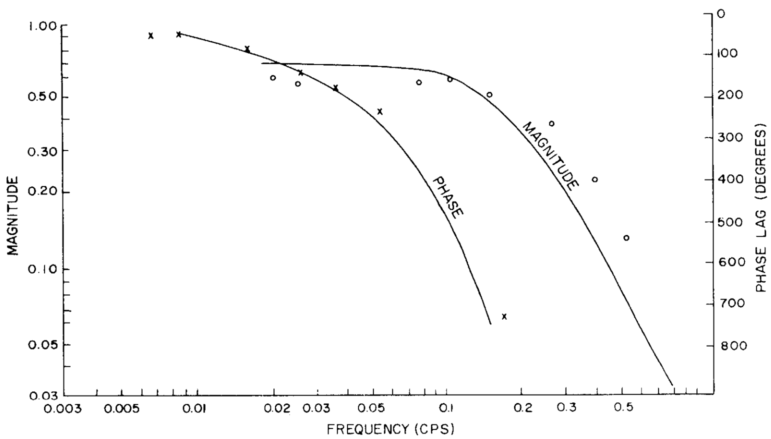

Fig. VIII-11

Transfer function of heat exchanger computed from correlation measurements (solid lines) and measured with sinusoidal excitation (points).

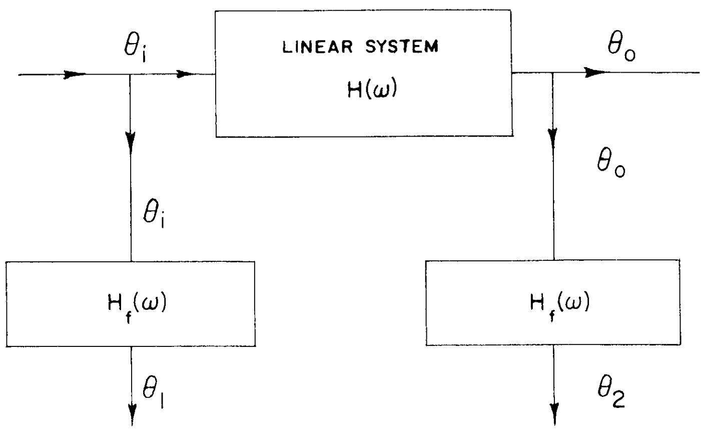

Fig. VIII-12

The use of filters to improve the form of the autocorrelation function.

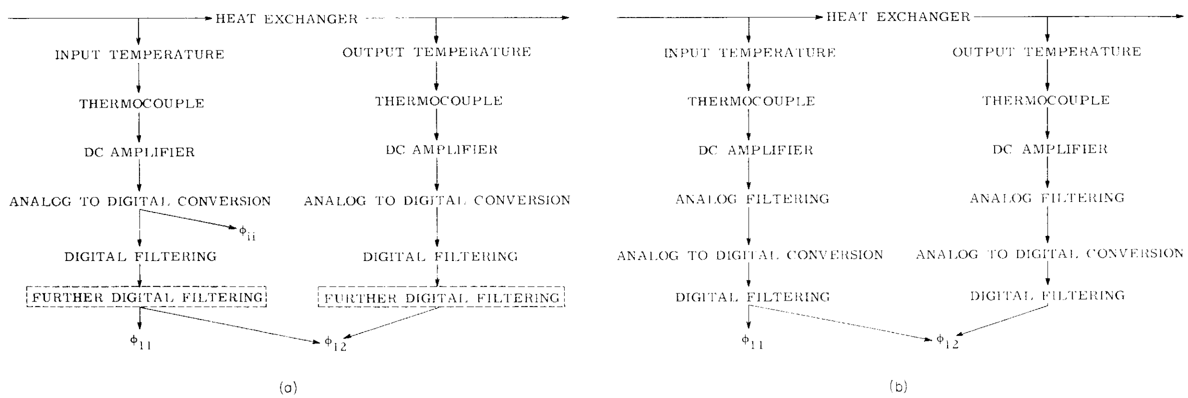

Fig. VIII-13

(a) Use of two stages of digital filtering to improve form of ϕ_{ii}; (b) use of combination of analog and digital filtering to improve form of ϕ_{ii}.

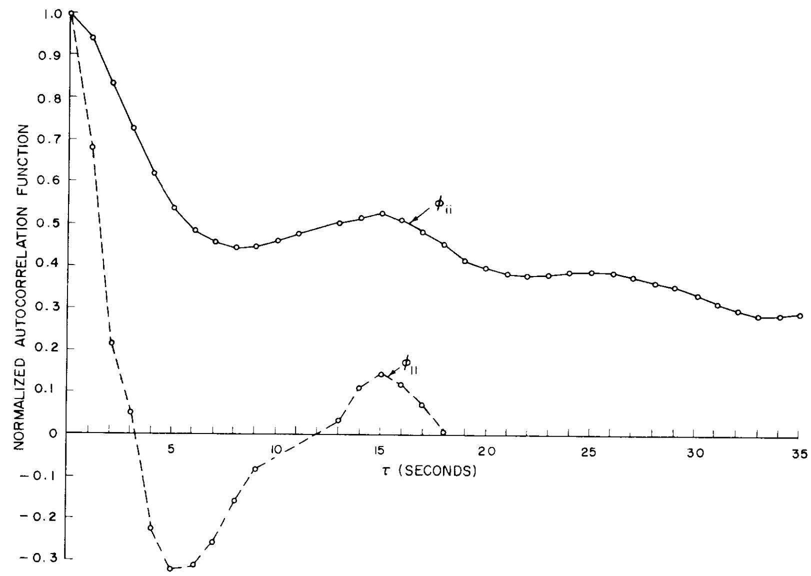

Fig. VIII-14

The effect of data filtering on the shape of the autocorrelation function.

filtering; the further filtering could not be carried out because of the loss of significant digits in the first stage. Loss of digits is avoided by use of the method of Fig. VIII-13(b), in which analog-to-digital conversion follows a stage of conventional analog filtering. An experiment using this method has proceeded through the analog-to-digital conversion stage.

The experimental data were obtained in the Process Control Laboratory through the cooperation of Professor D. P. Campbell and Professor L. A. Gould.

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